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LIFESPAN FOR RADIALLY SYMMETRIC SOLUTIONS TO SYSTEMS OF SEMILINEAR WAVE EQUATIONS WITH MULTIPLE SPEEDS

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Abstract

We consider the Cauchy problem for a system of semilinear wave equations with multiple propagation speeds in three space dimensions. We obtain the sharp lower bound for the lifespan of radially symmetric solutions to a class of these systems. We also show global existence of radially symmetric solutions to another class of systems with small initial data.

1. Introduction and the main results

For $c > 0$, we define

$$\Box_c = \partial_t^2 - c^2\Delta_x = \partial_0^2 - c^2\sum_{j=1}^{3}\partial_j^2,$$

where $\partial_0 = \partial_t = \partial/\partial t$, and $\partial_j = \partial/\partial x_j$ for $j = 1, 2, 3$. The above constant c is called the propagation speed. We simply write $\Box$ for $\Box_1 = \partial_t^2 - \Delta_x$.

This paper is devoted to a study on the Cauchy problem for systems of semilinear wave equations in three space dimensions of the type

$$\Box_{c_i} u_i = F_i(u, \partial u) \quad \text{for} \quad (t, x) \in (0, \infty) \times \mathbb{R}^3 \quad (i = 1, \dots, m) \tag{1.1}$$

with initial data

$$u_i(0, x) = \varepsilon f_i(x), \quad (\partial_t u_i)(0, x) = \varepsilon g_i(x) \tag{1.2}$$

for $x \in \mathbb{R}^3$ $(i = 1, \dots, m)$, where c_i $(1 \le i \le m)$ are given positive constants, $u = (u_j)_{1\le j\le m}$, and $\partial u = (\partial_a u_j)_{1\le j\le m, 0\le a\le 3}$, while ε is a small positive parameter. In the following, we assume that $F(u, v) = (F_j(u, v))_{1\le j\le m}$ is a smooth function of $(u, v) \in \mathbb{R}^m \times \mathbb{R}^{4m}$, vanishing together with its first derivatives at $(u, v) = (0, 0)$. We suppose

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$f = (f_j)_{1\le j\le m}$, $g = (g_j)_{1\le j\le m} \in C_0^\infty(\mathbb{R}^3;\mathbb{R}^m)$. For simplicity of exposition, we also suppose that the propagation speeds c_i $(1 \le i \le m)$ are distinct.

Let $T_\varepsilon = T_\varepsilon(f, g, F)$ be the supremum of all T such that the Cauchy problem (1.1)–(1.2) admits a C^∞-solution $u = (u_j)_{1\le j\le m}$ for $(t, x) \in [0, T) \times \mathbb{R}^3$. T_ε is called the lifespan of the classical solution to the Cauchy problem (1.1)–(1.2). We say that *small data global existence* (or (SDGE)) holds for (1.1)–(1.2) if for any $f, g \in C_0^\infty(\mathbb{R}^3;\mathbb{R}^m)$, there exists a positive constant ε_0 such that $T_\varepsilon(f, g, F) = \infty$ for all $\varepsilon \in (0, \varepsilon_0]$. When $T_\varepsilon(f, g, F) < \infty$, we say that the solution blows up in finite time.

For the following single wave equation

$$\begin{cases} \Box u = (\partial_t u)^2 \text{ (or } u(\partial_t u)) & \text{in } (0,\infty)\times\mathbb{R}^3, \\ u(0,x) = \varepsilon f(x),\ (\partial_t u)(0,x) = \varepsilon g(x) & \text{for } x \in \mathbb{R}^3, \end{cases} \tag{1.3}$$

it is known that there exist $f, g \in C_0^\infty(\mathbb{R}^3)$ and two positive constants C_1 and ε_1 such that

