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HYBRID MEAN VALUE RESULTS FOR A GENERALIZATION ON A PROBLEM OF D.H. LEHMER AND HYPER-KLOOSTERMAN SUMS

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Abstract

The main purpose of this paper is by using the Fourier expansion for character sums and the mean value theorems of Dirichlet L -functions to give some hybrid mean value results for a generalization on a problem of D.H. Lehmer and hyper-Kloosterman sums.

1. Introduction

Let $q > 2$ and c be two integers with $(c, q) = 1$. For each integer a with $1 \leq a \leq q$ and $(a, q) = 1$, we know that there exists one and only one b with $1 \leq b \leq q$ such that $ab \equiv c \pmod{q}$. Let

$$L(q, k, c) = \sum_{\substack{a=1 \\ ab \equiv c \pmod{q}}}^q \sum_{b=1}^q (a-b)^{2k},$$

where \sum'_a denotes the summation over all a such that $(a, q) = 1$. In reference [1], the second author used the estimates for Kloosterman sums and trigonometric sums to obtain a sharp asymptotic formula for $L(q, k, c)$, and prove the following:

Proposition 1. *Let $q > 2$ and c be two integers with $(c, q) = 1$. Then for any positive integer k , we have the asymptotic formula*

$$L(q, k, c) = \frac{1}{(2k+1)(k+1)} \phi(q) q^{2k} + O(4^k q^{(4k+1)/2} d^2(q) \ln^2 q),$$

where $\phi(q)$ is the Euler function, and $d(q)$ is the divisor function.

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The error terms in Proposition 1 is best possible. In fact for $k = 1$, let

$$L(q, 1, c) = \frac{1}{6}\phi(q)q^2 + \frac{1}{3}q \prod_{p|q} (1 - p) + G(q, c),$$

where $\prod_{p|q}$ denotes the product over all distinct prime divisors of q , the second author [2] used the properties of Dedekind sums and Cochrane sums to give a sharp mean value formula for $G(q, c)$. That is the following:

Proposition 2. *For any integer $q > 2$, we have the asymptotic formula*

$$\sum_{c=1}^q G^2(q, c) = \frac{5}{36}q^3 \phi^3(q) \prod_{p^\alpha \parallel q} \frac{(p+1)^3/(p(p^2+1)) - 1/(p^{3\alpha-1})}{1 + 1/p + 1/p^2} + O\left(q^5 \exp\left(\frac{4 \ln q}{\ln \ln q}\right)\right),$$

where $\exp(y) = e^y$, $\prod_{p^\alpha \parallel q}$ denotes the product over all prime divisors p of q with $p^\alpha | q$ and $p^{\alpha+1} \nmid q$.

Let $M(q, c)$ be the number of cases in which a and b are of opposite parity. That is,

$$M(q, c) = \sum_{\substack{a=1 \\ ab \equiv c \pmod q \\ 2 \nmid a+b}}^q \sum_{b=1}^q 1.$$

For $q = p$ an odd prime and $c = 1$, D.H. Lehmer [3] asked us to find $M(p, 1)$ or at least to say something nontrivial about it. For the sake of simplicity, we call such a number as a D.H. Lehmer Number. In references [4] and [5], the second author proved that

$$(1) \quad M(q, 1) = \frac{1}{2}\phi(q) + O(q^{1/2}d^2(q)\ln^2 q).$$

For any nonnegative integer n , let

$$M(q, 1, n) = \sum_{\substack{a=1 \\ ab \equiv 1 \pmod q \\ 2 \nmid a+b}}^q \sum_{b=1}^q (a - b)^{2n},$$

the second author [6] also proved the following asymptotic formula:

$$M(q, 1, n) = \frac{1}{(2n+1)(2n+2)}\phi(q)q^{2n} + O(4^n q^{2n+1/2}d^2(q)\ln^2 q).$$

For any fixed positive integer c with $(c, q) = 1$, define

$$F(q, c) = M(q, c) - \frac{1}{2}\phi(q).$$

Then the second author showed in [7] and [8] that for any odd number $q > 2$,

$$\sum_{c=1}^q |F(q, c)|^2 = \frac{3}{4}\phi^2(q) \prod_{p^{\alpha} \parallel q} \frac{(p+1)^3/(p(p^2+1)) - 1/p^{3\alpha-1}}{1 + 1/p + 1/p^2} + O\left(q \exp\left(\frac{4 \ln q}{\ln \ln q}\right)\right).$$

This proved that the error terms in (1) is also best possible.

In [9], the second author found that there exists some close relation between the error terms $F(q, c)$ and the classical Kloosterman sums:

$$K(m, n; q) = \sum_{b=1}^q e\left(\frac{mb + n\bar{b}}{q}\right),$$

where $e(y) = e^{2\pi iy}$, \bar{b} is defined by the equation $b\bar{b} \equiv 1 \pmod q$, and obtained the following hybrid mean value formula:

$$\sum_{c=1}^q F(q, c)K(\bar{4}c, 1; q) = \frac{4}{\pi^2}q\phi(q) \prod_{p \parallel q} \left(1 - \frac{1}{p(p-1)}\right) + O(q^{3/2+\epsilon}),$$

where ϵ is any fixed positive number.

In [10], Mordell introduced the hyper-Kloosterman sums as follows:

$$K(h, k, q) = \sum_{\substack{a_1, \dots, a_k \pmod q \\ (a_1, q) \dots (a_k, q) = 1}} e\left(\frac{a_1 + \dots + a_k + h\bar{a}_1 \dots \bar{a}_k}{q}\right),$$

which is the high-dimensional generalization of the Kloosterman sums. Some applications of the hyper-Kloosterman sums were found in the estimation of Fourier coefficients of Maass forms [11] and the work on Selberg’s eigenvalue conjecture [12]. Moreover, there exists some interesting connections between the hyper-Kloosterman sums and the Heibronn sums (see reference [13]).

Now we consider a generalization on this problem of D.H. Lehmer. For any integer $k \geq 1$, let

$$N(q, k, c) = \sum_{a_1=1}^q \dots \sum_{a_k=1}^q \sum_{\substack{b=1 \\ a_1 \dots a_k b \equiv c \pmod q \\ 2 \nmid a_1 + \dots + a_k + b}}^q (a_1 + \dots + a_k - b)^2$$

and

$$E(q, k, c) = N(q, k, c) - \frac{(3k^2 - 5k + 4)}{24}\phi^k(q)q^2 - \frac{(k+1)}{12}\phi^{k-1}(q)q \prod_{p \mid q} (1 - p).$$

In this paper, we use the Fourier expansion for character sums and the mean value theorems of Dirichlet L -functions to study the hybrid mean value of $E(q, k, c)$ and the hyper-Kloosterman sums, and give an interesting mean value formula. That is, we shall prove the following:

Theorem. *For any odd number $q \geq 3$ and integer $k \geq 1$, we have the asymptotic formula*

$$\sum_{c=1}^q E(q, k, c) K(\overline{2}^{-k+1} c, k, q) = \frac{c_k q^{k+2} \phi(q)}{\pi^{k+3}} \prod_{p \parallel q} \left(1 - \frac{p^k - 1}{p^k(p-1)^2} \right) + O(q^{k+5/2+\epsilon}),$$

where ϵ is any fixed positive number, $\prod_{p \parallel q}$ denotes the product over all prime divisors p of q with $p \mid q$ and $p^2 \nmid q$, and

$$c_k = \begin{cases} -6, & \text{if } k = 1, \\ i^{k+3} 2^{2k-2} [\pi^2(k^2 - k + 2) - 8(k + 1)], & \text{otherwise.} \end{cases}$$

2. Several lemmas

To complete the proof of the theorem, we need the following lemmas.

Lemma 1. *Let $q \geq 3$ be an odd number. Then for any positive integer c with $(c, q) = 1$, we have*

$$\begin{aligned} E(q, 1, c) &= \frac{1}{\phi(q)} \sum_{\substack{\chi(-1)=1 \\ \chi \neq \chi_0}} \overline{\chi}(c) \left(\sum_{a=1}^q (-1)^a a \chi(a) \right)^2 \\ &+ \frac{1}{\phi(q)} \sum_{\chi(-1)=-1} \overline{\chi}(c) (4\chi(4) - 4\chi(2)) \left(\sum_{a=1}^q a \chi(a) \right)^2 \\ &- \frac{2}{\phi(q)q} \sum_{\chi(-1)=-1} \overline{\chi}(c) (1 - 2\chi(2)) \left(\sum_{a=1}^q a \chi(a) \right) \left(\sum_{b=1}^q (-1)^b b^2 \chi(b) \right) + O(q), \end{aligned}$$

where $\sum_{\substack{\chi(-1)=1 \\ \chi \neq \chi_0}}$ denotes the summation over all non-principal even characters modulo q , and $\sum_{\chi(-1)=-1}$ denotes the summation over all odd characters modulo q .

