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Mathematical Formulations of Geologic Mapping Process —Algorithms for an Automatic System—

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(With 21 Figures and 2 Tables)

Abstract

The application of computer to geologic mapping offers several merits in practical use such as rapid generation and reduction of laborious manual procedures, as well as quantitative evaluation of geological data. This paper presents mathematical formulations for the fundamental process of geologic mapping, and the algorithms for the construction of computerized system. The principles of geology are formulated in terms of relations between strata and of relations between the relations. For primary definition, five axioms A1,··· A5 are postulated.

The inference rules for ordering the stratigraphic sequence from field-observation data are derived from first three axioms :

[A1] $W \cup W^{-1} \cup E = I$
[A2] $L \subset W$
[A3] $C \subset K$.

These axioms are based on the principle of original horizontality, the principle of original lateral extension and the law of superposition.

Contact surfaces between strata are represented by the boundary surfaces that divide the 3-D space X into two subspaces. Axioms A4 and A5 are postulated as the formulation for C1 and C2 types of boundary surfaces simplified from conformity and unconformity ; the successive sedimentation without erosion and the sedimentation after erosion, respectively. These two axioms provide several inference rules to determine uniquely the locational relation between strata and boundary surfaces based on field observations. The locational relation is represented by a function t called "a logical model for locational relation".

The five axioms A1, ···, A5 provide practical algorithms to construct a function $g{:}X \rightarrow B$ that assigns a unique stratum $b \in B$ to every point in the 3-D space X on the basis of field observations. Thus the logical structure of geologic mapping is formulated systematically based on the axiom system A1, ···, A5 modeling a geologic structure consisting of sedimentary layers without faulting nor overfolding. According to the formulation, computerized geologic mapping system "CIGMA" is constructed to create a geologic map based on the observations through automatic data processing. More complex geologic structures will be introduced into the geologic mapping system through further formulations of geologic principles and knowledge.

Key Words: Computerized geologic mapping system, Axiom system, Set theory, Binary relation, Logical models of geologic structures, Function g, CIGMA

1. Introduction

The computer is useful for processing of voluminous data and laborious repetition of operation in analyzing experimental and observational data (AGTERBERG, 1974; DAVIS,

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1986; FISHER *et al.*, 1987; SHIONO *et al.*, 1988a; 1988b; 1990a; 1990b; 1992 etc.) However, many problems have remained unresolved in computer processing of geologic data, compared with data processing in other sciences. Many difficulties of computer processing arise from the fact that geologic principles are described in the natural language which we use for ordinary communication.

This paper attempts a mathematical formulation of geologic methods which can be used as a theoretical basis to construct a computerized geologic mapping system.Computer drawing of a geologic map provides several merits as follows:

Automatic mapping
Consistency of geologic map
Reproducibility of the same map from same data
Easy revision of maps
Multiplicity of graphic presentation.

As basic studies for computer processing of a geologic map, SHIONO *et al.*(1987) presented the computer algorithm to determine geologic surfaces, and MASUMOTO *et al.*(1986; 1987) and SAKAMOTO *et al.*(1988; 1991) developed Basic programs for graphical outputs of geologic maps. However, these studies do not treat of fundamental works of geologic mapping such as determination of the stratigraphic sequence and inference of geologic structures.

For computer processing of geologic data, it is necessary to formulate mathematically geologic principles that are usually described in the natural language, and to establish the geologic inference system. The formulations lead to express inference rules explicitly in terms of mathematical formulae, which can be translated into the computer algorithm. Therefore, WADATSUMI *et al.*(1987) and SHIONO and WADATSUMI (1988) proposed an idea of GEO-LOGICS (Geology-Oriented Logical System) which directs the reconstruction of a logical system of geology in mathematical form. It is expected that studies on GEO-LOGICS provide theoretical bases for developments of effective and consistent inference algorithms.

The computer mapping system based on GEO-LOGICS provides additional merits as follows:

Effective usage of geologic principles
Automatic judgement by computer instead of expert geologist
Automatic inference of geologic structure appropriate to a given set of data .

This paper analyzes the working process for constructing a geologic map from a viewpoint of GEO-LOGICS. Geologic concepts and basic assumptions are defined strictly in terms of set theory, and the computer algorithms for data processing are derived from these assumptions.

The readers are requested to refer to textbooks of set theory and/or discrete mathematics (e.g., GILL, 1976; LIU, 1986) for details regarding the mathematical notations used in this paper.

2. Basic Framework of Geologic Mapping by Computer

Prior to the mathematical description of geologic concepts, several sets and a 3-D space are introduced and the outline of discussions is reviewed in this section.

2.1 Three-dimensional Space and Strata

Partitioning of a 3-D space into strata is one of main concerns in this paper.
Two basic sets X and B are defined as follows :

X is a 3-D Euclidean space in which the orientation and the distance are defined in ordinary sense.
$B=\{b_1, b_2, \cdots, b_n\}$ is a set of all names of strata distributed in X, where n is the number of strata.

If there is a rule to assign a unique name of stratum name in the set B to each point in X, then the rule can be said to be a function from a set X into a set B, which is denoted by $g:X \rightarrow B$. The function g defines the distribution of strata in the 3-D space, and:

$$g(p)=b_i$$

is interpreted to show that a point p is included in a stratum b_i.

The space where a stratum is distributed is represented by an inverse image of the function g.

$$g^{-1}(b_i)=\sigma(b_i)=\{p \mid g(p)=b_i,\ p \in X\}$$

shows the space where a stratum b_i is distributed. Let A be the range of the function σ:

$$A=\{\sigma(b_i) \mid b_i \in B\}.$$

Then, A is a partition of the set X, and the function $\sigma:B \rightarrow A$ is a bijective mapping from B to A. $\sigma(b_i)$ is called here a stratum named b_i.

Each stratum is bounded by some contact surfaces. Let $S=\{s_1, s_2, \cdots, s_m\}$ be a set of boundary surfaces defined as surfaces which include at least one contact surface between strata, and also divide a 3-D space into two subspaces. Then, it is possible to represent each stratum using the boundary surfaces. This indicates that a function $g:X \rightarrow B$ can be defined by some combinations of boundary surfaces.

2.2 Outline of Logical System for Deriving the Function g

Five axioms A1, $\cdots$, A5, as described later, provide basic algorithms to construct a function $g:X \rightarrow B$. Axioms A1, A2, A3 are basic principles of geology concerning with relations between strata. These axioms introduce a principle to infer the stratigraphic sequence from observed relations. Axioms A4 and A5 are postulates concerning with C1 and C2 types of boundary surfaces, respectively. These types correspond to a conformity and an unconformity, respectively. Axioms A4 and A5 introduce the locational relation between strata and boundary surfaces. The locational relation is represented by a function $t{:}B\times S\to\{-1,\ 0,\ +1\}$ called a logical model for locational relation.

For the use of inclinational data, inclinational relation along vertical axes is inferred. Inclinational relation between strata and boundary surfaces is represented by a function $p{:}B\times S\to\{0, 1\}$ called a logical model for inclinational relation. Data required to determine a boundary surface are selected using logical models both for locational relation and for inclinational relation. Each 3-D boundary surface is determined by the method of constrained optimization. Combining logical models and 3-D boundary surfaces, we can define the function $g{:}X\to B$ that assigns a unique name of stratum to every point in the 3-D space X. Finally we can draw a geologic map which illustrate the distributions of strata defined by the function g.

2.3 Main Flow of Data Processing

Figure 1 shows a flow diagram of data processing based on above theories.

(0) Preparation of input data :
Before starting to process, we prepare data obtained from the field survey. Observational data are translated for computer processing as follows :

$$
\begin{array}{ccccccccc}
x_1, & y_1, & z_1, & \xi_1, & \eta_1, & \alpha_1, & \beta_1, & \tau_1, & \pi_1 \\
\cdot\cdot, & \cdot\cdot, & \cdot\cdot, & \cdot\cdot, & \cdot\cdot, & \cdot\cdot, & \cdot\cdot, & \cdot\cdot, & \cdot\cdot \\
x_r, & y_r, & z_r, & \xi_r, & \eta_r, & \alpha_r, & \beta_t, & \tau_r, & \pi_r \\
\cdot\cdot, & \cdot\cdot, & \cdot\cdot, & \cdot\cdot, & \cdot\cdot, & \cdot\cdot, & \cdot\cdot, & \cdot\cdot, & \cdot\cdot \\
x_N, & y_N, & z_N, & \xi_N, & \eta_N, & \alpha_N, & \beta_N, & \tau_N, & \pi_N
\end{array}
$$

where x, y and z are coordinates of observation points in an orthogonal coordinate system in which x, y and z axes are oriented eastward, northward and upward, respectively, ξ and η are strike and dip, respectively, α and β are the names of strata, and τ and π are parameters to represent structural relations. τ is a parameter for contact type explained in Section 4, and π is a parameter for inclinational relation explained in Section 5.

(1) Inference of the stratigraphic sequence :
Observed stratigraphic relations of strata are represented in the form of a relation matrix to determine the stratigraphic sequence by matrix operations.

(2) Construction of the logical models of geologic structures :
According to given types of boundary surfaces, locational relations between strata and boundary surfaces are inferred to construct a logical model for locational relation and a logical model for inclinational relation.

(3) Determination of the boundary surfaces :
The boundary surfaces are determined as the smoothest surfaces that satisfy both locational and inclinational data selected from the observations referring to two kinds

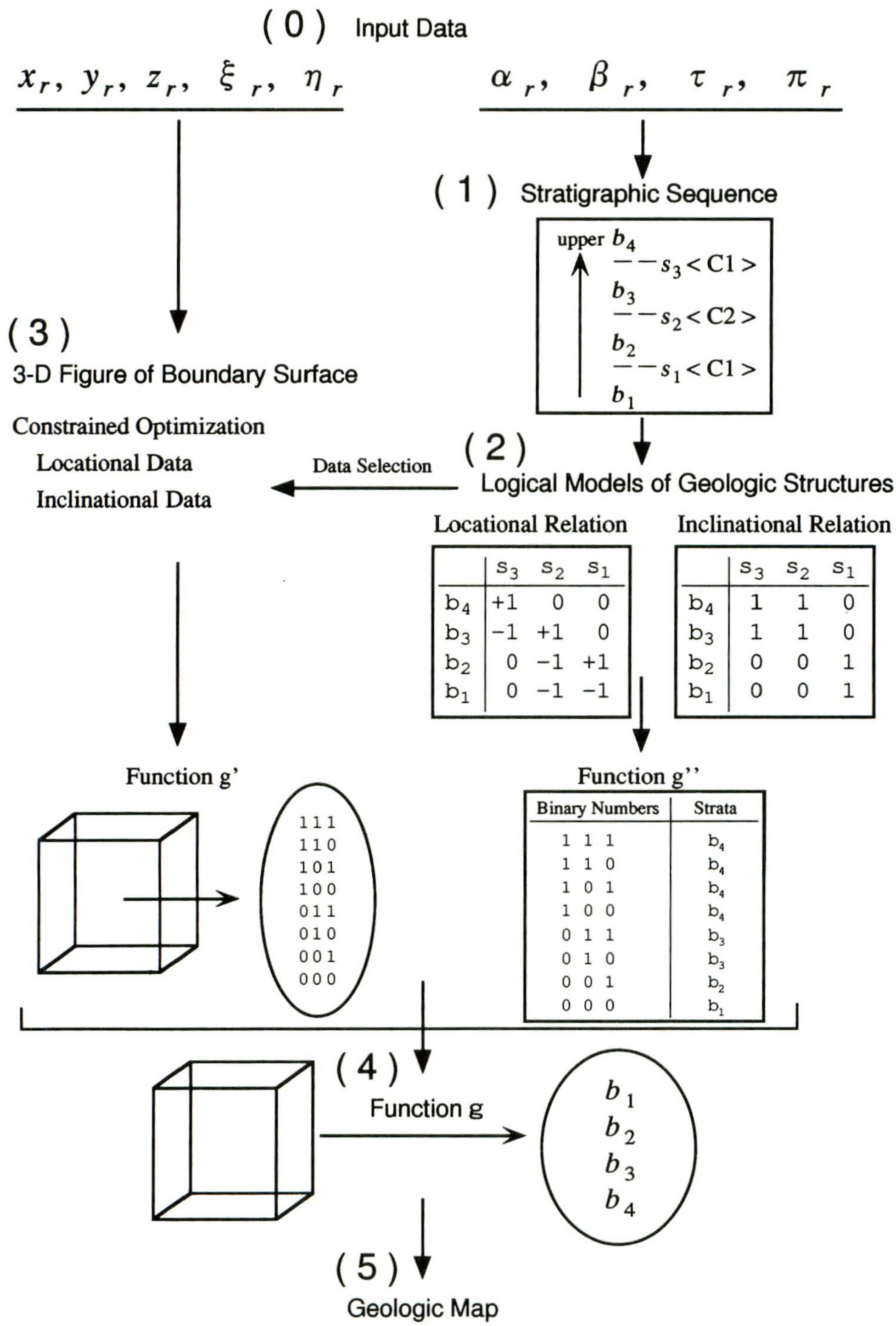


Fig. 1. Flow diagram of computerized geologic mapping process.

of logical models.

(4) Creation of a function $g:X \to B$:
The function $g:X \to B$ is constructed by combining 3-D figures of boundary surfaces and the logical model for locational relation.

(5) Graphical presentation :
Finally, a 3-D geologic map is drawn on a display by coloring the strata respective to all grid cells of the topographic surface and four side sections.

3. Inference of Stratigraphic Sequence

As a first step of geologic mapping, we consider the inference of the stratigraphic sequence.

3.1 Ordering of Strata

3.1.1 Formulation of Geologic Principles

In this section, let us discuss logical meanings of the following geologic principles :

- the law of superposition : The rocks in a given succession of strata decrease in age from the bottom to the top ;
- the principle of original horizontality : The upper surfaces of sedimentary deposits initially come to rest essentially parallel to the surface of deposition, which is usually parallel to the horizon or inclined to it at relatively low angles ;
- the principle of original lateral extension : A given stratum of rock resulting from the dumping of sediment into a basin must eventually thin out in all direction, unless it abuts a steep margin of preexisting matter.

These three principles were enunciated first in 1669 by N.STENO. His original statements are translated in English by J.G.WINTER and reprinted in CLOUD (1970).

According to SHIONO and WADATSUMI (1992), three principles are formulated as follows :

Axiom A1

$$(\forall b_i,\ b_j \in B)(b_i W b_j \lor b_j W b_i \lor b_i = b_j) \text{ i.e., } W \cup W^{-1} \cup E = I$$

Axiom A2

$$(\forall b_i,\ b_j \in B)(b_i L b_j \Rightarrow b_i W b_j) \text{ i.e., } L \subset W$$

Axiom A3

$$(\forall b_i,\ b_j \in B)(b_i C b_j \Rightarrow b_i K b_j) \text{ i.e., } C \subset K$$

Definitions of binary relations W, L, C, K on the set B are given in Table 1 and Fig. 2. Trivial relations I, E, O and properties of relations are listed in Table 2.

Axiom A1 shows that $b_i W b_j$, $b_j W b_i$ or $b_i = b_j$ holds true for all b_i and b_j. Axiom A2 shows that if b_i is stratigraphically lower than b_j, then $\sigma(b_i)$ is below $\sigma(b_j)$ along all vertical lines through both $\sigma(b_i)$ and $\sigma(b_j)$. Axiom A3 shows that if $\sigma(b_i)$ is under $\sigma(b_j)$, then b_i is older than b_j. Axioms A1 and A2 are formulations of both principles of original horizontality and original lateral extension, and Axiom A3 is a formulation of

Table 1. Relations between strata.

Symbol	Definition	Geologic interpretation
T	$b_iTb_j \Leftrightarrow \sigma(b_i)^- \cap \sigma(b_j)^- \neq \phi$	$\sigma(b_i)$ touches $\sigma(b_j)$.
V	$b_iVb_j \Leftrightarrow \exists l\,((l \cap \sigma(b_i) \neq \phi) \wedge (l \cap \sigma(b_j) \neq \phi)$	$\sigma(b_i)$ and $\sigma(b_j)$ are piled up along a vertical line l.
W	$b_iWb_j \Leftrightarrow \forall l\,(sup\{l \cap \sigma(b_i)\} \leqq inf\{l \cap \sigma(b_j)\}$	$\sigma(b_i)$ is below $\sigma(b_j)$ along all vertical number lines l through both strata.
C	$C = T \cap V \cap W$	$\sigma(b_i)$ is under $\sigma(b_j)$.
L	$L = C^* = C \cup C^2 \cup \cdots$	$\sigma(b_i)$ is stratigraphically lower than $\sigma(b_j)$.
L_E	$L_E = L \cup E$	(L_E is a reflexive and transitive closure of C .)
K	$b_iKb_j \Leftrightarrow sup\{\tau(\sigma(b_i))\} \leqq inf\{\tau(\sigma(b_j))\}$	Any points of $\sigma(b_i)$ are older than any points of $\sigma(b_j)$.
K_E	$K_E = K \cup E$	(K_E is reflexive.)
T_O	$b_iT_Ob_j \Leftrightarrow \exists l\,(l \cap \sigma(b_i)^- \cap \sigma(b_j)^- \neq \phi)$	$\sigma(b_i)$ touches $\sigma(b_j)$ at an outcrop along a vertical line through both strata.
W_O	$b_iW_Ob_j \Leftrightarrow (b_i \neq b_j) \wedge (\exists l\,((l \cap \sigma(b_i) \neq \phi)$ $\wedge (l \cap \sigma(b_j) \neq \phi)$ $\wedge (sup\{l \cap \sigma(b_i)\} \leqq inf\{l \cap \sigma(b_j)\})))$	$\sigma(b_i)$ is below $\sigma(b_j)$ along a vertical number line l through both strata.
C_O	$C_O = T_O \cap W_O$	A part of $\sigma(b_i)$ is under $\sigma(b_j)$ at an outcrop.
L_O	$L_O = C_O{}^* = C_O \cup C_O{}^2 \cup \cdots$	(L_O is a transitive closure of C_O .)
L_{OE}	$L_{OE} = L_O \cup E$	(L_{OE} is a reflexive and transitive closure of C_O .)

