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REGULAR ORBIT CLOSURES IN MODULE VARIETIES

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Abstract

Let A be a finitely generated associative algebra over an algebraically closed field. We characterize the finite dimensional modules over A whose orbit closures are regular varieties.

1. Introduction and the main result

Throughout the paper k denotes a fixed algebraically closed field. By an algebra we mean an associative finitely generated k-algebra with identity, and by a module a finite dimensional left module. Let d be a positive integer and denote by $\mathbb{M}(d)$ the algebra of $d \times d$-matrices with coefficients in k. For an algebra A the set $\operatorname{mod}_A(d)$ of the A-module structures on the vector space k^d has a natural structure of an affine variety. Indeed, if $A \simeq k\langle X_1, \dots, X_t\rangle/J$ for $t > 0$ and a two-sided ideal J, then $\operatorname{mod}_A(d)$ can be identified with the closed subset of $(\mathbb{M}(d))^t$ given by vanishing of the entries of all matrices $\rho(X_1, \dots, X_t)$ for $\rho \in J$. Moreover, the general linear group $\mathrm{GL}(d)$ acts on $\operatorname{mod}_A(d)$ by conjugation and the $\mathrm{GL}(d)$-orbits in $\operatorname{mod}_A(d)$ correspond bijectively to the isomorphism classes of d-dimensional A-modules. We shall denote by $\mathcal{O}_M$ the $\mathrm{GL}(d)$-orbit in $\operatorname{mod}_A(d)$ corresponding to (the isomorphism class of) a d-dimensional A-module M. It is an interesting task to study geometric properties of the Zariski closure $\overline{\mathcal{O}}_M$ of $\mathcal{O}_M$. We note that using a geometric equivalence described in [4], this is closely related to a similar problem for representations of quivers. We refer to [2], [3], [4], [5], [6], [9], [10], [11], [12], [13] and [14] for results concerning geometric properties of orbit closures in module varieties or varieties of representations.

The main result of the paper concerns the global regularity of such varieties. Let $\operatorname{Ann}(M)$ denote the annihilator of a module M. It is the kernel of the algebra homomorphism $A \to \operatorname{End}_k(M)$ induced by the module M, and therefore the algebra $B = A/\operatorname{Ann}(M)$ is finite dimensional. Obviously M can be considered as a B-module.

Theorem 1.1. *Let M be an A-module and let $B = A/\operatorname{Ann}(M)$. Then the orbit closure $\overline{\mathcal{O}}_M$ is a regular variety if and only if the algebra B is hereditary and* $\operatorname{Ext}^1_B(M, M) = 0$.

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Let $d = \dim_k M$. Observe that $\operatorname{mod}_B(d)$ is a closed $\mathrm{GL}(d)$-subvariety of $\operatorname{mod}_A(d)$ containing $\overline{\mathcal{O}}_M$. Moreover, M is faithful as a B-module. Hence we may reformulate Theorem 1.1 as follows:

Theorem 1.2. *Let M be a faithful module over a finite dimensional algebra B. Then the orbit closure $\overline{\mathcal{O}}_M$ is a regular variety if and only if the algebra B is hereditary and* $\mathrm{Ext}^1_B(M, M) = 0$.

The next section contains a reduction of the proof of Theorem 1.2 to Theorem 2.1 presented in terms of properties of regular orbit closures for representations of quivers. Sections 3 and 4 are devoted to the proof of Theorem 2.1. For basic background on the representation theory of algebras and quivers we refer to [1].

2. Representations of quivers

Let $Q = (Q_0, Q_1; s, t : Q_1 \to Q_0)$ be a finite quiver, i.e. Q_0 is a finite set of vertices, and Q_1 is a finite set of arrows $\alpha : s(\alpha) \to t(\alpha)$. By a representation of Q we mean a collection $V = (V_i, V_\alpha)$ of finite dimensional k-vector spaces V_i, $i \in Q_0$, together with linear maps $V_\alpha : V_{s(\alpha)} \to V_{t(\alpha)}$, $\alpha \in Q_1$. The dimension vector of the representation V is the vector

$$\mathbf{dim}\, V = (\dim_k V_i) \in \mathbb{N}^{Q_0}.$$

By a path of length $m \geq 1$ in Q we mean a sequence of arrows in Q_1:

$$\omega = \alpha_m \alpha_{m-1} \cdots \alpha_2 \alpha_1,$$

such that $s(\alpha_{l+1}) = t(\alpha_l)$ for $l = 1, \ldots, m-1$. In the above situation we write $s(\omega) = s(\alpha_1)$ and $t(\omega) = t(\alpha_m)$. We agree to associate to each $i \in Q_0$ a path ε_i in Q of length zero with $s(\varepsilon_i) = t(\varepsilon_i) = i$. The paths of Q form a k-linear basis of the path algebra kQ. We define

