CASSON-WALKER INvariants of CYClic COVERS
BRANched ALONG SATELLITE KNOTS

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Abstract

We express the Casson-Walker invariants for the cyclic covering spaces of the three-dimensional sphere branched along satellite knots in terms of companions, patterns, and winding numbers.

1. Introduction

Let $C$ be a knot in $S^3$, and $K$ a knot in a tubular neighborhood $N(C)$ of $C$. Let $\gamma$ be a generator of the kernel of $i_* : H_1(\partial N(C); \mathbb{Z}) \to H_1(S^3 - \tilde{N}(C); \mathbb{Z})$, where $i$ is the inclusion. We regard $\gamma$ as a simple closed curve on $\partial N(C)$. Then, there is a unique embedding $f : N(C) \to S^3$, up to isotopy, such that the exterior $E(f(C)) = S^3 - f(\tilde{N}(C))$ is the solid torus and $f(\gamma)$ bounds a disk in $E(f(C))$. Denote by $C_0$ the core circle of $E(f(C))$. The knot $P = f(K)$ is called a pattern knot for $K$ associated to the companion $C$. The 2-component link $P \cup C_0$ in $S^3$ is called a pattern link, and $w = |\text{lk}(P, C_0)|$ is called the winding number (cf. Fig. 1). We denote by $\Sigma^r_K$ the $r$-fold cyclic covering space over $S^3$ branched along a knot $K$. In this paper, we present the Casson-Walker invariant of the $r$-fold cyclic covering space $\Sigma^r_K$ of $S^3$ branched along a satellite knot $K$ in terms of patterns, companions, and the winding numbers.

A Laurent polynomial $\Lambda(t) \in \mathbb{Z}[t, t^{-1}]$ is called a knot-Alexander polynomial provided that $\Lambda(t^{-1}) = \Lambda(t)$ and $\Lambda(1) = 1$. Namely, a knot-Alexander polynomial can be written as a finite sum $\Lambda(t) = c_0 + \sum_{i > 0} c_i (t^i - t^{-i})$, where $c_i \in \mathbb{Z}$ and $c_0 = 1 - 2 \sum c_i$. The Alexander polynomial $\Delta_K(t)$ of a knot $K$ in a homology sphere is a knot-Alexander polynomial. Conversely, given a knot Alexander polynomial $\Lambda(t)$ there is a knot $K$ with $\Delta_K(t) = \Lambda(t)$. We also use Conway’s version of the Alexander polynomial $\nabla_K(z) = 1 + \sum_{i > 0} a_{2i} z^i$, where we denote by $a_{2i}(K)$ the $2i$-th coefficient of the Conway polynomial $\nabla_K(z)$, which is equivalent to $\Delta_K(t)$ via $z = t^{-1/2} - t^{1/2}$. Note that $a_2(K) = 1/2 \Delta''_K(1)$, where $\Delta''_K(t)$ is the second derivative of $\Delta_K(t)$. Note also that when $|H_1(\Sigma^1_K; \mathbb{Z})|$ is finite, the order is determined by the Alexander polynomial $\Delta_K(t)$.

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that H. Seifert [19] showed the equation $\Delta_K(t) = \Delta_P(t)\Delta_C(t^w)$ between the Alexander polynomials for a knot $K$ with a companion $C$ and a pattern $P$.

A compact orientable 3-manifold $M$ is called a rational homology circle if the homology group $H_*(M; \mathbb{Q})$ is isomorphic to that of the standard solid torus $D^2 \times S^1$. Note that each rational homology circle is homeomorphic to the exterior $E(K, H)$ of a knot $K$ in a rational homology sphere $H$ and the symmetrized Alexander polynomial $\Delta'_K(t) \in \mathbb{Q}[t, t^{-1}]$ is naturally defined so that $\Delta'_K(1) = 1$ (cf. [23]) and $a_2(K) = 1 / 2 \Delta''_K(1)$.

