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CONTACT CALABI-YAU MANIFOLDS AND SPECIAL LEGENDRIAN SUBMANIFOLDS

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Abstract

We consider a generalization of Calabi-Yau structures in the context of Sasakian manifolds. We study deformations of a special class of Legendrian submanifolds and classify invariant contact Calabi-Yau structures on 5-dimensional nilmanifolds. Finally we generalize to codimension r.

1. Introduction

In their celebrated paper [9] Harvey and Lawson introduced the concept of calibration and calibrated geometry. Namely, a *calibration* on an n-dimensional oriented Riemannian manifold (M, g) is a closed r-form ϕ such that for any $x \in M$

$$\phi_{x|V} \leq \mathrm{Vol}(V),$$

where V is an arbitrary oriented r-plane in $T_x M$. An oriented submanifold $p\colon L \hookrightarrow M$ is said to be *calibrated* by ϕ if $p^*(\phi) = \mathrm{Vol}(L)$. Compact calibrated submanifolds have the important property of minimizing volume in their homology class. As a typical example, the real part of holomorphic volume form of a Calabi-Yau manifold is a calibration; the corresponding calibrated submanifolds are said to be *special Lagrangian*. In [13] McLean studied special Lagrangian submanifolds (and other special calibrated geometries) showing that the Moduli space of deformations of special Lagrangian manifolds of a fixed compact one L is a smooth manifold of dimension equal to the first Betti number of L.

In this paper we consider a generalization of Calabi-Yau structures in the context of Sasakian manifolds. Recall that a *Sasakian structure* on a $2n+1$-dimensional manifold M is a pair (α, J), where α is a contact form on M and J is an integrable complex structure on $\xi = \ker\alpha$ calibrated by $\kappa = (1/2)\, d\alpha$. This is equivalent to require the following data: a quadruple (α, g, R, J), where α is a 1-form, g is a Riemannian

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metric, R is a unitary Killing vector field, $J \in \mathrm{End}(TM)$ satisfying

$$J^2 = -I + \alpha \otimes R, \quad g(J\,\cdot\,,\,J\,\cdot\,) = g(\,\cdot\,,\,\cdot\,) - \alpha \otimes \alpha, \quad \alpha(R) = 1$$

and such that the metric cone $(M \times \mathbb{R}^+, r^2 g + dr \otimes dr)$ endowed with the almost complex structure $\tilde{J} = J - r\alpha \otimes \partial_r + (1/r)\, dr \otimes R$ is Kähler, where we extend J by $J(\partial_r) = 0$ (see e.g. [1], [2], [12]). These manifolds have been studied by many authors (see e.g. [1], [3], [8], [11], [12] and the references included).

We consider contact Calabi-Yau manifolds which are a special class of Sasakian manifolds: namely a *contact Calabi-Yau manifold* is a $2n + 1$-dimensional Sasakian manifold (M, α, J) endowed with a closed basic complex volume form ϵ. It turns out that these manifolds are a special class of null-Sasakian α-Einstein manifolds. As a direct consequence of the above definition, in a contact Calabi-Yau manifold (M, α, J, ϵ) the real part of ϵ is a calibration. Furthermore, we have that an n-dimensional submanifold $p \colon L \hookrightarrow M$ of a contact Calabi-Yau manifold admits an orientation making it a calibrated submanifold by $\mathfrak{Re}\,\epsilon$ if and only if

$$p^*(\alpha) = 0, \quad p^*(\mathfrak{Im}\,\epsilon) = 0.$$

In such a case L is said to be a *special Legendrian submanifold*. We prove that:

The moduli space of deformations of special Legendrian submanifolds near a fixed compact one L is a smooth 1-dimensional manifold.

Moreover we get the following extension theorem:

Let $(M, \alpha_t, J_t, \epsilon_t)$ be a smooth family of contact Calabi-Yau manifolds and let $p \colon L \hookrightarrow (M, \alpha_0, J_0, \epsilon_0)$ be a compact special Legendrian submanifold. Then there exists a smooth family of special Legendrian submanifolds $p_t \colon L \hookrightarrow (M, \alpha_t, J_t, \epsilon_t)$ that extends $p \colon L \hookrightarrow M$ if and only if the cohomology class $[p^*(\mathfrak{Im}\,\epsilon)]$ vanishes.

This can be considered a contact version of a theorem of Lu Peng (see [10]) in Calabi-Yau manifolds (see also [14]).

In Section 2 we fix some notation on contact and Sasakian geometry. In Section 3 we define contact Calabi-Yau manifolds and we obtain some simple topological obstructions to the existence of contact Calabi-Yau structures on odd-dimensional manifolds. As a corollary, we get that there are no contact Calabi-Yau structures on odd-dimensional spheres. In Section 4 we study the moduli space of special Legendrian submanifolds, proving the theorems stated above. In Section 5 we classify the 5-dimensional nilmanifolds carrying an invariant contact Calabi-Yau structure. The proof is based on Theorems 21 and 23 of [5]. In the last section we generalize the previous definition to the case of codimension r proving an extension theorem. Some examples of contact Calabi-Yau manifolds and special Legendrian submanifolds are carefully described.

2. Preliminaries

Let M be a manifold of dimension $2n+1$. A *contact structure* on M is a distribution $\xi \subset TM$ of dimension $2n$, such that the defining 1-form α satisfies

$$\alpha \wedge (d\alpha)^n \neq 0. \tag{1}$$

A 1-form α satisfying (1) is said to be a *contact form* on M. Let α be a contact form on M; then there exists a unique vector field R_α on M such that

$$\alpha(R_\alpha) = 1, \quad \iota_{R_\alpha} d\alpha = 0,$$

where $\iota_{R_\alpha} d\alpha$ denotes the contraction of $d\alpha$ along R_α. By definition R_α is called the *Reeb vector field* of the contact form α. A *contact manifold* is a pair (M, ξ) where M is a $2n+1$-dimensional manifold and ξ is a contact structure. Let (M, ξ) be a contact manifold and fix a defining (contact) form α. Then the 2-form $\kappa = (1/2)\, d\alpha$ defines a symplectic form on the contact structure ξ; therefore the pair (ξ, κ) is a symplectic vector bundle over M. A *complex structure* on ξ is the datum of $J \in \mathrm{End}(\xi)$ such that $J^2 = -I_\xi$.