$$V_\omega = V_{\alpha_m} \circ V_{\alpha_{m-1}} \circ \cdots \circ V_{\alpha_2} \circ V_{\alpha_1} : V_{s(\omega)} \to V_{t(\omega)}$$

for a path $\omega = \alpha_m \cdots \alpha_1$ and extend easily this definition to $V_\rho : V_i \to V_j$ for any ρ in $\varepsilon_j \cdot kQ \cdot \varepsilon_i$, where $i, j \in Q_0$, as ρ is a k-linear combination of paths ω with $s(\omega) = i$ and $t(\omega) = j$. Finally, we set

$$\mathrm{Ann}(V) = \{\rho \in kQ \mid V_{\varepsilon_j \cdot \rho \cdot \varepsilon_i} = 0 \text{ for all } i, j \in Q_0\},$$

which is a two-sided ideal in kQ. In fact, it is the annihilator of the kQ-module induced by V with underlying k-vector space $\bigoplus_{i \in Q_0} V_i$.

Let $\mathbf{d} = (d_i)_{i\in Q_0} \in \mathbb{N}^{Q_0}$ be a dimension vector. Then the representations $V = (V_i, V_\alpha)$ of Q with $V_i = k^{d_i}$, $i \in Q_0$, form a vector space

$$\operatorname{rep}_Q(\mathbf{d}) = \bigoplus_{\alpha\in Q_1} \operatorname{Hom}_k(V_{s(\alpha)}, V_{t(\alpha)}) = \bigoplus_{\alpha\in Q_1} \mathbb{M}(d_{t(\alpha)} \times d_{s(\alpha)}),$$

where $\mathbb{M}(d' \times d'')$ stands for the space of $d' \times d''$-matrices with coefficients in k. For abbreviation, we denote the representations in $\operatorname{rep}_Q(\mathbf{d})$ by $V = (V_\alpha)$. The group $\mathrm{GL}(\mathbf{d}) = \bigoplus_{i\in Q_0} \mathrm{GL}(d_i)$ acts regularly on $\operatorname{rep}_Q(\mathbf{d})$ via

$$(g_i)_{i\in Q_0} * (V_\alpha)_{\alpha\in Q_1} = \big(g_{t(\alpha)} \cdot V_\alpha \cdot g_{s(\alpha)}^{-1}\big)_{\alpha\in Q_1}.$$

Given a representation $W = (W_i, W_\alpha)$ of Q with $\mathbf{dim}\, W = \mathbf{d}$, we denote by $\mathcal{O}_W$ the $\mathrm{GL}(\mathbf{d})$-orbit in $\operatorname{rep}_Q(\mathbf{d})$ of representations isomorphic to W.

Let M be a faithful module over a finite dimensional algebra B. It is well known that the algebra B is Morita-equivalent to the quotient algebra kQ/I, where Q is a finite quiver and I an admissible ideal in kQ, i.e. I is a two-sided ideal such that $(\mathcal{R}_Q)^r \subseteq I \subseteq (\mathcal{R}_Q)^2$ for some positive integer r, where $\mathcal{R}_Q$ denotes the two-sided ideal of kQ generated by the paths of length one (arrows) in Q. Furthermore, the algebra B is hereditary if and only if $I = \{0\}$ (in particular, the quiver Q has no oriented cycles, i.e. paths ω of positive lengths with $s(\omega) = t(\omega)$). According to the above equivalence, the faithful B-module M corresponds to a representation $N = (N_\alpha)$ in $\operatorname{rep}_Q(\mathbf{d})$ for some $\mathbf{d}$, such that $\operatorname{Ann}(N) = I$. Applying the geometric version of the Morita equivalence described by Bongartz in [4], $\overline{\mathcal{O}}_M$ is isomorphic to an associated fibre bundle $\mathrm{GL}(d) \times^{\mathrm{GL}(\mathbf{d})} \overline{\mathcal{O}}_N$. In particular, $\overline{\mathcal{O}}_M$ is regular if and only if $\overline{\mathcal{O}}_N$ is. By the Artin-Voigt formula (see [8]):

$$\operatorname{codim}_{\operatorname{rep}_Q(\mathbf{d})} \overline{\mathcal{O}}_N = \dim_k \operatorname{Ext}^1_Q(N, N),$$

the vanishing of $\operatorname{Ext}^1_Q(N, N)$ means that $\overline{\mathcal{O}}_N = \operatorname{rep}_Q(\mathbf{d})$. Consequently, one implication in Theorem 1.2 is proved and it suffices to show the following fact:

Theorem 2.1. *Let N be a representation in* $\operatorname{rep}_Q(\mathbf{d})$ *such that* $\operatorname{Ann}(N)$ *is an admissible ideal in kQ and $\overline{\mathcal{O}}_N$ is a regular variety. Then* $\operatorname{Ann}(N) = \{0\}$ *and* $\overline{\mathcal{O}}_N = \operatorname{rep}_Q(\mathbf{d})$.