Proof. From the definition of $N(q, 1, c)$ we can get

$$N(q, 1, c) = \sum_{\substack{a=1 \\ ab \equiv c \pmod q \\ 2 \nmid a+b}}^q \sum_{b=1}^q (a-b)^2 = \frac{1}{2} \sum_{a=1}^q \sum_{\substack{b=1 \\ ab \equiv c \pmod q}}^q [1 - (-1)^{a+b}] (a-b)^2$$

$$= \frac{1}{2} \sum_{a=1}^q \sum'_{\substack{b=1 \\ ab \equiv c \pmod q}} (a-b)^2 - \frac{1}{2} \sum_{a=1}^q \sum'_{\substack{b=1 \\ ab \equiv c \pmod q}} (-1)^{a+b} (a-b)^2.$$

By the orthogonality relation for character sums mod q we easily deduce

$$\begin{aligned} &N(q, 1, c) \\ &= \frac{1}{12} \phi(q)q^2 + \frac{1}{6}q \prod_{p|q} (1-p) - \frac{1}{\phi(q)} \sum_{\chi(-1)=-1} \bar{\chi}(c) \left(\sum_{a=1}^q a\chi(a) \right)^2 \\ &\quad - \frac{1}{\phi(q)} \sum_{\chi \neq \chi_0} \bar{\chi}(c) \left[\left(\sum_{a=1}^q (-1)^a a^2 \chi(a) \right) \left(\sum_{b=1}^q (-1)^b \chi(b) \right) - \left(\sum_{a=1}^q (-1)^a a\chi(a) \right)^2 \right] \\ &\quad + O(q), \end{aligned}$$

where we have used the identities (see reference [14])

$$(2) \quad \sum_{a=1}^q a^2 = \frac{1}{3} \phi(q)q^2 + \frac{1}{6}q \prod_{p|q} (1-p), \quad \sum_{a=1}^q a = \frac{\phi(q)q}{2}$$

and

$$\sum_{a=1}^q (-1)^a = 0, \quad \sum_{a=1}^q (-1)^a a = -\frac{1}{2} \prod_{p|q} (1-p), \quad \sum_{a=1}^q (-1)^a a^2 = -\frac{1}{2}q \prod_{p|q} (1-p)$$

for any odd number $q \geq 3$.

Note that if $\chi(-1) = 1$ then

$$(3) \quad \sum_{b=1}^q (-1)^b \chi(b) = 0,$$

and if $\chi(-1) = -1$, then

$$(4) \quad \sum_{a=1}^q (-1)^a a\chi(a) = \frac{q}{2} \sum_{a=1}^q (-1)^a \chi(a), \quad \sum_{b=1}^q (-1)^b \chi(b) = 2\chi(2) \sum_{b=1}^{(q-1)/2} \chi(b).$$

From [15] we also know that for any odd character $\chi \pmod q$, we have

$$(5) \quad (1 - 2\chi(2)) \sum_{c=1}^q c\chi(c) = \chi(2)q \sum_{c=1}^{(q-1)/2} \chi(c).$$

So from the above formulae we can have

$$\begin{aligned}
 E(q, 1, c) &= \frac{1}{\phi(q)} \sum_{\substack{\chi(-1)=1 \\ \chi \neq \chi_0}} \bar{\chi}(c) \left(\sum_{a=1}^q (-1)^a a \chi(a) \right)^2 \\
 &\quad + \frac{1}{\phi(q)} \sum_{\chi(-1)=-1} \bar{\chi}(c) (4\chi(4) - 4\chi(2)) \left(\sum_{a=1}^q a \chi(a) \right)^2 \\
 &\quad - \frac{2}{\phi(q)q} \sum_{\chi(-1)=-1} \bar{\chi}(c) (1 - 2\chi(2)) \left(\sum_{a=1}^q a \chi(a) \right) \left(\sum_{b=1}^q (-1)^b b^2 \chi(b) \right) + O(q).
 \end{aligned}$$

This proves Lemma 1. \square

Lemma 2. *Let $q \geq 3$ be an odd number and $k \geq 2$ be an integer. Then for any positive integer c with $(c, q) = 1$, we have the identity*

$$\begin{aligned}
 E(q, k, c) &= \frac{-2^{k-2}}{\phi(q)q^{k-1}} \sum_{\chi(-1)=-1} \bar{\chi}(c) (1 - 2\chi(2))^k \left(\sum_{a=1}^q a \chi(a) \right)^k \\
 &\quad \times \left[\frac{2(k+1)}{q} \left(\sum_{b=1}^q (-1)^b b^2 \chi(b) \right) + k(k-3)(1 - 2\chi(2)) \left(\sum_{b=1}^q b \chi(b) \right) \right].
 \end{aligned}$$

Proof. From the definition of $N(q, k, c)$ we can get

$$\begin{aligned}
 N(q, k, c) &= \sum_{\substack{a_1=1 \\ a_1 \cdots a_k b \equiv c \pmod{q} \\ 2 \nmid a_1 + \cdots + a_k + b}}^{q'} \cdots \sum_{\substack{a_k=1 \\ a_1 \cdots a_k b \equiv c \pmod{q}}}^{q'} \sum_{b=1}^q (a_1 + \cdots + a_k - b)^2 \\
 &= \frac{1}{2} \sum_{\substack{a_1=1 \\ a_1 \cdots a_k b \equiv c \pmod{q}}}^{q'} \cdots \sum_{\substack{a_k=1 \\ a_1 \cdots a_k b \equiv c \pmod{q}}}^{q'} \sum_{b=1}^q [1 - (-1)^{a_1 + \cdots + a_k + b}] (a_1 + \cdots + a_k - b)^2 \\
 &= \frac{1}{2} \sum_{\substack{a_1=1 \\ a_1 \cdots a_k b \equiv c \pmod{q}}}^{q'} \cdots \sum_{\substack{a_k=1 \\ a_1 \cdots a_k b \equiv c \pmod{q}}}^{q'} \sum_{b=1}^q (a_1 + \cdots + a_k - b)^2 \\
 &\quad - \frac{1}{2} \sum_{\substack{a_1=1 \\ a_1 \cdots a_k b \equiv c \pmod{q}}}^{q'} \cdots \sum_{\substack{a_k=1 \\ a_1 \cdots a_k b \equiv c \pmod{q}}}^{q'} \sum_{b=1}^q (-1)^{a_1 + \cdots + a_k + b} (a_1 + \cdots + a_k - b)^2 \\
 &:= P(q, k, c) + E(q, k, c).
 \end{aligned}$$

Then by (2) and the properties of character sums mod q we have

$$\begin{aligned}
 P(q, k, c) &= \frac{1}{2} \sum'_{a_1=1}^q \cdots \sum'_{a_k=1}^q \sum'_{b=1}^q (a_1 + \cdots + a_k - b)^2 \\
 &\quad a_1 \cdots a_k b \equiv c \pmod{q} \\
 &= \frac{1}{2\phi(q)} \sum_{\chi \pmod{q}} \bar{\chi}(c) \sum'_{a_1=1}^q \cdots \sum'_{a_k=1}^q \sum'_{b=1}^q \chi(a_1) \cdots \chi(a_k) \chi(b) (a_1 + \cdots + a_k - b)^2 \\
 &= \frac{1}{2\phi(q)} \sum'_{a_1=1}^q \cdots \sum'_{a_k=1}^q \sum'_{b=1}^q (a_1 + \cdots + a_k - b)^2 \\
 &= \frac{1}{2\phi(q)} \sum'_{a_1=1}^q \cdots \sum'_{a_k=1}^q \sum'_{b=1}^q \left(\sum_{1 \leq i \leq k} a_i^2 + b^2 + 2 \sum_{1 \leq i < j \leq k} a_i a_j - 2 \sum_{1 \leq i \leq k} a_i b \right) \\
 &= \frac{(k+1)}{2\phi(q)} \sum'_{a_1=1}^q \cdots \sum'_{a_k=1}^q \sum'_{b=1}^q b^2 \\
 &\quad + \frac{1}{\phi(q)} ((k-1) + (k-2) + \cdots + 1 - k) \sum'_{a_1=1}^q \cdots \sum'_{a_k=1}^q \sum'_{b=1}^q a_1 b \\
 &= \frac{(k+1)}{2\phi(q)} \sum'_{a_1=1}^q \cdots \sum'_{a_k=1}^q \sum'_{b=1}^q b^2 + \frac{k(k-3)}{2\phi(q)} \sum'_{a_1=1}^q \cdots \sum'_{a_k=1}^q \sum'_{b=1}^q a_1 b \\
 &= \frac{(k+1)\phi^{k-1}(q)}{2} \left[\frac{1}{3}\phi(q)q^2 + \frac{1}{6}q \prod_{p|q} (1-p) \right] + \frac{k(k-3)}{8}\phi^k(q)q^2 \\
 &= \frac{(3k^2 - 5k + 4)}{24}\phi^k(q)q^2 + \frac{(k+1)}{12}\phi^{k-1}(q)q \prod_{p|q} (1-p).
 \end{aligned}$$