In 1985, A. Casson introduced an integer valued invariant $\lambda$ for integral homology spheres that counts the number of the conjugacy classes of irreducible $SU(2)$-representations of the fundamental group in some sense (cf. [1], [18]). For two homology spheres $H$ and $H'$, if $H'$ is obtained from $H$ by the $(1/n)$-surgery along a knot $K$ in $H$, then $\lambda(H') = \lambda(H) + n / 2 \Delta''_K(1)$. The Casson invariant $\lambda$ was extended to rational homology spheres by K. Walker [23], who provided a method of defining this invariant in a combinatorial way. Casson’s surgery formula is generalized as $\lambda(M') = \lambda(M) + q / p a_2(K \subset M) + \lambda(L(p, q))$, where $M'$ is obtained from $M$ by the $q/p$-surgery on a knot $K$ in $M$, and the value of the Casson-Walker invariant of the Lens space $L(p, q)$ is written as the Dedekind sum. C. Lescop [13] extended the Casson-Walker invariant to all closed 3-manifolds.

To state our theorem, we need the following notation: For a 2-component link $l \cup k$, we define $\alpha'_r(k)$ as follows: Let $k^+$ denote the preferred longitude for $k$. Let $\psi: \Sigma^+_r \rightarrow S^3$ be the $r$-fold cyclic cover branched along $l$. Let $k'$ denote a component of $\psi^{-1}(k)$ and $k'^+$ the component of $\psi^{-1}(k)$ corresponding to $k^+$. Then, when $\Sigma^+_r$ is a rational homology sphere, $\text{lk}(k'^+, k)$ is an integer and we put $\alpha'_r(k) = \text{lk}(k'^+, k)$. It is easy to see that $|\alpha'_r(K)| = 2$ for the Whitehead link $W = k \cup l$. Then we show that $\lambda(\Sigma^+_K)$ is written in terms of $P$ and $C$ as follows:

**Theorem 1.1.** For a satellite knot $K$ with a pattern $P$ associated to a companion $C$ whose winding number is $w$, where $\mu = \gcd(r, w)$, $\lambda(\Sigma'_K) = \lambda(\Sigma'_P) + \mu \lambda(\Sigma'_C^\mu) - \mu a'_P(C_0)a_2(\tilde{C} \subset \Sigma'_C^\mu)$ when $\Sigma'_K$ is a rational homology sphere.
We notice that $\Sigma'_K$ is a rational homology sphere if and only if each of $\Sigma'_P$ and $\Sigma^m_C$ is a rational homology sphere. As a special case, we have the following corollary:

**Corollary 1.2.** For a satellite knot $K$ with a pattern $P$ associated to a companion $C$ whose winding number is zero, $\lambda(\Sigma'_K) = \lambda(\Sigma'_P) - \alpha'_P(C_0)a_2(C)$ when $\Sigma'_P$ is a rational homology sphere.

Several authors investigate the Casson-Walker invariant of $\Sigma'_K$ for some satellite knots. A satellite knot is called an untwisted doubled knot if $P \cup C_0$ forms the Whitehead link. J. Hoste proved that for untwisted doubled knots $DP$ about $P$ in $S^3$, $\lambda(\Sigma'_D) = 2a_2(D)r$ [9, Theorem 3.2]. In [21], Yasuyoshi Tsutsumi generalized this result for satellite knots whose patterns are two-bridge knots and companions are based on certain Conway’s rational tangles. See Ishibe [10] for general doubled knots. In [6], Fujita gave a formula for the Casson-Walker invariant of the manifold obtained from two rational homology circles by gluing their boundaries. See [17] for a simple proof. See also [7] and [3].