3. Tangent spaces of orbit closures and nilpotent representations

From now on, N is a representation in $\operatorname{rep}_Q(\mathbf{d})$ such that $\operatorname{Ann}(N)$ is an admissible ideal in kQ and $\overline{\mathcal{O}}_N$ is a regular variety. The aim of the section is to prove that the quiver Q has no oriented cycles.

Let $S[j] = (S[j]_i, S[j]_\alpha)$ stand for the simple representation of Q such that $S[j]_j = k$ is the only non-zero vector space and all linear maps $S[j]_\alpha$ are zero, for any vertex

$j \in Q_0$. Observe that the point 0 in $\mathrm{rep}_Q(\mathbf{d})$ is the semisimple representation $\bigoplus_{i\in Q_0} S[i]^{d_i}$. A representation $W = (W_i, W_\alpha)$ of Q is said to be nilpotent if one of the following equivalent conditions is satisfied:

(1) The endomorphism $W_\omega \in \mathrm{End}_k(W_{s(\omega)})$ is nilpotent for any oriented cycle ω in Q.
(2) The ideal $\mathrm{Ann}(W)$ contains $(\mathcal{R}_Q)^r$ for some positive integer r.
(3) Any composition factor of W is isomorphic to some $S[i]$, $i \in Q_0$.
(4) The orbit closure $\overline{\mathcal{O}}_W$ in $\mathrm{rep}_Q(\mathbf{dim}\, W)$ contains 0.

Obviously the representation N is nilpotent. Thus the set $\mathcal{N}_Q(\mathbf{d})$ of nilpotent representations in $\mathrm{rep}_Q(\mathbf{d})$ is a closed $\mathrm{GL}(\mathbf{d})$-invariant subset which contains $\overline{\mathcal{O}}_N$. Furthermore, $\mathcal{N}_Q(\mathbf{d})$ is a cone, i.e. it is invariant under multiplication by scalars in the vector space $\mathrm{rep}_Q(\mathbf{d})$.

We shall identify the tangent space $\mathcal{T}_{\mathrm{rep}_Q(\mathbf{d}),0}$ of $\mathrm{rep}_Q(\mathbf{d})$ at 0 with $\mathrm{rep}_Q(\mathbf{d})$ itself. Thus the tangent space $\mathcal{T}_{\overline{\mathcal{O}}_N,0}$ is a subspace of $\mathrm{rep}_Q(\mathbf{d})$ and is invariant under the action of $\mathrm{GL}(\mathbf{d})$, i.e. it is a $\mathrm{GL}(\mathbf{d})$-subrepresentation of $\mathrm{rep}_Q(\mathbf{d})$. Since $\overline{\mathcal{O}}_N$ is a regular variety, the tangent space $\mathcal{T}_{\overline{\mathcal{O}}_N,0}$ is the tangent cone of $\overline{\mathcal{O}}_N$ at 0 (see [7, III. 4]), and the latter is contained in the tangent cone of $\mathcal{N}_Q(\mathbf{d})$ at 0. Therefore

$$\mathcal{T}_{\overline{\mathcal{O}}_N,0} \subseteq \mathcal{N}_Q(\mathbf{d}). \tag{3.1}$$

Lemma 3.1. *Let $W = (W_\alpha)$ be a tangent vector in $\mathcal{T}_{\overline{\mathcal{O}}_N,0}$. Then $W_\gamma = 0$ for any loop $\gamma \in Q_1$.*

Proof. Suppose that the nilpotent matrix $W_\gamma \in \mathbb{M}(d_j)$ is non-zero for some loop $\gamma : j \to j$ in Q_1. Then there are two linearly independent vectors $v_1, v_2 \in k^{d_j}$ such that $W_\gamma \cdot v_1 = v_2$ and $W_\gamma \cdot v_2 = 0$. We choose $g = (g_i)$ in $\mathrm{GL}(\mathbf{d})$ such that $g_j \cdot v_1 = v_2$ and $g_j \cdot v_2 = v_1$. Then $U = W + g * W$ belongs to $\mathcal{T}_{\overline{\mathcal{O}}_N,0}$. Observe that $U_\gamma \cdot v_1 = v_2$ and $U_\gamma \cdot v_2 = v_1$. Hence the representation U is not nilpotent, contrary to (3.1). □