On the other hand, from (3), (4) and (5) we also have

$$\begin{aligned}
 E(q, k, c) &= -\frac{1}{2} \sum'_{a_1=1}^q \cdots \sum'_{a_k=1}^q \sum'_{b=1}^q (-1)^{a_1 + \cdots + a_k + b} (a_1 + \cdots + a_k - b)^2 \\
 &\quad a_1 \cdots a_k b \equiv c \pmod{q} \\
 &= -\frac{1}{2\phi(q)} \sum_{\chi \pmod{q}} \bar{\chi}(c) \sum'_{a_1=1}^q \cdots \sum'_{a_k=1}^q \sum'_{b=1}^q (-1)^{a_1 + \cdots + a_k + b} \\
 &\quad \times \chi(a_1) \cdots \chi(a_k) \chi(b) (a_1 + \cdots + a_k - b)^2
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{-1}{2\phi(q)} \sum_{\chi(-1)=-1} \bar{\chi}(c) \left[(k+1) \left(\sum_{a=1}^q (-1)^a \chi(a) \right)^k \left(\sum_{b=1}^q (-1)^b b^2 \chi(b) \right) \right. \\
 &\quad \left. + k(k-3) \left(\sum_{a=1}^q (-1)^a a \chi(a) \right)^2 \left(\sum_{b=1}^q (-1)^b \chi(b) \right)^{k-1} \right] \\
 &= \frac{-2^{k-2}}{\phi(q)q^{k-1}} \sum_{\chi(-1)=-1} \bar{\chi}(c) (1-2\chi(2))^k \left(\sum_{a=1}^q a \chi(a) \right)^k \\
 &\quad \times \left[\frac{2(k+1)}{q} \left(\sum_{b=1}^q (-1)^b b^2 \chi(b) \right) + k(k-3)(1-2\chi(2)) \left(\sum_{b=1}^q b \chi(b) \right) \right].
 \end{aligned}$$

This completes the proof of Lemma 2. □

Lemma 3. *Let χ be a primitive character modulo m , and let $m \geq 3$ be an odd number. Then we have*

$$\begin{aligned}
 \sum_{a=1}^m a \chi(a) &= \frac{mi}{\pi} \tau(\chi) L(1, \bar{\chi}), \quad \text{if } \chi(-1) = -1; \\
 \sum_{a=1}^m (-1)^a a \chi(a) &= \frac{m\tau(\chi)(1-4\chi(2))}{\pi^2} L(2, \bar{\chi}) + O(m), \quad \text{if } \chi(-1) = 1; \\
 \sum_{a=1}^m (-1)^a a^2 \chi(a) &= \frac{m^2(1-2\chi(2))i}{\pi} \tau(\chi) L(1, \bar{\chi}) \\
 &\quad + \frac{m^2(8\chi(2)-1)i}{\pi^3} \tau(\chi) L(3, \bar{\chi}) + O(m^2), \quad \text{if } \chi(-1) = -1,
 \end{aligned}$$

where $L(1, \chi)$ is the Dirichlet L -function corresponding to χ , $\tau(\chi) = \sum_{a=1}^m \chi(a) \epsilon(a/m)$ is the Gauss sum, and $|\tau(\chi)| = \sqrt{m}$.

Proof. From Theorem 12.11 and Theorem 12.20 of [14] we know that if χ is a primitive character modulo m with $\chi(-1) = -1$, then

$$(6) \quad \frac{1}{m} \sum_{b=1}^m b \chi(b) = \frac{i}{\pi} \tau(\chi) L(1, \bar{\chi}),$$

so we get the first formula.

Now we prove the third formula, since similarly we can deduce the second one. For any odd primitive character χ modulo m , we have

$$\begin{aligned}
 \sum_{a=1}^m (-1)^a a^2 \chi(a) &= \sum_{\substack{a=1 \\ 2|a}}^m a^2 \chi(a) - \sum_{\substack{a=1 \\ 2 \nmid a}}^m a^2 \chi(a) = \sum_{\substack{a=1 \\ 2|a}}^m a^2 \chi(a) - \sum_{\substack{a=1 \\ 2|a}}^m (m-a)^2 \chi(m-a) \\
 (7) \qquad \qquad \qquad &= 2 \sum_{\substack{a=1 \\ 2|a}}^m a^2 \chi(a) - 2m \sum_{\substack{a=1 \\ 2|a}}^m a \chi(a) + m^2 \sum_{\substack{a=1 \\ 2|a}}^m \chi(a).
 \end{aligned}$$

Note that

$$\sum_{a=1}^m a \chi(a) = \sum_{\substack{a=1 \\ 2|a}}^m a \chi(a) + \sum_{\substack{a=1 \\ 2 \nmid a}}^m (m-a) \chi(m-a) = 2 \sum_{\substack{a=1 \\ 2|a}}^m a \chi(a) - m \sum_{\substack{a=1 \\ 2|a}}^m \chi(a),$$

so from (6) and (7) we can get

$$\begin{aligned}
 \sum_{a=1}^m (-1)^a a^2 \chi(a) &= 2 \sum_{\substack{a=1 \\ 2|a}}^m a^2 \chi(a) - m \sum_{a=1}^m a \chi(a) \\
 (8) \qquad \qquad \qquad &= 8\chi(2) \sum_{1 \leq a \leq m/2} a^2 \chi(a) - \frac{m^2 i}{\pi} \tau(\chi) L(1, \bar{\chi}).
 \end{aligned}$$

Notice the Fourier expansion for character sums which was first given by Pólya [16]:

$$\sum_{0 < n \leq my} \chi(n) = \begin{cases} \frac{\tau(\chi)}{\pi} \sum_{n=1}^{\infty} \frac{\bar{\chi}(n) \sin(2\pi ny)}{n} + O(1), & \text{if } \chi(-1) = 1; \\ \frac{\tau(\chi)L(1, \bar{\chi})}{\pi i} - \frac{\tau(\chi)}{\pi i} \sum_{n=1}^{\infty} \frac{\bar{\chi}(n) \cos(2\pi ny)}{n} + O(1), & \text{if } \chi(-1) = -1, \end{cases}$$

where χ is a primitive character modulo m , and $y > 0$ is a real number. Then by Abel's identity we have

$$\begin{aligned}
 \sum_{0 < n \leq m/2} n^2 \chi(n) &= \frac{m^2}{4} \sum_{0 < n \leq m/2} \chi(n) - 2 \int_0^{m/2} u \sum_{0 < n \leq u} \chi(n) du \\
 &= \frac{m^2(\bar{\chi}(2) - 2)i}{4\pi} \tau(\chi)L(1, \bar{\chi}) - 2 \int_0^{m/2} u \sum_{0 < n \leq u} \chi(n) du + O(m^2) \\
 &= \frac{m^2(\bar{\chi}(2) - 2)i}{4\pi} \tau(\chi)L(1, \bar{\chi}) - 2m^2 \int_0^{1/2} s \sum_{0 < n \leq ms} \chi(n) ds + O(m^2)
 \end{aligned}$$