$$T_\varepsilon \le \exp(C_1\varepsilon^{-1}) \tag{1.4}$$

for any $\varepsilon \in (0, \varepsilon_1]$ (see John [3], Sideris [18], and Kubo [14]). In other words, for such f and g, the solution to (1.3) blows up in finite time no matter how small ε is. The above upper bound for the lifespan T_ε is sharp in the sense that for any $f, g \in C_0^\infty(\mathbb{R}^3)$, there exist two positive constants C_2 and ε_2 such that

$$T_\varepsilon \ge \exp(C_2\varepsilon^{-1}) \tag{1.5}$$

for any $\varepsilon \in (0, \varepsilon_2]$ (see John-Klainerman [4], and Klainerman [10]; see also Lindblad [16] for the case $m = 1$ and $F(u, 0) = O(|u|^3)$ for small u, and the author [6] for the case $m \ge 2$ and $F(u, \partial u) = O(|u|^3 + |\partial u|^2)$ around $(u, \partial u) = (0, 0)$).

The above example (1.3) shows that some restriction on F is necessary for (SDGE). To recall known results for (SDGE), we introduce several types of nonlinearities. Let $\phi = (\phi_i)_{1\le i\le m}$ and $\psi = (\psi_i)_{1\le i\le m}$ be C^2-functions. In the following, $\alpha^{j}_{N,i}$, $\alpha^{jk,ab}_{\mathrm{I},i}$, $\alpha^{j,ab}_{\mathrm{II},i}$, $\alpha^{jk,a}_{\mathrm{III},i}$ and $\alpha^{j,a}_{\mathrm{IV},i}$ are arbitrary constants. First of all, we introduce *null terms*

$$N_i(\partial\phi, \partial\psi) = \sum_{j=1}^{m} \alpha^{j}_{N,i}\, Q_0(\phi_j, \psi_j; c_j), \tag{1.6}$$

where $Q_0(v, w; c)$ is the null form defined by

$$Q_0(v, w; c) = (\partial_t v)(\partial_t w) - c^2 \sum_{k=1}^{3} (\partial_k v)(\partial_k w)$$

(see Klainerman [11]; note that another type of the null form

$$Q_{ab}(v, w) = (\partial_a v)(\partial_b w) - (\partial_b v)(\partial_a w),$$

which was also introduced in [11], does not appear here, just because we have restricted our attention to the simplified situation of semilinear systems with distinct speeds). Next we introduce

$$R_i^{\mathrm{I}}(\partial\phi, \partial\psi) = \sum_{\substack{1\le j,k\le m\\ j\ne k}} \sum_{0\le a,b\le 3} \alpha_{\mathrm{I},i}^{jk,ab}(\partial_a\phi_j)(\partial_b\psi_k), \tag{1.7}$$

$$R_i^{\mathrm{II}}(\partial\phi, \partial\psi) = \sum_{\substack{1\le j\le m\\ j\ne i}} \sum_{0\le a,b\le 3} \alpha_{\mathrm{II},i}^{j,ab}(\partial_a\phi_j)(\partial_b\psi_j), \tag{1.8}$$

which we call *nonresonant terms of types* (I) *and* (II), respectively. Similarly we define *nonresonant terms of types* (III) *and* (IV) by

$$R_i^{\mathrm{III}}(\phi, \partial\psi) = \sum_{\substack{1\le j,k\le m\\ j\ne k}} \sum_{0\le a\le 3} \alpha_{\mathrm{III},i}^{jk,a}\phi_j(\partial_a\psi_k), \tag{1.9}$$

$$R_i^{\mathrm{IV}}(\phi, \partial\psi) = \sum_{\substack{1\le j\le m\\ j\ne i}} \sum_{0\le a\le 3} \alpha_{\mathrm{IV},i}^{j,a}\phi_j(\partial_a\psi_j), \tag{1.10}$$

respectively. Finally, let H_i be a smooth function of $(u, \partial u)$, satisfying

$$H_i(u, \partial u) = O(|u|^3 + |\partial u|^3) \quad \text{near} \quad (u, \partial u) = (0, 0). \tag{1.11}$$