DEFINITION 2.1. Let α be a contact form on M, with $\xi = \ker\alpha$ and let $\kappa = (1/2)\, d\alpha$. A complex structure J on ξ is said to be *κ-calibrated* if

$$g_J[x](\,\cdot\,,\ \cdot\,) := \kappa[x](\,\cdot\,,\ J_x\,\cdot\,)$$

is a J_x-Hermitian inner product on ξ_x for any $x \in M$.

The set of κ-calibrated complex structures on ξ will be denoted by $\mathfrak{C}_\alpha(M)$. If J is a complex structure on $\xi = \ker\alpha$, then we extend it to an endomorphism of TM by setting

$$J(R_\alpha) = 0.$$

Note that such a J satisfies

$$J^2 = -I + \alpha \otimes R_\alpha.$$

If J is κ-calibrated, then it induces a Riemannian metric g on M given by

$$g := g_J + \alpha \otimes \alpha. \tag{2}$$

Furthermore the Nijenhuis tensor of J is defined by

$$N_J(X, Y) = [JX, JY] - J[X, JY] - J[Y, JX] + J^2[X, Y]$$

for any $X, Y \in TM$. We recall the following

DEFINITION 2.2. A Sasakian structure on a $2n+1$-dimensional manifold M is a pair (α, J), where

- α is a contact form;
- $J \in \mathfrak{C}_\alpha(M)$ satisfies $N_J = -d\alpha \otimes R_\alpha$.

The triple (M, α, J) is said to be a *Sasakian manifold*.

For other characterizations of Sasakian structure see e.g. [1] and [2].

We recall now the definition of basic r-forms.

DEFINITION 2.3. Let (M, ξ) be a contact manifold. A differential r-form γ on M is said to be *basic* if

$$\iota_{R_\alpha}\gamma = 0, \quad \mathcal{L}_{R_\alpha}\gamma = 0,$$

where $\mathcal{L}$ denotes the Lie derivative and R_α is the Reeb vector field of an arbitrary contact form defining ξ.

We will denote by $\Lambda^r_B(M)$ the set of basic r-forms on (M, ξ). Note that

$$d\Lambda^r_B(M) \subset \Lambda^{r+1}_B(M).$$

The cohomology $H^\bullet_B(M)$ of this complex is called the *basic cohomology* of (M, ξ).

If (M, α, J) is a Sasakian manifold, then

$$J(\Lambda^r_B(M)) = \Lambda^r_B(M),$$

where, as usual, the action of J on r-forms is defined by

$$J\phi(X_1, \ldots, X_r) = \phi(JX_1, \ldots, JX_r).$$

Consequently $\Lambda^r_B(M) \otimes \mathbb{C}$ splits as

$$\Lambda^r_B(M) \otimes \mathbb{C} = \bigoplus_{p+q=r} \Lambda^{p,q}_J(\xi)$$

and, according with this gradation, it is possible to define the cohomology groups $H^{p,q}_B(M)$. The r-forms belonging to $\Lambda^{p,q}_J(\xi)$ are said to be of *type* (p, q) *with respect* *to* J. Note that $\kappa = (1/2)\, d\alpha \in \Lambda^{1,1}_J(\xi)$ and it determines a non-vanishing cohomology class in $H^{1,1}_B(M)$. The Sasakian structure (α, J) also induces a natural connection ∇^ξ on ξ given by

$$\nabla^\xi_X Y = \begin{cases} (\nabla_X Y)^\xi & \text{if } X \in \xi \\ [R_\alpha, Y] & \text{if } X = R_\alpha, \end{cases}$$

where the subscript ξ denotes the projection onto ξ. One easily gets

$$\nabla^\xi_X J = 0, \quad \nabla^\xi_X g_J = 0, \quad \nabla^\xi_X d\alpha = 0, \quad \nabla^\xi_X Y - \nabla^\xi_Y X = [X, Y]^\xi,$$

for any $X, Y \in TM$. Consequently we have

$$\mathrm{Hol}(\nabla^\xi) \subseteq \mathrm{U}(n).$$

Moreover the *transverse Ricci tensor* Ric^T is defined as

$$\mathrm{Ric}^T(X, Y) = \sum_{i=1}^{2n} g\big(\nabla^\xi_X \nabla^\xi_{e_i} e_i - \nabla^\xi_{e_i} \nabla^\xi_X e_i - \nabla^\xi_{[X,e_i]} e_i, Y\big),$$

for any $X, Y \in \xi$, where $\{e_1, \ldots, e_{2n}\}$ is an arbitrary orthonormal frame of ξ. It is known that Ric^T satisfies

$$\mathrm{Ric}^T(X, Y) = \mathrm{Ric}(X, Y) + 2g(X, Y),$$

for any $X, Y \in \xi$, where Ric denotes the Ricci tensor of the Riemannian metric $g = g_J + \alpha \otimes \alpha$. Let us denote by ρ^T the Ricci form of Ric^T, i.e.

$$\rho^T(X, Y) = \mathrm{Ric}^T(JX, Y) = \mathrm{Ric}(JX, Y) + 2\kappa(X, Y),$$

for any $X, Y \in \xi$. We recall that ρ^T is a closed form such that $(1/(2\pi))\rho$ represents the first Chern class of (ξ, J) (see e.g. [7]); this form is called the *transverse Ricci form* of (α, J).

DEFINITION 2.4. The basic cohomology class

$$c^B_1(M) = \frac{1}{2\pi}[\rho^T] \in H^{1,1}_B(M)$$

is called the *first basic Chern class* of (M, α, J) and, if it vanishes, then (M, α, J) is said to be *null-Sasakian*.

Furthermore we recall that a Sasakian manifold is called *α-Einstein* if there exist $\lambda, \nu \in C^\infty(M, \mathbb{R})$ such that

$$\mathrm{Ric} = \lambda g + \nu \alpha \otimes \alpha.$$

For general references on these topics see e.g. [4] and [3].

Finally, recall that a submanifold $p\colon L \hookrightarrow M$ of a $2n+1$-dimensional contact manifold (M, ξ) is said to be *Legendrian* if:

1) $\dim_{\mathbb{R}} L = n$,
2) $p_*(TL) \subset \xi$.

Observe that, if α is a defining form of the contact structure ξ, then condition 2) is equivalent to say that $p^*(\alpha) = 0$. Hence Legendrian submanifolds are the analogue of Lagrangian submanifolds in contact geometry.