$$\begin{aligned}
&= \frac{m^2(\bar{\chi}(2) - 2)i}{4\pi} \tau(\chi)L(1, \bar{\chi}) \\
&\quad - 2m^2 \int_0^{1/2} s \left[\frac{\tau(\chi)L(1, \bar{\chi})}{\pi i} - \frac{\tau(\chi)}{\pi i} \sum_{n=1}^{\infty} \frac{\bar{\chi}(n) \cos(2\pi ns)}{n} + O(1) \right] ds \\
&\quad + O(m^2) \\
&= \frac{m^2(\bar{\chi}(2) - 1)i}{4\pi} \tau(\chi)L(1, \bar{\chi}) + 2m^2 \frac{\tau(\chi)}{\pi i} \sum_{n=1}^{\infty} \frac{\bar{\chi}(n)}{n} \int_0^{1/2} s \cos(2\pi ns) ds \\
&\quad + O(m^2) \\
&= \frac{m^2(\bar{\chi}(2) - 1)i}{4\pi} \tau(\chi)L(1, \bar{\chi}) + \frac{m^2 \tau(\chi)}{2\pi^3 i} \sum_{n=1}^{\infty} \frac{\bar{\chi}(n)(\cos(\pi n) - 1)}{n^3} + O(m^2) \\
&= \frac{m^2(\bar{\chi}(2) - 1)i}{4\pi} \tau(\chi)L(1, \bar{\chi}) - \frac{m^2 \tau(\chi)}{\pi^3 i} \sum_{\substack{n=1 \\ 2|n}}^{\infty} \frac{\bar{\chi}(n)}{n^3} + O(m^2) \\
&= \frac{m^2(\bar{\chi}(2) - 1)i}{4\pi} \tau(\chi)L(1, \bar{\chi}) + \frac{m^2(\bar{\chi}(2) - 8)}{8\pi^3 i} \tau(\chi)L(3, \bar{\chi}) + O(m^2).
\end{aligned}$$

Hence from (8) we have

$$\sum_{a=1}^m (-1)^a a^2 \chi(a) = \frac{m^2(1 - 2\chi(2))i}{\pi i} \tau(\chi)L(1, \bar{\chi}) + \frac{m^2(8\chi(2) - 1)i}{\pi^3} \tau(\chi)L(3, \bar{\chi}) + O(m^2).$$

This proves Lemma 3. \square

Lemma 4. *Suppose that χ is an even character modulo q , generated by the primitive character χ_m modulo m , and $q \geq 3$ is an odd number. Let l be the largest divisor of q with $(l, m) = 1$. Then we have*

$$\sum_{a=1}^q (-1)^a a \chi(a) = \frac{m\tau(\chi_m)(1 - 4\chi_m(2))}{\pi^2} L(2, \bar{\chi}_m) \left(\sum_{d|l} d\mu(d)\chi_m(d) \right) + O(q),$$

where $\mu(n)$ is the Möbius function.

Proof. Note that m and l both are odd numbers, then from (3) we have

$$\begin{aligned}
\sum_{a=1}^q (-1)^a a \chi(a) &= \sum_{i=0}^{(q/ml)-1} \sum_{j=1}^{ml} (-1)^{iml+j} (iml+j) \chi(iml+j) \\
&= \sum_{i=0}^{(q/ml)-1} (-1)^i \sum_{j=1}^{ml} (-1)^j (iml+j) \chi(iml+j) = \sum_{i=0}^{(q/ml)-1} (-1)^i \sum_{j=1}^{ml} (-1)^j j \chi(j)
\end{aligned}$$

$$\begin{aligned} &= \sum_{j=1}^{ml} (-1)^j j \chi(j) = \sum_{\substack{a=1 \\ (a,l)=1}}^{ml} (-1)^a a \chi_m(a) = \sum_{a=1}^{ml} (-1)^a a \chi_m(a) \sum_{\substack{d|a \\ d|l}} \mu(d) \\ &= \sum_{d|l} \mu(d) \sum_{\substack{a=1 \\ d|a}}^{lm} (-1)^a a \chi_m(a) = \sum_{d|l} \mu(d) d \chi_m(d) \sum_{b=1}^{lm/d} (-1)^{db} b \chi_m(b). \end{aligned}$$

Since d is odd, we have $(-1)^{db} = (-1)^b$. Therefore

$$\begin{aligned} \sum_{a=1}^q (-1)^a a \chi(a) &= \sum_{d|l} \mu(d) d \chi_m(d) \sum_{b=1}^{lm/d} (-1)^b b \chi_m(b) \\ &= \sum_{d|l} \mu(d) d \chi_m(d) \sum_{i=0}^{(l/d)-1} \sum_{j=1}^m (-1)^{im+j} (im+j) \chi_m(im+j) \\ &= \sum_{d|l} \mu(d) d \chi_m(d) \sum_{i=0}^{(l/d)-1} (-1)^i \sum_{j=1}^m (-1)^j (im+j) \chi_m(j) \\ &= \sum_{d|l} \mu(d) d \chi_m(d) \sum_{j=1}^m (-1)^j j \chi_m(j). \end{aligned}$$

Note that

$$\left| \sum_{d|l} \mu(d) d \chi_m(d) \right| = \left| \prod_{p|l} (1 - p \chi_m(p)) \right| < \prod_{p|l} p \leq l,$$

so from Lemma 3 we get

$$\sum_{a=1}^q (-1)^a a \chi(a) = \frac{m \tau(\chi_m)(1 - 4\chi_m(2))}{\pi^2} L(2, \overline{\chi}_m) \left(\sum_{d|l} d \mu(d) \chi_m(d) \right) + O(q).$$

This completes the proof of Lemma 4. □

Lemma 5. *Suppose that χ is an odd character modulo q , generated by the primitive character χ_m modulo m , and $q \geq 3$ is an odd number. Let l be the largest divisor of q with $(l, m) = 1$. Then we have*

$$\sum_{a=1}^q a \chi(a) = \frac{qi}{\pi} \tau(\chi_m) L(1, \overline{\chi}_m) \left(\sum_{d|l} \mu(d) \chi_m(d) \right).$$

Furthermore, for $q = lm$, we also have

$$\begin{aligned} \sum_{a=1}^q (-1)^a a^2 \chi(a) &= \frac{q^2(1 - 2\chi_m(2))i}{\pi} \tau(\chi_m)L(1, \bar{\chi}_m) \left(\sum_{d|l} \mu(d)\chi_m(d) \right) \\ &\quad + \frac{m^2(8\chi_m(2) - 1)i}{\pi^3} \tau(\chi_m)L(3, \bar{\chi}_m) \left(\sum_{d|l} d^2 \mu(d)\chi_m(d) \right) + O(q^2). \end{aligned}$$

Proof. The first formula can be easily deduced from Lemma 6 of [5] and Lemma 3. Now let $q = lm$, we have

$$\begin{aligned} \sum_{a=1}^q (-1)^a a^2 \chi(a) &= \sum_{a=1}^{lm} (-1)^a a^2 \chi(a) = \sum_{\substack{a=1 \\ (a,l)=1}}^{lm} (-1)^a a^2 \chi_m(a) = \sum_{d|l} \mu(d) \sum_{\substack{a=1 \\ d|a}}^{lm} (-1)^a a^2 \chi_m(a) \\ &= \sum_{d|l} \mu(d) d^2 \chi_m(d) \sum_{b=1}^{lm/d} (-1)^{db} b^2 \chi_m(b) \\ &= \sum_{d|l} d^2 \mu(d) \chi_m(d) \sum_{b=1}^{lm/d} (-1)^b b^2 \chi_m(b). \end{aligned}$$

Since

$$\begin{aligned} \sum_{b=1}^{lm/d} (-1)^b b^2 \chi_m(b) &= \sum_{i=0}^{(l/d)-1} \sum_{j=1}^m (-1)^{im+j} (im+j)^2 \chi_m(im+j) \\ &= \sum_{i=0}^{(l/d)-1} (-1)^i \sum_{j=1}^m (-1)^j (i^2 m^2 + 2imj + j^2) \chi_m(j) \\ &= m^2 \sum_{i=0}^{(l/d)-1} (-1)^i i^2 \sum_{j=1}^m (-1)^j \chi_m(j) + 2m \sum_{i=0}^{(l/d)-1} (-1)^i i \sum_{j=1}^m (-1)^j j \chi_m(j) \\ &\quad + \sum_{i=0}^{(l/d)-1} (-1)^i \sum_{j=1}^m (-1)^j j^2 \chi_m(j) \\ &= \frac{m^2(l-d)}{2d^2} \sum_{j=1}^m (-1)^j \chi_m(j) \\ &\quad + \frac{m(l-d)}{d} \sum_{j=1}^m (-1)^j j \chi_m(j) + \sum_{j=1}^m (-1)^j j^2 \chi_m(j), \end{aligned}$$

and note that

$$\left| \sum_{d|l} d^2 \mu(d) \chi_m(d) \right| = \left| \prod_{p|l} (1 - p^2 \chi_m(p)) \right| < \prod_{p|l} p^2 \leq l^2,$$