Now the known results for (SDGE) can be summarized as follows. If F_i has the form

$$F_i(u, \partial u) = N_i(\partial u, \partial u) + R_i^{\mathrm{I}}(\partial u, \partial u) + R_i^{\mathrm{II}}(\partial u, \partial u) + H_i(u, \partial u) \quad \text{for all} \quad i \in \{1, \ldots, m\}, \tag{1.12}$$

then (SDGE) holds for (1.1)–(1.2) (see the author [6]; see also Klainerman [11], Christodoulou [2], Kovalyov [13], Yokoyama [21], Sideris-Tu [19], Kubota-Yokoyama [15], and Sogge [20]). Note that in (1.12), quadratic terms of F_i depend only on ∂u. On the other hand, even if u is involved in quadratic terms, we also have (SDGE) for (1.1)–(1.2), if F_i can be written as

$$F_i(u, \partial u) = N_i(\partial u, \partial u) + R_i^{\mathrm{I}}(\partial u, \partial u) + R_i^{\mathrm{III}}(u, \partial u) + R_i^{\mathrm{IV}}(u, \partial u) + H_i(u, \partial u) \quad \text{for all} \quad i \in \{1, \ldots, m\} \tag{1.13}$$

(see Katayama-Yokoyama [9]; see also the author [5, 7]).

From (1.12) and (1.13), it seems reasonable to conjecture that if F_i can be written as

$$\begin{aligned} F_i(u, \partial u) = {} & N_i(\partial u, \partial u) + R_i^{\mathrm{I}}(\partial u, \partial u) + R_i^{\mathrm{II}}(\partial u, \partial u) \\ & + R_i^{\mathrm{III}}(u, \partial u) + R_i^{\mathrm{IV}}(u, \partial u) + H_i(u, \partial u) \end{aligned} \tag{1.14}$$

for all $i \in \{1, \dots, m\}$, then (SDGE) holds. But this conjecture turns out to be false because of the following counterexample by Ohta [17]:

$$\begin{cases} \Box_{c_1} u_1 = F_1(u, \partial u) := u_2(\partial_t u_1), \\ \Box_{c_2} u_2 = F_2(u, \partial u) := (\partial_t u_1)^2. \end{cases} \tag{1.15}$$

Note that F_1 and F_2 in (1.15) are the nonresonant terms of types (III) and (II), respectively. Hence (1.14) holds for $F = (F_1, F_2)$ in (1.15), but neither (1.12) nor (1.13) is satisfied. In [17], it was proved that (SDGE) does not hold for the above system (1.15) in general. More precisely, for the system (1.15) with $c_1 < c_2$, it was shown that there exist radially symmetric data $f, g \in C_0^\infty(\mathbb{R}^3; \mathbb{R}^2)$ and two positive constants C_3 and ε_3 such that

$$T_\varepsilon \le \exp(C_3 \varepsilon^{-2}) \tag{1.16}$$

for all $\varepsilon \in (0, \varepsilon_3]$.

Since the upper bound of the lifespan obtained in (1.16) is somewhat longer than (1.4), it is interesting to investigate sharpness of (1.16). Our first aim in this paper is to get the lower bound of the lifespan for (1.15). Unfortunately, because it is difficult to obtain energy estimates for (1.15) in large time, we restrict our consideration to radial solutions. Note that the upper bound (1.16) was also obtained for radial solutions.

Before stating our results, we introduce some notation. We say that ϕ is a radially symmetric C_0^∞-function if ϕ belongs to $C_0^\infty(\mathbb{R}^3)$ and there exists a function $\tilde{\phi} \in C^\infty([0, \infty))$ such that $\phi(x) = \tilde{\phi}(|x|)$ for any $x \in \mathbb{R}^3$. We say $F = F(u, \partial u)$ is rotationally invariant if

$$F(u_O(t, x), \partial u_O(t, x)) = F(u(t, O(x)), (\partial u)(t, O(x)))$$

holds for any C^1-function $u = u(t, x)$ and any orthogonal transformation $O = O(x)$ on $\mathbb{R}^3$, where u_O is defined by $u_O(t, x) = u(t, O(x))$. It is easy to see that if $F = F(u, \partial u)$ is rotationally invariant, and the initial data f and g are radially symmetric C_0^∞-functions, then the solution u to (1.1)–(1.2) is radial, namely $u(t, x) = \tilde{u}(t, |x|)$ with some function $\tilde{u} = \tilde{u}(t, r)$.