Let $V_i = k^{d_i}$ and $R_{i,j}$ be the vector space of formal linear combinations of arrows $\alpha \in Q_1$ with $s(\alpha) = i$ and $t(\alpha) = j$, for any $i, j \in Q_0$. We shall identify:

$$\mathrm{rep}_Q(\mathbf{d}) = \bigoplus_{i,j\in Q_0} \mathrm{Hom}_k(R_{i,j}, \mathrm{Hom}_k(V_i, V_j)) \quad \text{and} \quad \mathrm{GL}(\mathbf{d}) = \bigoplus_{i\in Q_0} \mathrm{GL}(V_i).$$

Applying Lemma 3.1 we get

$$\mathcal{T}_{\overline{\mathcal{O}}_N,0} \subseteq \bigoplus_{\substack{i,j\in Q_0\\ i\neq j}} \mathrm{Hom}_k(R_{i,j}, \mathrm{Hom}_k(V_i, V_j)).$$

Since the GL($\mathbf{d}$)-representations $\mathrm{Hom}_k(V_i, V_j)$, $i \neq j$, are simple and pairwise non-isomorphic, we have

$$\mathcal{T}_{\overline{\mathcal{O}}_N,0} = \bigoplus_{\substack{i,j\in Q_0\\ i\neq j}} \{\varphi\colon R_{i,j} \to \mathrm{Hom}_k(V_i, V_j) \mid \varphi(U_{i,j}) = 0\}$$

for some subspaces $U_{i,j}$ of $R_{i,j}$, $i \neq j$.

The spaces $U_{i,j}$ are not necessarily spanned by arrows α: $i \to j$ in Q_1, and we are going to replace N by a "better" representation in $\mathrm{rep}_Q(\mathbf{d})$. The group $\widetilde{G} = \bigoplus_{i,j\in Q_0} \mathrm{GL}(R_{i,j})$ can be identified naturally with a subgroup of automorphisms of the path algebra kQ which change linearly the paths of length 1 but do not change the paths of length 0. Let $\widetilde{g} = (\widetilde{g}_{i,j})$ be an element of $\widetilde{G}$. Then $\widetilde{g} \star (\mathcal{R}_Q)^p = (\mathcal{R}_Q)^p$ for any positive integer p, where $\star$ denotes the action of $\widetilde{G}$ on kQ. For a representation W of Q presented in the form

$$W = (W_i, W_{i,j}\colon R_{i,j} \to \mathrm{Hom}_k(W_i, W_j))_{i,j\in Q_0},$$

we define the representation

$$\widetilde{g} \star W = (W_i, W_{i,j} \circ (\widetilde{g}_{i,j})^{-1})_{i,j\in Q_0}.$$

Hence $\widetilde{G}$ acts regularly on $\mathrm{rep}_Q(\mathbf{d})$ and this action commutes with the GL($\mathbf{d}$)-action. Therefore the orbit closure $\overline{\mathcal{O}}_{\widetilde{g}\star N} = \widetilde{g} \star \overline{\mathcal{O}}_N$ is a regular variety, $\mathcal{T}_{\overline{\mathcal{O}}_{\widetilde{g}\star N},0} = \widetilde{g} \star \mathcal{T}_{\overline{\mathcal{O}}_N,0}$ and the ideal $\mathrm{Ann}(\widetilde{g} \star N) = \widetilde{g} \star \mathrm{Ann}(N)$ is admissible as

$$(\mathcal{R}_Q)^r = \widetilde{g} \star (\mathcal{R}_Q)^r \subseteq \widetilde{g} \star \mathrm{Ann}(N) \subseteq \widetilde{g} \star (\mathcal{R}_Q)^2 = (\mathcal{R}_Q)^2.$$

Hence, replacing N by $\widetilde{g} \star N$ for an appropriate $\widetilde{g}$, we may assume that the spaces $U_{i,j}$, $i \neq j$, are spanned by arrows in Q_1. Consequently,

$$\mathcal{T}_{\overline{\mathcal{O}}_N,0} = \mathrm{rep}_{Q'}(\mathbf{d}) \subseteq \mathrm{rep}_Q(\mathbf{d}) \tag{3.2}$$

for some subquiver Q' of Q such that $Q'_0 = Q_0$ and Q'_1 has no loops.