D. Mullins [14] gave a relation among the first derivative of the Jones polynomial at $-1$, $J'_K(-1)$, the Casson-Walker invariant of the double branched cover $\lambda(\Sigma^2_Z)$ and the ordinal signature $\sigma(K)$. Shinohara [20] gave a relation between the signatures of satellite knots and the companion, patterns. It is easy to see that when the winding number $w = 0$, $K$ and the pattern have the same signature. Combining Corollary 1.2 with Mullins’ result, we have the following:

**Corollary 1.3.** $J'_K(-1) = J'_P(-1) - 6\alpha^2_P(C_0)a_2(C)$ if the winding number is zero.

There are several relation between $\alpha_i^r(k)$ and some other invariants. In fact, $\alpha_i^2(k)$ is related to Cochran’s beta-invariant [4] of the 2-component link and the derived links [2]. One also notices that $\alpha'_i(k)$ can be derived from Kojima-Yamasaki’s function [12] and equivariant linking numbers. In §3, we show the following:

**Proposition 1.4.** Let $N \geq 2$ be a natural number greater than one. Given $N$ integers $\lambda_2, \lambda_3, \ldots, \lambda_N, \lambda$, there are knots $K$ in $S^3$ such that $\Delta_K(t) = 1$ and

$$
\lambda(\Sigma'_K) = \begin{cases} 
2r\lambda_i & \text{if } 2 \leq r \leq N \\
2r\lambda & \text{if } N < r 
\end{cases}.
$$

We note that Collin and Saveliev [5, Theorem 1] showed that the equivariant Casson invariant $\lambda^{Z/n}$ of $\Sigma^n_K$ is determined by a Tristram-Levine equivariant signature of $K$, that is, the signature of Hermitian form $(1 - e^{2\pi i k/n})S + (1 - e^{-2\pi i k/n})S^T$ for $k = 0, 1, \ldots, n - 1$, where $S$ is a Seifert form of $K$. This implies that $\lambda^{Z/n}(\Sigma^n_K) = 0$ if $\Delta_K(t) = 1$ since any knots with trivial Alexander polynomials are $S$-equivalent to the unknot. They also showed that $\lambda^{Z/n}(\Sigma^n_K) = \lambda(\Sigma^n_K)$ for a graph knot $K$ [5, Theorem 3].
Hence, most knots constructed in the proof of Proposition 1.4 cannot be a graph knot, and the difference between \( \lambda^{\mathbb{Z}/n}(\Sigma^n_{\mathbb{Z}}) \) and \( \lambda(\Sigma^n_{\mathbb{Z}}) \) is due to the variation of \( \sigma^n_x \) for certain links. According to Garoufalidis and Kricker [8], this difference can be written in terms of residues of the 2-loop polynomials related to the Kontsevich integral of the knot.

2. Proof of Theorem 1.1

Let \( V \) be a rational homology circle. Then, \( \partial V \) is a single torus and a generator of the kernel of \( i_*: H_1(\partial V; \mathbb{Z}) \to H_1(V; \mathbb{Z}) \) is called a **longitude** for \( V \), where \( i: \partial V \to V \) is the inclusion. There are a primitive element \( \rho \in H_1(\partial V; \mathbb{Z}) \) and a positive integer \( \delta(V) \) such that \( \delta(V)\rho \in H_1(\partial V; \mathbb{Z}) \) is the longitude for \( V \). Then, there is a properly embedded orientable surface \( S \) in \( V \) such that \( [\partial S] = \delta(V)\rho \) in \( H_1(\partial V; \mathbb{Z}) \) and \( \partial S \) consists of \( \delta(V) \) components. It is well-known that \( \delta(V) \) is unique and \( \partial S \) is also unique up to isotopy. Such an \( S \) is called a **characteristic surface** for \( V \).