For $\phi = (\phi_i)_{1 \le i \le m}$ and $\psi = (\psi_i)_{1 \le i \le m}$, we define

$$r_i^{\mathrm{I}}(\partial\phi, \partial\psi) = \sum_{\{(j,k); j \ne k\}} (\beta_{\mathrm{I},i}^{jk,0}(\partial_t \phi_j)(\partial_t \psi_k) + \beta_{\mathrm{I},i}^{jk,1}(\nabla_x \phi_j) \cdot (\nabla_x \psi_k)),$$

$$r_i^{\mathrm{II}}(\partial\phi, \partial\psi) = \sum_{\{j; j \ne i\}} (\beta_{\mathrm{II},i}^{j,0}(\partial_t \phi_j)(\partial_t \psi_j) + \beta_{\mathrm{II},i}^{j,1}(\nabla_x \phi_j) \cdot (\nabla_x \psi_j)),$$

$$r_i^{\mathrm{III}}(\phi, \partial_t\psi) = \sum_{\{(j,k); j \ne k\}} \beta_{\mathrm{III},i}^{jk} \phi_j (\partial_t \psi_k),$$

$$r_i^{\mathrm{IV}}(\phi, \partial_t\psi) = \sum_{\{j; j \ne i\}} \beta_{\mathrm{IV},i}^{j} \phi_j (\partial_t \psi_j)$$

for $i = 1, \dots, m$, where $\beta^{jk,a}_{\mathrm{I},i}$, $\beta^{j,a}_{\mathrm{II},i}$ $(a = 0, 1)$, $\beta^{jk}_{\mathrm{III},i}$ and $\beta^{j}_{\mathrm{IV},i}$ are arbitrary constants. Here $\nabla_x\phi = (\partial_1\phi, \partial_2\phi, \partial_3\phi)$ for a C^1-function ϕ, and $\cdot$ denotes the inner product of $\mathbb{R}^3$. Note that $r^{\mathrm{I}}_i(\partial u, \partial u)$, $r^{\mathrm{II}}_i(\partial u, \partial u)$, $r^{\mathrm{III}}_i(u, \partial u)$, and $r^{\mathrm{IV}}_i(u, \partial u)$ are rotationally invariant nonresonant terms of types (I), (II), (III), and (IV), respectively. It is easy to see that the null terms $N_i(\partial u, \partial u)$ are also rotationally invariant.

Theorem 1.1. *Let the propagation speeds $c_1, \dots, c_m$ be distinct. Assume that for each $i \in \{1, \dots, m\}$, F_i has the form*

$$F_i(u, \partial u) = N_i(\partial u, \partial u) + r^{\mathrm{I}}_i(\partial u, \partial u) + r^{\mathrm{II}}_i(\partial u, \partial u) + r^{\mathrm{III}}_i(u, \partial_t u) + H_i(u, \partial u), \tag{1.17}$$

where H_i is rotationally invariant, and satisfies (1.11).

Then, for any radially symmetric C^∞_0-functions f and g, there exist two positive constants ε_0 and C such that the lifespan T_ε for (1.1)–(1.2) *satisfies*

$$T_\varepsilon(f, g, F) \geq \exp(C\varepsilon^{-2}) \tag{1.18}$$

for any $\varepsilon \in (0, \varepsilon_0]$.