Lemma 3.2. *The quiver Q' has no oriented cycles.*

Proof. Suppose there is an oriented cycle ω in Q'. Let $W = (W_\alpha)$ be a tangent vector in $\mathcal{T}_{\overline{\mathcal{O}}_N,0} = \mathrm{rep}_{Q'}(\mathbf{d})$ such that each W_α, $\alpha \in (Q')_1$, is the matrix whose (1, 1)-entry is 1, while the other entries are 0. Then the matrix W_ω has the same form, contrary to (3.1). □

Let $W = (W_i, W_\alpha)$ be a representation of Q. We denote by $\mathrm{rad}(W)$ the radical of W. In case W is nilpotent, $\mathrm{rad}(W) = \sum_{\alpha\in Q_1} \mathrm{Im}(W_\alpha)$. We write $\langle w\rangle$ for the subrepresentation of W generated by a vector $w \in \bigoplus_{i\in Q_0} W_i$.

Lemma 3.3. *Let $\alpha: i \to j$ be an arrow in Q_1 such that $N_\alpha(v)$ does not belong to* $\mathrm{rad}^2\langle v\rangle$ *for some $v \in V_i$. Then $\alpha \in Q_1'$.*

Proof. Let $d = \sum_{i\in Q_0} d_i$ and $c = \dim_k\langle v\rangle$. Then $\dim_k \mathrm{rad}\langle v\rangle = c - 1$ and $d \geq c \geq 2$. Since $N_\alpha(v)$ does not belong to $\mathrm{rad}(\mathrm{rad}\langle v\rangle)$, there is a codimension one subrepresentation W of $\mathrm{rad}\langle v\rangle$ which does not contain $N_\alpha(v)$. We choose a basis $\{\epsilon_1, \ldots, \epsilon_d\}$ of the vector space $\bigoplus_{i\in Q_0} V_i$ such that:

- the vector ϵ_b belongs to V_{i_b} for some vertex $i_b \in Q_0$, for any $b \leq d$;
- the vectors $\epsilon_1, \ldots, \epsilon_b$ span a subrepresentation, say $N(b)$, of N for any $b \leq d$;
- $N(c-2) = W$, $\epsilon_{c-1} = N_\alpha(v)$, $N(c-1) = \mathrm{rad}\langle v\rangle$, $\epsilon_c = v$ and $N(c) = \langle v\rangle$.

In fact, $0 = N(0) \subset N(1) \subset N(2) \subset \cdots \subset N(d) = N$ is a composition series of N. In particular, $N_\beta(\epsilon_b)$ belongs to $N(b-1)$, for any $b \leq d$ and any arrow $\beta: i_b \to j$ in Q_1. We take a decreasing sequence of integers

$$p_1 > p_2 > \cdots > p_d$$

and define a group homomorphism $\varphi: k^* \to \mathrm{GL}(\mathbf{d}) = \bigoplus_{i\in Q_0} \mathrm{GL}(V_i)$ such that $\varphi(t)(\epsilon_b) = t^{p_b} \cdot \epsilon_b$ for any $b \leq d$. Observe that

$$N_\beta(\epsilon_b) = \sum_{i<b} \lambda_i \cdot \epsilon_i, \quad \lambda_i \in k, \quad \text{implies} \quad (\varphi(t) * N)_\beta(\epsilon_b) = \sum_{i<b} t^{p_i - p_b} \lambda_i \cdot \epsilon_i$$

for any $b \leq d$ and any arrow $\beta: i_b \to j$ in Q_1. This leads to a regular map $\psi: k \to \overline{\mathcal{O}}_N$ such that $\psi(t) = \varphi(t) * N$ for $t \neq 0$ and $\psi(0) = 0$.

Assume now that $p_{c-1} - p_c = 1$. Applying the induced linear map $\mathcal{T}_{\psi,0}: \mathcal{T}_{k,0} \to \mathcal{T}_{\overline{\mathcal{O}}_N,0}$ and using the fact that $N_\alpha(\epsilon_c) = \epsilon_{c-1}$, we obtain a tangent vector $W = (W_\alpha) \in \mathcal{T}_{\overline{\mathcal{O}}_N,0}$ such that $W_\alpha(\epsilon_c) = \epsilon_{c-1} \neq 0$. Thus $\alpha \in Q_1'$. □

Lemma 3.4. *For any arrow $\alpha: i \to j$ in Q_1, there exists a path ω in Q' of positive length such that $s(\omega) = i$ and $t(\omega) = j$.*