$\sigma(x)^-$ is a closure of $\sigma(x)$, i.e., closed space including boundary points.
sup and *inf* are upper and lower limits, respectively.
τ is a function which assign an age $\tau(p)$ to a point in X.

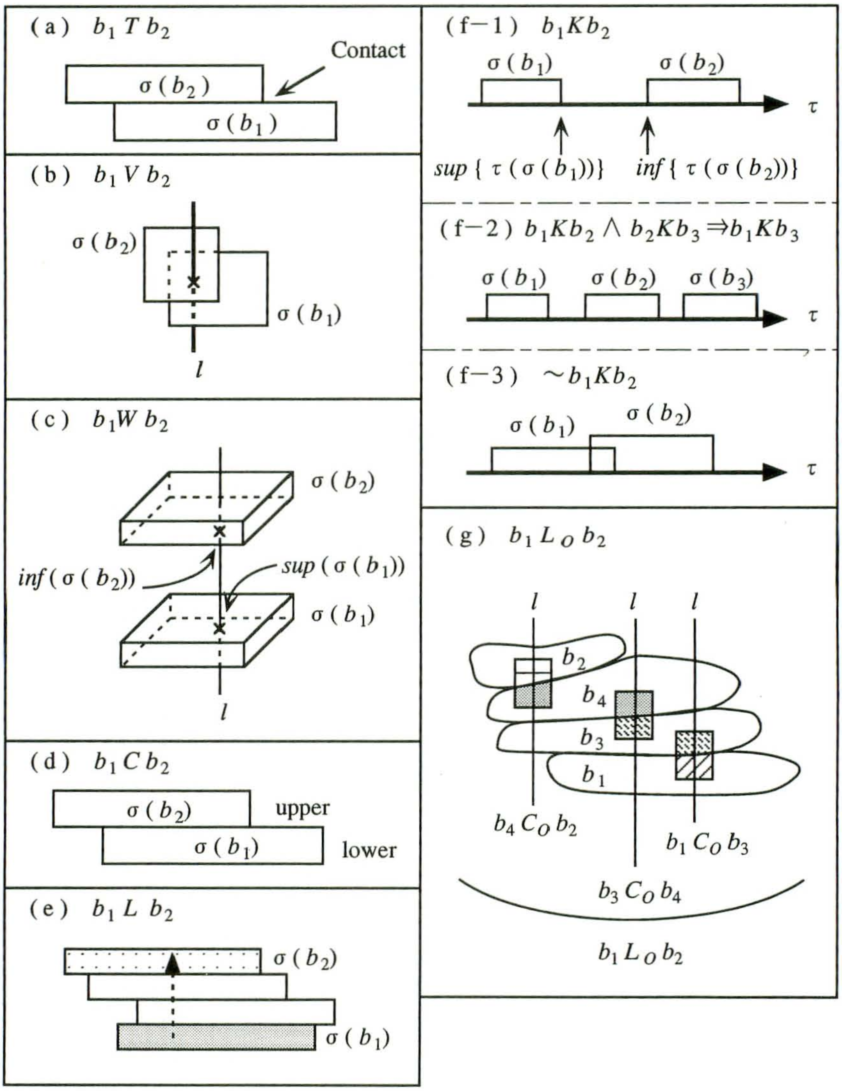


Fig. 2. Relations between strata.
(a) Relation T. (b) Relation V. (c) Relation W. (d) Relation C. (e) Relation L. (f) Relation K and the characters. (f-1) Definition of K. (f-2) Transitive property of K. (f-3) If the ages of b_1 and b_2 are overlapped, then neither b_1Kb_2 nor b_2Kb_1 is satisfied. (g) Relation L_O.

the law of superposition.

3.1.2 Stratigraphic Sequence

Axioms A1, A2 and A3 introduce the following theorems. Detailed proofs are reported by SHIONO and WADATSUMI (1992).

[Property introduced from Axiom A1]

Table 2. Trivial relations and properties.

Mathematical term	Symbolic expression	
universal relation	$I : B \times B$	
identity relation	$E : \{(b_i, b_i) \mid b_i \in B\}$	
null relation	$O : \phi$ (empty set)	
reflexive property	$R = R \cup E$	$b_i R b_i$ for all $b_i \in B$
symmetric property	$R = R^{-1}$	$b_i R b_j \Rightarrow b_j R b_i$
antisymmetric property	$R \cap R^{-1} \subset E$	$b_i R b_j$ and $b_j R b_i \Rightarrow b_i = b_j$
transitive property	$R \cdot R \subset R$	$b_i R b_k$ and $b_k R b_j \Rightarrow b_i R b_j$
transitive closure	$R^* = R \cup R^2 \cup \cdots$	

We can translate Axiom A1 into :

$$b_iWb_j \vee b_jWb_i \vee b_i = b_j \Leftrightarrow ((b_i \neq b_j \wedge \sim b_jWb_i) \Rightarrow b_iWb_j)$$

and the definition of W_O into :

$$b_iW_Ob_j \Leftrightarrow b_i \neq b_j \wedge \sim b_iW^{-1}b_j$$

where $\sim P$ represents the negation of a proposition P. From these two formulae, we obtain :

Theorem 3.1

$$W_O \subset W.$$

Since T_O has a property such that :

$$b_iT_Ob_j \Rightarrow b_iTb_j \wedge b_iVb_j,$$

we obtain :

$$\begin{aligned} b_iC_Ob_j &\Leftrightarrow b_iW_Ob_j \wedge b_iT_Ob_j \\ &\Rightarrow b_iW_Ob_j \wedge b_iTb_j \wedge b_iVb_j \\ &\Rightarrow b_iWb_j \wedge b_iTb_j \wedge b_iVb_j \\ &\Rightarrow b_iCb_j. \end{aligned}$$

Thus, we have:

Theorem 3.2

(i) $$C_O \subset C,$$

(ii) $$L_O \subset L.$$

Theorems 3.1 and 3.2 provide an inference rule that we can know relations W, C and L on the set B from observations of strata partially exposed at outcrops.

[Property introduced from Axiom A2]
From the definition of W, we have :

$$W \cap W^{-1} = \sim V,$$

and from Axiom A2, we have :

$$\begin{aligned} b_i(L\cap C^{-1})b_j &\Rightarrow b_iWb_j\cap b_iC^{-1}b_j \\ &\Rightarrow b_iWb_j\cap(b_jT^{-1}b_j\cap b_iV^{-1}b_j\cap b_iW^{-1}b_j) \\ &\Rightarrow b_iOb_j. \end{aligned}$$

Hence, we have :

$$L\cap C^{-1}=O.$$

Since for all b_i, $b_j\in B$, b_iLb_j and $b_jC^{-1}b_i$ are not simultaneously satisfied, we have :

Theorem 3.3

(i) $$L\cap L^{-1}=O,$$

(ii) $$L_E\cap L_E^{-1}=E.$$

Thus, we have :

Theorem 3.4

The relation L_E on the set B is reflexive, antisymmetric and transitive, that is, L_E is a partial ordering.

Theorem 3.4 shows that strata can be arranged in a linear sequence :

$$b'_1,\ b'_2,\ \cdots,\ b'_n \quad (b'_iL_Eb'_j\Rightarrow i\leqq j).$$

Axiom A2 is important for arranging strata in an order of piling from the bottom to the top.

[Property introduced from Axioms A1 and A2]
Theorems 3.2 and 3.3 derive :

$$L_O\cap L_O^{-1}\subset L\cap L^{-1}=O$$
$$\therefore L_{OE}\cap L_{OE}^{-1}=E.$$

Thus, we have :

Theorem 3.5

The relation $L_{OE}=L_O\cup E$ on the set B is a partial ordering.

Theorem 3.5 shows that the elements of B can be arranged in a linear sequence in the same manner as the case of L_E. From Theorems 3.2 and 3.5, it is clear that :

Theorem 3.6

$$L_{OE}\subset L_E.$$

Theorem 3.6 provides a theoretical basis to infer the piling order of strata from the observations at outcrops.

When we have either $b_iL_Eb_j$ or $b_jL_Eb_i$ for all b_i, $b_j\in B$, that is, we have :

$$L_E\cup L_E^{-1}=I,$$

L_E is a total ordering. Then, the set B can be rearranged linearly such that :

$$b'_1,\ b'_2,\ \cdots,\ b'_n \quad (b'_i L_E b'_j \Leftrightarrow i \leqq j).$$

This shows the order in which strata are piled up from lowest stratum to the upper most stratum.

[Property introduced from Axiom A3]

From the definition of K, it is clear that the relation K_E is a partial ordering. Directly from Axiom A3, we have :

Theorem 3.7

$$L_E \subset K_E.$$

[Property introduced from Axioms A1, A2 and A3]

Theorems 3.6 and 3.7 introduce a rule to infer the stratigraphic sequence from observations at outcrops, as follows :

Theorem 3.8

$$L_{OE} \subset K_E.$$

Theorem 3.8 provides a rule to infer the age relationK_E from the relation L_{OE} obtained at outcrops. Thus, we have a very important geological inference rule which states that the piling order of strata from the lowest one to the uppermost one determined from field observations represents the order of formation of strata from the oldest one to the youngest one.

Figure 3 illustrates the derivational process from Axioms A1, A2 and A3 to the rule of inference.

3.2 Algorithm for Inference of Stratigraphic Sequence

The logical operation of binary relations can be performed by the two methods ; one is the method based on relation matrices (Burns, 1975; Shiono and Wadatsumi, 1988, 1991; Sakamoto and Shiono, 1992) and the other is one based on a symbolic operation language (Sakamoto and Shiono, 1990). The former method is more useful to construct a geologic mapping system because it is easy to combine other numerical calculations such as determinations of surfaces, and also it is convenient to inspect the results of operations. The following describes an algorithm for inference of the stratigraphic sequence using relation matrices based on Sakamoto and Shiono (1992). Since the entry of relation matrix has either 0 or 1, addition and multiplication are calculated as follows :

$$0+0=0,\ 0+1=1+0=1+1=1,$$
$$0 \cdot 0=0 \cdot 1=1 \cdot 0=0,\ 1 \cdot 1=1.$$

Suppose that the relation C_O obtained at outcrops is represented in the form as follows :

$$—,\ —,\ —,\ —,\ —,\ \alpha_r,\ \beta_r,\ —,\ —$$

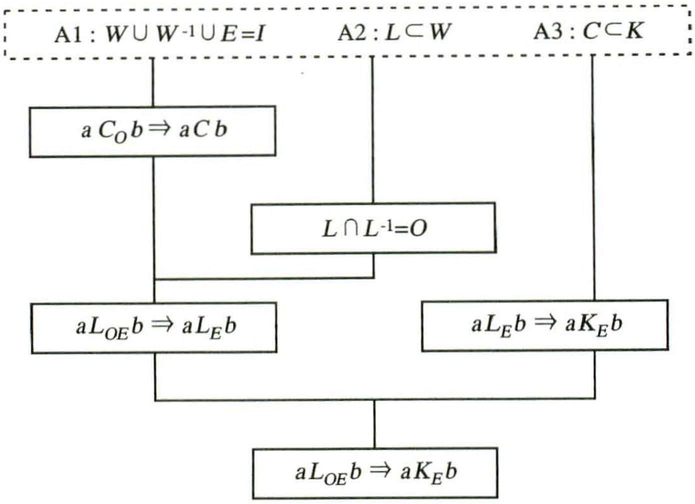


Fig. 3. Relations between theorems introduced from Axioms A1, A2 and A3 (after SHIONO and WADATSUMI, 1992).

where α_r and β_r are names of strata and gives the relation $\alpha_t C_O \beta_r$, i.e., a part of α_r is under a part of β_r at the r-th outcrop. The stratigraphic sequence is inferred through the following steps :

(i) Let B be a set of all strata listed in input data.

(ii) Construct a relation matrix $\mathbf{C}$ by assigning **1** to (i,j) entry of the matric $\mathbf{C}$ if $b_i C_O b_j$, based on a inference rule $C_O \subset C$.

(iii) Construct a relation matrix $\mathbf{L_E}=\mathbf{E}+\mathbf{C}+\mathbf{C}^2+\cdots+\mathbf{C}^{n-1}$.

(iv) Let l'_{ij} be (i,j) entry of the matrix $\mathbf{L_E}$. If $l'_{ij}=0$ or $l'_{ji}=0$ for $i\neq j$, that is, $\mathbf{L_E}$ is antisymmetric, then go to step (v). If not, halt the processing after showing the pair (b_i, b_j) of $l'_{ij}=l'_{ji}=1$.

(v) Arrange the element of the set B in such an order that 1 entries concentrate in lower triangle of matrix $\mathbf{L_E}$.

(vi) If $l'_{ij}=l'_{ji}=0$ for some i, j $(i\neq j)$, then L_E is not a total ordering. If L_E is a total ordering, then the order in (v) is the stratigraphic sequence.

For example, observations at outcrops shown by circles in Fig. 4(a) are described as follows :

$$—, —, —, —, —, b_1, b_3, —, —$$
$$—, —, —, —, —, b_2, b_4, —, —$$
$$—, —, —, —, —, b_3, b_5, —, —$$
$$—, —, —, —, —, b_4, b_1, —, —$$
$$—, —, —, —, —, b_4, b_3, —, —$$

Then, the relation matrix $\mathbf{C}$ in step (ii) is :

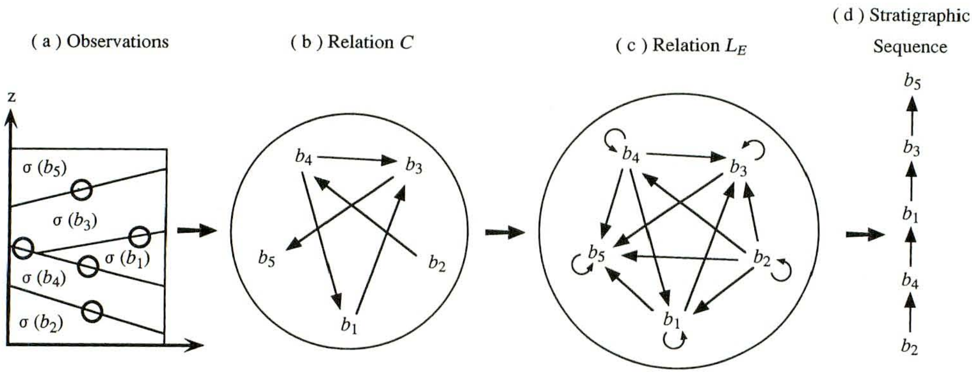


Fig. 4. Inference process of stratigraphic sequence.
Relation C_o observed at the circle point in (a) is represented by the graph as shown in (b). $b_4 \rightarrow b_5$ means $b_4 C b_5$. (c) shows relation L_E inferred from C. If L_E is a total ordering, then we obtained stratigraphic sequence as shown in (d).

$$\begin{array}{c} \\ b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{array} \begin{array}{c} \begin{array}{ccccc} b_1 & b_2 & b_3 & b_4 & b_5 \end{array} \\ \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

Fig. 4(b) shows the graph of **C**. $\mathbf{L_E}$ in step (iii) becomes :

$$\begin{array}{c} \\ b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{array} \begin{array}{c} \begin{array}{ccccc} b_1 & b_2 & b_3 & b_4 & b_5 \end{array} \\ \begin{pmatrix} 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

Fig. 4(c) shows the graph of $\mathbf{L_E}$. Finally, $\mathbf{L_E}$ in step (v) becomes :

$$\begin{array}{c} \\ b_5 \\ b_3 \\ b_1 \\ b_4 \\ b_2 \end{array} \begin{array}{c} \begin{array}{ccccc} b_5 & b_3 & b_1 & b_4 & b_2 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} \end{array}$$

Then, the order $b_2 \rightarrow b_4 \rightarrow b_1 \rightarrow b_3 \rightarrow b_5$ shows the stratigraphic sequence from the bottom to the top.

4. Construction of Logical Models of Geologic Structures

Logical models of geologic structures represent logical relations between strata and boundary surfaces. In this paper, we consider two models ; a logical model for locational relation explained in this section and a logical model for inclinational relation explained in next section.