A **meridian** \( \gamma \) for a rational homology circle \( V \) is a non-trivial simple closed curve (or its isotopy class) on \( \partial V \) which intersects the longitude \( \rho \) transversely in \( \delta(V) \) points. A rational homology circle \( V \) equipped with a pair \( (\rho, \gamma) \) on \( \partial V \) is called a **framed homology circle**, where the pair \( (\rho, \gamma) \) is also regarded as a pair of homology classes in \( H_1(\partial V; \mathbb{Z}) \). We denote by \( \hat{V} \) the rational homology sphere obtained from \( V \) by attaching a solid torus \( D^2 \times S^1 \) so that the boundary of a meridian disk \( \partial D^2 \times \{*\} \) is identified with \( \gamma \). The linking number \( \text{lk}(\cdot, \cdot) \) for a link \( K_1 \cup K_2 \) is defined as \( \text{lk}(\cdot, \cdot) \) for two disjoint oriented knots in \( V \) and the Alexander-Conway polynomials of knots in \( V \) are well-defined as in \( \hat{V} \). And, we put \( \lambda(V) = \lambda(\hat{V}) \). We note that, if \( M = V \cup V' \) is a rational homology sphere obtained from \( V \) by gluing a rational homology circle \( V' \) so that the meridian of \( V' \) is identified with the longitude of \( V \), then the linking numbers of links in \( V \subset M \) and the Alexander polynomials of knots in \( V \subset M \) coincide with that of \( \hat{V} \).

Let \( K \) be an oriented knot in \( V \), and \( K^+ \) the preferred longitude for \( K \). That is, \( K^+ \) is an oriented knot in \( \partial N(K) \) with \( \text{lk}(K, K^+) = 0 \). We denote the **winding number** by \( w = w(K) \) which is defined as the absolute value of the algebraic intersection number of \( K \) with a characteristic surface \( S \), which is independent of the choice of \( S \). For a framed rational homology circle \( C \) in a rational homology sphere (or in a framed rational homology circle), we denote by \( \text{fr}(C) \) the framing number of \( C \), namely the linking number of the framing \( \gamma \) and the knot \( \gamma^+ \) parallel to \( \gamma \) in \( \partial C \). Notice that \( \text{fr}(C) \) is an integer. The following lemma can be shown by Fujita’s formula [6] and Walker’s surgery formula [23].

**Lemma 2.1.** Let \( H \) be a rational homology sphere, and \( C \) a framed rational homology circle with \( \delta(C) = 1 \) embedded in \( H \). Let \( V \) be a framed rational homology circle with \( \delta(V) = 1 \). Denote by \( M \) the manifold obtained from \( H \) by replacing \( C \) with
V by a meridian-longitude preserving map. Then, \( \lambda(M) = \lambda(H) + \lambda(V) - \text{fr}(C)\alpha_2(V) \).

Proof. Let \( M' \) denote the homology sphere obtained from \( H \) by replacing \( C \) with \( V \) by a longitude preserving map such that the image of the meridian of \( V \) is preferred in \( M' \). Denote by \((\gamma, \rho)\) the meridian-longitude pair of \( V \) in \( M' \). Notice that \( M = \chi(M'; (\gamma, -1/\text{fr}(C))) \). By Walker’s surgery formula, we have that \( \lambda(M) = \lambda(M') - \text{fr}(C)\alpha_2(\gamma) \). Since \( \gamma \) bounds a characteristic surface as a Seifert surface in \( M' \), we see that \( \Delta_\gamma \cup M'(t) = \Delta_V(t) \) and thus \( \alpha_2(\gamma) \subset M' = \alpha_2(V) \). On the other hand, it follows from Fujita’s splicing formula [6] that \( \lambda(M') = \lambda(H) + \lambda(V) \). Now we have that \( \lambda(M) = \lambda(H) + \lambda(V) - \text{fr}(C)\alpha_2(V) \). This completes the proof.