3. Contact Calabi-Yau manifolds

In this section we study contact Calabi-Yau manifolds. As already explained in the introduction, these manifolds are a natural generalization of the Calabi-Yau ones in the context of contact geometry. Roughly speaking a contact Calabi-Yau manifold is a Sasakian manifold endowed with a basic closed complex volume form. We can give now the following

DEFINITION 3.1. A *contact Calabi-Yau manifold* is a quadruple (M, α, J, ϵ), where

- (M, α, J) is a $2n+1$-dimensional Sasakian manifold;
- $\epsilon \in \Lambda^{n,0}_J(\xi)$ is a nowhere vanishing basic form on $\xi = \ker\alpha$ such that

$$\begin{cases} \epsilon \wedge \overline{\epsilon} = c_n \kappa^n \\ d\epsilon = 0, \end{cases}$$

where $c_n = (-1)^{n(n+1)/2}(2i)^n$ and $\kappa = (1/2)\, d\alpha$.

Now we will describe a couple of examples.

EXAMPLE 3.2. Consider $\mathbb{R}^{2n+1}$ endowed with the standard Euclidean coordinates $\{x_1, \dots, x_n, y_1, \dots, y_n, t\}$. Let

$$\alpha_0 = 2\, dt - 2\sum_{i=1}^{n} y_i\, dx_i$$

be the *standard contact form* on $\mathbb{R}^{2n+1}$ and let $\xi_0 = \ker\alpha_0$. Then ξ_0 is spanned by

$$\{y_1\partial_t + \partial_{x_1}, \dots, y_n\partial_t + \partial_{x_n}, \partial_{y_1}, \dots, \partial_{y_n}\}.$$

For simplicity, set $V_i = y_i\partial_t + \partial_{x_i}$, $W_j = \partial_{y_j}$, $i, j = 1, \dots, n$ and

$$\begin{cases} J_0(V_r) = W_r \\ J_0(W_r) = -V_r \end{cases} \quad r = 1, \dots, n.$$

Then J_0 defines a complex structure in $\mathfrak{C}_\alpha(M)$. Since the space of transverse 1-forms is spanned by $\{dx_1, \dots, dx_n, dy_1, \dots, dy_n\}$, then the complex valued form

$$\epsilon_0 := (dx_1 + i\, dy_1) \wedge \cdots \wedge (dx_n + i\, dy_n)$$

is of type $(n, 0)$ with respect to J_0 and it satisfies

$$\begin{cases} \epsilon_0 \wedge \overline{\epsilon_0} = c_n \kappa_0^n \\ d\epsilon_0 = 0, \end{cases}$$

where $\kappa_0 = (1/2)\, d\alpha_0$. Therefore $(\mathbb{R}^{2n+1}, \alpha_0, J_0, \epsilon_0)$ is a contact Calabi-Yau manifold.

The following will describe a compact contact Calabi-Yau manifold.

EXAMPLE 3.3. Let

$$H(3) := \left\{ A = \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} \middle|\, x, y, z \in \mathbb{R} \right\}$$

be the 3-dimensional Heisenberg group and let $M = H(3)/\Gamma$, where Γ denotes the subgroup of $H(3)$ given by the matrices with integral entries. The 1-forms $\alpha_1 = dx$, $\alpha_2 = dy$, $\alpha_3 = x\, dy - dz$ are $H(3)$-invariant and therefore they define a global coframe on M. Then $\alpha = 2\alpha_3$ is a contact form whose contact distribution ξ is spanned by $V = \partial_x$, $W = \partial_y + x\partial_z$. Again

$$\begin{cases} J(V) = W \\ J(W) = -V \end{cases}$$

defines a κ-calibrated complex structure on ξ and $\epsilon = \alpha_1 + i\alpha_2$ is a $(1, 0)$-form on ξ such that (M, α, J, ϵ) is a contact Calabi-Yau manifold.

The last example gives an invariant contact Calabi-Yau structure on a nilmanifold. It can be generalized to the dimension $2n + 1$ in this way: let $\mathfrak{g}$ be the Lie algebra spanned by $\{X_1, \dots, X_{2n+1}\}$ with

$$[X_{2k-1}, X_{2k}] = -X_{2n+1}$$

for $k = 1, \dots, n$ and the other brackets are zero. Then $\mathfrak{g}$ is a $2n + 1$-dimensional nilpotent Lie algebra with rational constant structures and, by Malcev theorem, it follows that if G is the simply connected Lie group with Lie algebra $\mathfrak{g}$, then G has a compact quotient. Let $\{\alpha_1, \ldots, \alpha_{2n+1}\}$ be the dual basis of $\{X_1, \ldots, X_{2n+1}\}$. Then we immediately get

$$d\alpha_1 = 0, \ldots, d\alpha_{2n} = 0, \quad d\alpha_{2n+1} = \sum_{k=1}^{n} \alpha_{2k-1} \wedge \alpha_{2k}.$$

Hence

$$\alpha = 2\alpha_{2n+1},$$

the endomorphism J of $\xi = \ker\alpha$ defined by

$$\begin{cases} J(X_{2k-1}) = X_{2k} \\ J(X_{2k}) = -X_{2k-1} \end{cases}$$

for $k = 1, \ldots, n$ and the complex form

$$\epsilon = (\alpha_1 + i\alpha_2) \wedge \cdots \wedge (\alpha_{2n-1} + i\alpha_{2n})$$

define a contact Calabi-Yau structure on any compact nilmanifold associated with $\mathfrak{g}$.

The following proposition gives simple topological obstructions in order that a compact $2n+1$-dimensional manifold M carries a contact Calabi-Yau structure.

Proposition 3.4. *Let M be a $2n+1$-dimensional compact manifold. Assume that M admits a contact Calabi-Yau structure; then the following hold*
1. *if n is* even, *then* $b_{n+1}(M) > 0$;
2. *if n is* odd, *then*

$$\begin{cases} b_n(M) \geq 2 \\ b_{n+1}(M) \geq 2, \end{cases}$$

where $b_j(M)$ denotes the j^{th} Betti number of M.