so from (4), (5), (6) and Lemma 3 we have

$$\begin{aligned} \sum_{a=1}^q (-1)^a a^2 \chi(a) &= \frac{q^2}{2} \left(\sum_{d|l} \mu(d) \chi_m(d) \right) \left(\sum_{j=1}^m (-1)^j \chi_m(j) \right) \\ &\quad - \frac{m^2}{2} \left(\sum_{d|l} d^2 \mu(d) \chi_m(d) \right) \left(\sum_{j=1}^m (-1)^j \chi_m(j) \right) \\ &\quad + \left(\sum_{d|l} d^2 \mu(d) \chi_m(d) \right) \left(\sum_{j=1}^m (-1)^j j^2 \chi_m(j) \right) \\ &= \frac{q^2(1 - 2\chi_m(2))i}{\pi} \tau(\chi_m) L(1, \bar{\chi}_m) \left(\sum_{d|l} \mu(d) \chi_m(d) \right) \\ &\quad + \frac{m^2(8\chi_m(2) - 1)i}{\pi^3} \tau(\chi_m) L(3, \bar{\chi}_m) \left(\sum_{d|l} d^2 \mu(d) \chi_m(d) \right) + O(q^2). \end{aligned}$$

This proves Lemma 5. □

Lemma 6. *Let χ be a character modulo q , generated by the primitive character χ_m modulo m . Then we have the identity*

$$\tau(\chi) = \chi_m\left(\frac{q}{m}\right) \mu\left(\frac{q}{m}\right) \tau(\chi_m).$$

Proof. See Lemma 1.3 of reference [17]. □

Lemma 7. *Let m and r be integers with $m \geq 2$ and $(r, m) = 1$, and let χ be a Dirichlet character modulo m . Then we have the identities*

$$\sum_{\chi \bmod m}^* \chi(r) = \sum_{d|(m, r-1)} \mu\left(\frac{m}{d}\right) \phi(d)$$

and

$$J(m) = \sum_{d|m} \mu(d) \phi\left(\frac{m}{d}\right),$$

where $\sum_{\chi \bmod m}^*$ denotes the summation over all primitive characters modulo m and $J(m)$ denotes the number of primitive characters modulo m .

Proof. This is Lemma 3 of [18]. □

Lemma 8. *Let $q = uv$, where $(u, v) = 1$, u be a square-full number or $u = 1$, v be a square-free number. Then we have the asymptotic formulae*

$$\begin{aligned} \Psi_1 &:= \sum_{d|v} u^{k+1} d^{k+1} \sum_{d_1|(v/d)} \cdots \sum_{d_{k+1}|(v/d)} \mu(d_1) \cdots \mu(d_{k+1}) \sum_{\substack{\chi \bmod ud \\ \chi(-1)=-1}}^* \bar{\chi}(d_1 \cdots d_{k+1}) L^{k+1}(1, \bar{\chi}) \\ &= \frac{q^k \phi^2(q)}{2} \prod_{p|q} \left(1 - \frac{p^k - 1}{p^k(p-1)^2} \right) + O(q^{k+1+\epsilon}); \\ \Psi_2 &:= \sum_{d|v} u^{k+1} d^{k+1} \sum_{d_1|(v/d)} \cdots \sum_{d_{k+1}|(v/d)} \frac{\mu(d_1) \cdots \mu(d_{k+1})}{d_1 \cdots d_{k+1}} \sum_{\substack{\chi \bmod ud \\ \chi(-1)=1}}^* \bar{\chi}(d_1 \cdots d_{k+1}) L^{k+1}(2, \bar{\chi}) \\ &= \frac{q^k \phi^2(q)}{2} \prod_{p|q} \left(1 - \frac{p^k - 1}{p^k(p-1)^2} \right) + O(q^{k+1+\epsilon}); \\ \Psi_3 &:= \sum_{d|v} u^{k+1} d^{k+1} \sum_{d_1|(v/d)} \cdots \sum_{d_{k+1}|(v/d)} \frac{\mu(d_1) \cdots \mu(d_{k+1})}{d_{k+1}^2} \sum_{\substack{\chi \bmod ud \\ \chi(-1)=-1}}^* \bar{\chi}(d_1 \cdots d_{k+1}) L^k(1, \bar{\chi}) L(3, \bar{\chi}) \\ &= \frac{q^k \phi^2(q)}{2} \prod_{p|q} \left(1 - \frac{p^k - 1}{p^k(p-1)^2} \right) + O(q^{k+1+\epsilon}). \end{aligned}$$

Proof. We only prove the first formula, since similarly we can get the others. For any non-principal character χ modulo ud , and parameter $N \geq ud$, applying Abel’s identity we have

$$\begin{aligned} L(s, \bar{\chi}) &= \sum_{n=1}^{\infty} \frac{\bar{\chi}(n)}{n^s} = \sum_{1 \leq n \leq N} \frac{\bar{\chi}(n)}{n^s} + s \int_N^{\infty} \frac{\sum_{N < n \leq y} \bar{\chi}(n)}{y^{s+1}} dy \\ &= \sum_{1 \leq n \leq N} \frac{\bar{\chi}(n)}{n^s} + O\left(\frac{\sqrt{ud} \log(ud)}{N^s}\right). \end{aligned}$$

Let $\tau_{k+1}(n)$ be the $(k + 1)$ -th divisor function (i.e., the number of positive integer solutions of the equation $n_1 n_2 \cdots n_{k+1} = n$). Then we have

$$\begin{aligned} \Psi_1 &= \sum_{d|v} u^{k+1} d^{k+1} \sum_{d_1|(v/d)} \cdots \sum_{d_{k+1}|(v/d)} \mu(d_1) \cdots \mu(d_{k+1}) \sum_{\substack{\chi \bmod ud \\ \chi(-1)=-1}}^* \bar{\chi}(d_1 \cdots d_{k+1}) \\ &\quad \times \left(\sum_{1 \leq n \leq N} \frac{\bar{\chi}(n)}{n} + O\left(\frac{\sqrt{ud} \log(ud)}{N}\right) \right)^{k+1} \end{aligned}$$

$$\begin{aligned}
 &= \sum_{d|v} u^{k+1} d^{k+1} \sum_{d_1|(v/d)} \cdots \sum_{d_{k+1}|(v/d)} \mu(d_1) \cdots \mu(d_{k+1}) \sum_{\substack{\chi \bmod ud \\ \chi(-1)=-1}}^* \bar{\chi}(d_1 \cdots d_{k+1}) \\
 &\quad \times \left(\sum_{1 \leq n \leq N} \frac{\bar{\chi}(n)}{n} \right)^{k+1} + O\left(\frac{q^{k+5/2+\epsilon} \log^k N}{N} \right) \\
 &= \sum_{d|v} u^{k+1} d^{k+1} \sum_{d_1|(v/d)} \cdots \sum_{d_{k+1}|(v/d)} \mu(d_1) \cdots \mu(d_{k+1}) \sum_{1 \leq n \leq N^{k+1}} \frac{\tau_{k+1}(n)}{n} \\
 &\quad \times \sum_{\substack{\chi \bmod ud \\ \chi(-1)=-1}}^* \bar{\chi}(d_1 \cdots d_{k+1}) \bar{\chi}(n) + O\left(\frac{q^{k+5/2+\epsilon} \log^k N}{N} \right) \\
 &= \sum_{d|v} u^{k+1} d^{k+1} \sum_{d_1|(v/d)} \cdots \sum_{d_{k+1}|(v/d)} \sum_{1 \leq n \leq N^{k+1}} \frac{\mu(d_1) \cdots \mu(d_{k+1}) d_1 \cdots d_{k+1} \tau_{k+1}(n)}{d_1 \cdots d_{k+1} n} \\
 &\quad \times \sum_{\substack{\chi \bmod ud \\ \chi(-1)=-1}}^* \bar{\chi}(d_1 \cdots d_{k+1}) \bar{\chi}(n) + O\left(\frac{q^{k+5/2+\epsilon} \log^k N}{N} \right) \\
 &:= \Omega + O\left(\frac{q^{k+5/2+\epsilon} \log^k N}{N} \right).
 \end{aligned}$$