Note that (1.17) contains the null terms, nonresonant terms of types (I), (II) and (III), and terms of higher order. Since F in (1.15) has the form (1.17), the upper bound (1.16) and the lower bound (1.18) guarantee the sharpness of one another, as far as radially symmetric solutions are considered.

To get (1.18), we follow a similar strategy to that in Katayama-Matsumura [8], where the sharp lower bound of the lifespan for the system

$$\begin{cases} \Box_{c_1} u_1 = u_1 u_2 & \text{in} \quad (0, \infty) \times \mathbb{R}^3, \\ \Box_{c_2} u_2 = u_1^3 & \text{in} \quad (0, \infty) \times \mathbb{R}^3 \end{cases}$$

with $c_1 \neq c_2$ was obtained. The proof of Theorem 1.1 will be given in Section 3.

Now we turn our attention to another problem. Ohta's counterexample (1.15) says that (SDGE) does not hold for (1.14), especially for a combination of nonresonant terms of types (II) and (III). Our next question is what happens for other combinations. Here we give an example which suggests (SDGE) may hold in general for a combination of null terms, nonresonant terms of types (I), (II) and (IV).

Theorem 1.2. *Let $m = 2$, and consider the Cauchy problem* (1.1)–(1.2) *with*

$$F_1(u, \partial u) := N_1(\partial u, \partial u) + r^{\mathrm{I}}_1(\partial u, \partial u) + r^{\mathrm{II}}_1(\partial u, \partial u) + r^{\mathrm{IV}}_1(u, \partial_t u), \tag{1.19}$$

$$F_2(u, \partial u) := N_2(\partial u, \partial u) + r^{\mathrm{I}}_2(\partial u, \partial u) + r^{\mathrm{II}}_2(\partial u, \partial u). \tag{1.20}$$

Assume $c_1 \neq c_2$. *Then, for any radially symmetric* C_0^∞*-functions* $f = (f_1, f_2)$ *and* $g = (g_1, g_2)$, *there exists a positive constant* ε_0 *such that*

$$T_\varepsilon(f, g, F) = \infty \quad \text{for any} \quad \varepsilon \in (0, \varepsilon_0]. \tag{1.21}$$

Nonresonant terms of types (II) and (IV) are involved in (1.19)–(1.20). Hence neither (1.12) nor (1.13) holds for this $F = (F_1, F_2)$. Nonetheless (SDGE) holds for (1.1) with this F as far as we consider radial solutions. Theorem 1.2 suggests that there may be a certain sufficient condition for (SDGE) other than (1.12) and (1.13). Of course, even for (1.19)–(1.20), it may possibly happen that (SDGE) does not hold for general C_0^∞ data. This problem is still open. The proof of Theorem 1.2 will be given in Section 4.

Throughout this paper, various positive constants, which may change line by line, are denoted just by the same letter C.

2. Basic decay estimates

In this section, we derive basic L^∞-L^∞ decay estimates.

For $\phi, \psi \in C_0^\infty(\mathbb{R}^3)$ and a positive constant c, we write $U_c^*[\phi, \psi]$ for the solution to the Cauchy problem for

$$\begin{cases} \Box_c U_c^*[\phi, \psi](t, x) = 0 & \text{for} \quad (t, x) \in (0, \infty) \times \mathbb{R}^3, \\ U_c^*[\phi, \psi](0, x) = \phi(x), \ (\partial_t U_c^*[\phi, \psi])(0, x) = \psi(x) & \text{for} \quad x \in \mathbb{R}^3. \end{cases}$$

Similarly, for a continuous function $G = G(t, x)$ on $(0, \infty) \times \mathbb{R}^3$, we write $U_c[G]$ for the solution to the Cauchy problem for

$$\begin{cases} \Box_c U_c[G](t, x) = G(t, x) & \text{for} \quad (t, x) \in (0, \infty) \times \mathbb{R}^3, \\ U_c[G](0, x) = (\partial_t U_c[G])(0, x) = 0 & \text{for} \quad x \in \mathbb{R}^3. \end{cases}$$