4.1 Locational Relation between Strata and Boundary Surfaces

4.1.1 Types of Boundary Surfaces

Let B be a set of all names of strata distributed in the 3-D space X, and suppose that all elements of B are enumerated linearly as follows :

$$b_i K_E b_j \Leftrightarrow 1 \leqq i \leqq j \leqq n.$$

Let A be a set of all subspaces $\sigma(b_1), \cdots, \sigma(b_n)$, where each stratum is distributed. Since $\sigma(b_i)$ do not include its boundary, $\sigma(b_i)^-$ including boundary points is used in this section. When the 3-D space X consists of $\sigma(b_1), \cdots, \sigma(b_n)$, we have :

$$\sigma(b_1)^- \cup \sigma(b_2)^- \cup \cdots \cup \sigma(b_n)^- = X.$$

Let S be a set of all boundary surfaces $s_1, \cdots, s_{n-1}$, where $s_k (1 \leqq k \leqq n-1)$ is the surface which includes the contact surface between two successive strata b_k and b_{k+1} and divides X into two subspaces s_k^{+1} and s_k^{-1}. In order to simplify the computer algorithms, we assume here that every boundary surface s_k $(k=1, \cdots, n-1)$ is represented by a single-valued function $z=s_k(x, y)$. Then, subspaces s_k^{+1} and s_k^{-1} give half spaces above and below the surface $z=s_k(x, y)$, respectively :

$$s_k^{+1} = \{(x, y, z) | z \geqq s_k(x, y)\},$$

$$s_k^{-1} = \{(x, y, z) | z \leqq s_k(x, y)\},$$

Then, the intersection of two subspaces s_k^{+1} and s_k^{-1} gives a surface s_k :

$$s_k^{+1} \cap s_k^{-1} = s_k$$

and the union of these is the universal space X :

$$s_k^{+1} \cup s_k^{-1} = X.$$

From the definition of s_k, it is clear that :

$$\sigma(b_{k+1})^- = (\sigma(b_k)^- \cup \sigma(b_{k+1})^-) \cap s_k^{+1}$$

$$\sigma(b_k)^- = (\sigma(b_k)^- \cup \sigma(b_{k+1})^-) \cap s_k^{-1} \quad (k=1, \cdots, n-1).$$

Similarly, a fact that a boundary surface s_t divides a subspace $\sigma(b_r)^- \cup \cdots \cup \sigma(b_k)^-$ into two subspaces $\sigma(b_r)^- \cup \cdots \cup \sigma(b_t)^-$ in the lower side and $\sigma(b_{t+1})^- \cup \cdots \cup \sigma(b_k)^-$ in

(a) $\sigma(b_2) \cup \sigma(b_3) = (\sigma(b_1) \cup \sigma(b_2) \cup \sigma(b_3) \cap s_1^{+1}$
$\sigma(b_1) = (\sigma(b_1) \cup \sigma(b_2) \cup \sigma(b_3)) \cap s_1^{-1}$

(b) $\sigma(b_2) \cup \sigma(b_3) \cup \sigma(b_4) = (\sigma(b_1) \cup \sigma(b_2) \cup \sigma(b_3) \cup \sigma(b_4)) \cap s_1^{+1}$
$\sigma(b_1) = (\sigma(b_1) \cup \sigma(b_2) \cup \sigma(b_3) \cup \sigma(b_4)) \cap s_1^{-1}$

Fig. 5. Examples of C1 type boundary surfaces.
(a) s_1 is a surface dividing a series of successive strata into (b_1) and (b_2, b_3).
(b) s_1 divides a series of successive strata into (b_1) and (b_2, b_3, b_4).

(a) $\sigma(b_5) = (\sigma(b_1) \cup \sigma(b_2) \cup \sigma(b_3) \cup \sigma(b_4) \cup \sigma(b_5)) \cap s_4^{+1}$
$\sigma(b_1) \cup \sigma(b_2) \cup \sigma(b_3) \cup \sigma(b_4) = (\sigma(b_1) \cup \sigma(b_2) \cup \sigma(b_3) \cup \sigma(b_4) \cup \sigma(b_5)) \cap s_4^{-1}$

(b) $\sigma(b_3) = (\sigma(b_1) \cup \sigma(b_2) \cup \sigma(b_3)) \cap s_2^{+1}$
$\sigma(b_1) \cup \sigma(b_2) = (\sigma(b_1) \cup \sigma(b_2) \cup \sigma(b_3)) \cap s_2^{-1}$

Fig. 6. Examples of C2 type boundary surfaces.
(a) s_4 is a surface dividing a series of successive strata $(b_1, \cdots, b_5)$ into (b_1, b_2, b_3, b_4) and (b_5).
(b) s_2 divides a series of successive strata (b_1, b_2, b_3) into (b_1, b_2) and (b_3).

the upper side can be expressed by :

$$\left.\begin{aligned} \sigma(b_{t+1})^- \cup \cdots \cup \sigma(b_k)^- &= (\sigma(b_r)^- \cup \cdots \cup \sigma(b_t)^- \cup \sigma(b_{t+1})^- \cup \cdots \cup \sigma(b_k)^-) \cap s_t^{+1} \\ \sigma(b_r)^- \cup \cdots \cup \sigma(b_t)^- &= (\sigma(b_r)^- \cup \cdots \cup \sigma(b_t)^- \cup \sigma(b_{t+1})^- \cup \cdots \cup \sigma(b_k)^-) \cap s_t^{-1}. \end{aligned}\right\} \quad (4.1)$$

Then we say that a series of successive strata $(b_r, b_{r+1}, \cdots, b_k)$ $(1 \leqq r < k \leqq n)$ is divided into two series of successive strata $(b_r, \cdots, b_t)$ and $(b_{t+1}, \cdots, b_k)$ by a boundary surface s_t $(r \leqq t < k)$. Using this notation, we postulate Axioms A4 and A5 for C1 and C2 types of boundary surfaces, respectively. The C1 and C2 types of boundary surfaces provide simplified models for the conformity (Fig. 5) and the unconformity (Fig. 6), respectively.

Axiom A4 : C1 type of boundary surface

A surface s_k called a C1 type of boundary surface or simply "C1" implies that there exists s_t ($r \leqq t < k$) such that divides a series of successive strata (b_r, b_{r+1}, $\cdots$, b_k, b_{k+1}) into two series of successive strata (b_r, $\cdots$, b_t) and (b_{t+1}, $\cdots$, b_{k+1}), when s_t is a boundary surface dividing a series of successive strata (b_r,b_{r+1}, $\cdots$, b_k) ($1 \leqq r < k+1 \leqq n$) into two series of successive strata (b_r, $\cdots$, b_t) and (b_{t+1}, $\cdots$, b_k).

Therefore, if we have equations (4.1) for some series of successive strata (b_r, b_{r+1}, $\cdots$, b_k), and find that s_k is a C1 type of boundary surface, then for some s_t we can infer that the following equations hold true:

$$\left.\begin{aligned}&\sigma(b_{t+1})^- \cup \cdots \cup \sigma(b_k)^- \cup \sigma(b_{k+1})^- \\ &\quad = (\{\sigma(b_r)^- \cup \cdots \cup \sigma(b_t)^-\} \cup \{\sigma(b_{t+1})^- \cup \cdots \cup \sigma(b_k)^- \cup \sigma(b_{k+1})^-\}) \cap s_t^{+1} \\ &\sigma(b_r)^- \cup \cdots \cup \sigma(b_t)^- \\ &\quad = (\{\sigma(b_r)^- \cup \cdots \cup \sigma(b_t)^-\} \cup \{\sigma(b_{t+1})^- \cup \cdots \cup \sigma(b_k)^- \cup \sigma(b_{k+1})^-\}) \cap s_t^{-1}\end{aligned}\right\} \quad (4.2)$$

It is noted that we have equations (4.2) by substituting $(b_k)^- \cup \sigma(b_{k+1})^-$ for $\sigma(b_k)^-$ in equations (4.1).

Axiom A5 : C2 type of boundary surface

The surface s_k called a C2 type of boundary surface or simply "C2" implies that a boundary surface s_k divides a series of successive strata (b_1, b_2, $\cdots$, b_k, b_{k+1}) ($1 < k+1 \leqq n$) into two series of successive strata (b_1, $\cdots$, b_k) and (b_{k+1}).

Therefore, if s_k is a C2 type of boundary surface, we have :

$$\begin{aligned}\sigma(b_{k+1})^- &= (\sigma(b_1)^- \cup \cdots \cup \sigma(b_k)^- \cup \cdots \cup \sigma(b_{k+1})^-) \cap s_k^{+1} \\ \sigma(b_1)^- \cup \cdots \cup \sigma(b_k)^- &= (\sigma(b_1)^- \cup \cdots \cup \sigma(b_k)^- \cup \cdots \cup \sigma(b_{k+1})^-) \cap s_k^{-1}.\end{aligned}$$

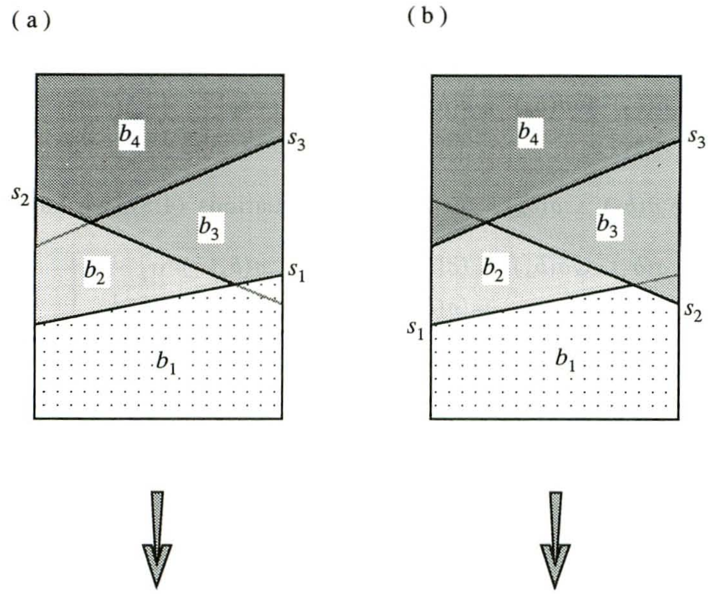
Based on above two axioms, we consider the geologic structure bounded by "C1" and "C2".

4.1.2 Relation between Strata and Boundary Surface

Formulating two types of boundary surfaces, we can define the distribution of every stratum uniquely by boundary surfaces. For example, we consider the case that the 3-D space X consists of four strata named b_1, b_2, b_3 and b_4. Let s_1, s_2 and s_3 be boundary surfaces. Regardless of the type of boundary surface, we have :

$$\sigma(b_1)^- \cup \sigma(b_2)^- \cup \sigma(b_3)^- \cup \sigma(b_4)^- = X \quad (4.3)$$

$$\left.\begin{aligned}\sigma(b_4)^- &= (\sigma(b_3)^- \cup \sigma(b_4)^-) \cap s_3^{+1} \\ \sigma(b_3)^- &= (\sigma(b_3)^- \cup \sigma(b_4)^-) \cap s_3^{-1}\end{aligned}\right\} \quad (4.4)$$



(c)

	s_3	s_2	s_1
b_4	+1	+1	+1
b_3	−1	+1	+1
b_2	0	−1	+1
b_1	0	0	−1

(d)

	s_3	s_2	s_1
b_4	+1	0	0
b_3	−1	+1	0
b_2	−1	−1	+1
b_1	−1	−1	−1

Fig. 7. C1 and C2 types of boundary surfaces.
(a) Geologic structure bounded only by C1 type of boundary surfaces. (b) Geologic structure bounded only by C2 type of boundary surfaces. Boundary surface is considered as a surface dividing X into two subspaces. (c) Logical model for locational relation representing a structure (a). (d) Logical model for locational relation representing a structure (b).

$$\left.\begin{aligned}\sigma(b_3)^- &= (\sigma(b_2)^- \cup \sigma(b_3)^-) \cap s_2{}^{+1}\\ \sigma(b_2)^- &= (\sigma(b_2)^- \cup \sigma(b_3)^-) \cap s_2{}^{-1}\end{aligned}\right\} \tag{4.5}$$

$$\left.\begin{aligned}\sigma(b_2)^- &= (\sigma(b_1)^- \cup \sigma(b_2)^-) \cap s_1{}^{+1}\\ \sigma(b_1)^- &= (\sigma(b_1)^- \cup \sigma(b_2)^-) \cap s_1{}^{-1}\end{aligned}\right\} \tag{4.6}$$

At first, let us consider the case that all boundary surfaces are "C1" (Fig. 7(a)). Since s_2 is "C1", substituting $\sigma(b_2)^- \cup \sigma(b_3)^-$ for $\sigma(b_2)^-$ in equations (4.6), we have :

$$\left.\begin{aligned}\sigma(b_2)^- \cup \sigma(b_3)^- &= (\sigma(b_1)^- \cup \sigma(b_2)^- \cup \sigma(b_3)^-) \cap s_1{}^{+1}\\ \sigma(b_1)^- &= (\sigma(b_1)^- \cup \sigma(b_2)^- \cup \sigma(b_3)^-) \cap s_1{}^{-1}\end{aligned}\right\} \tag{4.7}$$

In the same way, substituting $\sigma(b_3)^- \cup \sigma(b_4)^-$ for $\sigma(b_3)^-$ in equations (4.7), we have :

$$\sigma(b_2)^- \cup \sigma(b_3)^- \cup \sigma(b_4)^- = (\sigma(b_1)^- \cup \sigma(b_2)^- \cup \sigma(b_3)^- \cup \sigma(b_4)^-) \cap {s_1}^{+1}$$
$$\sigma(b_1)^- = (\sigma(b_1)^- \cup \sigma(b_2)^- \cup \sigma(b_3)^- \cup \sigma(b_4)^-) \cap {s_1}^{-1}$$

Thus, we obtain :

$$\left.\begin{aligned} \sigma(b_2)^- \cup \sigma(b_3)^- \cup \sigma(b_4)^- &= X \cap {s_1}^{+1} = {s_1}^{+1} \\ \sigma(b_1)^- &= X \cap {s_1}^{-1} = {s_1}^{-1}. \end{aligned}\right\} \tag{4.8}$$

And, substituting $\sigma(b_3)^- \cup \sigma(b_4)^-$ for $\sigma(b_3)^-$ in equations (4.5), we have :

$$\left.\begin{aligned} \sigma(b_3)^- \cup \sigma(b_4)^- &= (\sigma(b_2)^- \cup \sigma(b_3)^- \cup \sigma(b_4)^-) \cap {s_2}^{+1} \\ \sigma(b_2)^- &= (\sigma(b_2)^- \cup \sigma(b_3)^- \cup \sigma(b_4)^-) \cap {s_2}^{-1}. \end{aligned}\right\} \tag{4.9}$$

From equations (4.9) and (4.8), we obtain :

$$\left.\begin{aligned} \sigma(b_3)^- \cup \sigma(b_4)^- &= {s_2}^{+1} \cap {s_1}^{+1} \\ \sigma(b_2)^- &= {s_2}^{-1} \cap {s_1}^{+1} \end{aligned}\right\} \tag{4.10}$$

From equations (4.10) and (4.4), we obtain :

$$\sigma(b_4)^- = {s_3}^{+1} \cap {s_2}^{+1} \cap {s_1}^{+1}$$
$$\sigma(b_3)^- = {s_3}^{-1} \cap {s_2}^{+1} \cap {s_1}^{+1}.$$

Thus, distributions of strata are defined by s_1, s_2, s_3 as follows :

$$\left.\begin{aligned} \sigma(b_4)^- &= {s_3}^{+1} \cap {s_2}^{+1} \cap {s_1}^{+1} \\ \sigma(b_3)^- &= {s_3}^{-1} \cap {s_2}^{+1} \cap {s_1}^{+1} \\ \sigma(b_2)^- &= {s_2}^{-1} \cap {s_1}^{+1} \\ \sigma(b_1)^- &= {s_1}^{-1}. \end{aligned}\right\} \tag{4.11}$$

Generalizing the result, we have the following theorem.

Theorem 4.1

Suppose that the 3-D space X consists of strata named $b_1, \cdots, b_n$, and that all boundary surfaces $s_1, \cdots, s_{n-1}$ are "C1". Then, each stratum is represented as follows :

$$\sigma(b_n)^- = {s_{n-1}}^{+1} \cap {s_{n-2}}^{+1} \cap \cdots \cap {s_1}^{+1}$$
$$\sigma(b_i)^- = {s_i}^{-1} \cap {s_{i-1}}^{+1} \cap \cdots \cap {s_1}^{+1} \qquad (i = n-1, \cdots, 2)$$
$$\sigma(b_1)^- = {s_1}^{-1}$$

Next, let us consider the case that all boundary surfaces are "C2" (Fig. 7(b)). Since s_2 is "C2", we have :

$$\left.\begin{aligned} \sigma(b_3)^- &= (\sigma(b_1)^- \cup \sigma(b_2)^- \cup \sigma(b_3)^-) \cap {s_2}^{+1} \\ \sigma(b_1)^- \cup \sigma(b_2)^- &= (\sigma(b_1)^- \cup \sigma(b_2)^- \cup \sigma(b_3)^-) \cap {s_2}^{-1}. \end{aligned}\right\} \tag{4.12}$$

Similarly :

$$\left.\begin{aligned} \sigma(b_4)^- &= (\sigma(b_1)^- \cup \sigma(b_2)^- \cup \sigma(b_3)^- \cup \sigma(b_4)^-) \cap s_3{}^{+1} \\ \sigma(b_1)^- \cup \sigma(b_2)^- \cup \sigma(b_3)^- &= (\sigma(b_1)^- \cup \sigma(b_2)^- \cup \sigma(b_3)^- \cup \sigma(b_4)^-) \cap s_3{}^{-1}. \end{aligned}\right\} \tag{4.13}$$

The substitution of equations (4.3) into (4.13) gives :

$$\begin{aligned} \sigma(b_4)^- &= X \cap s_3{}^{+1} = s_3{}^{+1} \\ \sigma(b_1)^- \cup \sigma(b_2)^- \cup \sigma(b_3)^- &= X \cap s_3{}^{-1} = s_3{}^{-1} \end{aligned}$$

and the substitution into (4.12) gives :

$$\begin{aligned} \sigma(b_3)^- &= s_3{}^{-1} \cap s_2{}^{+1} \\ \sigma(b_1)^- \cup \sigma(b_2)^- &= s_3{}^{-1} \cap s_2{}^{-1}. \end{aligned}$$

Further, the substitution into (4.6) gives :

$$\begin{aligned} \sigma(b_2)^- &= s_3{}^{-1} \cap s_2{}^{-1} \cap s_1{}^{+1} \\ \sigma(b_1)^- &= s_3{}^{-1} \cap s_2{}^{-1} \cap s_1{}^{-1} \end{aligned}$$

Thus, we obtain :

$$\begin{aligned} \sigma(b_4)^- &= s_3{}^{+1} \\ \sigma(b_3)^- &= s_3{}^{-1} \cap s_2{}^{+1} \\ \sigma(b_2)^- &= s_3{}^{-1} \cap s_2{}^{-1} \cap s_1{}^{+1} \\ \sigma(b_1)^- &= s_3{}^{-1} \cap s_2{}^{-1} \cap s_1{}^{-1} \end{aligned}$$

Generalizing the result, we get the following theorem.