Proof. Let (α, J, ϵ) be a contact Calabi-Yau structure on M and let $\xi = \ker\alpha$. Set $\Omega = \mathfrak{Re}\,\epsilon$; then, since $\epsilon \in \Lambda_J^{n,0}(\xi)$, we have $\epsilon = \Omega + iJ\Omega$. In view of the assumption $d\epsilon = 0$, we obtain $d\Omega = dJ\Omega = 0$ and since $d\alpha \in \Lambda_J^{1,1}(\xi)$ it follows that

$$\Omega \wedge d\alpha = J\Omega \wedge d\alpha = 0.$$

Hence

$$d(\Omega \wedge \alpha) = d(J\Omega \wedge \alpha) = 0.$$

Furthermore we have

$$\begin{aligned} \epsilon \wedge \overline{\epsilon} &= \Omega \wedge \Omega + J\Omega \wedge J\Omega \quad && \text{if } n \text{ is even;} \\ \epsilon \wedge \overline{\epsilon} &= -2i\Omega \wedge J\Omega && \text{if } n \text{ is odd.} \end{aligned}$$

1. If n is even, then $\alpha\wedge(\Omega\wedge\Omega+J\Omega\wedge J\Omega)$ is a volume form on M. Assume that the cohomology classes $[\Omega\wedge\alpha]$, $[J\Omega\wedge\alpha]$ vanish; then there exist $\beta,\gamma\in\Lambda^n(M)$ such that

$$\alpha\wedge\Omega=d\beta,\quad \alpha\wedge J\Omega=d\gamma.$$

By Stokes theorem we have

$$\begin{aligned}0\neq\int_M\alpha\wedge\Omega\wedge\Omega+\alpha\wedge J\Omega\wedge J\Omega&=\int_M d\beta\wedge\Omega+d\gamma\wedge J\Omega\\&=\int_M d(\beta\wedge\Omega)+d(\gamma\wedge J\Omega)=0,\end{aligned}$$

which is absurd. Therefore one of $[\Omega\wedge\alpha]$, $[J\Omega\wedge\alpha]$ does not vanish. Consequently $b_{n+1}(M)>0$.
2. Let n be odd. We prove that the cohomology classes $[\Omega]$ and $[J\Omega]$ are $\mathbb{R}$-independent. Assume that there exist $a,b\in\mathbb{R}$ such that $a[\Omega]+b[J\Omega]=0$, $(a,b)\neq(0,0)$. Then there exists $\beta\in\Lambda^{n-1}(M)$ such that

$$a\Omega+bJ\Omega=d\beta.$$

We may assume that $a=1$, so that $\Omega=d\beta-bJ\Omega$. Stokes theorem implies

$$0\neq\int_M\alpha\wedge\Omega\wedge J\Omega=\int_M\alpha\wedge d\beta\wedge J\Omega=-\int_M d(\alpha\wedge\beta\wedge J\Omega)=0$$

which is a contradiction. Hence $b_n(M)\geq 2$. With the same argument, it is possible to prove that $b_{n+1}(M)\geq 2$ by showing that $[\Omega\wedge\alpha]$ and $[J\Omega\wedge\alpha]$ are $\mathbb{R}$-independent in $H^{n+1}(M,\mathbb{R})$. □

The following is an immediate consequence of Proposition 3.4.

Corollary 3.5. *A 3-dimensional compact manifold M admitting contact Calabi-Yau structure has $b_1(M)\geq 2$. In particular, there are no compact 3-dimensional simply connected contact Calabi-Yau manifolds. Moreover, the $2n+1$-dimensional sphere has no contact Calabi-Yau structures.*

The following proposition implies that the transverse Ricci tensor of a contact Calabi-Yau manifold vanishes

Proposition 3.6. *Let (M,α,J) be a $2n+1$-dimensional Sasakian manifold and $\xi=\ker\alpha$. The following facts are equivalent:*
1. $\mathrm{Hol}^0(\nabla^\xi)\subseteq \mathrm{SU}(n)$
2. $\mathrm{Ric}^T=0$.

Proof. The connection ∇^{ξ} induces a connection ∇^{K} on $\Lambda^{n,0}_{J}(\xi)$ which has $\mathrm{Hol}(\nabla^{K}) \subseteq \mathrm{U}(1)$. Since $\mathrm{Hol}^{0}(\nabla^{K})$ and $\mathrm{Hol}^{0}(\nabla^{\xi})$ are related by

$$\mathrm{Hol}^{0}(\nabla^{K}) = \det(\mathrm{Hol}^{0}(\nabla^{\xi})),$$

where det is the map induced by the determinant $\mathrm{U}(n) \to \mathrm{U}(1)$, then it follows that $\mathrm{Hol}^{0}(\nabla^{\xi}) \subseteq \mathrm{SU}(n)$ if and only if $\mathrm{Hol}^{0}(\nabla^{K}) = \{1\}$ and in this case ∇^{K} is flat. As in the Kähler case it can be showed using transverse holomorphic coordinates (see e.g. [7], [8]) that the curvature form of ∇^{K} coincides with the transverse Ricci form of (α, J). Hence $\mathrm{Hol}^{0}(\nabla^{\xi}) \subseteq \mathrm{SU}(n)$ if and only if $\mathrm{Ric}^{T} = 0$. □

As a consequence of the last proposition we have the following

Corollary 3.7. *Let (M, α, J, ϵ) be a contact Calabi-Yau manifold. Then (M, α, J) is null-Sasakian and the metric g induced by (α, J) is α-Einstein with $\lambda = -2$ and $\nu = 2n + 2$. In particular the scalar curvature of the metric g associated to (α, J) is equal to $-2n$.*

4. Deformations of special Legendrian submanifolds

In this section we are going to study the geometry of Legendrian submanifolds in a contact Calabi-Yau ambient. We will prove a contact version of McLean and Lu Peng theorems (see [13] and [10]).

Let (M, α, J, ϵ) be a contact Calabi-Yau manifold of dimension $2n+1$. It easy to see that for any oriented n-plane $V \subset T_x M$

$$\mathfrak{Re}\, \epsilon_{|V} \leq \mathrm{Vol}(V),$$

where $\mathrm{Vol}(V)$ is computed with respect to the metric g induced by (α, J) on M. Hence $\mathfrak{Re}\, \epsilon$ is a calibration on (M, g) (see [9]). We have the following

Proposition 4.1. *Let $p\colon L \hookrightarrow M$ be an n-dimensional submanifold. The following facts are equivalent*

1. *the submanifold satisfies*

$$\begin{cases} p^*(\alpha) = 0 \\ p^*(\mathfrak{Im}\, \epsilon) = 0, \end{cases}$$

2. *there exists an orientation on L making it calibrated by $\mathfrak{Re}\, \epsilon$.*

We can give the following

DEFINITION 4.2. An n-dimensional submanifold $p\colon L \hookrightarrow M$ is said to be *special Legendrian* if

$$\begin{cases} p^*(\alpha) = 0 \\ p^*(\Im\mathfrak{m}\,\epsilon) = 0. \end{cases}$$

It follows that compact special Legendrian submanifolds minimize volume in their homology class and that there are no compact special Legendrian submanifolds in $(\mathbb{R}^{2n+1}, \alpha_0, J_0, \epsilon_0)$.