Let \( V \) be a rational homology circle with \( \delta(V) = 1 \). Then, the \( r \)-fold cyclic covering space \( V' \) of \( V \) is obtained from \( r \) copies of \( V - S \) by gluing up cyclically, where \( S \) is a characteristic surface for \( V \) with a single boundary. Denote by \( \varphi_r : V' \to V \) the covering projection. Let \( \rho_r \) denote a component of \( \varphi_r^{-1}(\rho) \) for the longitude \( \rho \) of \( V \). Notice that \( \gamma_r = \varphi_r^{-1}(\gamma) \) is connected and intersects \( \rho_r \) transversely in a single point. Then, the pair \((\gamma_r, \rho_r)\) is regarded as a meridian-longitude pair on \( \partial V' \). Hereafter we assume that \( V' \) is a homology circle. Let \( K \) be a knot in \( V \). It is elementary to see that \( \varphi_r^{-1}(K) \) consists of \( \mu_r = \text{gcd}(w, r) \) components \( K_{r,1}, \ldots, K_{r,\mu_r} \). We number them so that \( K_{r,i} \) is mapped to \( K_{r,i+1} \) by a natural generator of the group of covering translations \( V' \to V' \), that is, \( K_{r,i+1} \) is next to \( K_{r,i} \). Then, \( \varphi_r|_{K_{r,i}} : K_{r,i} \to K \) is an \( r/\mu \)-fold cyclic cover. For the preferred longitude \( K^+ \) for \( K \), we denote by \( K_{r,i}^+ \) the component of \( \varphi_r^{-1}(K^+) \) that corresponds to \( K_{r,i} \). Here \( \text{gcd}(0, r) = r \) and \( 0 \)-fold cyclic covers mean the infinite cyclic covers. Put \( \alpha'_r(K) = \text{lk}(K_{r,i}, K_{r,i}^+) \), which is independent of the choice of \( i \).

For a 2-component link \( K \cup k \), we regard the exterior \( E(k, S^3) \) as a framed solid torus, \( K \) as a knot in \( E(k, S^3) \). Then we put \( \alpha'_r(K) = \alpha'_r(k, S^3)(K) \).

Notice that \( \Sigma'_K \) is a rational homology sphere if and only if each of \( \Sigma'_C, \Sigma'_P \) is a rational homology sphere. Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. We denote by \( \psi_K : \Sigma'_K \to S^3, \psi_P : \Sigma'_P \to S^3 \) the covering projections. Note that \( C_0 \) is regarded as a knot in the homology circle \( E(P, S^3) \) with winding number \( w \), and note also that \( E(\psi_P^{-1}(P), \Sigma'_P) \) is the \( r \)-fold cyclic covering space of \( E(P, S^3) \). Hence, \( \psi_P^{-1}(N(C_0, S^3)) \) is the union of \( \mu = \text{gcd}(w, r) \) solid tori \( U_1, \ldots, U_{\mu} \) and \( \psi_P|_{U_i} : U_i \to N(C_0, S^3) \) is the \( r/\mu \)-fold cyclic cover. Note that \( E(K, S^3) \) is obtained from \( E(P, S^3) \) by replacing the solid torus \( N(C_0, S^3) \) with the homology circle \( E(C, S^3) \) via a meridian-longitude preserving map. Then, \( \Sigma'_K \) is obtained from \( \Sigma'_P \) by replacing each solid torus \( U_i \) with a copy \( U'_i \) of the \( r/\mu \)-fold cyclic covering space \( E(\tilde{C}, \Sigma'_C) \) of \( E(C, S^3) \) via a meridian-longitude preserving map, where \( \psi_C : \Sigma'_C \to S^3 \) is the \( r/\mu \) cyclic branched cover along \( C \) and \( \tilde{C} = \psi_C^{-1}(C) \). Recall that \( \text{fr}(U_i) = \alpha'_{r/\mu}(C_0) \). By applications of Lemma 2.1, we have that \( \lambda(\Sigma'_K) = \lambda(M) = \lambda(H) + \lambda(V) - \text{fr}(C)\alpha_2(V) \).
\[ \lambda(\Sigma^t_{\mu}) + \mu \lambda(\Sigma^r_{\nu}) - \mu \alpha^r_{\nu}(C_0) \alpha^t_{\mu}(\tilde{C}). \] This completes the proof. \[ \square \]