Theorem 4.2

Suppose that the 3-D space X consists of strata named $b_1, \cdots, b_n$, and that all boundary surfaces $s_1, \cdots, s_{n-1}$ are "C2". Then, each stratum is represented as follows :

$$\begin{aligned} \sigma(b_n)^- &= s_{n-1}{}^{+1} \\ \sigma(b_i)^- &= s_{n-1}{}^{-1} \cap \cdots \cap s_i{}^{-1} \cap s_{i-1}{}^{+1} \qquad (i = n-1, \cdots, 2) \\ \sigma(b_1)^- &= s_{n-1}{}^{-1} \cap \cdots \cap s_2{}^{-1} \cap s_1^{-1} \end{aligned}$$

If the boundary surface is either "C1" or "C2", then we can formulate the relation between strata and boundary surfaces.

Theorem 4.3

Suppose that the 3-D space X consists of strata named $b_1, \cdots, b_n$, and that each boundary surface s_i $(i=1, \cdots, n-1)$ is either "C1" or "C2". Then each stratum is defined uniquely by $s_1, \cdots, s_{n-1}$

Proof

The theorem holds true in the cases that all surfaces are either "C1" or "C2" as shown in Theorems 4.1 and 4.2. Let us consider the case that there exist both "C1"

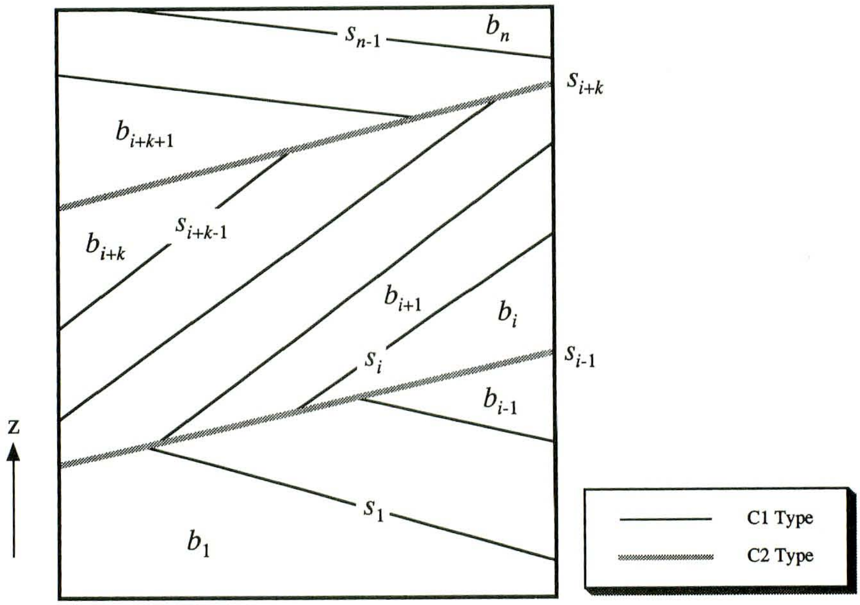


Fig. 8. Geologic structure including both C1 and C2 types of boundary surfaces. s_{i-1} and s_{i+k} are "C2", and others are "C1".

and "C2".

Let s_i, $\cdots$, s_{i+k-1} ($k>0$) be "C1", and let s_{i-1} and s_{i+k} be "C2" (Fig. 8). Since s_{i-1} is "C2", we have :

$$\sigma(b_i)^- = (\sigma(b_1)^- \cup \cdots \cup \sigma(b_{i-1})^- \cup \sigma(b_i)^-) \cap s_{i-1}{}^{+1}$$

$$\sigma(b_1)^- \cup \cdots \cup \sigma(b_{i-1})^- = (\sigma(b_1)^- \cup \cdots \cup \sigma(b_{i-1})^- \cup \sigma(b_i)^-) \cap s_{i-1}{}^{-1}$$

Since s_i is "C1", substituting $\sigma(b_i)^- \cup \sigma(b_{i+1})^-$ for $\sigma(b_i)^-$ of above equations gives :

$$\sigma(b_i)^- \cup \sigma(b_{i+1})^-$$
$$= (\sigma(b_1)^- \cup \cdots \cup \sigma(b_{i-1})^- \cup \sigma(b_i)^- \cup \sigma(b_{i+1})^-) \cap s_{i-1}{}^{+1}$$

$$\sigma(b_1)^- \cup \cdots \cup \sigma(b_{i-1})^-$$
$$= (\sigma(b_1)^- \cup \cdots \cup \sigma(b_{i-1})^- \cup \sigma(b_i)^- \cup \sigma(b_{i+1})^-) \cap s_{i-1}{}^{-1}$$

And, since s_{i+1} is also "C1", substituting $\sigma(b_{i+1})^- \cup \sigma(b_{i+2})^-$ for $\sigma(b_{i+1})^-$ gives similar equations. Repeating such operations, we finally obtain :

$$\sigma(b_i)^- \cup \cdots \cup \sigma(b_{i+k})^-$$
$$= \sigma(b_1)^- \cup \cdots \cup \sigma(b_{i-1})^- \cup \sigma(b_i)^- \cup \cdots \cup \sigma(b_{i+k})^-) \cap s_{i-1}{}^{+1}$$

$$\sigma(b_1)^- \cup \cdots \cup \sigma(b_{i-1})^-$$
$$= \sigma(b_1)^- \cup \cdots \cup \sigma(b_{i-1})^- \cup \sigma(b_i)^- \cup \cdots \cup \sigma(b_{i+k})^-) \cap s_{i-1}{}^{-1}$$

On the other hand, since s_{i+k} is "C2", we obtain :

$$\begin{aligned}\sigma(b_{i+k+1})^- &= (\sigma(b_1)^- \cup \cdots \cup \sigma(b_{i+k+1})^-) \cap s_{i+k}{}^{+1} \\ \sigma(b_1)^- \cup \cdots \cup \sigma(b_i)^- \cup \cdots \cup \sigma(b_{i+k})^- &= (\sigma(b_1)^- \cup \cdots \cup \sigma(b_{i+k+1})^-) \cap s_{i+k}{}^{-1}\end{aligned}$$

Thus, we have :

$$\left.\begin{aligned}&\sigma(b_{i+k+1})^- \\ &\quad = \sigma(b_1)^- \cup \cdots \cup (\sigma(b_i)^- \cup \cdots \cup \sigma(b_{i+k})^-) \cup \sigma(b_{i+k+1})^-) \cap s_{i+k}{}^{+1} \\ &\sigma(b_1)^- \cup \cdots \cup \sigma(b_{i-1})^- \cup (\sigma(b_i)^- \cup \cdots \cup \sigma(b_{i+k})^-) \\ &\quad = (\sigma(b_1)^- \cup \cdots \cup (\sigma(b_i)^- \cup \cdots \cup \sigma(b_{i+k})^-) \cup \sigma(b_{i+k+1})^-) \cap s_{i+k}{}^{-1} \\ &(\sigma(b_i)^- \cup \cdots \cup \sigma(b_{i+k})^-) \\ &\quad = \sigma(b_1)^- \cup \cdots \cup \sigma(b_{i-1})^- \cup (\sigma(b_i)^- \cup \cdots \cup \sigma(b_{i+k})^-)) \cap s_{i-1}{}^{+1} \\ &\sigma(b_1)^- \cup \cdots \cup \sigma(b_{i-1})^- \\ &\quad = \sigma(b_1)^- \cup \cdots \cup \sigma(b_{i-1})^- \cup (\sigma(b_i)^- \cup \cdots \cup \sigma(b_{i+k})^-)) \cap s_{i-1}{}^{-1}.\end{aligned}\right\} \quad (4.14)$$

Set :

$$\sigma'^- = \sigma(b_1)^- \cup \cdots \cup \sigma(b_{i+k})^-.$$

Then, equations (4.14) become :

$$\left.\begin{aligned}\sigma(b_{i+k+1})^- &= (\sigma(b_1)^- \cup \cdots \cup \sigma'^- \cup \sigma(b_{i+k+1})^-) \cap s_{i+k}{}^{+1} \\ \sigma(b_1)^- \cup \cdots \cup \sigma(b_{i-1})^- \cup \sigma'^- &= (\sigma(b_1)^- \cup \cdots \cup \sigma'^- \cup \sigma(b_{i+k+1})^-) \cap s_{i+k}{}^{-1} \\ \sigma'^- &= (\sigma(b_1)^- \cup \cdots \cup \sigma(b_{i-1})^- \cup \sigma'^-) \cap s_{i-1}{}^{+1} \\ \sigma(b_1)^- \cup \cdots \cup \sigma(b_{i-1})^- &= (\sigma(b_1)^- \cup \cdots \cup \sigma(b_{i-1})^- \cup \sigma'^-) \cap s_{i-1}{}^{-1}.\end{aligned}\right\} \quad (4.15)$$

Equations (4.15) indicate that a series of successive strata (b_i, $\cdots$, b_{i+k}) bounded by "C1" can be managed as one group of strata which is bounded by "C2" types of boundary surfaces s_{i-1} and s_{i+k}. From Theorem 4.1, it is clear that every stratum in the group of strata σ'^- is represented by :

$$\begin{aligned}\sigma(b_{i+k})^- &= (\sigma(b_i)^- \cup \cdots \cup \sigma(b_{i+k})^-) \cap s_i{}^{+1} \cap \cdots \cap s_{i+k-2}{}^{+1} \cap s_{i+k-1}{}^{+1} \\ \sigma(b_j)^- &= (\sigma(b_i)^- \cup \cdots \cup \sigma(b_{i+k})^-) \cap s_i{}^{+1} \cap \cdots \cap s_{j-1}{}^{+1} \cap s_j{}^{-1} \\ &\qquad (j = i+k-1, \cdots, i+1) \\ \sigma(b_j)^- &= (\sigma(b_i)^- \cup \cdots \cup \sigma(b_{i+k}) -) \cap s_i{}^{-1}.\end{aligned}$$

Let b'_1, $\cdots$ b'_m be names of such groups of strata, and let s'_j be a boundary surface between b'_j and b'_{j+1}. Since s'_1, s'_2, $\cdots$, s'_{m-1} are the C2 type of boundary surfaces, from Theorem 4.2, $\sigma(b'_1)^-$, $\cdots$, $\sigma(b'_m)^-$ shows :

$$\begin{aligned}\sigma(b'_m)^- &= s'_{m-1}{}^{+1} \\ \sigma(b'_j)^- &= s'_{m-1}{}^{+1} \cap \cdots \cap s'_j{}^{-1} \cap s'_{j-1}{}^{+1} \qquad (j = m-1, \cdots, 2) \\ \sigma(b'_1)^- &= s'_{m-1}{}^{+1} \cap \cdots \cap s'_2{}^{-1} \cap s'_1{}^{-1}\end{aligned}$$

Thus, distributions of all strata $b_1, \cdots, b_n$ are defined uniquely by bounded surfaces. □

4.1.3 Logical Model for Locational Relation

Theorem 4.3 indicates that strata named $b_1, \cdots, b_n$ are defined uniquely by the boundary surfaces $s_1, \cdots, s_{n-1}$. The relation between strata and boundary surfaces can be represented by a function $t: B \times S \rightarrow \{-1, 0, +1\}$ called "a logical model for locational relation", where $t(b_i, s_j) = +1$ and -1 indicate that i-th stratum $\sigma(b_i)$ is above and below the j-th boundary surface s_j, respectively. $t(b_i, s_j) = 0$ shows that a stratum $\sigma(b_i)$ has no specific relation with the surface s_j. An example of the logical model for locational relation is shown in Fig. 7.

The logical model for locational relation will be used when we prepare locational data required to determine the boundary surface (Section 6) and also when we define a function which assigns a stratum to each subspace divided by boundary surfaces (Section 7).

4.2 Algorithm for Determination of Logical Model for Locational Relation

Data required to determine the logical model for locational relation are the stratigraphic sequence and the type of boundary surface. The stratigraphic sequence is determined by the method described in the previous section. The type of boundary surface is given by a parameter τ_r in input data :

$$—, —, —, —, —, \alpha_r, \beta_r, \tau_r, —$$

where $\tau_r = 1$ and 2 if the contact surface between strata named α_r and β_r is "C1" and "C2", respectively :

$$\tau_r = \begin{cases} 1, & \text{C1 type of boundary surface} \\ 2, & \text{C2 type of boundary surface}. \end{cases}$$

The type of every boundary surface s_i ($i = 1, \cdots, n-1$) can be found by searching input data which describe natures of pairs of successive strata. For example, if we find that $\alpha_r = b_i$ and $\beta_r = b_{i+1}$, τ_r gives the type of s_i.

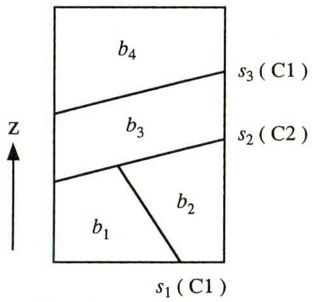
The following steps show an algorithm to determine a function t representing the logical model for locational relation, after all elements of B are enumerated linearly using pairs (α_r, β_r) based on the algorithm shown in the previous section.

(i) Let all the values of a function t be 0 as the initial value.

(ii) Repeat step (iii) or (iv) for every surface s_i ($i = 1, \cdots, n-1$). If the surface s_i is "C2", then go to step (iii). If not, that is, the surface s_i is "C1", then go to step (iv).

(iii) Set $t(b_j, s_i) = -1$ ($j = 1, \cdots, i$). If there exists the other C2 type of surface s_k among $s_{i+1}, \cdots, s_{n-1}$ ($i < k \leqq n-1$), then set $t(b_j, s_i) = +1$ ($j = \mathrm{i}+1, \cdots, k$). If not, then $t(b_j, s_i) = +1$ ($j = i+1, \cdots, n$).

(a)



(b)

	s3	s2	s1
b4	+1	+1	0
b3	-1	+1	0
b2	0	-1	+1
b1	0	-1	-1

(c)

	b4	b3	b2	b1
b4	*	s3	s2	s2
b3	s3	*	s2	s2
b2	s2	s2	*	s1
b1	s2	s2	s1	*

Fig. 9. Contact surfaces between strata.
(a) A section of geologic structure consists of four strata. (b) Logical model for locational relation $t:B\times S\to\{-1, 0, +1\}$. (c) Table representing a function $u:B\times S\to S$.

(iv) Set $t(b_i, s_i)=-1$. If there exists the C2 type of surface s_k among $s_{i+1}, \cdots, s_{n-1}$ $(i<k\leqq n-1)$, then set $t(b_j, s_i)=+1$ $(j=i+1, \cdots, k)$. If not, then $t(b_j, s_i)=+1$ $(j=i+1, \cdots, n)$.

For example, if we have input data as shown in Fig. 9(a) :

$$—, —, —, —, —, b_3, b_4, 1, —$$
$$—, —, —, —, —, b_2, b_3, 2, —$$
$$—, —, —, —, —, b_1, b_2, 1, —,$$

then we obtain the logical model for locational relation as shown in Fig. 9(b).

4.3 Contact Relation between Strata

As mentioned above, each boundary surface s_i $(i=1, \cdots, n-1)$ is defined as a surface between successive strata b_i and b_{i+1}, dividing the 3-D space. This definition does not directly mention about the contact surface between arbitrary two strata. However, the logical model for locational relation gives us information about the contact surface. For example, when we have equations (4.11), we get :

$$\sigma(b_3)^-\cap\sigma(b_4)^- =(s_3{}^{-1}\cap s_2{}^{+1}\cap s_1{}^{+1})\cap(s_3{}^{+1}\cap s_2{}^{+1}\cap s_1{}^{+1})$$
$$=s_3\cap s_2{}^{+1}\cap s_1{}^{+1} \qquad (\because\ s_k=s_k{}^{+1}\cap s_k{}^{-1})$$
$$\therefore\ \sigma(b_3)^-\cap\sigma(b_4)^-\subset s_3$$

This indicates that the contact surface between $\sigma(b_3)$ and $\sigma(b_4)$ is s_3. Generally, we get the following theorem.

Theorem 4.4

Suppose that a 3-D space X consists of strata $b_1, \cdots, b_n$, and let each boundary surface s_i $(i=1, \cdots, n-1)$ be either "C1" or "C2". Then, a contact surface between two strata is one of these boundary surfaces.