EXAMPLE 4.3. Let $(M = H(3)/\Gamma, \alpha, J, \epsilon)$ be the contact Calabi-Yau manifold considered in the Example 3.3. Then the submanifold

$$L := \left\{ [A] \in M \;\middle|\; A = \begin{pmatrix} 1 & x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\} \simeq S^1$$

is a compact special Legendrian submanifold.

Now we define the moduli space of special Legendrian submanifolds.

DEFINITION 4.4. Let (M, α, J, ϵ) be a contact Calabi-Yau manifold and let $p_0\colon L \hookrightarrow M$, $p_1\colon L \hookrightarrow M$ be two special Legendrian submanifolds. Then $p_1\colon L \hookrightarrow M$ is said to be a *deformation* of $p_0\colon L \hookrightarrow M$ if there exists a smooth map $F\colon L \times [0,1] \to M$ such that

- $F(\,\cdot\,, t)\colon L \times \{t\} \to M$ is a special Legendrian embedding for any $t \in [0, 1]$;
- $F(\,\cdot\,, 0) = p_0$, $F(\,\cdot\,, 1) = p_1$.

Let (M, α, J, ϵ) be a contact Calabi-Yau manifold and let $p\colon L \hookrightarrow M$ be a fixed compact special Legendrian submanifold. Set

$$\begin{aligned}\mathfrak{M}(L) := \{&\text{special Legendrian submanifolds of } (M, \alpha, J, \epsilon) \\ &\text{which are deformations of } p\colon L \hookrightarrow M\}/\sim,\end{aligned}$$

where two embeddings are considered equivalent if they differ by a diffeomorphism of L; then by definition $\mathfrak{M}(L)$ is the *moduli space of special Legendrian submanifolds* which are deformations of $p\colon L \hookrightarrow M$. We have the following

Theorem 4.5. *Let (M, α, J, ϵ) be a contact Calabi-Yau manifold and let $p\colon L \hookrightarrow M$ be a compact special Legendrian submanifold. Then the moduli space $\mathfrak{M}(L)$ is a 1-dimensional manifold.*

The next lemma will be useful in the proof of Theorem 4.5:

Lemma 4.6 ([13], [6]). *Let (V,κ) be a symplectic vector space and let $i\colon W \hookrightarrow V$ be a Lagrangian subspace. Then*

1. $\tau\colon V/W \to W^*$ *defined as* $\tau([v]) = i^*(\iota_v\kappa)$ *is an isomorphism*;
2. *let J be a κ-calibrated complex structure on V and let $\epsilon \in \Lambda^{n,0}_J(V^*)$ satisfy*

$$i^*(\mathfrak{Im}\,\epsilon) = 0, \quad \epsilon \wedge \overline{\epsilon} = c_n \frac{\kappa^n}{n!}.$$

Then $\theta\colon V/W \to \Lambda^{n-1}(W^*)$ defined as $\theta([v]) := i^*(\iota_v\,\mathfrak{Im}\,\epsilon)$ is an isomorphism. Moreover for any $v \in V$, we have

$$\theta([v]) = - * \tau([v]),$$

where $*$ is computed with respect to $i^*(g_J(\,\cdot\,,\,\cdot\,)) := i^*(\kappa(\,\cdot\,, J\,\cdot\,))$ and the volume form $\mathrm{Vol}(W) := i^*(\mathfrak{Re}\,\epsilon)$.

For the proof of Lemma 4.6 we refer to [13] and [6].

Proof of Theorem 4.5. Let $\mathcal{N}(L)$ be the normal bundle to L. Then

$$\mathcal{N}(L) = \langle R_\alpha\rangle \oplus J(p_*(TL)),$$

where R_α is the Reeb vector field of α. Let Z be a vector field normal to L and let $\exp_Z\colon L \to M$ be defined as

$$\exp_Z(x) := \exp_x(Z(x)).$$

Let U be a neighborhood of 0 in $C^{2,\alpha}(\langle R_\alpha\rangle) \oplus C^{1,\alpha}(J(p_*(TL)))$ and let

$$F\colon U \to C^{1,\alpha}(\Lambda^1(L)) \oplus C^{0,\alpha}(\Lambda^n(L)),$$

be defined as

$$F(Z) = (\exp_Z^*(\alpha),\, 2\exp_Z^*(\mathfrak{Im}\,\epsilon)).$$

We obviously have

$$Z \in F^{-1}((0,0)) \cap C^\infty(\mathcal{N}(L)) \iff \exp_Z(L) \text{ is a special Legendrian submanifold.}$$

Note that since $\exp_Z$ and p are homotopic via $\exp_{tZ}$, we have

$$[\exp_Z^*(\mathfrak{Im}\,\epsilon)] = [p^*(\mathfrak{Im}\,\epsilon)] = 0.$$

Therefore

$$F\colon U \to C^{1,\alpha}(\Lambda^1(L)) \oplus dC^{1,\alpha}(\Lambda^{n-1}(L)).$$

Let us compute the differential of the map F.

$$F_*[0](Z) = \frac{d}{dt}(\exp^*_{tZ}(\alpha),\, 2\exp^*_{tZ}(\mathfrak{Im}\,\epsilon))_{|t=0} = (p^*(\mathcal{L}_Z\alpha),\, 2p^*(\mathcal{L}_Z\,\mathfrak{Im}\,\epsilon)),$$

where $\mathcal{L}$ denotes the Lie derivative. We may write $Z = JX + fR_\alpha$; then applying Cartan formula we obtain

$$\begin{aligned} F_*[0](Z) &= (p^*(\mathcal{L}_Z\alpha),\, 2p^*(\mathcal{L}_Z\,\mathfrak{Im}\,\epsilon)) \\ &= (p^*(d\iota_Z\alpha + \iota_Z\,d\alpha),\, 2p^*(d\iota_Z\,\mathfrak{Im}\,\epsilon)) \\ &= (p^*(d\iota_{JX+fR_\alpha}\alpha + \iota_{JX+fR_\alpha}d\alpha),\, 2p^*(d\iota_{JX+fR_\alpha}\,\mathfrak{Im}\,\epsilon)) \\ &= (p^*(d\iota_{fR_\alpha}\alpha + \iota_{JX}\,d\alpha),\, 2p^*(d\iota_{JX}\,\mathfrak{Im}\,\epsilon)) \\ &= (p^*(df + \iota_{JX}\,d\alpha),\, 2dp^*(\iota_{JX}\,\mathfrak{Im}\,\epsilon)). \end{aligned}$$