3. Proof of Proposition 1.4

Let \( l \cup k(m, n) \) be the link illustrated in Fig. 2. Notice that \( k(m, n) \) is trivial in \( S^3 \) and \( \text{lk}(l, k(m, n)) = 0 \). Then, it is direct to see that

\[ \alpha^r_t(k(m, n)) = \begin{cases} 0 & (r \mid n) \\ -2m & (r \nmid n) \end{cases}. \]

In particular, \( \alpha^n_t(k(m, n)) = 0 \) and \( \alpha^{r>n}_t(k(m, n)) = -2m \).

Let \( l \cup K(m, n) \) denote the link obtained from \( l \cup k(m, n) \) by replacing \( k(m, n) \) with the untwisted Whitehead double about \( k(m, n) \). See Fig. 3-(1) for \( l \cup K(-1, 1) \). Note that \( K(m, n) \) is also a trivial knot in \( S^3 \). Note also that \( l \cup K(m, n) \) is a boundary link. Let \( K^*_t(m, n) \) denote a component of \( \psi_t^{-1}(K(m, n)) \), where \( \psi_t : S^3 \rightarrow S^3 \) is the \( t \)-fold cyclic cover branched along \( l \). Then,

\[ \Delta_{K^*_t(m, n)}(t) = \begin{cases} 1 & (r \mid n) \\ 1 + 2m(t^{1/2} - t^{-1/2})^2 & (r \nmid n) \end{cases}. \]

Put \( K^*(m, n) = \chi(l; (K(m, n), -1)) \# \chi(l; (K(-1, 1), 1)) \). We see that \( \Delta_{K^*(m, n)}(t) = 1 \) since each factor is obtained from a trivial knot \( l \) and \( l \cup K(m, n) \) is a boundary link. Further we have that \( \Sigma^r_t(K^*(m, n)) = \Sigma^r_t(l; (K(m, n), -1)) \# \Sigma_t^r(l; (K(-1, 1), 1)) \) and

\[ \lambda(\Sigma^r_t(K^*(m, n))) = \begin{cases} 2mr & (r \mid n) \\ 0 & (r \nmid n) \end{cases}. \]

by the additivity of the Casson invariant.

Now we have the following claim.

Claim 3.1. For an integer number \( n \geq 2 \), there is a knot \( K_n \) such that \( \Delta_{K_n}(t) = 1 \), \( \lambda(\Sigma^r_{K_n}) = 2n \), \( \lambda(\Sigma^{r>n}_{K_n}) = 0 \) and \( \lambda(\Sigma^{r<r<n}_{K_n}) \) is divisible by \( 2r \).

Proof of Proposition 1.4. By taking connected-sums of \( K^*(m, n) \)'s according to the given integers for \( n = N, N - 1, N - 2, \ldots, 2 \), we get a desired knot. \[ \square \]

Since the exterior \( E(K) \) of a knot \( K \) in \( S^3 \) is a Haken manifold, \( E(K) \) is uniquely decomposed into hyperbolic manifolds and Seifert fibered spaces by characteristic tori. Hence, if we define the volume \( \text{vol}(K) \) of \( K \) as the sum of volumes of the hyperbolic manifolds in the JSJ-family, it is a topological invariant of a knot. Then, we can define a graph knot as that of whose volume is zero. The construction of knots in the proof of Proposition 1.4 always yields satellite knots. There are several ways to prove the next proposition. Here we refer the reader to Kawauchi’s techniques [11] or Myers’ gluing lemma [15].
Fig. 2. $I \cup k(m, n)$.

(1): $I \cup K(-1, 1)$.

(2): $\psi^{-1}_2(K(-1, 1))$.

Fig. 3.
Proposition 3.2. Given a knot $K$ in $S^3$, there are infinitely many hyperbolic knots $K^{**}$ such that $K^{**}$ have the same Seifert form as $K$, $\Sigma^{r}_{K^{**}}$ is hyperbolic, and $\lambda(\Sigma^{r}_{K^{**}}) = \lambda(\Sigma^{r}_{K})$ for any $r$.