For $(a, m) = 1$, by Lemma 7 we have

$$\begin{aligned}
 \sum_{\substack{\chi \bmod m \\ \chi(-1)=-1}}^* \chi(a) &= \frac{1}{2} \sum_{\chi \bmod m}^* (1 - \chi(-1)) \chi(a) = \frac{1}{2} \sum_{\chi \bmod m}^* \chi(a) - \frac{1}{2} \sum_{\chi \bmod m}^* \chi(-a) \\
 (9) \quad &= \frac{1}{2} \sum_{s|(m, a-1)} \mu\left(\frac{m}{s}\right) \phi(s) - \frac{1}{2} \sum_{s|(m, a+1)} \mu\left(\frac{m}{s}\right) \phi(s).
 \end{aligned}$$

Therefore

$$\begin{aligned}
 \Omega &= \frac{1}{2} \sum_{d|v} u^{k+1} d^{k+1} \sum_{s|ud} \mu\left(\frac{ud}{s}\right) \phi(s) \\
 &\quad \times \sum_{d_1|(v/d)} \cdots \sum_{d_{k+1}|(v/d)} \sum_{\substack{1 \leq n \leq N^{k+1} \\ (n, ud)=1 \\ d_1 \cdots d_{k+1} n \equiv 1 \pmod s}} \frac{\mu(d_1) \cdots \mu(d_{k+1}) d_1 \cdots d_{k+1} \tau_{k+1}(n)}{d_1 \cdots d_{k+1} n} \\
 &\quad - \frac{1}{2} \sum_{d|v} u^{k+1} d^{k+1} \sum_{s|ud} \mu\left(\frac{ud}{s}\right) \phi(s) \\
 &\quad \times \sum_{d_1|(v/d)} \cdots \sum_{d_{k+1}|(v/d)} \sum_{\substack{1 \leq n \leq N^{k+1} \\ (n, ud)=1 \\ d_1 \cdots d_{k+1} n \equiv -1 \pmod s}} \frac{\mu(d_1) \cdots \mu(d_{k+1}) d_1 \cdots d_{k+1} \tau_{k+1}(n)}{d_1 \cdots d_{k+1} n}
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{2} \sum_{d|v} u^{k+1} d^{k+1} \sum_{s|ud} \mu\left(\frac{ud}{s}\right) \phi(s) \\
 &\times \sum_{d_1|(v/d)} \cdots \sum_{d_{k+1}|(v/d)} \sum_{\substack{1 \leq n \leq N^{k+1} \\ (n, ud)=1 \\ d_1 \cdots d_{k+1} n = ls+1}} \sum_{0 \leq l \leq ((Nv/d)^{k+1}-1)/s} \frac{\mu(d_1) \cdots \mu(d_{k+1}) d_1 \cdots d_{k+1} \tau_{k+1}(n)}{ls+1} \\
 &- \frac{1}{2} \sum_{d|v} u^{k+1} d^{k+1} \sum_{s|ud} \mu\left(\frac{ud}{s}\right) \phi(s) \\
 &\times \sum_{d_1|(v/d)} \cdots \sum_{d_{k+1}|(v/d)} \sum_{\substack{1 \leq n \leq N^{k+1} \\ (n, ud)=1 \\ d_1 \cdots d_{k+1} n = ls-1}} \sum_{2/s \leq l \leq ((Nv/d)^{k+1}+1)/s} \frac{\mu(d_1) \cdots \mu(d_{k+1}) d_1 \cdots d_{k+1} \tau_{k+1}(n)}{ls-1} \\
 &= \frac{1}{2} \sum_{d|v} u^{k+1} d^{k+1} \sum_{s|ud} \mu\left(\frac{ud}{s}\right) \phi(s) \\
 &+ O\left(\sum_{d|v} u^{k+1} d^{k+1} \sum_{s|ud} \frac{\phi(s)}{s} \sum_{d_1|(v/d)} \cdots \sum_{d_{k+1}|(v/d)} \sum_{1 \leq l \leq ((Nv/d)^{k+1}-1)/s} \frac{d_1 \cdots d_{k+1} N^\epsilon}{l+1/s}\right) \\
 &+ O\left(\sum_{d|v} u^{k+1} d^{k+1} \sum_{s|ud} \frac{\phi(s)}{s} \sum_{d_1|(v/d)} \cdots \sum_{d_{k+1}|(v/d)} \sum_{2/s \leq l \leq ((Nv/d)^{k+1}+1)/s} \frac{d_1 \cdots d_{k+1} N^\epsilon}{l-1/s}\right) \\
 &= \frac{1}{2} \sum_{d|v} u^{k+1} d^{k+1} J(ud) + O(q^{k+1+\epsilon} N^\epsilon) = \frac{u^k \phi^2(u)}{2} \sum_{d|v} d^{k+1} J(d) + O(q^{k+1+\epsilon} N^\epsilon) \\
 &= \frac{u^k \phi^2(u)}{2} \prod_{p|v} \left[p^k (p-1)^2 \left(1 - \frac{p^k-1}{p^k(p-1)^2} \right) \right] + O(q^{k+1+\epsilon} N^\epsilon) \\
 &= \frac{q^k \phi^2(q)}{2} \prod_{p|q} \left(1 - \frac{p^k-1}{p^k(p-1)^2} \right) + O(q^{k+1+\epsilon} N^\epsilon),
 \end{aligned}$$

where we have used the estimate $\tau_{k+1}(n) \ll n^\epsilon$, the fact that v is a square-free number, u is a square-full number, and the identity $J(u) = \phi^2(u)/u$, if u is a square-full number.

Now taking $N = q^{3/2}$ in the above, we immediately get

$$\Psi_1 = \frac{q^k \phi^2(q)}{2} \prod_{p|q} \left(1 - \frac{p^k-1}{p^k(p-1)^2} \right) + O(q^{k+1+\epsilon}).$$

This completes the proof of Lemma 8. □

Lemma 9. *Let $q = uv$, where $(u, v) = 1$, u be a square-full number or $u = 1$, v be a square-free number. Then we have the following estimates*

$$\begin{aligned} \Psi_4 &:= \sum_{d|v} u^{k+1} d^{k+1} \sum_{d_1|(v/d)} \cdots \sum_{d_{k+1}|(v/d)} \mu(d_1) \cdots \mu(d_{k+1}) \\ &\quad \times \sum_{\substack{\chi \bmod ud \\ \chi(-1)=-1}}^* \bar{\chi}^j(2) \bar{\chi}(d_1 \cdots d_{k+1}) L^{k+1}(1, \bar{\chi}) \\ &\ll q^{k+1+\epsilon}, \quad j = 1, 2, \dots, k+1; \\ \Psi_5 &:= \sum_{d|v} u^{k+1} d^{k+1} \sum_{d_1|(v/d)} \cdots \sum_{d_{k+1}|(v/d)} \frac{\mu(d_1) \cdots \mu(d_{k+1})}{d_1 \cdots d_{k+1}} \\ &\quad \times \sum_{\substack{\chi \bmod ud \\ \chi(-1)=1}}^* \bar{\chi}^j(2) \bar{\chi}(d_1 \cdots d_{k+1}) L^{k+1}(2, \bar{\chi}) \\ &\ll q^{k+1+\epsilon}, \quad j = 1, 2, \dots, k+1; \\ \Psi_6 &:= \sum_{d|v} u^{k+1} d^{k+1} \sum_{d_1|(v/d)} \cdots \sum_{d_{k+1}|(v/d)} \frac{\mu(d_1) \cdots \mu(d_{k+1})}{d_{k+1}^2} \\ &\quad \times \sum_{\substack{\chi \bmod ud \\ \chi(-1)=-1}}^* \bar{\chi}^j(2) \bar{\chi}(d_1 \cdots d_{k+1}) L^k(1, \bar{\chi}) L(3, \bar{\chi}) \\ &\ll q^{k+1+\epsilon}, \quad j = 1, 2, \dots, k+1. \end{aligned}$$

Proof. We only prove the first formula, since similarly we can get the others. For parameter $N \geq ud$, using the method of Lemma 8 we have

$$\begin{aligned} \Psi_4 &= \sum_{d|v} u^{k+1} d^{k+1} \sum_{d_1|(v/d)} \cdots \sum_{d_{k+1}|(v/d)} \sum_{1 \leq n \leq N^{k+1}} \frac{\mu(d_1) \cdots \mu(d_{k+1}) 2^j d_1 \cdots d_{k+1} \tau_{k+1}(n)}{2^j d_1 \cdots d_{k+1} n} \\ &\quad \times \sum_{\substack{\chi \bmod ud \\ \chi(-1)=-1}}^* \bar{\chi}(2^j) \bar{\chi}(d_1 \cdots d_{k+1}) \bar{\chi}(n) + O\left(\frac{q^{k+5/2+\epsilon} \log^k N}{N}\right) \\ &:= \Upsilon + O\left(\frac{q^{k+5/2+\epsilon} \log^k N}{N}\right). \end{aligned}$$