For $\rho \in \mathbb{R}$, we write $\langle \rho \rangle = \sqrt{1 + \rho^2}$. For a continuous function ϕ, a non-negative constant ν, and $t, r \in [0, \infty)$, $\|\phi\|_{\nu,t,r}$ is defined by

$$\|\phi\|_{\nu,t,r} = \sup_{y \in \mathbb{R}^3 \text{ with } |t-r| \le |y| \le t+r} \langle |y| \rangle^\nu |\phi(y)|.$$

For $U_c^*[\phi, \psi]$, we have the following.

Lemma 2.1. *Let* $c > 0$ *and* $\kappa > 0$. *Then there exists a positive constant* C *such that we have*

$$\begin{aligned} &\langle t + |x| \rangle \langle ct - |x| \rangle^\kappa |U_c^*[\phi, \psi](t, x)| \\ &\le C(\|\phi\|_{2+\kappa,ct,|x|} + \|\nabla_x \phi\|_{2+\kappa,ct,|x|} + \|\psi\|_{2+\kappa,ct,|x|}) \end{aligned} \tag{2.1}$$

for $(t, x) \in [0, \infty) \times \mathbb{R}^3$.

For the proof, see Kubota-Yokoyama [15]. More precisely, the estimate stated in [15] is not exactly (2.1), but we can easily obtain (2.1) by investigating their proof (see also Asakura [1]).

Let $c_1, \dots, c_m$ be given positive constants, and let $c_0 = 0$. We define

$$w(t, r) = \min_{0 \le j \le m} \langle c_j t - r \rangle$$

for $(t, r) \in [0, \infty) \times [0, \infty)$. We also define

$$\Phi_\kappa(t, r) = \begin{cases} \langle t + r \rangle^{-\kappa} & \text{if} \quad \kappa < 0, \\ \log\left(2 + \dfrac{\langle t + r \rangle}{\langle t - r \rangle} \right) & \text{if} \quad \kappa = 0, \\ \langle t - r \rangle^{-\kappa} & \text{if} \quad \kappa > 0, \end{cases} \tag{2.2}$$

$$\Psi_\mu(t) = \begin{cases} \log(2 + t) & \text{if} \quad \mu = 0, \\ 1 & \text{if} \quad \mu > 0. \end{cases} \tag{2.3}$$

Note that we have $\Phi_0(t, r) \le C \log(2 + t)$ for any $(t, r) \in [0, \infty) \times [0, \infty)$. For $1 \le i \le m$, $(t, x) \in [0, \infty) \times \mathbb{R}^3$ and $(t, r) \in [0, \infty) \times [0, \infty)$, we put

$$\Theta_i(t, x) = \left\{ (\tau, y) \in [0, t] \times \mathbb{R}^3 ;\ \big| |x| - c_i(t - \tau) \big| \le |y| \le |x| + c_i(t - \tau) \right\},$$
$$\Theta_i^*(t, r) = \{ (\tau, \lambda) \in [0, t] \times [0, \infty);\ |r - c_i(t - \tau)| \le \lambda \le r + c_i(t - \tau) \}.$$

Then, for $U_{c_i}[G]$ we have

Lemma 2.2. *Let* $i \in \{1, \dots, m\}$. *For* $\rho > 0$ *and* $\mu \ge 0$, *there exists a positive constant* C *such that*

$$\begin{aligned} &\langle t + |x| \rangle \Phi_{\rho - 1}(c_i t, |x|)^{-1} |U_{c_i}[G](t, x)| \\ &\le C \Psi_\mu(t) \sup_{(\tau, y) \in \Theta_i(t, x)} \langle \tau + |y| \rangle^\rho w(\tau, |y|)^{1 + \mu} |y|\, |G(\tau, y)| \end{aligned} \tag{2.4}$$

for $(t, x) \in [0, \infty) \times \mathbb{R}^3$.