By applying Lemma 4.6 we get

$$F_*[0](Z) = (d(f\circ p) + p^*(\iota_{JX}\,d\alpha),\, -d*p^*(\iota_{JX}\,d\alpha)), \tag{3}$$

where $*$ is the Hodge star operator with respect to the metric $p^*(g_J)$ and the volume form $p^*(\mathfrak{Re}\,\epsilon)$. Now we show that $F_*[0]$ is surjective. Let $(\eta, d\gamma) \in C^{1,\alpha}(\Lambda^1(L)) \oplus dC^{1,\alpha}(\Lambda^{n-1}(L))$. By the Hodge decomposition theorem we may assume

$$d\gamma = -d*du \quad \text{with} \quad u \in C^{3,\alpha}(L)$$

and we have

$$\eta = dv + d^*\beta + h(\eta)$$

where $v \in C^{2,\alpha}(L)$, $\beta \in C^{2,\alpha}(\Lambda^2(L))$ and $h(\eta)$ denotes the harmonic component of η. Then we get

$$\begin{aligned} (\eta, d\gamma) &= (du - du + dv + d^*\beta + h(\eta),\, -d*du) \\ &= (dv - du + du + d^*\beta + h(\eta),\, -d*(du + d^*\beta + h(\eta))). \end{aligned}$$

We can find $f \in C^{2,\alpha}(p(L))$ and $X \in C^{1,\alpha}(p_*(TL))$ such that

$$\begin{aligned} &f\circ p = v - u \\ &p^*(\iota_{JX}d\alpha) = du + d^*\beta + h(\eta). \end{aligned}$$

Hence

$$(\eta, d\gamma) = (d(f\circ p) + p^*(\iota_{JX}\,d\alpha),\, -d*p^*(\iota_{JX}\,d\alpha))$$

and $F_*[0]$ is surjective. Therefore $(0, 0)$ is a regular value of F. Now we compute $\ker F_*[0]$. Formula (3) implies that $Z \in \ker F_*[0]$ if and only if

$$d(f \circ p) + p^*(\iota_{JX}\, d\alpha) = 0 \tag{4}$$

$$d^* p^*(\iota_{JX}\, d\alpha) = 0. \tag{5}$$

By applying d^* to both sides of (4) and taking into account (5) we get

$$0 = d^*d(f \circ p) + d^* p^*(\iota_{JX}\, d\alpha) = d^*d(f \circ p),$$

i.e.

$$\Delta(f \circ p) = 0.$$

Since L is compact f is constant. Hence (4) reduces to

$$p^*(\iota_{JX}\, d\alpha) = 0. \tag{6}$$

The map

$$\Theta\colon\, p_*(TL) \to \Lambda^1(L)$$

defined by

$$\Theta(X) = p^*(\iota_{JX}\, d\alpha)$$

is an isomorphism; hence equation (6) implies $X = 0$. Therefore $Z = W + f R_\alpha$ belongs to $\ker F_*[0]$ if and only if

$$\begin{cases} W = 0 \\ f = \text{constant}. \end{cases}$$

It follows that $\ker F_*[0] = \mathrm{Span}_{\mathbb{R}}(R_\alpha) \subset C^\infty(\mathcal{N}(L))$. The implicit function theorem between Banach spaces implies that the moduli space $\mathfrak{M}(L)$ is a 1-dimensional smooth manifold. □

REMARK 4.7. Note that the dimension of $\mathfrak{M}(L)$ does not depend on that of L. This is quite different from the Calabi-Yau case, where the dimension of the moduli space of deformations of special Lagrangian submanifolds near a fixed compact L is equal to the first Betti number of L. This difference can be explained in the following way: the deformations parameterized by curves tangent to the contact structure are trivial, while those one along the Reeb vector field R_α parameterize the moduli space.

Now we study the following

Extension problem. Let $(M, \alpha_t, J_t, \epsilon_t)$, $t \in (-\delta, \delta)$, be a smooth family of contact Calabi-Yau manifolds. Given a compact special Legendrian submanifold $p\colon L \hookrightarrow M$ of $(M, \alpha_0, J_0, \epsilon_0)$ does it exist a family $p_t\colon L \hookrightarrow M$ of special Legendrian submanifolds of $(M, \alpha_t, J_t, \epsilon_t)$ such that $p_0\colon L \hookrightarrow M$ coincides with p?

This is a contact version of the extension problem in the Calabi-Yau case (see [10] and [14]). We can state the following

Theorem 4.8. *Let $(M, \alpha_t, J_t, \epsilon_t)_{t\in(-\delta,\delta)}$ be a smooth family of contact Calabi-Yau manifolds. Let $p\colon L \hookrightarrow M$ be a compact special Legendrian submanifold of $(M, \alpha_0, J_0, \epsilon_0)$. Then there exists, for small t, a family of compact special Legendrian submanifolds $p_t\colon L \hookrightarrow (M, \alpha_t, J_t, \epsilon_t)$ such that $p_0 = p$ if and only if the condition*

$$[p^*(\mathfrak{Im}\,\epsilon_t)] = 0 \tag{7}$$

holds for t small enough.