Proof

(i) Case that all boundaries are "C1".
From Theorem 4.1 the distributions of two strata b_i and b_j $(i<j)$ are represented as follows :

$$\sigma(b_j)^- = s_j^{-1} \cap s_{j-1}^{+1} \cap \cdots \cap s_i^{+1} \cap \cdots \cap s_1^{+1} \qquad (2 \leqq i < j \leqq n-1)$$
$$\sigma(b_i)^- = s_i^{-1} \cap s_{i-1}^{+1} \cap \cdots \cap s_1^{+1}$$

Then, we obtain :

$$\sigma(b_j)^- \cap \sigma(b_i)^- = (s_j^{-1} \cap s_{j-1}^{+1} \cap \cdots \cap s_i^{+1} \cap \cdots \cap s_1^{+1}) \cap (s_i^{-1} \cap s_{i-1}^{+1} \cap \cdots \cap s_1^{+1})$$
$$= s_j^{-1} \cap s_{j-1}^{+1} \cap \cdots \cap s_{i+1}^{+1} \cap s_i \cap s_{i-1}^{+1} \cap \cdots \cap s_1^{+1}$$
$$\therefore \sigma(b_j)^- \cap \sigma(b_i)^- \subset s_i$$

Therefore, if b_i is in contact with b_j, i.e., $\sigma(b_j)^- \cap \sigma(b_i)^- \neq \phi$, then s_i is the only one contact surface between them. In the same way, we can determine the contact surface between b_n and b_i and one between b_j and b_1.

(ii) Case that there exists "C2" among a series of surfaces s_i, s_{i+1}, $\cdots$, s_{j-1}.
As shown in the proof of Theorem 4.3, composite strata b'_1, $\cdots$, b'_m bounded by "C1" can be grouped into one set of strata bounded by "C2" s'_i ($i=1$, $\cdots$, $m-1$). Let $\sigma(b_j)$ and $\sigma(b_i)$ be included in $\sigma(b'_J)$ and $\sigma(b'_I)$, respectively. Then, since $\sigma(b'_J)^-$ and $\sigma(b'_I)^-$ are represented as :

$$\sigma(b'_J)^- = s'_{m-1}{}^{-1} \cap \cdots \cap s'_J{}^{-1} \cap s'_{J-1}{}^{+1} \qquad (2 \leqq I < J \leqq m-1)$$
$$\sigma(b'_I)^- = s'_{m-1}{}^{-1} \cap \cdots \cap s'_J{}^{-1} \cap s'_{J-1}{}^{-1} \cap \cdots \cap s'_I{}^{-1} \cap s'_{I-1}{}^{+1},$$

we obtain :

$$\sigma(b'_J)^- \cap \sigma(b'_I)^-$$
$$= (s'_m{}^{-1} \cap \cdots \cap s'_J{}^{-1} \cap s'_{J-1}{}^{+1}) \cap (s'_{m-1}{}^{-1} \cap \cdots \cap s'_J{}^{-1} \cap s'_{J-1}{}^{-1} \cap \cdots \cap s'_I{}^{-1} \cap s'_{I-1}{}^{+1})$$
$$= s'_{m-1}{}^{-1} \cap \cdots \cap s'_J{}^{-1} \cap s'_{J-1} \cap s'_{j+1}{}^{-1} \cap \cdots \cap s'_I{}^{-1} \cap s'_{I-1}{}^{+1}$$
$$\therefore \sigma(b'_J)^- \cap \sigma(b'_I)^- \subset s'_{J-1}.$$

Therefore, s'_{J-1} includes the contact surface between $\sigma(b'_J)$ and $\sigma(b'_I)$. It is clear that the contact surface between $\sigma(b_j)$ and $\sigma(b_i)$ is identical with the contact surface between $\sigma(b'_J)$ and $\sigma(b'_I)$. □

We can consider this rule which assigns a boundary surface to a pair of strata as a function, denoted by $u{:}B \times B \rightarrow S$.

4.4 Contact Relation Derived from Logical Model for Locational Relation

The fact that a pair (b_i, b_j) satisfies :

$$t(b_i, s_k) \times t(b_j, s_k) = -1$$

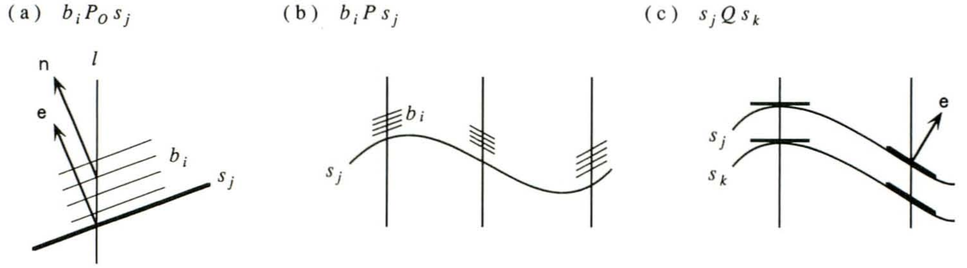


Fig. 10. Definition of inclinational relation.
(a) Relation P_o. (b) Relation P. (c) Relation Q.

for some s_k, implies that s_k is a contact surface between strata b_i and b_j. Thus, we can construct the function $u{:}B \times B \rightarrow S$ from the logical model for locational relation as follows.

If :

$$t(b_i,\ s_k) \times t(b_j,\ s_k) = -1$$

then :

$$u(b_i,\ b_j) = s_k \text{ and } u(b_j,\ b_i) = s_k \ .$$

Figure 9(b) shows the logical model for locational relation for geologic structure given in Fig. 9(a), and Fig. 9(c) shows the function u in tabular form.

5. Logical Model for Inclinational Relation

Inclinational data (e.g., strike and dip) are useful for the determination of the boundary surface. If strata are parallel layered, we can effectively use strikes and dips obtained from a contact surface or the bedding plane of a stratum for the determination of other surfaces. In practical situations, strata distributed finitely are rarely parallel. However, boundary surfaces often show similar tendencies to each other, even if they are not parallel. Therefore, an inclinational relation is introduced.

5.1 Inclinational Relation between Strata and Boundary Surface

We consider the inclinational relation between strata and boundary surfaces as a theoretical basis for the effective use of inclinational data. We introduce two unit vectors $\mathbf{n}(l,\ b_i)$ and $\mathbf{e}(l,\ s_j)$; $\mathbf{n}(l,\ b_i)$ is a normal vector of the bedding plane of b_i if the bedding planes have same inclinations to each other along a vertical line l, and $(0,\ 0,\ -1)$ in other cases, and $\mathbf{e}(l,\ s_j)$ is a normal vector of a boundary surface s_j at a point of the intersection of a vertical line l and the surface s_j (Fig. 10).

Definition : *P*

Let P be an inclinational relation between a stratum b_i and a boundary surface s_j such that :

$$b_iPs_j \Leftrightarrow \forall l(\mathbf{n}(l,\ b_i)=\mathbf{e}(l,\ s_j))\ .$$

Definition : Q

Let Q be an inclinational relation between boundary surfaces s_j and s_k such that :

$$s_jQs_k \Leftrightarrow \forall l(\mathbf{e}(l,\ s_j)=\mathbf{e}(l,\ s_k))\ .$$

Theorem 5.1

An inclinational relation Q is an equivalence relation on a set S of boundary surfaces.

Proof

(i) Reflexive property

It is clear that for all boundary surfaces s_j $(1 \leqq j \leqq n-1)$, we have :

$$s_jQs_j \Leftrightarrow \forall l(\mathbf{e}(l,\ s_j)=\mathbf{e}(l,\ s_j))\ .$$

(ii) Symmetric property

For any pair of boundary surfaces s_j and s_k, we have :

$$\begin{aligned} s_jQs_k &\Leftrightarrow \forall l(\mathbf{e}(l,\ s_j)=\mathbf{e}(l,\ s_k)) \\ &\Leftrightarrow \forall l(\mathbf{e}(l,\ s_k)=\mathbf{e}(l,\ s_j)) \\ &\Leftrightarrow s_kQs_j\ . \end{aligned}$$

(iii) Transitive property

For any boundary surfaces s_i, s_j and s_k, we have :

$$\begin{aligned} s_iQs_j \wedge s_jQs_k &\Leftrightarrow \forall l(\mathbf{e}(l,\ s_i)=\mathbf{e}(l,\ s_j)) \wedge \forall l(\mathbf{e}(l,\ s_j)=\mathbf{e}(l,\ s_k)) \\ &\Leftrightarrow \forall l((\mathbf{e}(l,\ s_i)=\mathbf{e}(l,\ s_j)) \wedge (\mathbf{e}(l,\ s_j)=\mathbf{e}(l,\ s_k))) \\ &\Leftrightarrow \forall l(\mathbf{e}(l,\ s_j)=(\mathbf{e}(l,\ s_i)=\mathbf{e}(l,\ s_k))) \\ &\Leftrightarrow \forall l(\mathbf{e}(l,\ s_i)=\mathbf{e}(l,\ s_k)) \\ &\Leftrightarrow s_iQs_k\ . \end{aligned}$$

From (i), (ii) and (iii), the relation Q is reflexive, symmetric and transitive. Therefore, Q is an equivalence relation on S. □

Theorem 5.2

(i) $$P^{-1}\cdot P \subset Q$$

(ii) $$\mathrm{P}\cdot\mathrm{Q} \subset P\ .$$

Proof

(i) $s_i(P^{-1}\cdot P)s_j$ implies that there exists b_k such that satisfies both b_kPs_i and b_kPs_j. Further :

$$\begin{aligned} b_kPs_i \wedge b_kPs_j &\Leftrightarrow \forall l(\mathbf{n}(l,\ b_k)=\mathbf{e}(l,\ s_i)) \wedge \forall l(\mathbf{n}(l,\ b_k)=\mathbf{e}(l,\ s_j)) \\ &\Leftrightarrow \forall((\mathbf{n}(l,\ b_k)=\mathbf{e}(l,\ s_i)) \wedge (\mathbf{n}(l,\ b_k)=\mathbf{e}(l,\ s_j))) \\ &\Leftrightarrow \forall l(\mathbf{e}(l,\ s_i)=\mathbf{e}(l,\ s_j)) \\ &\Leftrightarrow s_iQs_j\ . \end{aligned}$$

Thus, we get :

$$P^{-1} \cdot P \subset Q \ .$$

(ii) $b_i(P \cdot Q)s_j$ implies that there exists s_k such that satisfies both b_iPs_k and s_kQs_j. Further :

$$\begin{aligned} b_iPs_k \wedge s_kQs_j &\Leftrightarrow \forall l(\mathbf{n}(l,\ b_i)=\mathbf{e}(l,\ s_k)) \wedge \forall l(\mathbf{e}(l,\ s_k)=\mathbf{e}(l,\ s_j)) \\ &\Leftrightarrow \forall l((\mathbf{n}(l,\ b_i)=\mathbf{e}(l,\ s_k)) \wedge (\mathbf{e}(l,\ s_k)=\mathbf{e}(l,\ s_j))) \\ &\Leftrightarrow \forall l(\mathbf{n}(l,\ b_i)=\mathbf{e}(l,\ s_j)) \\ &\Leftrightarrow b_iPs_j \ . \end{aligned}$$

Thus, we get :

$$P \cdot Q \subset P \ . \qquad \square$$

It is noted that we cannot directly observe relations P and Q, but only a relation between exposed parts of strata which are defined as follows.

Definition : P_o

$$b_iP_os_j \Leftrightarrow \exists l(\mathbf{n}(l,\ b_i)=\mathbf{e}(l,\ s_j)) \ .$$

Directly from the above definition, we obtain the following relations between P and P_o.

Theorem 5.3

$$\sim P_o \subset P \ .$$

Proof

For $b_i \in B$ and $s_j \in S$, we have :

$$\begin{aligned} \sim b_iP_os_j &\Leftrightarrow \sim(\exists l(\mathbf{n}(l,\ b_i)=\mathbf{e}(l,\ s_j))) \\ &\Leftrightarrow \forall l(\sim(\mathbf{n}(l,\ b_i)=\mathbf{e}(l,\ s_j))) \\ &\Rightarrow \exists l(\sim(\mathbf{n}(l,\ b_i)=\mathbf{e}(l,\ s_j))) \\ &\Leftrightarrow \sim(\forall l(\mathbf{n}(l,\ b_i)=\mathbf{e}(l,\ s_j))) \\ &\Leftrightarrow \sim b_iPs_j \qquad \square \end{aligned}$$

Here, we introduce the following assumption.

Assumption 5.1

$$P_o \subset P$$

as far as this assumption does not cause any contradictions.

Assumption 5.1 derives a rule to infer inclinational relations P and Q from the observable relation P_o.

Theorem 5.4

(i) $Q_o \subset Q$

(ii) $P_o \cdot Q_o \subset P$

where :

$$Q_o = E \cup (P_o^{-1} \cdot P_o)^* .$$

Proof

(i) From Assumption 5.1, we have :

$$P_o \subset P$$
$$P_o^{-1} \subset P^{-1}$$

which imply :

$$P_o^{-1} \cdot P_o \subset P^{-1} \cdot P. \tag{5.1}$$

Theorem 5.2(i) and (5.1) give :

$$P_o^{-1} \cdot P_o \subset Q$$

and therefore, we have :

$$E \cup (P_o^{-1} \cdot P_o)^* \subset E \cup Q^* .$$

Since Q is reflexive and transitive, we have :

$$E \cup Q^* \subset Q .$$

Hence, we have :

$$E \cup (P_o^{-1} \cdot P_o)^* \subset Q .$$

Let :

$$Q_o = E \cup (P_o^{-1} \cdot P_o)^* ,$$

then :

$$Q_o \subset Q .$$

(ii) Assumption 5.1 and Theorem 5.4(i) give :

$$P_o \cdot Q_o \subset P \cdot Q .$$

From Theorem 5.2(ii), finally we have :

$$P_o \cdot Q_o \subset P . \qquad \square$$

5.2 Algorithm for Logical Model for Inclinational Relation

5.2.1 Input Data

The relation P_o observed at the r-th outcrop is given by a parameter π_r in input data :

$$x_r,\ y_r,\ z_r,\ \xi_r,\ \eta_r,\ \alpha_r,\ \beta_r,\ \tau_r,\ \pi_r.$$

π_r represents the inclinational relation of the contact surface between α_r and β_r relative to the bedding planes within α_r and β_r as follows :

$$\pi_r = \begin{cases} 0, & \text{if the inclination of the contact surface is different with both the lower stratum } \alpha_r \text{ and the upper one } \beta_r \\ 1, & \text{if the contact surface has the same inclination as only the lower stratum } \alpha_r \text{ along a vertical line} \\ 2, & \text{if the contact surface has the same inclination as only the upper stratum } \beta_r \text{ along a vertical line} \\ 3, & \text{if the contact surface has the same inclination as both the lower stratum } \alpha_r \text{ and the upper one } \beta_r \text{ along a vertical line} \end{cases}$$

For example, if $\alpha_r = b_i$, $\beta_r = b_j$ and s_k is the boundary surface between b_i and b_j $(i<j)$, then $\pi_r = 1$ implies :

$$b_i P_o s_k \text{ and } \sim b_j P_o s_k\ .$$

As this example shows, it is noted that in order to infer the relation P from input data we must know previously which boundary surface becomes the contact surface between any pairs of strata by a function $u{:}B \times B \rightarrow S$ as mentioned in Section 4.3.

We should note that the inference rule of $P_o \subset P$ is applicable as far as there are no contradictions. For example, in the case that we observe $b_k P_o s_i$ at one outcrop and $\sim b_k P_o s_i$ at another place, it is clear from Theorem 5.3 that $b_k P s_i$ does not hold true because we have $\sim b_k P_o s_i$. Nevertheless, if we apply the rule $P_o \subset P$, then we have both $b_k P s_i$ and $\sim b_k P s_i$. This is a contradiction. In order to avoid this type of contradiction, the inference rule $P_o \subset P$ should be applied carefully.

Suppose that for the contact surface between same pair of strata, we have different sets of data :

$$\begin{array}{lllllllll} x_r, & y_r, & z_r, & \xi_r, & \eta_r, & \alpha_r, & \beta_r, & \tau_r, & \pi_r \\ x_{r'}, & y_{r'}, & z_{r'}, & \xi_{r'}, & \eta_{r'}, & \alpha_{r'}, & \beta_{r'}, & \tau_{r'}, & \pi_{r'}. \end{array}$$

where $\alpha_r = \alpha_{r'}$' and $\beta_r = \beta_{r'}$. Then, one method to avoid contradictions is to apply the rule $P_o \subset P$ after adjusting the input parameter π_r as follows :

(i) Set $\pi_r = 0$ if $\pi_{r'} = 0$
if $\pi_r = 1$ and $\pi_{r'} = 2$
if $\pi_r = 2$ and $\pi_{r'} = 1$

(ii) Set $\pi_r = 1$ if $\pi_{r'} = 1$ and $\pi_{r'} = 3$

if $\pi_r=3$ and $\pi_{r'}=1$

(iii) Set $\pi_r=2$ if $\pi_r=2$ and $\pi_{r'}=3$

if $\pi_r=3$ and $\pi_{r'}=2$

5.2.2 Construction of Relation Matrix

The relation P_o can be represented by an $n\times(n-1)$ matrix $\mathbf{P_o}$ with the row $b_n, \cdots, b_1$ and the column $s_{n-1}, \cdots, s_1$. Let p_{oij} be the (i, j) entry of the matrix $\mathbf{P_o}$. Then $p_{oij}=1$ if $b_iP_os_j$, and $p_{oij}=0$ if not. Based on Theorem 5.4, the relation matrix $\mathbf{Q_o}$ of the relation Q_o is inferred from the matrix $\mathbf{P_o}$ through matrix operations as follows :

$$\mathbf{Q_o}=\mathbf{E}+(\mathbf{P_o}^t\cdot\mathbf{P_o})+(\mathbf{P_o}^t\cdot\mathbf{P_o})^2+\cdots.$$

Let q_{oij} be the (i, j) entry of the matrix $\mathbf{Q_o}$. Then the inclinations of boundary surfaces s_i and s_j along a vertical line are the same if $q_{oij}=1$, and those of s_i and s_j are different if $q_{oij}=0$. The relation matrix $\mathbf{Q_o}$ can be represented by a function $q:S\times S\rightarrow\{1, 0\}$.