Proof. The case where $\rho > 1$ and $\mu > 0$ was proved by Katayama-Yokoyama [9, Section 8]. Other cases can be proved by apparent modifications of the proof in [9], and we only give a sketch of the proof here (see also the author [6] for the case $\rho = 1$ and $\mu = 0$, and Katayama-Matsumura [8] for the case $\rho > 1$ and $\mu = 0$).

Without loss of generality, we may assume $c_i = 1$. Then, following [9, Section 8], we find

$$\begin{aligned} |U_{c_i}[G](t, x)| \le{}& C r^{-1} \int_0^t \int_{|r - t + \tau|}^{t - \tau + r} \mathcal{W}(\tau, \lambda)^{-1}\, d\lambda\, d\tau \\ &\times \sup_{(\tau, y) \in \Theta_i(t, x)} \mathcal{W}(\tau, |y|) |y|\, |G(\tau, y)| \end{aligned} \tag{2.5}$$

for any positive function $\mathcal{W} = \mathcal{W}(\tau, \lambda)$, where $r = |x|$. Therefore our task is to show

$$r^{-1} \int_0^t \int_{|r-t+\tau|}^{t-\tau+r} \mathcal{W}(\tau, \lambda)^{-1}\, d\lambda\, d\tau \le C\langle t+r\rangle^{-1}\Phi_{\rho-1}(t, r)\Psi_\mu(t) \tag{2.6}$$

with $\mathcal{W}(t, r) = \langle t + r\rangle^\rho w(t, r)^{1+\mu}$. Since $w(t, r)^{-1} \le \sum_{j=0}^m \langle c_j t - r\rangle^{-1}$, it suffices to evaluate the integral

$$J_{\rho,\mu,a}(t, r) := r^{-1} \int_0^t \int_{|r-t+\tau|}^{t-\tau+r} (1 + \tau + \lambda)^{-\rho}(1 + |a\tau - \lambda|)^{-1-\mu}\, d\lambda\, d\tau \tag{2.7}$$

with $a \ge 0$. Performing a change of variables $\alpha = \tau + \lambda$ and $\beta = \lambda - a\tau$, we obtain

$$J_{\rho,\mu,a}(t, r) = \frac{1}{(a+1)r} \int_{|t-r|}^{t+r} (1+\alpha)^{-\rho}\, d\alpha \int_{\hat\beta}^{\alpha} (1+|\beta|)^{-1-\mu}\, d\beta \tag{2.8}$$

with $\hat\beta = \{(1-a)\alpha + (1+a)(r-t)\}/2$ (see [9, (8.6) and (8.7)]).

For example, if $\mu > 0$, it is easy to see

$$J_{1,\mu,a}(t, r) \le Cr^{-1} \int_{|t-r|}^{t+r} (1+\alpha)^{-1}\, d\alpha. \tag{2.9}$$

By a direct calculation, we get

$$\begin{aligned} r^{-1} \int_{|t-r|}^{t+r} (1+\alpha)^{-1}\, d\alpha &= Cr^{-1} \log\left(\frac{1+t+r}{1+|t-r|}\right) \\ &\le C\langle t+r\rangle^{-1}\Phi_0(t, r) \end{aligned} \tag{2.10}$$

for $r \ge (t+1)/2$. On the other hand, we also have

$$r^{-1} \int_{|t-r|}^{t+r} (1+\alpha)^{-1}\, d\alpha \le C(1+|t-r|)^{-1} \le C(1+t+r)^{-1} \tag{2.11}$$

for $r \le (t+1)/2$. (2.9), (2.10) and (2.11) imply (2.6) for the case $\rho = 1$ and $\mu > 0$ immediately. Other cases can be treated similarly. □