Proof. The condition (7) is necessary. Indeed if we can extend L, then $\mathfrak{Im}\,\epsilon_t$ is a closed form such that $p_t^*(\mathfrak{Im}\,\epsilon_t) = 0$. Since p_t is homotopic to p_0 we have

$$[p_0^*(\mathfrak{Im}\,\epsilon_t)] = [p_t^*(\mathfrak{Im}\,\epsilon_t)] = 0.$$

In order to prove that condition (7) is sufficient, we can consider the map

$$G\colon (-\sigma, \sigma) \times C^{1,\alpha}(J(p_*TL)) \to C^{0,\alpha}(\Lambda^2(L)) \oplus C^{0,\alpha}(\Lambda^n(L))$$

defined as

$$G(t, Z) = (\exp_Z^*(d\alpha_t), 2\exp_Z^*(\mathfrak{Im}\,\epsilon_t)).$$

By our assumption it follows that

$$\mathrm{Im}(G) \subset dC^{1,\alpha}(\Lambda^1(L)) \oplus dC^{(1,\alpha)}(\Lambda^{n-1}(L)).$$

Let $X \in C^{1,\alpha}(p_*(TL))$; a direct computation and Lemma 4.6 give

$$\begin{aligned} G_*[(0,0)](0, JX) &= (dp^*(\iota_{JX}\, d\alpha_0), 2dp^*(\iota_{JX}\,\mathfrak{Im}\,\epsilon)) \\ &= (dp^*(\iota_{JX}\, d\alpha_0), -d * p^*(\iota_{JX}\, d\alpha_0)), \end{aligned}$$

where $*$ is the Hodge operator of the metric $p^*(g_J)$ with respect to the volume form $p^*(\mathfrak{Re}\,\epsilon)$. It follows that $G_*[(0,0)](0, \cdot)$ is surjective and that

$$\ker G_*[(0,0)]_{\{0\}\times C^{1,\alpha}(p_*(J(TL)))} \equiv \mathcal{H}^1(L),$$

where $\mathcal{H}^1(L)$ denotes the space of harmonic 1-forms on L. Let

$$A = \{X \in C^{1,\alpha}(p_*(TL)) \mid p^*(\iota_{JX} d\alpha) \in dC^{1,\alpha}(L) \oplus d^* C^{1,\alpha}(\Lambda^2(L))\}$$

and

$$\hat{G} = G_{|(-\delta,\delta)\times A}.$$

Then by the Hodge decomposition of $\Lambda(L)$ it follows that

$$G_*[(0,0)]_{\{0\}\times A} \colon A \to dC^{1,\alpha}(L) \oplus d^* C^{1,\alpha}(\Lambda^2(L))$$

is an isomorphism. Again by the implicit function theorem and the elliptic regularity there exists a local smooth solution of the equation

$$\hat{G}(t, \psi(t)) = 0.$$

The extension of $p \colon L \hookrightarrow M$ is obtained by considering

$$p_t := \exp_{\psi(t)}.$$

□

5. The 5-dimensional nilpotent case

In this section we study invariant contact Calabi-Yau structures on 5-dimensional nilmanifolds. We will prove that a compact 5-dimensional nilmanifold carrying an invariant Calabi-Yau structure is covered by a Lie group whose Lie algebra is isomorphic to

$$\mathfrak{g} = (0, 0, 0, 0, 12 + 34),$$

just described in Section 2. Notation $\mathfrak{g} = (0, 0, 0, 0, 12 + 34)$ means that there exists a basis $\{\alpha_1, \dots, \alpha_5\}$ of the dual space of the Lie algebra $\mathfrak{g}$ such that

$$d\alpha_1 = d\alpha_2 = d\alpha_3 = d\alpha_4 = 0, \quad d\alpha_5 = \alpha_1 \wedge \alpha_2 + \alpha_3 \wedge \alpha_4.$$

First of all we note that 5-dimensional contact Calabi-Yau manifolds are in particular hypo. Recall that an *hypo structure* on a 5-dimensional manifold is the datum of $(\alpha, \omega_1, \omega_2, \omega_3)$, where $\alpha \in \Lambda^1(M)$ and $\omega_i \in \Lambda^2(M)$ and

1. $\omega_i \wedge \omega_j = \delta_{ij} v$, for some $v \in \Lambda^4(M)$ satisfying $v \wedge \alpha \neq 0$;
2. $\iota_X \omega_1 = \iota_Y \omega_2 \iff \omega_3(X, Y) \geqslant 0$:
3. $d\omega_1 = 0$, $d(\omega_2 \wedge \alpha) = 0$, $d(\omega_3 \wedge \alpha) = 0$.

These structures have been introduced and studied by D. Conti and S. Salamon in [5]. Let (M, α, J, ϵ) be a contact Calabi-Yau manifold of dimension 5. Then

$$\alpha, \quad \omega_1 = \frac{1}{2} d\alpha, \quad \omega_2 = \Re\mathfrak{e}\, \epsilon, \quad \omega_3 = \Im\mathfrak{m}\, \epsilon,$$

define an hypo structure on M.

The following lemma, whose proof is immediate, will be useful in the sequel

Lemma 5.1. *Let $M = G/\Gamma$ be a nilmanifold of dimension* 5. *If M admits an invariant contact form, then the Lie algebra of G is isomorphic to one of the following*

- (0, 0, 12, 13, 14 + 23);
- (0, 0, 0, 12, 13 + 24);
- (0, 0, 0, 0, 12 + 34).

Let $\mathfrak{g}$ be a non-trivial 5-dimensional nilpotent Lie algebra and denote by $V = \mathfrak{g}^*$ the dual vector space of $\mathfrak{g}$. There exists a filtration of V

$$V^1 \subset V^2 \subset V^3 \subset V^4 \subset V^5 = V,$$

with $dV^i \subset \Lambda^2 V^{i-1}$ and $\dim_{\mathbb{R}} V^i = i$. We may choose the filtration V in such a way that $V^2 \subset \ker d \subset V^4$.

Let $(M = G/\Gamma, \alpha, \omega_1, \omega_2, \omega_3)$ be a nilmanifold endowed with an invariant hypo structure $(\alpha, \omega_1, \omega_2, \omega_3)$

1. Assume that $\alpha \in V^4$. Then we have the following (see [5])

Theorem 5.2. *If $\alpha \in V^4$, then $\mathfrak{g}$ is either* (0, 0, 0, 0, 12), (0, 0, 0, 12, 13), *or* (0, 0, 12, 13, 14).

In particular if (M, α, J, ϵ) is contact Calabi-Yau, then $\alpha \in V^4$.

2. Assume that $\alpha \notin V^4$. We have (see again [5])

Lemma 5.3. *If $\alpha \notin V^4$ and all ω_i are closed, then α is orthogonal to V^4.*

Theorem 5.4. *If α is orthogonal to V^4, then $\mathfrak{g}$ is one of*

$$(0, 0, 0, 0, 12), \quad (0, 0, 0, 0, 12 + 34).$$

Let (M, α, J, ϵ) be a contact Calabi-Yau manifold of dimension 5 endowed with an invariant contact Calabi-Yau structure; then by 1. α does not belong to V^4. By Lemma 5.3 α is orthogonal to V^4 and by Theorem 5.4 $\mathfrak{g}$ = (0, 0, 0, 0, 12 + 34). Hence we have proved the following

Theorem 5.5. *Let $M = G/\Gamma$ be a nilmanifold of dimension* 5 *admitting an invariant contact Calabi-Yau structure. Then $\mathfrak{g}$ is isomorphic to*

$$(0, 0, 0, 0, 12 + 34).$$

6. Calabi-Yau manifolds of codimension r

In this section we extend the definition of contact Calabi-Yau manifold to codimension r showing the analogous of Theorem 4.8.