Further, the relation matrix $\mathbf{P}$ of the relation P can be derived from $\mathbf{P_o}$ and $\mathbf{Q_o}$:

$$\mathbf{P}=\mathbf{P_o}\cdot\mathbf{Q_o} .$$

We call this relation matrix $\mathbf{P}$ "the logical model for inclinational relation". The (i, j) entry p_{ij} means that b_i and s_j satisfy the inclinational relation if $p_{ij}=1$, and b_i and s_j do not satisfy the inclinational relation if $p_{ij}=0$. The logical model for inclinational relation may be represented by a function p from $B\times S$ to $\{1, 0\}$. Then $p(b_i, s_j)=1$ and 0 represent that b_i and s_j satisfy and do not satisfy the inclinational relation, respectively.

5.2.3 Algorithm to Construct Logical Model for Inclinational Relation

Using ordered pairs (α_r, β_r) in input data :

$$—, —, —, —, —, \alpha_r, \beta_r, \tau_r, \pi_r.$$

we can define a set of strata $B=\{b_1, \cdots, b_n\}$ whose elements are enumerated linearly from the lowest stratum b_1 to the uppermost one b_n, and a set of boundary surface $S=\{s_1, \cdots, s_{n-1}\}$. Further, as mentioned in Section 4, we can construct the logical model for locational relation from the given parameter τ_r, and determine the function $u:B\times \mathrm{B}\rightarrow S$.

The following is an algorithm to construct the logical model for inclinational relation including the adjustment of data.

(i) Set $p_{oij}=2$ as the initial values of a relation matrix $\mathbf{P_o}$ $(i=1, \cdots, n; j=1, \cdots, n-1)$.

(ii) Repeat the following operations for $r=1, \cdots, N$(N : the number of data).

(ii-1) Find numbers i and j which satisfy :

$b_i=\alpha_r$

$b_j=\beta_r$.

(ii-2) Determine the boundary surface $s_k=u(b_i, b_j)$.

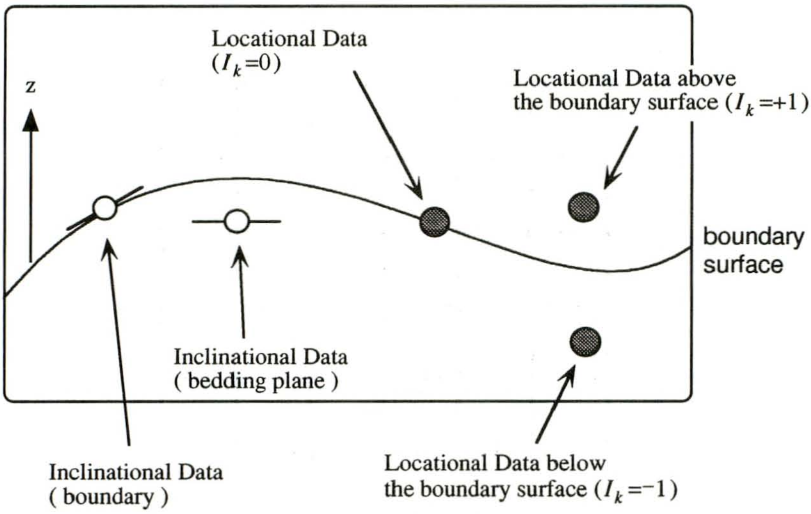


Fig. 11. Data for determination of boundary surface.

(ii-3) Determine the values p_{oik} and p_{ojk} depending on π_r :

If $\pi_r=0$, then set $p_{oik}=0$ and $p_{ojk}=0$.

If $\pi_r=1$, then set $p_{ojk}=0$.

If $\pi_r=1$, and $p_{oik}=2$ then set $p_{oik}=1$.

If $\pi_r=2$, then set $p_{oik}=0$.

If $\pi_r=2$, and $p_{ojk}=2$ then set $p_{ojk}=1$.

If $\pi_r=3$, and $p_{oik}=2$ then set $p_{oik}=1$.

If $\pi_r=3$, and $p_{ojk}=2$ then set $p_{ojk}=1$.

(iii) If the value 2 is still remained in the matrix, replace 2 with 0.

(iv) Construct the relation matrix $\mathbf{Q}_o$ by :

$$\mathbf{Q}_o=\mathbf{E}+(\mathbf{P}_o{}^t\cdot\mathbf{P}_o)+(\mathbf{P}_o{}^t\cdot\mathbf{P}_o)^2+\cdots+(\mathbf{P}_o{}^t\cdot\mathbf{P}_o)^{n-2}.$$

(v) Construct the relation matrix $\mathbf{P}$ by :

$$\mathbf{P}=\mathbf{P}_o\cdot\mathbf{Q}_o\ .$$

6. Determination of Boundary Surface

6.1 Method for Determination of Boundary Surfaces

SHIONO *et al.*(1987) presents a method to determine 3-D shapes of boundary surface $z=s(x, y)$ as the geologic application of constrained optimization problem. In this method, two kinds of field data are used (Fig. 11) :

· locational data

$$x_k,\ y_k,\ z_k,\ I_k\ (k=1,\ 2,\ \cdots)$$

· inclinational data

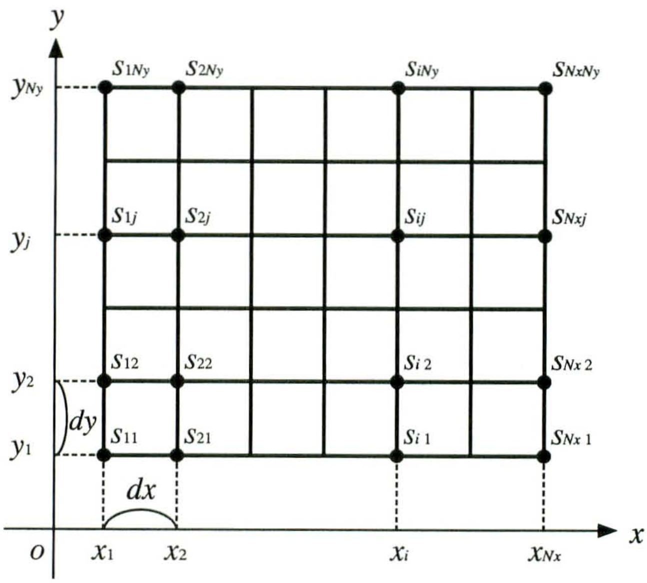


Fig. 12. Grid Data.

$$x_k,\ y_k,\ z_k,\ \xi_k,\ \eta_k \quad (k=1,\ 2,\ \cdots)$$

where $(x_k,\ y_k,\ z_k)$ is the coordinate of the outcrop, and ξ_k and η_k are strike and dip, respectively. I_k in locational data is an index assigning the spatial relationship between the outcrop $(x_k,\ y_k,\ z_k)$ and the surface $s(x,\ y)$. The variable $I_k=-1$, 0, and $+1$ shows that the outcrop is below, just on, and above the surface, respectively. These data provide constraints that the surface $s(x,\ y)$ should satisfy as follows :

$$\begin{aligned}
&s(x_k, y_k) \geqq z_k && (I_k=-1)\\
&s(x_k, y_k) = z_k && (I_k=0)\\
&s(x_k, y_k) \leqq z_k && (I_k=+1)\\
&s_x(x_k,\ y_k) = -\cos\ \xi_k\ \tan\ \eta_k\\
&s_y(x_k,\ y_k) = \sin\ \xi_k\ \tan\ \eta_k
\end{aligned}$$

where $s_x(x_k,\ y_k)$ and $s_y(x_k,\ y_k)$ are the partial derivatives of $s(x_k,\ y_k)$ with respect to x and y, respectively.

Then, the residual sums of squares are evaluated by $\phi_H(s)$ and $\phi_D(s)$ as follows :

$$\phi_H(s)=\Sigma^-[\min\{0,\ s(x_k,\ y_k)-z_k\}]^2+\Sigma^0(s(x_k,\ y_k)-z_k)^2+\Sigma^+[\max\{0,\ s(x_k,\ y_k)-z_k\}]^2$$
$$\phi_D(s)=\Sigma\{[s_x(x_k,\ y_k)+\cos\xi_k\ \tan\eta_k]^2+[s_y(x_k,\ y_k)-\sin\ \xi_k\ \tan\ \eta_k]^2\}.$$

where Σ^-, Σ^0 and Σ^+ are summation signs for data of $I_k=-1$, 0 and $+1$, respectively. When the smoothness of $s(x,\ y)$ is evaluated by:

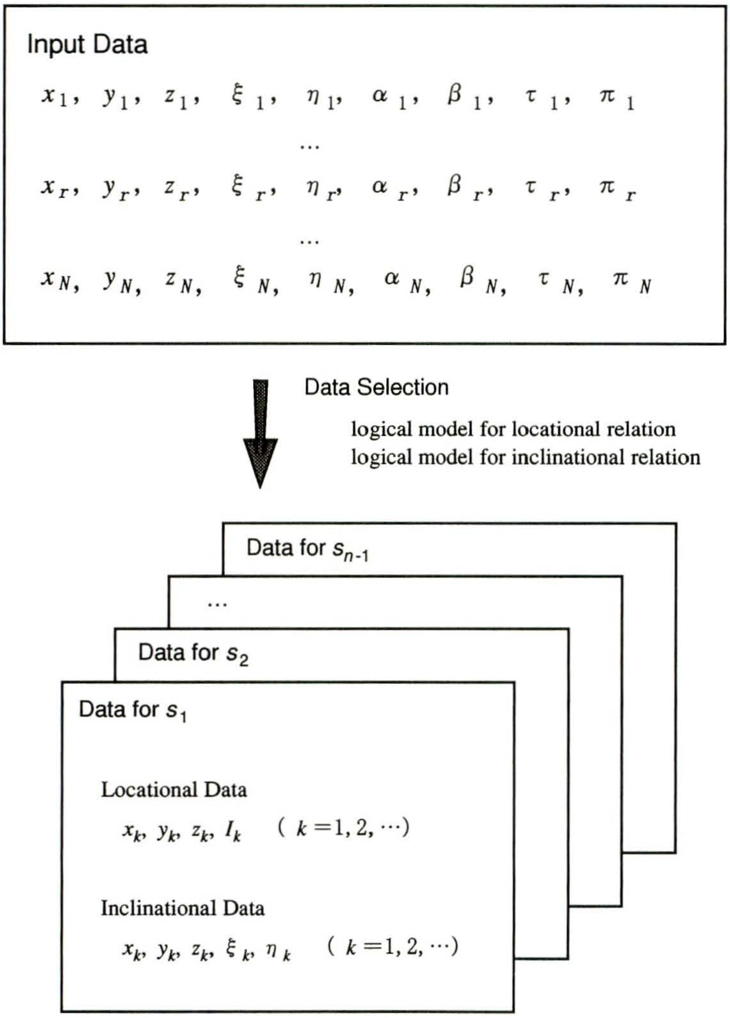


Fig. 13. Selection of input data for determination of boundary surface.

$$J(s)=m_1\int\{[s_x(x,y)]^2+[s_y(x,y)]^2\}dxdy+m_2\int\{[s_{xx}(x,y)]^2+2[s_{xy}(x,y)]^2+s_{yy}(x,y)]^2\}dxdy\ ,$$

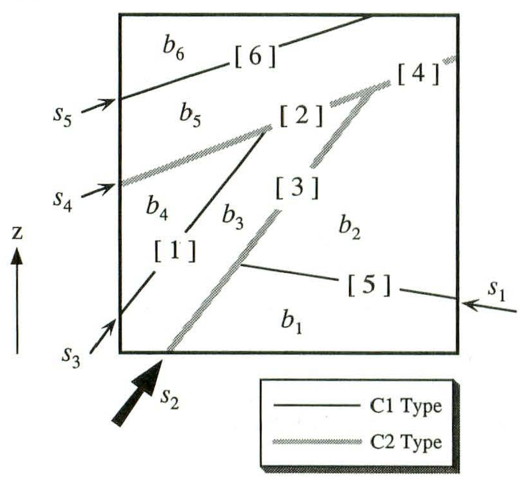
the smoothest surface $s(x, y)$ consistent with given constraints should minimize :

$$\Omega(s;\alpha)=J(s)+\alpha[\phi_H(s)+\gamma\phi_D(s)]$$

where α and γ are parameters to control the relative weights of $\phi_H(s)$ and $\phi_D(s)$, respectively.

In this paper, we represent the topographic surface and the geologic boundary surface in the form of the grid data (Fig. 12), which are arranged in a regular pattern. Similarly we approximate $s(x,y)$ by the discrete values $\mathbf{s}=(s_{11}, \cdots, s_{N_xN_y})$ on an $N_x \times N_y$ grid. Then $J(s)$, $\phi_H(s)$ and $\phi_D(s)$ are evaluated in a quadratic form of

(a)



(b)

	s_5	s_4	s_3	s_2	s_1
b_6	+1	+1	0	0	0
b_5	−1	+1	0	0	0
b_4	0	−1	+1	+1	0
b_3	0	−1	−1	+1	0
b_2	0	−1	0	−1	+1
b_1	0	−1	0	−1	−1

Fig. 14. Example for selection of locational data.
(a) Geologic section. [1] to [6] are outcrops. (b) Logical model for locational relation representing a structure (a).

s. We can obtain the optimal solution s^* through successive approximation of $s^{(k)}$, which minimizes $\Omega(s;\alpha_k)$ for the increasing sequence $\{\alpha_k|\alpha_1<\alpha_2<\cdots<\alpha_K\}$ (refer to SHIONO *et al.*,1987 for details).

6.2 Algorithm for Selection of Data

The set of data required to determine each 3-D boundary surface s_i $(i=1, \cdots, n-1)$ is prepared from data given in the form :

$$x_r,\ y_r,\ z_r,\ \xi_r,\ \eta_r,\ \alpha_r,\ \beta_r,\ \tau_r,\ \pi_r,$$

through mechanical procedures using the logical model for locational relation and the logical model for inclinational relation (Fig. 13).

6.2.1 Selection of Locational Data

Referring to a function $t:B\times S\to\{-1, 0, +1\}$ representing logical model for locational relation, we have a complete set of locational data required to determine the boundary surface s_i $(i=1, \cdots, n-1)$, after we repeat the following judgements for all input data :

$$x_r,\ y_r,\ z_r,\ —,\ —,\ \alpha_r,\ \beta_r,\ —,\ — \qquad (r=1,\ \cdots,\ N).$$

Case (i) : $t(\alpha_r,\ s_i)=+1$ and $t(\beta_r,\ s_i)=+1$
Both strata α_r and β_r are upper than a boundary surface s_i. Therefore, $(x_r,\ y_r,\ z_r)$ constrains the upper limit of the surface s_i, providing an inequality datum for s_i :

$$x_r,\ y_r,\ z_r,\ +1\ .$$

For example, Fig. 14(a) shows that a contact surface between two strata ($\alpha_1=b_3$,

$\beta_1 = b_4$) are observed at the outcrop [1]. Figure 14(b) shows that the logical model for locational relation gives $t(b_3, s_2) = +1$ and $t(b_4, s_2) = +1$. Therefore, we have a datum :

$$x_1,\ y_1,\ z_1,\ +1$$

to determine the surface s_2.

Case (ii) : $t(\alpha_r,\ s_i) = +1$ and $t(\beta_r,\ s_i) = 0$
A stratum α_r is upper than a boundary surface s_i, and a stratum β_r is independent of s_i. Since (x_r, y_r, z_r) constrains the upper limit of s_i, we have an inequality datum for s_i :

$$x_r,\ y_r,\ z_r,\ +1\ .$$

For example, the outcrop [2] in Fig. 14(a) provides a datum for s_2 as follows :

$$x_2,\ y_2,\ z_2,\ +1\ .$$

Similarly in the case of $t(\alpha_r,\ s_i) = 0$ and $t(\beta_r,\ s_i) = +1$, we have a datum :

$$x_r,\ y_r,\ z_r,\ +1\ .$$

Case (iii) : $t(\alpha_r,\ s_i) = -1$ and $t(\beta_r,\ s_i) = +1$

A stratum α_r is lower than a boundary surface s_i, and a stratum β_r is upper than s_i. This indicates that $(x_r,\ y_r,\ z_r)$ is on the boundary surface s_i. Therefore we have an equality data for s_i :

$$x_r,\ y_r,\ z_r,\ 0\ .$$

For example, the outcrop [3] in Fig. 14(a) gives a datum for s_2 :

$$x_3,\ y_3,\ z_3,\ 0\ .$$

It should be noted that there are no pairs $(\alpha_r,\ \beta_r)$ which satisfy $t(\alpha_r,\ s_i) = +1$ and $t(\beta_r,\ s_i) = -1$ because α_r is always lower than β_r.