Proof. Since $\mathrm{Ann}(N)$ is an admissible ideal in kQ, there is a vector $v \in V_i$ such that $N_\alpha(v) \neq 0$. Let $\omega = \alpha_m \cdots \alpha_2\alpha_1$ be a longest path from i to j with $N_\omega(v) \neq 0$. Hence $N_\rho(v) = 0$ for any $\rho \in \epsilon_j \cdot (\mathcal{R}_Q)^{m+1} \cdot \epsilon_i$. We show that the path ω satisfies the claim. Let $v_0 = v$ and $v_l = N_{\alpha_l}(v_{l-1})$ for $l = 1, \ldots, m$. According to Lemma 3.3, it is enough to show that $v_l \notin \mathrm{rad}^2\langle v_{l-1}\rangle$ for any $1 \leq l \leq m$. Indeed, if $v_l \in \mathrm{rad}^2\langle v_{l-1}\rangle$ for some l, then $v_m \in \mathrm{rad}^{m+1}\langle v_0\rangle$, or equivalently, $N_\omega(v) = N_\rho(v)$ for some $\rho \in \epsilon_j \cdot (\mathcal{R}_Q)^{m+1} \cdot \epsilon_i$, a contradiction. □

Combining Lemmas 3.2 and 3.4, we get

Corollary 3.5. *The quiver Q does not contain oriented cycles.*

4. Gradings of polynomials on $\mathrm{rep}_Q(\mathbf{d})$

Let $\pi\colon \mathrm{rep}_Q(\mathbf{d}) \to \mathrm{rep}_{Q'}(\mathbf{d})$ denote the obvious $\mathrm{GL}(\mathbf{d})$-equivariant linear projection and let $N' = \pi(N)$. Then $\pi(\mathcal{O}_N) = \mathcal{O}_{N'}$ and we get a dominant morphism

$$\eta = \pi|_{\overline{\mathcal{O}}_N} \colon \overline{\mathcal{O}}_N \to \overline{\mathcal{O}}_{N'}.$$

Lemma 4.1. $\overline{\mathcal{O}}_{N'} = \mathrm{rep}_{Q'}(\mathbf{d})$.

Proof. Since $\mathrm{Ker}(\pi) \cap \mathcal{T}_{\overline{\mathcal{O}}_N,0} = \{0\}$, the morphism η is étale at 0. This implies that the variety $\overline{\mathcal{O}}_{N'}$ is regular at $\eta(0) = 0$ (see [7, III. 5] for basic information about étale morphisms). Since it is contained in $\mathrm{rep}_{Q'}(\mathbf{d})$, it suffices to show that $\mathcal{T}_{\overline{\mathcal{O}}_{N'},0} = \mathrm{rep}_{Q'}(\mathbf{d})$. The latter can be concluded from the induced linear map $\mathcal{T}_{\eta,0}\colon \mathcal{T}_{\overline{\mathcal{O}}_N,0} \to \mathcal{T}_{\overline{\mathcal{O}}_{N'},0}$, which is the restriction of $\mathcal{T}_{\pi,0} = \pi$. $\square$

Let $R = k[X_{\alpha,p,q}]_{\alpha \in Q_1, p \le d_{t(\alpha)}, q \le d_{s(\alpha)}}$ denote the algebra of polynomial functions on the vector space $\mathrm{rep}_Q(\mathbf{d})$ and $\mathfrak{m} = (X_{\alpha,p,q})$ be the maximal ideal in R generated by variables. Here, $X_{\beta,p,q}$ maps a representation $W = (W_\alpha)$ to the (p,q)-entry of the matrix W_β. Using π, the polynomial functions on $\mathrm{rep}_{Q'}(\mathbf{d})$ form the subalgebra $R' = k[X_{\alpha,p,q}]_{\alpha \in Q'_1, p \le d_{t(\alpha)}, q \le d_{s(\alpha)}}$ of R. By Lemma 4.1,

$$I(\overline{\mathcal{O}}_N) \cap R' = \{0\}, \tag{4.1}$$

where $I(\overline{\mathcal{O}}_N)$ stands for the ideal of the set $\overline{\mathcal{O}}_N$ in R.

Let X_α denote the $d_{t(\alpha)} \times d_{s(\alpha)}$-matrix whose (p,q)-entry is the variable $X_{\alpha,p,q}$, for any arrow α in Q_1. We define the $d_j \times d_i$-matrix X_ρ for $\rho \in \varepsilon_j \cdot kQ \cdot \varepsilon_i$, with coefficients in R, in a similar way as for representations of Q.