Proof of Proposition 3.2. Let $S_0$ be a Seifert surface for $K_0 = K$ of positive genus. Take two disjoint trivial knots $K_1$, $K_2$ in $E(S_0)$ so that $K_1$ and $K_2$ bound disjoint disks in $E(S_0)$, and take disjoint genus one Seifert surfaces $S_1$, $S_2$ contained in disjoint 3-balls $B_1$ and $B_2$ for $K_1$, $K_2$. Let $\Gamma_0$, $\Gamma_1$, $\Gamma_2$ be spin of $S_0$, $S_1$, $S_2$. Then, $\Gamma_0 \cup \Gamma_1 \cup \Gamma_2$ forms a 3-component graph embedded in $S^2$. By Kawauchi [11, Theorem 1.1] there are graphs $\Gamma^i_0 \cup \Gamma^i_1 \cup \Gamma^i_2$ such that $E(\Gamma^i_0 \cup \Gamma^i_1 \cup \Gamma^i_2)$ is hyperbolic and $\Gamma^i_0 \cup \Gamma^i_1 \cup \Gamma^i_2 - \Gamma^i_i$ is isotopic to $\Gamma_0 \cup \Gamma_1 \cup \Gamma_2 - \Gamma_i$ for $i = 0, 1, 2$. We construct Seifert surfaces $S^*_0$, $S^*_1$, $S^*_2$ along $\Gamma^*_i$’s for knots $K^*_0 = \partial S^*_0$, $K^*_1 = \partial S^*_1$, $K^*_2 = \partial S^*_2$ corresponding to $S_0$, $S_1$, $S_2$ respectively such that $K^*_0 \cup K^*_1 \cup K^*_2 - K^*_i$ is equivalent to $K_0 \cup K_1 \cup K_2 - K_i$. Since $K^*_0 \cup K^*_1 \cup K^*_2$ bounds a disconnected Seifert surface $S^*_0 \cup S^*_1 \cup S^*_2$ whose exterior is hyperbolic and since each of the components has positive genus, it follows from [22, Proposition 3.1] that $K^*_0 \cup K^*_1 \cup K^*_2$ is a hyperbolic link in $S^3$. By Thurston’s hyperbolic surgery theorem [16], twisting along $K^*_1$ and $K^*_2$ produce infinitely many hyperbolic knots $K^{**}$ of distinct volumes such that $E(S^*_0^{**})$ is hyperbolic, where $S^*_0$ is the Seifert surface obtained from $S^*_0$ by twisting along $K^*_1$ and $K^*_2$. Then $\Sigma^{r}_{K^{**}}$ is hyperbolic by [22, Proposition 3.2]. We shall show that the sequence of the knots $K^{**}$ is a desired one. Let $\psi: \Sigma^{r}_K \to S^3$ be the $r$-fold cyclic branched cover, and $\phi: \Sigma^{r}_{K^{**}} \to S^3$ the $r$-fold cyclic branched cover. Note that $\Sigma^{r}_{K^{**}}$ is obtained from $\Sigma^{r}_K$ by $1/n$-surgeries on all components of $\phi^{-1}(K^*_1 \cup K^*_2)$. Since $S^*_1 \cup S^*_2$ is disjoint from $S^*_0$, $\phi^{-1}(S^*_1 \cup S^*_2)$ consists of $r$-copies of $S^*_1 \cup S^*_2$. Since $S^*_1$ is contained in a 3-ball which lifts to $\Sigma^{r}_K$, we see that $\phi^{-1}(K^*_1 \cup K^*_2)$ is a boundary link such that the Alexander polynomial of each component is trivial. Then we see that $\lambda(\Sigma^{r}_{K^{**}}) = \lambda(\Sigma^{r}_{K})$. This completes the proof. \(\square\)

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