Case (iv) : $t(\alpha_r,\ s_i) = -1$ and $t(\beta_r,\ s_i) = 0$
A stratum α_r is lower than a boundary surface s_i, and a stratum β_r is independent of s_i. Since $(x_r,\ y_r,\ z_r)$ constrains the lower limit of s_i, we have a datum for s_i :

$$x_r,\ y_r,\ z_r,\ -1\ .$$

For example, the outcrop [4] in Fig. 14(a) gives a datum for s_2 :

$$x_4,\ y_4,\ z_4,\ -1\ .$$

Similarly in the case of $t(\alpha_r,\ s_i) = 0$ and $t(\beta_r,\ s_i) = -1$, we have a datum :

$$x_r,\ y_r,\ z_r,\ -1\ .$$

Case (v) : $t(\alpha_r,\ s_i) = -1$ and $t(\beta_r,\ s_i) = -1$

Both strata α_r and β_r are lower than a boundary surface s_i. Since (x_r, y_r, z_r) constrains the lower limit of s_i, we have a datum for s_i :

$$x_r,\ y_r,\ z_r,\ -1\ .$$

For example, the outcrop [5] in Fig. 14(a) gives a datum for s_2 :

$$x_5,\ y_5,\ z_5,\ -1\ .$$

Case (vi) : $t(\alpha_r, s_i)=0$ and $t(\beta_r, s_i)=0$
Both strata α_r and β_r are independent of a boundary surface s_i. Therefore, location (x_r, y_r, z_r) cannot be used for the determination of the surface s_i.

For example, Fig. 14(a) shows that two strata ($\alpha_6=b_5$, $\beta_6=b_6$) at the outcrop [6]. Since $t(b_1, s_2)=0$ and $t(b_2, s_2)=0$, the location of [6] is independent of s_2.

Finally, observations at outcrops [1], ⋯, [6] in Fig. 14(a) provide a set of data for s_2 as follows :

$$\begin{array}{l} x_1,\ y_1,\ z_1,\ +1 \\ x_2,\ y_2,\ z_2,\ +1 \\ x_3,\ y_3,\ z_3,\ 0 \\ x_4,\ y_4,\ z_4,\ -1 \\ x_5,\ y_5,\ z_5,\ -1 \end{array}$$

6.2.2 Selection of inclinational data

Referring to logical model for inclinational relation, we can select a proper set of inclinational data from input data :

$$x_r,\ y_r,\ z_r,\ \xi_r,\ \eta_r,\ \alpha_r,\ \beta_r,\ \tau_r,\ \pi_r,$$

where :

(i) if $\alpha_r=\beta_r$, then ξ_r and η_r give strike and dip of the bedding plane in a stratum α_r, respectively.

(ii) if $\alpha_r\neq\beta_r$, then ξ_r and η_r give strike and dip of a contact surface between α_r and β_r, respectively.

(i) Case of $\alpha_r=\beta_r$

$$x_r,\ y_r,\ z_r,\ \xi_r,\ \eta_r$$

is used to determine a surface s_i which satisfies $p(\alpha_r, s_i)=1$.

(ii) Case of $\alpha_r\neq\beta_r$

$$x_r,\ y_r,\ z_r,\ \xi_r,\ \eta_r$$

is used to determine a surface s_j if $u(\alpha_r, \beta_r)=s_j$, that is, if s_j is a boundary surface between α_r and β_r. Further the inclinational data is also used to determine every surface

s_i which satisfies $q(s_i, s_j)=1$.

7. Construction of Function $g:X\rightarrow B$

Combining the logical models for locational relation $t: B\times S\rightarrow\{-1, 0, +1\}$ and the 3-D figures of the boundary surfaces $z=s_i(x, y)$ $(i=1, \cdots, n-1)$, we can define a function $g: X\rightarrow B$ which assigns a unique stratum $b\in B$ to every point $p\in X$.

7.1 Function g' — from 3-D space to set of binary numbers—

Let (x_p, y_p, z_p) be a coordinate of a point $p\in X$. Then comparing the elevation z_p of the point p with the height $s_i(x_p, y_p)$ of the i-th boundary surface defines a number δ_i such that :

$$\delta_i=\begin{cases}1, & \text{if } z_p\geqq s_i(x_p, y_p)\\ 0, & \text{if } z_p<s_i(x_p, y_p).\end{cases} \qquad (i=1, \cdots, n-1)$$

Thus, an $(n-1)$-digit binary number $(\delta_{n-1}\delta_{n-2}\cdots\delta_2\delta_1)_2$ is assigned to each point p. This rule represents a function $g':X\rightarrow Y$, where Y is the set of all $(n-1)$-digit binary numbers $\{(\delta_{n-1}\delta_{n-2}\cdots\delta_2\delta_1)_2|\delta_k=0, 1; k=1, \cdots, n-1\}$. For example, when there are three surfaces s_1, s_2, s_3 in the 3-D space X, $(011)_2$ is assigned to the point which is below s_3, above s_2 and above s_1 (Fig. 15(a)).

The set of points to which binary number $(\delta_{n-1}\delta_{n-2}\cdots\delta_2\delta_1)_2$ is assigned can be represented by the inverse image of g' as follows :

$$g'^{-1}((\delta_{n-1}\delta_{n-2}\cdots\delta_2\delta_1)_2) = \bigcap_{k=1}^{n-1} S_k=S_{n-1}\cap\cdots\cap S_1$$

where :

$$S_k=\begin{cases} s_k^{+1}, & \text{if } \delta_k=1\\ s_k^{-1}, & \text{if } \delta_k=0.\end{cases}$$

Thus, we can consider the binary number $(\delta_{n-1}\delta_{n-2}\cdots\delta_2\delta_1)_2$ as the code number of the subspace divided by boundary surfaces.

7.2 Function g'' —from set of binary numbers to set of strata—

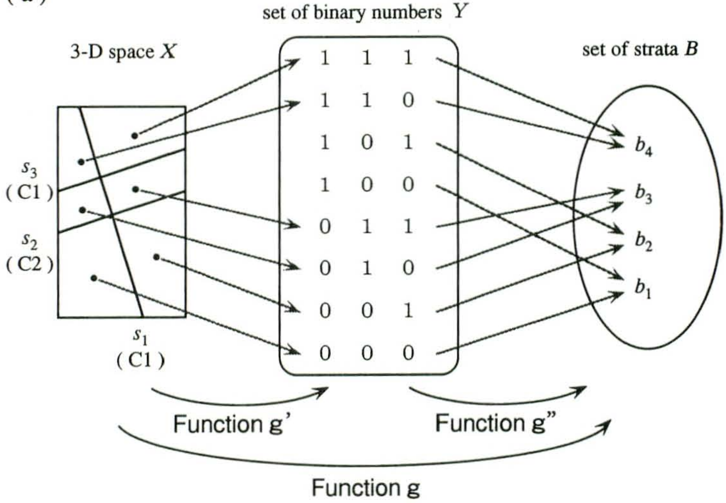
From the logical model for locational relation, we can determine a function $g'': Y\rightarrow B$ which assign a stratum to a binary number.

The logical model for locational relation represents the distribution of stratum. For example, the logical model given in Fig. 15(b) shows that the distribution of b_3 is represented by :

$$\sigma(b_3)=s_3^{-1}\cap s_2^{+1}.$$

This indicates that the surface s_1 is independent of the distribution of b_3. However, considering that $\sigma(b_3)$ is included in both the upper subspace s_1^{+1} and the lower subspace

(a)



(b)

	s_3	s_2	s_1
b_4	+1	+1	0
b_3	−1	+1	0
b_2	0	−1	+1
b_1	0	−1	−1

(c)

δ_3	δ_2	δ_1	Strata
1	1	1	b_4
1	1	0	b_4
1	0	1	b_2
1	0	0	b_1
0	1	1	b_3
0	1	0	b_3
0	0	1	b_2
0	0	0	b_1

Fig. 15. Function $g{:}X\rightarrow B$.
(a) Function $g{:}X\rightarrow B$ is constructed by functions $g'{:}X\rightarrow Y$ and $g''{:}Y\rightarrow B$. (b) Logical relation for locational relation representing a structure (a). (c) Function g''.

s_1^{-1}, we introduce an expression :

$$s_1^{0}=s_1^{+1}\cup s_1^{-1} .$$

Then, we have a formal expression for $\sigma(b_3)$ as follows :

$$\sigma(b_3)=s_3^{-1}\cap s_2^{+1}\cap s_1^{0} . \tag{7.1}$$

It is noted that the superscripts -1, $+1$ and 0 correspond formally to the components of the logical model for locational relation shown in Fig. 15(b).

Substituting $(s_1^{+1}\cup s_1^{-1})$ for s_1^{0} in equation (7.1), we have :

$$\begin{aligned}\sigma(b_3)&=s_3^{-1}\cap s_2^{+1}\cap(s_1^{+1}\cup s_1^{-1})\\&=(s_3^{-1}\cap s_2^{+1}\cap s_1^{+1})\cup(s_3^{-1}\cap s_2^{+1}\cap s_1^{-1}) .\end{aligned}$$

First and second terms of right-hand side represent subspaces to which binary numbers $(011)_2$ and $(010)_2$ are assigned, respectively. This suggests that the points to which the function g' assigns $(011)_2$ or $(010)_2$ are included in $\sigma(b_3)$.

Generalizing the above discussion, let us give the formal expression for every $\sigma(b_i)$ as follows :

$$\sigma(b_i)=\bigcap_{k=1}^{n-1} S'_{ik}=S'_{i1}\cap\cdots\cap S'_{in-1} \tag{7.2}$$

where :

$$S'_{ik}=\begin{cases} s_k^{+1}, & \text{if } t(b_i,\ s_k)=+1 \\ s_k^{0}=s_k^{+1}\cup s_k^{-1}, & \text{if } t(b_i,\ s_k)=0 \\ s_k^{-1}, & \text{if } t(b_i,\ s_k)=-1. \end{cases}$$

For s_k^0, let the set $J(i)$ be :

$$J(i)=\{k|S'_{ik}=s_k^{0}\},$$

and the set J be :

$$J=\{f|f{:}N\times N'\to\{-1,\ +1\},\ f(i,k)=t(b_i,\ s_k)\ \text{if}\ k\notin J(i)\}$$

where N and N' are the sets of integer as follows :

$$N=\{1,\ \cdots,\ n\}$$
$$N'=\{1,\ \cdots,\ n-1\}\ .$$

Then, we have an equation representing $\sigma(b_i)$ as the union of some subspaces divided by boundary surfaces $s_1,\ \cdots,\ s_{n-1}$ as follows :

$$\sigma(b_i)=\bigcup_{f\in J}(\bigcap_{k=1}^{n-1} s_k^{f(i,k)})$$

Using one-digit number δ_{ik}:

$$\delta_{ik}=\begin{cases} 1, & \text{if}\, f(i,k)=+1 \\ 0, & \text{if}\, f(i,k)=-1 \end{cases}$$

we have another expression of $\sigma(b_i)$:

$$\sigma(b_i)=\bigcup_{f\in J} g^{-1}((\delta_{in-1}\delta_{in-2}\cdots\delta_{i2}\delta_{i1})_2)\,. \tag{7.3}$$

Thus, we define a function $g''{:}\,Y\to B$ which assigns the stratum b_i to every binary numbers which appear in left-hand side of equation (7.3). Then, $g''((\delta_{n-1}\cdots\delta_2\delta_1)_2)\ =b_i$ represents that a subspace to which a binary number $(\delta_{n-1}\cdots\delta_2\delta_1)_2$ is assigned is included in $\sigma(b_i)$.

7.3 Algorithm for Construction of Function g''

The following is an algorithm for the construction of the function $g'': Y \rightarrow B$.

(i) Repeat steps (ii) to (iv) for $i=0, \cdots, 2^{n-1}-1$.
(ii) Translate the value i into a binary number $(\delta_{n-1}\delta_{n-2}\cdots\delta_2\delta_1)_2$.
(iii) Repeat step (iv) for $j=1, \cdots, n$.
(iv) Set $g''((\delta_{n-1}\cdots\delta_2\delta_1)_2)=b_i$ if for all $k=1, \cdots, n-1$, we have either :

$$t(b_j, s_k)=+1 \text{ or } 0 \text{ when } \delta_k=1$$

or :

$$t(b_j, s_k)=-1 \text{ or } 0 \text{ when } \delta_k=0 .$$

The function g'' is represented in a tabular form as shown in Fig. 15(c).

7.4 Function g

The function $g': X \rightarrow Y$ assigns a binary number $(\delta_{n-1}\cdots\delta_2\delta_1)_2$ to a point p in the 3-D space X, and the function $g'': Y \rightarrow B$ assigns a stratum name to the binary number. Therefore, the function $g: X \rightarrow B$ defined by :

$$\begin{aligned} g(p) &= g''(g'(p)) \\ &= (g''\cdot g')(p) \end{aligned}$$

or :

$$g=g''\cdot g',$$

assigns a stratum name to a point in X (Fig. 15(a)).

The algorithm to define $g(p)$ for any point p in X is simply described as follows :

Let (x_p, y_p, z_p) be coordinates of a point p in X, then :

$$g(p)=g''((\delta_{n-1}\cdots\delta_2\delta_1)_2)$$

where :

$$\delta_i=\begin{cases} 1, & \text{if } z_p \geqq s_i(x_p, y_p) \\ 0, & \text{if } z_p < s_i(x_p, y_p). \end{cases} \qquad (i=1, \cdots, n-1)$$

8. Computer System "CIGMA"

8.1 Outline of Computerized Mapping System

The computerized mapping system is developed, based on theory and algorithm described in the previous sections. Figure 1 summarizes the logical framework of the system. We call this system "CIGMA", abbreviated from Computer-Inferred Geologic Map.

(1) Inference of stratigraphic sequence
From given data, the set B of strata distributed in the studied area are determined, and observed stratigraphic relations of strata are represented in the form of a relation matrix. The stratigraphic sequence is determined by matrix operations.

(2) Construction of logical models of geologic structures
Two kinds of logical models of geologic structures are constructed according to the character of boundary surface described in given data : a logical model for locational relation and a logical model for inclinational relation.

(3) Determination of boundary surface
In order to determine each boundary surface, locational and inclinational data are selected from given data set using logical models of geologic structures. Boundary surface is determined as the smoothest surface by the method of constrained optimization.

(4) Assignment of strata by a function $g{:}X \to B$
Every point to be drawn in a geologic map is transformed to a binary number representing locational relation through a function $g'{:}X \to Y$, and is assigned a unique stratum through a function $g''{:}Y \to B$.

(5) Graphical presentation
Finally, CIGMA projects a 3-D geologic map on a display by coloring the strata respective to all grid cells on the topographic surface and four side sections.

8.2 Programs and Data

CIGMA is written in Fortran77 and the graphics functions of GKS (ISO, 1985), and is implemented under UNIX and X-Window environment. The source code of the program and sample data sets are available from the author.

CIGMA consists of the six routines. Figure 16 shows the flowchart. These procedures are performed automatically by the following routines.

Routine (i) : **ORDER**
Using the relation between strata in the input data, the routine ORDER infers the stratigraphic sequence and logical models of geologic structures, and selects sets of data to determine boundary surfaces. The subroutine to infer the stratigraphic sequence is based on the algorithm given by Sakamoto and Shiono (1992).

Routine (ii) : **BOUNDARY**
Using sets of data selected by ORDER, the routine BOUNDARY determines all boundary surfaces in forms of the grid data based on the algorithm given by Shiono *et al.*(1987).

Routine (iii) : **SOLID**

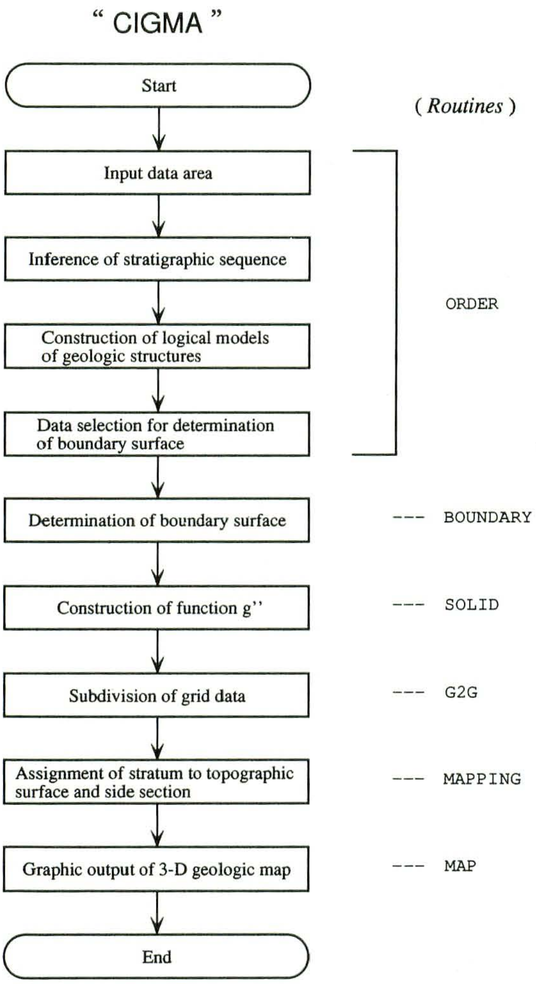


Fig. 16. Flow chart of CIGMA.