Then from (9) we get

$$\begin{aligned}
 \Upsilon &= \frac{1}{2} \sum_{d|v} u^{k+1} d^{k+1} \sum_{s|ud} \mu\left(\frac{ud}{s}\right) \phi(s) \\
 &\quad \times \sum_{d_1|(v/d)} \cdots \sum_{d_{k+1}|(v/d)} \sum_{\substack{1 \leq n \leq N^{k+1} \\ (n, ud)=1 \\ 2^j d_1 \cdots d_{k+1} n \equiv 1 \pmod{s}}} \frac{\mu(d_1) \cdots \mu(d_{k+1}) 2^j d_1 \cdots d_{k+1} \tau_{k+1}(n)}{2^j d_1 \cdots d_{k+1} n} \\
 &\quad - \frac{1}{2} \sum_{d|v} u^{k+1} d^{k+1} \sum_{s|ud} \mu\left(\frac{ud}{s}\right) \phi(s) \\
 &\quad \times \sum_{d_1|(v/d)} \cdots \sum_{d_{k+1}|(v/d)} \sum_{\substack{1 \leq n \leq N^{k+1} \\ (n, ud)=1 \\ 2^j d_1 \cdots d_{k+1} n \equiv -1 \pmod{s}}} \frac{\mu(d_1) \cdots \mu(d_{k+1}) 2^j d_1 \cdots d_{k+1} \tau_{k+1}(n)}{2^j d_1 \cdots d_{k+1} n} \\
 &= \frac{1}{2} \sum_{d|v} u^{k+1} d^{k+1} \sum_{s|ud} \mu\left(\frac{ud}{s}\right) \phi(s) \\
 &\quad \times \sum_{d_1|(v/d)} \cdots \sum_{d_{k+1}|(v/d)} \sum_{\substack{1 \leq n \leq N^{k+1} \\ (n, ud)=1 \\ 2^j d_1 \cdots d_{k+1} n = ls+1}} \sum_{(2^j-1)/s \leq l \leq (2^j(Nv/d)^{k+1}-1)/s} \frac{\mu(d_1) \cdots \mu(d_{k+1}) 2^j d_1 \cdots d_{k+1} \tau_{k+1}(n)}{ls+1} \\
 &\quad - \frac{1}{2} \sum_{d|v} u^{k+1} d^{k+1} \sum_{s|ud} \mu\left(\frac{ud}{s}\right) \phi(s) \\
 &\quad \times \sum_{d_1|(v/d)} \cdots \sum_{d_{k+1}|(v/d)} \sum_{\substack{1 \leq n \leq N^{k+1} \\ (n, ud)=1 \\ 2^j d_1 \cdots d_{k+1} n = ls-1}} \sum_{(2^j+1)/s \leq l \leq (2^j(Nv/d)^{k+1}+1)/s} \frac{\mu(d_1) \cdots \mu(d_{k+1}) 2^j d_1 \cdots d_{k+1} \tau_{k+1}(n)}{ls-1} \\
 &\ll \sum_{d|v} u^{k+1} d^{k+1} \sum_{s|ud} \frac{\phi(s)}{s} \sum_{d_1|(v/d)} \cdots \sum_{d_{k+1}|(v/d)} \sum_{(2^j-1)/s \leq l \leq (2^j(Nv/d)^{k+1}-1)/s} \frac{d_1 \cdots d_{k+1} N^\epsilon}{l+1/s} \\
 &\quad + \sum_{d|v} u^{k+1} d^{k+1} \sum_{s|ud} \frac{\phi(s)}{s} \sum_{d_1|(v/d)} \cdots \sum_{d_{k+1}|(v/d)} \sum_{(2^j+1)/s \leq l \leq (2^j(Nv/d)^{k+1}+1)/s} \frac{d_1 \cdots d_{k+1} N^\epsilon}{l-1/s} \\
 &\ll q^{k+1+\epsilon} N^\epsilon.
 \end{aligned}$$

Now taking $N = q^{3/2}$ in the above, we immediately get

$$\Psi_4 \ll q^{k+1+\epsilon}.$$

This proves Lemma 9. □

3. Proof of the theorem

In this section, we complete the proof of the theorem. Let $q \geq 3$ be an odd number and $k \geq 1$ be an integer, for any character $\chi \pmod q$, we have

$$\begin{aligned}
 \sum_{c=1}^q \bar{\chi}(c) K(\bar{2}^{k+1}c, k, q) &= \sum_{a_1=1}^{q'} \cdots \sum_{a_k=1}^{q'} \sum_{c=1}^q \bar{\chi}(c) e\left(\frac{a_1 + \cdots + a_k + \bar{2}^{k+1}c \cdot \bar{a}_1 \cdots \bar{a}_k}{q}\right) \\
 &= \bar{\chi}^{k+1}(2) \tau(\bar{\chi}) \left(\sum_{a=1}^q \bar{\chi}(a) e\left(\frac{a}{q}\right)\right)^k = \bar{\chi}^{k+1}(2) \tau^{k+1}(\bar{\chi}).
 \end{aligned}$$

We first treat the case $k = 1$ of the theorem. From Lemma 1 we get

$$\begin{aligned}
 &\sum_{c=1}^{q'} E(q, 1, c) K(\bar{4}c, 1, q) \\
 &= \frac{1}{\phi(q)} \sum_{\substack{\chi(-1)=1 \\ \chi \neq \chi_0}} \bar{\chi}(4) \tau^2(\bar{\chi}) \left(\sum_{a=1}^q (-1)^a a \chi(a)\right)^2 + \frac{1}{\phi(q)} \sum_{\chi(-1)=-1} (4 - 4\bar{\chi}(2)) \tau^2(\bar{\chi}) \left(\sum_{a=1}^q a \chi(a)\right)^2 \\
 &\quad - \frac{2}{\phi(q)q} \sum_{\chi(-1)=-1} (\bar{\chi}(4) - 2\bar{\chi}(2)) \tau^2(\bar{\chi}) \left(\sum_{a=1}^q a \chi(a)\right) \left(\sum_{b=1}^q (-1)^b b^2 \chi(b)\right) + O(q^{5/2+\epsilon}).
 \end{aligned}$$

Let $q = uv$, where $(u, v) = 1$, u be a square-full number or $u = 1$, v be a square-free number. Suppose that χ is a character modulo q , generated by the primitive character χ_m modulo m . Note that $\tau(\bar{\chi}) = \bar{\chi}_m(q/m) \mu(q/m) \tau(\bar{\chi}_m) \neq 0$ if and only if $m = ud$, where $d|v$. Then from Lemmas 4, 5, 6 we have

$$\begin{aligned}
 &\sum_{c=1}^{q'} E(q, 1, c) K(\bar{4}c, 1, q) \\
 &= \frac{1}{\pi^4 \phi(q)} \sum_{d|v} u^4 d^4 \sum_{\substack{\chi \pmod{ud} \\ \chi(-1)=1}}^* [16 - 8\bar{\chi}(2) + \bar{\chi}(4)] \\
 &\quad \times \bar{\chi}^2\left(\frac{v}{d}\right) \mu^2\left(\frac{v}{d}\right) \left[\sum_{d_1|(v/d)} \frac{v}{dd_1} \mu\left(\frac{v}{dd_1}\right) \chi\left(\frac{v}{dd_1}\right)\right]^2 L^2(2, \bar{\chi}) \\
 &\quad + \frac{q^2}{\pi^2 \phi(q)} \sum_{d|v} u^2 d^2 \sum_{\substack{\chi \pmod{ud} \\ \chi(-1)=-1}}^* [4 - 4\bar{\chi}(2) + 2\bar{\chi}(4)] \bar{\chi}^2\left(\frac{v}{d}\right) \mu^2\left(\frac{v}{d}\right) \\
 &\quad \times \left[\sum_{d_1|(v/d)} \mu\left(\frac{v}{dd_1}\right) \chi\left(\frac{v}{dd_1}\right)\right]^2 L^2(1, \bar{\chi})
 \end{aligned}$$

$$\begin{aligned}
 &+ \frac{2}{\pi^4 \phi(q)} \sum_{d|v} u^4 d^4 \sum_{\substack{\chi \bmod ud \\ \chi(-1)=-1}}^* [-16 + 10\bar{\chi}(2) - \bar{\chi}(4)] \bar{\chi}^2\left(\frac{v}{d}\right) \mu^2\left(\frac{v}{d}\right) \\
 &\times \left[\sum_{d_1|(v/d)} \mu\left(\frac{v}{dd_1}\right) \chi\left(\frac{v}{dd_1}\right) \right] \left[\sum_{d_2|(v/d)} \frac{v^2}{d^2 d_2^2} \mu\left(\frac{v}{dd_2}\right) \chi\left(\frac{v}{dd_2}\right) \right] L(1, \bar{\chi}) L(3, \bar{\chi}) \\
 &+ O(q^{7/2+\epsilon}).
 \end{aligned}$$