Let us consider the following

DEFINITION 6.1. Let M be a $2n+r$-dimensional manifold. An *r-contact structure* on M is the datum $\mathcal{D}=\{\alpha_1,\dots,\alpha_r\}$, where $\alpha_i\in\Lambda^1(M)$, such that
- $d\alpha_1=d\alpha_2=\cdots=d\alpha_r$;
- $\alpha_1\wedge\cdots\wedge\alpha_r\wedge(d\alpha_1)^n\neq 0$.

Note that if $\mathcal{D}=\{\alpha_1,\dots,\alpha_r\}$ is an r-contact structure and $\xi:=\bigcap\ker\alpha_i$, then $(\xi,d\alpha_1)$ is a symplectic vector bundle on M and there exists a unique set of vector fields $\{R_1,\dots,R_r\}$ satisfying

$$\alpha_i(R_j)=\delta_{ij},\quad \iota_{R_i}d\alpha_i=0\quad\text{for any}\quad i,\ j=1,\dots,r.$$

Let us denote by $\mathfrak{C}_\kappa(\xi)$ the set of complex structures on ξ calibrated by the symplectic form $\kappa=(1/2)\,d\alpha_1$ and by $\Lambda^r_0(M)$ the set of r-forms γ on M satisfying

$$\iota_{R_i}\gamma=0\quad\text{for any}\quad i=1,\dots,r.$$

If $J\in\mathfrak{C}_\kappa(\xi)$, then we extend it to TM by defining

$$J(R_i)=0.$$

Note that such a J satisfies

$$J^2=-I+\sum_{i=1}^r\alpha_i\otimes R_i.$$

Consequently, for any $J\in\mathfrak{C}_\kappa(\xi)$, we have $J(\Lambda^r_0(M))\subset\Lambda^r_0M$ and a natural splitting of $\Lambda^r_0(M)\otimes\mathbb{C}$ in

$$\Lambda^r_0(M)\otimes\mathbb{C}=\bigoplus_{p+q=r}\Lambda^{p,q}_J(\xi).$$

We can give the following

DEFINITION 6.2. An r-contact Calabi-Yau manifold is the datum of $(M,\mathcal{D},J,\epsilon)$, where
- M is a $2n+r$-dimensional manifold;
- $\mathcal{D}=\{\alpha_1,\dots,\alpha_r\}$ is an r-contact structure;

- $J \in \mathfrak{C}_\kappa(\xi)$
- $\epsilon \in \Lambda_J^{n,0}(\xi)$ satisfies

$$\begin{cases} \epsilon \wedge \bar{\epsilon} = c_n \kappa^n \\ d\epsilon = 0. \end{cases}$$

EXAMPLE 6.3. Let $M = H(3)/\Gamma \times S^1$ be the Kodaira-Thurston manifold, where $H(3)$ is the 3-dimensional Heisenberg group and Γ is the lattice of $H(3)$ of matrices with integers entries. Let

$$\alpha_1 = -2\,dz + 2x\,dy,$$
$$\alpha_2 = -2\,dz + 2x\,dy + 2\,dt.$$

One easily gets

$$d\alpha_1 = d\alpha_2 = 2\,dx \wedge dy$$

and that $\mathcal{D} = \{\alpha_1, \alpha_2\}$ is a 2-contact structure on M. Note that $\xi = \ker\alpha_1 \cap \ker\alpha_2$ is spanned by $\{X_1 = \partial_x, X_2 = \partial_y + x\partial_z\}$. Moreover the Reeb fields of $\mathcal{D}$ are

$$R_1 = -\frac{1}{2}\partial_z - \frac{1}{2}\partial_t,$$
$$R_2 = \frac{1}{2}\partial_t.$$

Therefore $\Lambda_0^1(M)$ is generated by $\{dx, dy\}$. Let $J \in \mathrm{End}(\xi)$ be the complex structure given by

$$J(X_1) = X_2, \quad J(X_2) = -X_1$$

and let $\epsilon \in \Lambda_J^{1,0}(\xi)$ be the form

$$\epsilon = dx + i\,dy.$$

Then $(M, \mathcal{D}, J, \epsilon)$ is a 2-contact Calabi-Yau structure.

As in the contact Calabi-Yau case if $(M, \mathcal{D}, J, \epsilon)$ is an r-contact Calabi-Yau manifold, then the n-form $\Omega = \mathfrak{Re}\,\epsilon$ is a calibration on M. Moreover an n-dimensional submanifold $p\colon L \hookrightarrow M$ admits an orientation making it calibrated by Ω if and only if

$$p^*(\alpha_i) = 0 \quad \text{for any} \quad \alpha_i \in \mathcal{D},$$
$$p^*(\mathfrak{Im}\,\epsilon) = 0.$$

A submanifold satisfying these equations will be called *special Legendrian*.

EXAMPLE 6.4. Let $(M, \mathcal{D}, J, \epsilon)$ be the 2-contact Calabi-Yau structure described in Example 6.3. Then

$$L := \left\{ [A] \in H(3)/\Gamma \;\middle|\; A = \begin{pmatrix} 1 & x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},\ x \in \mathbb{R} \right\} \times \{q\} \simeq S^1$$

is a compact special Legendrian submanifold for any $q \in S^1$.

The proof of the next theorem is very similar to that of Theorem 4.8 and it is omitted.

Theorem 6.5. *Let $(M, \mathcal{D}_t, J_t, \epsilon_t)_{t\in(-\delta,\delta)}$ be a smooth family of r-contact Calabi-Yau manifolds. Let $p\colon L \hookrightarrow M$ be a compact special Legendrian submanifold of $(M, \mathcal{D}_0, J_0, \epsilon_0)$. Then there exists, for small t, a family of compact special Legendrian submanifolds $p_t\colon L \hookrightarrow (M, \mathcal{D}_t, J_t, \epsilon_t)$ extending $p\colon L \hookrightarrow M$ if and only if the condition*

$$[p^*(\mathfrak{Im}\,\epsilon_t)] = 0$$

holds for t small enough.

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