Using the logical model for locational relation inferred by ORDER, the routine SOLID constructs a function $g'':Y\to B$.

Routine (iv) : **G2G**
In order to obtain a fine graphics output, the topographic surface is interpolated using the bi-cubic spline functions (deBOOR, 1962).

Routine (v) : **MAPPING**
Using grid data of boundary surfaces created by BOUNDARY and the function $g':Y\to B$ constructed by SOLID, the routine MAPPING assigns a stratum to every grid node of the topographic surface, and to every grid node of four gridded side sections.

Routine (vi) : **MAP**

The routine MAP projects the strata assigned to all grid nodes by MAPPING on the computer screen. Thus a 3-D geologic map is completed. This routine is based on MASUMOTO *et al.*(1986).

The CIGMA uses observation data described in the format :

$$x_r,\ y_r,\ z_r,\ \xi_r,\ \eta_r,\ \alpha_r,\ \beta_r,\ \tau_r,\ \pi_r$$

as mentioned in Section 2.3. This format can describe various types of data as follows.

(i) Case that relations between different strata are observed at r-th outcrop

$$x_r,\ y_r,\ z_r,\ (\xi_r),\ (\eta_r),\ \alpha_r,\ \beta_r,\ \tau_r,\ \pi_r.$$

Null values are given to ξ_r and η_r in the case that strike and dip of the boundary surface are not observed.

(ii) Case that only one stratum is observed at r-th outcrop

$$x_r,\ y_r,\ z_r,\ (\xi_r),\ (\eta_r),\ \alpha_r,\ \alpha_r,\ —,\ —$$

where — shows a null value. ξ_r and η_r are given in the case that strike and dip of the layered structure are observed.

(iii) Case that the expert knowledge is introduced

$$—,\ —,\ —,\ —,\ —,\ \alpha_r,\ \beta_r,\ (\tau_r),\ (\pi_r)$$

τ_r and π_r may be given from some assumptions on the geologic structure.

In addition to observation data mentioned above, the topographic surface must be given in the form of grid data.

8.3 Example of Application

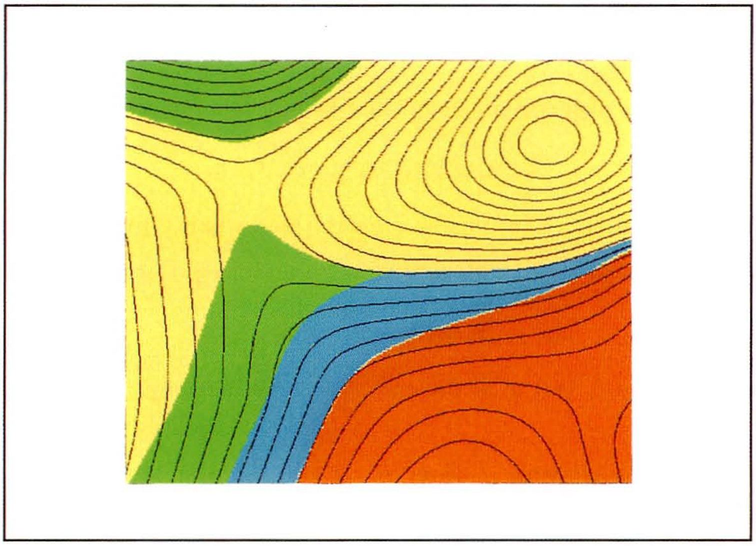
Figures 17 and 18 show examples of geologic maps drawn by CIGMA. Input data for the geologic map shown in Fig. 17 are :

```
100,  80, 180, 270, 10, b3, b4, 2, 2
100,  50, 157, 210, 15, b1, b2, 1, 3
 60,  50, 169, 210, 30, b2, b3, 1, 2
```

and data for Fig. 18 are :

```
 48,  24, 172, 190, 30, b2, b3, 1, 3
 72,  24, 157, 190, 30, b1, b2, 1, 3
128,  24, 138,  10, 30, b1, b2, 1, 3
160,  24, 142,  10, 30, b2, b3, 1, 3
 24, 144, 159, 190, 30, b2, b3, 1, 3
```

(a)



(b)

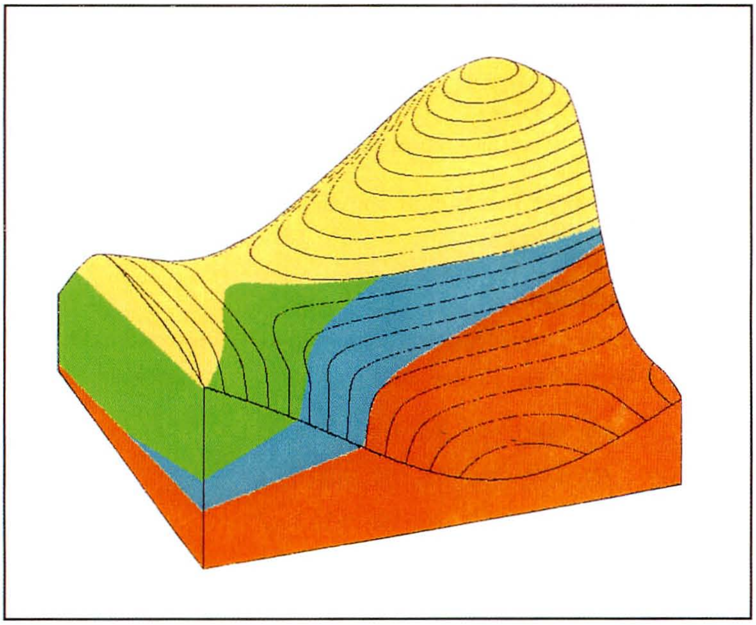


Fig. 17. Examples of output (1).
Strata are b1, b2, b3, and b4 in ascending order. All boundaries are plane surfaces. (a) 2-D geologic map. (b) 3-D geologic map.

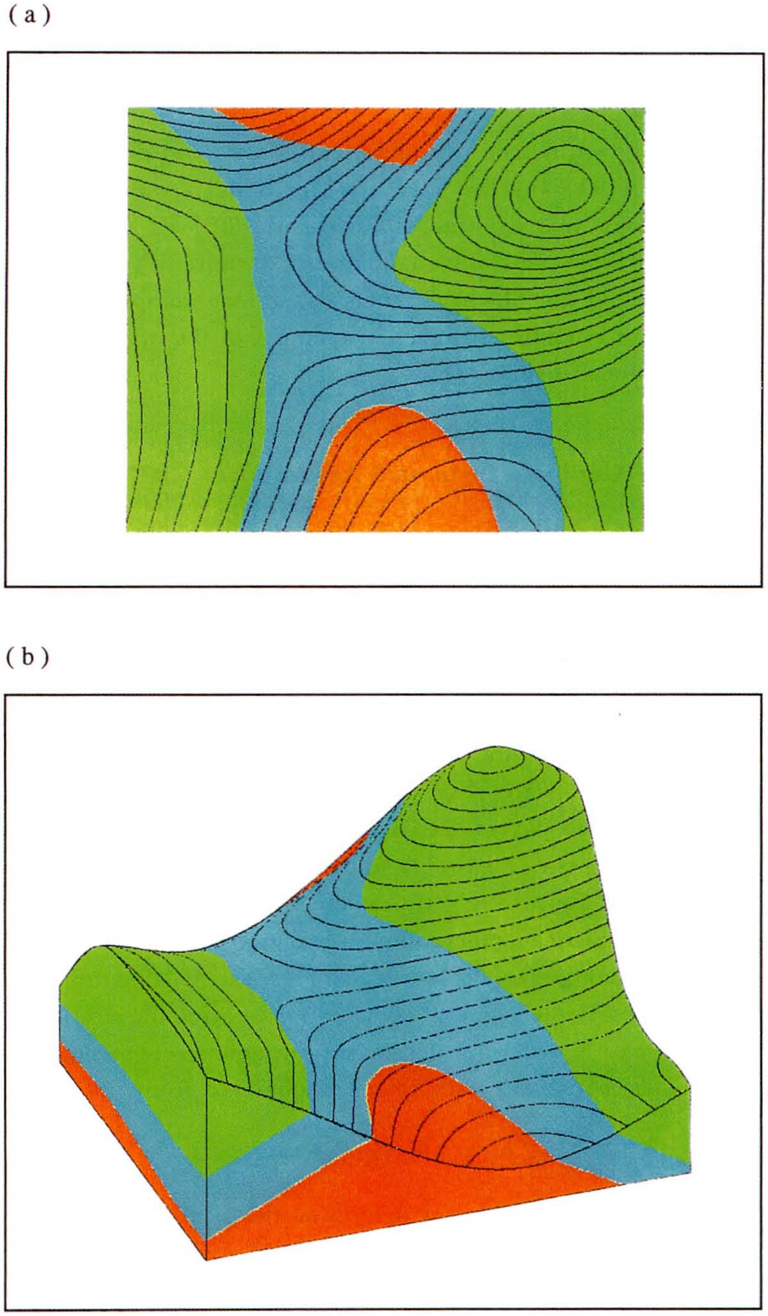


Fig. 18. Examples of output (2).
Strata are b1, b2, and b3 in ascending order. All boundaries are folded surfaces.
(a) 2-D geologic map. (b) 3-D geologic map.

80, 144, 172, 190, 30, b1, b2, 1, 3
112, 144, 193, 10, 30, b1, b2, 1, 3
128, 144, 205, 10, 30, b2, b3, 1, 3

where coordinates are given in arbitrary units, and strikes and dips are in degree. The same topographic data (192×160 area) are used for both examples. Figures 17(a) and 18(a) are the 2-D geologic maps. Figures 17(b) and 18(b) are the 3-D geologic maps, and the azimuth and inclination of a view point are 20° and 30°, respectively. Contour lines are drawn from 135 to 230 at 5 intervals. Most computational processes are automatically carried out after the manual arrangement of the original data and selection of parameters for graphical display.

9. Geologic Structure Assumed in CIGMA

CIGMA is a computer software system to draw geologic maps automatically according to data obtained directly from field observations. The system is constructed based on several inference rules derived from five Axioms A1 to A5. The axioms are introduced as tentative formulations for natures of an idealized geologic structures, i.e., accumulations of eroded and/or non-eroded sedimentary layers without faulting nor overfolding. Therefore, it should be noted that there are limitations in applicability of CIGMA (Fig. 19).

Axioms A1, A2 and A3 provide a theoretical basis for computer algorithms to infer the stratigraphic sequence from observations on spatial relations between exposed rocks. Most of sedimentary layers satisfy Axioms A1, A2 and A3. However, strata displaced by fault movements and strongly folded strata may not satisfy the axioms, but some stratified lava flows may satisfy the axioms. As seen from this example, it is noted that geologic bodies satisfying the axioms are not necessarily sedimentary layers.

Fact that geologic bodies in the surveyed area satisfy the three axioms do not guarantee that L_{OE} is a total ordering, that is, we can determine the stratigraphic sequence, but only that L_{OE} is a partial ordering, that is, we can enumerate geologic bodies linearly in such an order that is consistent with relations obtained from the observations :

$$b_1, b_2, \cdots, b_n \ (b_i L_{OE} b_j \Rightarrow i \leqq j).$$

There are two cases that we can not determine the stratigraphic sequence. One is the case that we have not observed sufficiently enough to make L_{OE} a total ordering, as shown in Fig. 20(a). In this case, we must continue to search outcrops which expose relations between incomparable pairs of geologic bodies. The other is the case that the geologic structure itself includes at least one pair of geologic bodies which are not comparable. In both cases, since the stratigraphic sequence is not fixed from given data, we cannot proceed to the next step to create the logical models of geologic structures. Therefore, CIGMA is designed to stop after showing the incomparable

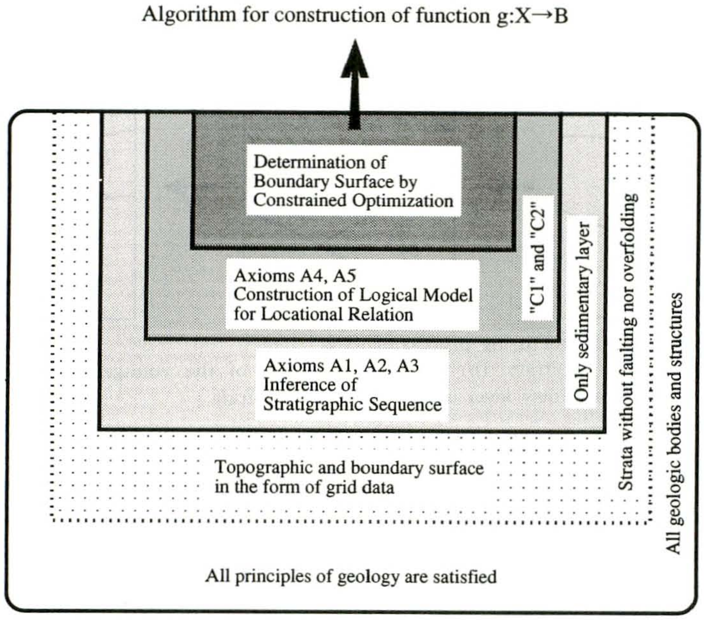


Fig. 19. Limitation in geologic structure for construction of function g.

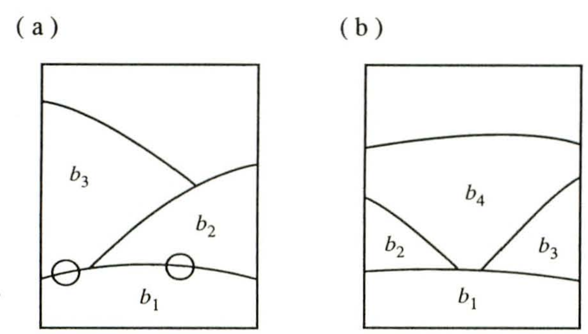


Fig. 20. Examples that b_2 and b_3 are incomparable.
(a) A relation L_{OE} among b_2 and b_3 is not observed. Open circles are outcrops.
(b) b_2 and b_3 are not comparable.

pairs of geologic bodies when L_{OE} is not a total ordering.

Axioms A4 and A5 provide a theoretical basis to create logical models of geologic structures based on field observations. However, the axioms introduce additional limitations. "C1" idealizes a contact surface between layers created by a successive sedimentation without any erosion. On the other hand, "C2" represents a contact surface between a new layer and eroded one. Strictly speaking, "C2"assumes a history such that a new layer overlies the preexisting ones after the preexisting ones are removed partially by erosion to the extent that the upper surface of the youngest layer among

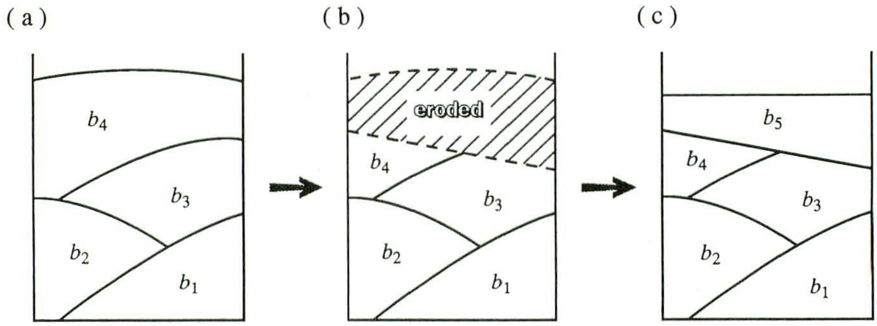


Fig. 21. An image of structuring process of "C2"
(a) Preexisting strata. (b) The upper surface of the youngest layer is completely removed. (c) A new layer overlies preexisting strata.

the preexisting ones is completely removed (Fig. 21). In the real situation, there are some cases that only a part of the upper surface is removed by erosion. Further, Axioms A4 and A5 assume that sedimentation and erosion occur at distinct intervals. Sedimentation and erosion may occur simultaneously in some cases. For example, considering the area around seashore, geologic bodies are eroded on land and eroded particles form a sedimentary layer under the water. Therefore, we need to generalize Axioms A4 and A5 in order to approach more realistic situations.

Assumption 5.1 is a tentative assumption which is introduced to make effective use of inclinational data for determination of boundary surfaces. The assumption is unnecessary when we have no inclinational data. The assumption is introduced from an idea that if surfaces are parallel to each other then inclinational data observed on one surface may be used as the inclination of other surfaces. As the assumption is used only for arrangement of data required to determine boundary surfaces by the routine BOUNDARY in CIGMA, surfaces determined by BOUNDARY may not necessarily be parallel to each other. We also need to develop new theories to reconstruct a various types of folding structures according to their behaviours.

10. Conclusion

In this paper, a basic theory for computerized geologic mapping is formulated systematically based on five Axioms A1 to A5. The results are summarized as follows :

1) Axioms A1, A2 and A3 introduce inference rules to determine the stratigraphic sequence from the field observations.

2) Axioms A4 and A5 provide a theoretical basis to construct logical models of geologic structures based on field observations.

3) Assumption 5.1 is a tentative assumption to create the logical model for inclinational relation so that inclinational data are used for determining boundary surfaces.

4) As the result, the computer algorithms for construction of function $g:X \rightarrow B$ which assigns a stratum to every point in the 3-D space are constructed.

5) A Fortran 77 computer program CIGMA is coded according to presented algorithms to draw a 3-D geologic map automatically.

As CIGMA draws a geologic map quickly based on the field observations, it will be useful for all steps of field survey from rough drafting to final mapping. Further it is expected that CIGMA will receive wide application including geologic educations.

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