Note that

$$(11) \quad \bar{\chi}\left(\frac{v}{d}\right) = \bar{\chi}\left(\frac{v}{dd_1}\right) \bar{\chi}(d_1), \quad \mu\left(\frac{v}{d}\right) = \mu\left(\frac{v}{dd_1}\right) \mu(d_1),$$

so by Lemma 8 and Lemma 9 we have

$$\begin{aligned}
 &\sum_{c=1}^q E(q, 1, c) K(\bar{4}c, 1, q) \\
 &= \frac{q^2}{\pi^4 \phi(q)} \sum_{d|v} u^2 d^2 \sum_{d_1|(v/d)} \sum_{d_2|(v/d)} \frac{\mu(d_1) \mu(d_2)}{d_1 d_2} \\
 &\times \sum_{\substack{\chi \bmod ud \\ \chi(-1)=1}}^* [16 - 8\bar{\chi}(2) + \bar{\chi}(4)] \bar{\chi}(d_1 d_2) L^2(2, \bar{\chi}) \\
 &+ \frac{q^2}{\pi^2 \phi(q)} \sum_{d|v} u^2 d^2 \sum_{d_1|(v/d)} \sum_{d_2|(v/d)} \mu(d_1) \mu(d_2) \\
 &\times \sum_{\substack{\chi \bmod ud \\ \chi(-1)=-1}}^* [4 - 4\bar{\chi}(2) + 2\bar{\chi}(4)] \bar{\chi}(d_1 d_2) L^2(1, \bar{\chi}) \\
 &+ \frac{2q^2}{\pi^4 \phi(q)} \sum_{d|v} u^2 d^2 \sum_{d_1|(v/d)} \sum_{d_2|(v/d)} \frac{\mu(d_1) \mu(d_2)}{d_2^2} \\
 &\times \sum_{\substack{\chi \bmod ud \\ \chi(-1)=-1}}^* [-16 + 10\bar{\chi}(2) - \bar{\chi}(4)] \bar{\chi}(d_1 d_2) L(1, \bar{\chi}) L(3, \bar{\chi}) + O(q^{7/2+\epsilon}) \\
 &= -\frac{6q^3 \phi(q)}{\pi^4} \prod_{p|q} \left(1 - \frac{p-1}{p(p-1)^2}\right) + O(q^{7/2+\epsilon}).
 \end{aligned}$$

This proves the theorem with $k = 1$.

Now let $k \geq 2$ be an integer. Then from formulae (10), (11), and Lemmas 2, 4, 5, 6, 8, 9 we can have

$$\begin{aligned}
 & \sum_{c=1}^q E(q, k, c) K(\bar{2}^{k+1} c, k, q) \\
 &= \frac{-2^{k-2}}{\phi(q)q^{k-1}} \sum_{\chi(-1)=-1} \tau^{k+1}(\bar{\chi}) \bar{\chi}^{k+1}(2)(1-2\chi(2))^k \\
 & \quad \times \left(\sum_{a=1}^q a\chi(a) \right)^k \left[\frac{2(k+1)}{q} \left(\sum_{b=1}^q (-1)^b b^2 \chi(b) \right) + k(k-3)(1-2\chi(2)) \left(\sum_{b=1}^q b\chi(b) \right) \right] \\
 &= \frac{i^{k+1}(-1)^{k+2} 2^{k-2} (k^2 - k + 2) q^2}{\pi^{k+1} \phi(q)} \sum_{d|v} u^{k+1} d^{k+1} \sum_{\substack{\chi \bmod ud \\ \chi(-1)=-1}}^* \bar{\chi}^{k+1} \left(\frac{v}{d} \right) \mu^{k+1} \left(\frac{v}{d} \right) \\
 & \quad \times [\bar{\chi}(2) - 2]^{k+1} \left[\sum_{d_1|(v/d)} \mu \left(\frac{v}{dd_1} \right) \chi \left(\frac{v}{dd_1} \right) \right]^{k+1} L^{k+1}(1, \bar{\chi}) \\
 & \quad + \frac{i^{k+1}(-1)^{k+2} 2^{k-1} (k+1)}{\pi^{k+3} \phi(q)} \sum_{d|v} u^{k+3} d^{k+3} \sum_{\substack{\chi \bmod ud \\ \chi(-1)=-1}}^* \bar{\chi}^{k+1} \left(\frac{v}{d} \right) \mu^{k+1} \left(\frac{v}{d} \right) [\bar{\chi}(2) - 2]^k \\
 & \quad \times [8 - \bar{\chi}(2)] \left[\sum_{d_1|(v/d)} \mu \left(\frac{v}{dd_1} \right) \chi \left(\frac{v}{dd_1} \right) \right]^k \left[\sum_{d_2|(v/d)} \frac{v^2}{d^2 d_2^2} \mu \left(\frac{v}{dd_2} \right) \chi \left(\frac{v}{dd_2} \right) \right] \\
 & \quad \times L^k(1, \bar{\chi}) L(3, \bar{\chi}) + O(q^{k+5/2+\epsilon}) \\
 &= \frac{i^{k+1}(-1)^{k+2} 2^{k-2} (k^2 - k + 2) q^2}{\pi^{k+1} \phi(q)} \sum_{d|v} u^{k+1} d^{k+1} \sum_{d_1|(v/d)} \dots \sum_{d_{k+1}|(v/d)} \mu(d_1) \dots \mu(d_{k+1}) \\
 & \quad \times \sum_{\substack{\chi \bmod ud \\ \chi(-1)=-1}}^* [\bar{\chi}(2) - 2]^{k+1} \bar{\chi}(d_1 \dots d_{k+1}) L^{k+1}(1, \bar{\chi}) \\
 & \quad + \frac{i^{k+1}(-1)^{k+2} 2^{k-1} (k+1) q^2}{\pi^{k+3} \phi(q)} \sum_{d|v} u^{k+1} d^{k+1} \sum_{d_1|(v/d)} \dots \sum_{d_{k+1}|(v/d)} \frac{\mu(d_1) \dots \mu(d_{k+1})}{d_{k+1}^2} \\
 & \quad \times \sum_{\substack{\chi \bmod ud \\ \chi(-1)=-1}}^* [\bar{\chi}(2) - 2]^k [8 - \bar{\chi}(2)] \bar{\chi}(d_1 \dots d_{k+1}) L^k(1, \bar{\chi}) L(3, \bar{\chi}) + O(q^{k+5/2+\epsilon}) \\
 &= \frac{i^{k+1}(-1) 2^{2k-2} (k^2 - k + 2) q^{k+2} \phi(q)}{\pi^{k+1}} \prod_{p|q} \left(1 - \frac{p^k - 1}{p^k(p-1)^2} \right) \\
 & \quad + \frac{i^{k+1} 2^{2k+1} (k+1) q^{k+2} \phi(q)}{\pi^{k+3}} \prod_{p|q} \left(1 - \frac{p^k - 1}{p^k(p-1)^2} \right) + O(q^{k+5/2+\epsilon})
 \end{aligned}$$

$$= \frac{i^{k+3} 2^{2k-2} q^{k+2} \phi(q)}{\pi^{k+3}} [\pi^2(k^2 - k + 2) - 8(k + 1)] \prod_{p \parallel q} \left(1 - \frac{p^k - 1}{p^k(p - 1)^2}\right) + O(q^{k+5/2+\epsilon}).$$

So from the above we have

$$\begin{aligned} & \sum_{c=1}^q E(q, k, c) K(\bar{2}^{k+1} c, k, q) \\ &= \frac{c_k q^{k+2} \phi(q)}{\pi^{k+3}} \prod_{p \parallel q} \left(1 - \frac{p^k - 1}{p^k(p - 1)^2}\right) + O(q^{k+5/2+\epsilon}), \quad \text{for } k \geq 1. \end{aligned}$$

This completes the proof of the theorem.

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