The action of $\mathrm{GL}(\mathbf{d})$ on $\mathrm{rep}_Q(\mathbf{d})$ induces an action on the algebra R by $(g * f)(W) = f(g^{-1} * W)$ for $g \in \mathrm{GL}(\mathbf{d})$, $f \in R$ and $W \in \mathrm{rep}_Q(\mathbf{d})$. We choose a standard maximal torus T in $\mathrm{GL}(\mathbf{d})$ consisting of $g = (g_i)$, where all $g_i \in \mathrm{GL}(d_i)$ are diagonal matrices. Let $\widetilde{Q}_0$ denote the set of pairs (i,p) with $i \in Q_0$ and $1 \le p \le d_i$. Then the action of T on R leads to a $\mathbb{Z}^{\widetilde{Q}_0}$-grading on R with

$$\deg(X_{\alpha,p,q}) = e_{s(\alpha),q} - e_{t(\alpha),p}, \tag{4.2}$$

where $\{e_{i,p}\}_{(i,p)\in\widetilde{Q}_0}$ is the standard basis of $\mathbb{Z}^{\widetilde{Q}_0}$.

Proposition 4.2. $Q' = Q$.

Proof. Suppose the contrary, which means there is an arrow β in $Q_1 \setminus Q'_1$. Since the quiver Q has no oriented cycles, we can choose β minimal in the sense that any path ω in Q of length greater than 1 with $s(\omega) = s(\beta)$ and $t(\omega) = t(\beta)$ is in fact a path in Q'. We conclude from (3.2) that $X_{\beta,u,v} \in \mathfrak{m}^2 + I(\overline{\mathcal{O}}_N)$ for $u \le d_{t(\beta)}$ and $v \le d_{s(\beta)}$. Since the polynomials $X_{\beta,u,v}$ as well as the ideals $\mathfrak{m}^2$ and $I(\overline{\mathcal{O}}_N)$ are homogeneous with respect to the above grading, there are homogeneous polynomials $f_{\beta,u,v}$ in the ideal $\mathfrak{m}^2$ such that

$$X_{\beta,u,v} - f_{\beta,u,v} \in I(\overline{\mathcal{O}}_N) \quad \text{and} \quad \deg(f_{\beta,u,v}) = e_{s(\beta),v} - e_{t(\beta),u}.$$

Let $\prod_{l \le n} X_{\alpha_l, p_l, q_l}$ be a monomial in R of degree $e_{s(\beta),v} - e_{t(\beta),u}$. Then

$$\begin{aligned}
&\#\{1 \le l \le n \mid s(\alpha_l) = i,\ q_l = r\} - \#\{1 \le l \le n \mid t(\alpha_l) = i,\ p_l = r\} \\
&= \begin{cases} 1 & (i,r) = (s(\beta), v), \\ -1 & (i,r) = (t(\beta), u), \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

Thus by (4.2), up to a permutation of the above variables, we get that $\omega = \alpha_m \cdots \alpha_1$ is a path in Q for some $m \le n$ such that $(s(\alpha_1), q_1) = (s(\beta), v)$, $(t(\alpha_m), p_m) = (t(\beta), u)$ and $q_l = p_{l-1}$ for $l = 2, \dots, m$. Consequently, $\deg(X_{\alpha_{m+1}, p_{m+1}, q_{m+1}} \cdot \dots \cdot X_{\alpha_n, p_n, q_n}) = 0$. Since Q has no oriented cycles, the only monomial in R with degree zero is the constant function 1. Hence $m = n$ and the homogenous polynomial $f_{\beta,u,v}$ is the following linear combination:

$$\begin{aligned}
f_{\beta,u,v} = \sum \lambda(u, \alpha_m, p_{m-1}, \alpha_{m-1}, \dots, p_1, \alpha_1, v) \\
\cdot X_{\alpha_m, u, p_{m-1}} \cdot X_{\alpha_{m-1}, p_{m-1}, p_{m-2}} \cdot \dots \cdot X_{\alpha_2, p_2, p_1} \cdot X_{\alpha_1, p_1, v},
\end{aligned}$$

where the sum runs over all paths $\omega = \alpha_m \cdots \alpha_1$ in Q with $s(\omega) = s(\beta)$, $t(\omega) = t(\beta)$ and positive integers $p_l \le d_{t(\alpha_l)}$ for $l = 1, \dots, m-1$. Since $f_{\beta,u,v}$ belongs to the ideal $\mathfrak{m}^2$, we may assume that $m \ge 2$. Then the arrows $\alpha_1, \dots, \alpha_m$ belong to Q_1', by the minimality of β. In particular, $f_{\beta,u,v}$ belongs to R'.

We claim that the scalars $\lambda(u, \alpha_m, p_{m-1}, \alpha_{m-1}, \dots, p_1, \alpha_1, v)$ do not depend on the integers u, $p_{m-1}, \dots, p_1$ and v. Indeed, take $u' \le d_{t(\beta)}$, $v' \le d_{s(\beta)}$ and $p_l' \le d_{t(\alpha_l)}$ for $l = 1, \dots, m-1$. We choose $g = (g_i)$ in $\mathrm{GL}(\mathbf{d})$ with each g_i being the permutation matrix associated to a specific permutation $\sigma_i \in S_{d_i}$. Then the multiplication by g in the algebra R permutes the monomials in R. We assume that

$$\begin{aligned}
&\sigma_{s(\beta)}(v) = v', \quad \sigma_{s(\beta)}(v') = v, \quad \sigma_{t(\beta)}(u) = u', \quad \sigma_{t(\beta)}(u') = u, \\
&\sigma_{t(\alpha_l)}(p_l) = p_l' \quad \text{and} \quad \sigma_{t(\alpha_l)}(p_l') = p_l, \quad \text{for} \quad l = 1, \dots, m-1.
\end{aligned}$$

Since $g * X_{\beta,u',v'} = X_{\beta,u,v}$, the polynomial

$$f_{\beta,u,v} - g * f_{\beta,u',v'} = g * (X_{\beta,u',v'} - f_{\beta,u',v'}) - (X_{\beta,u,v} - f_{\beta,u,v})$$

belongs to the ideal $I(\overline{\mathcal{O}}_N)$, as the latter is GL(**d**)-invariant. Thus $f_{\beta,u,v} = g * f_{\beta,u',v'}$, by (4.1). Hence the claim follows from the fact that the monomial

$$X_{\alpha_m,u,p_{m-1}} \cdot X_{\alpha_{m-1},p_{m-1},p_{m-2}} \cdot \dots \cdot X_{\alpha_2,p_2,p_1} \cdot X_{\alpha_1,p_1,v}$$

appears in $g * f_{\beta,u',v'}$ with coefficient $\lambda(u', \alpha_m, p'_{m-1}, \alpha_{m-1}, \dots, p'_1, \alpha_1, v')$.

Let Ξ denote the set of all paths ξ in Q' of length greater than 1 with $s(\xi) = s(\beta)$ and $t(\xi) = t(\beta)$. Then there are scalars $\lambda(\xi)$, $\xi \in \Xi$, such that

$$f_{\beta,u,v} = \sum_{\xi=\alpha_m\dots\alpha_1\in\Xi} \lambda(\xi) \cdot \sum_{p_1\le d_{t(\alpha_1)}} \cdots \sum_{p_{m-1}\le d_{t(\alpha_{m-1})}} X_{\alpha_m,u,p_{m-1}} \cdot \dots \cdot X_{\alpha_1,p_1,v}$$

for any $u \le d_{t(\beta)}$ and $v \le d_{s(\beta)}$. This equality means that $f_{\beta,u,v}$ is the (u, v)-entry of the matrix X_ρ, where $\rho = \sum_{\xi\in\Xi} \lambda(\xi) \cdot \xi \in kQ'$. Consequently, the entries of the matrix $X_{\beta-\rho}$ belong to the ideal $I(\overline{\mathcal{O}}_N)$. This implies that $\beta - \rho$ belongs to Ann(N). Since $\beta - \rho$ does not belong to $(\mathcal{R}_Q)^2$, the ideal Ann(N) is not admissible, a contradiction. □

Combining Lemma 4.1 and Proposition 4.2 we get

$$\overline{\mathcal{O}}_N = \mathrm{rep}_Q(\mathbf{d}). \tag{4.3}$$

Hence the following lemma finishes the proof of Theorem 2.1.

Lemma 4.3. Ann(N) = {0}.

Proof. Suppose the contrary, that there is a non-zero element ρ in $\varepsilon_j \cdot \mathrm{Ann}(N) \cdot \varepsilon_i$ for some vertices i and j. Observe that the set of representations $W = (W_\alpha)$ in $\mathrm{rep}_Q(\mathbf{d})$ such that $W_\rho = 0$ is closed and GL(**d**)-invariant. Hence $W_\rho = 0$ for any representation $W = (W_\alpha)$ in $\mathrm{rep}_Q(\mathbf{d})$, by (4.3). Of course, ρ is a linear combination of paths in Q of length greater than 1 with $s(\omega) = i$ and $t(\omega) = j$. Let ω_0 be a path appearing in ρ with coefficient $\lambda \neq 0$. We choose a representation $W = (W_\alpha)$ in $\mathrm{rep}_Q(\mathbf{d})$ such that W_α is the matrix whose (1, 1)-entry is 1 and the other entries are 0 if the arrow α appears in the path ω_0, and $W_\alpha = 0$ otherwise. Then the (1, 1)-entry of W_ρ equals λ, a contradiction. □

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