By (2.5) and similar lines to (2.6)–(2.11), we also have

Corollary 2.3. *Let $i \in \{1, \dots, m\}$ and*

$$\mathcal{W}(t, r)^{-1} = A_1\langle t+r\rangle^{-1} w(t, r)^{-1} + A_2\langle t+r\rangle^{-1} w(t, r)^{-2}$$

with some positive constants A_1 and A_2. Then we have

$$\begin{aligned} &\langle t+|x|\rangle\Phi_0(c_i t, |x|)^{-1}|U_{c_i}[G](t, x)| \\ &\le C(A_1 \log(2+t) + A_2) \sup_{(\tau,y)\in\Theta_i(t,x)} \mathcal{W}(\tau, |y|)|y|\,|G(\tau, y)|. \end{aligned}$$

For $c > 0$, $(t, r) \in [0, \infty) \times [0, \infty)$, and a C^1-function $G = G(t, r)$ on $[0, \infty) \times [0, \infty)$, we define

$$L_c[G](t, r) = \frac{1}{2c} \int_0^t \left(\int_{\lambda_c^-(\tau;t,r)}^{\lambda_c^+(\tau;t,r)} \check{G}(\tau, \lambda)\, d\lambda \right) d\tau, \tag{2.12}$$

where

$$\lambda_c^{\pm}(\tau; t, r) = r \pm c(t - \tau), \tag{2.13}$$

and $\check{G}$ is defined by $\check{G}(t, r) = rG(t, |r|)$ for $(t, r) \in [0, \infty) \times \mathbb{R}$. Then easy calculations lead to

$$(\partial_t \pm c\partial_r)L_c[G](t, r) = I_c^{\pm}[\check{G}](t, r), \tag{2.14}$$

$$\partial_t(\partial_t \pm c\partial_r)L_c[G](t, r) = \check{G}(t, r) \pm cI_c^{\pm}[\partial_r \check{G}](t, r), \tag{2.15}$$

$$\partial_r(\partial_t \pm c\partial_r)L_c[G](t, r) = I_c^{\pm}[\partial_r \check{G}](t, r), \tag{2.16}$$

where $I_c^{\pm}[H](t, r)$ is defined by

$$I_c^{\pm}[H](t, r) = \int_0^t H(\tau, \lambda_c^{\pm}(\tau; t, r))\, d\tau, \quad (t, r) \in [0, \infty) \times \mathbb{R}$$

for a function $H = H(t, r)$. Note that we have $(\partial_r \check{G})(t, r) = G(t, |r|) + |r|(\partial_r G)(t, |r|)$ for $r \in \mathbb{R}$. It is also easy to verify that a classical solution v to

$$\begin{cases} \Box_c v(t, x) = G(t, |x|) & \text{in} \quad (0, \infty) \times \mathbb{R}^3, \\ v(0, x) = \partial_t v(0, x) = 0 & \text{for} \quad x \in \mathbb{R}^3 \end{cases} \tag{2.17}$$

can be written as

$$v(t, x) = |x|^{-1} L_c[G](t, |x|) \quad \text{for} \quad (t, x) \in [0, \infty) \times (\mathbb{R}^3 \setminus \{0\}). \tag{2.18}$$

Before we proceed to estimate derivatives of solutions to wave equations, we give two technical lemmas.

Lemma 2.4. *Let $c > 0$, $\alpha \neq 0$ and $p \geq 0$. Then we have*

$$\int_0^t \left(1 + \left|\alpha\tau - |r \pm ct|\right|\right)^{-(1+p)} d\tau \leq C\Psi_p(t) \tag{2.19}$$

for $(t, r) \in [0, \infty) \times [0, \infty)$, where Ψ_μ is defined by (2.3).

Proof. It is very easy to treat the case $p > 0$, and we only consider the case $p = 0$ here. Suppose $r \geq (|\alpha| + c)t$. Then we get

$$\begin{aligned}\int_0^t \left(1 + \left|\alpha\tau - |r \pm ct|\right|\right)^{-1} d\tau &= \int_0^t (1 + r \pm ct - \alpha\tau)^{-1} d\tau \\ &= \frac{1}{\alpha} \log\left(\frac{1 + r \pm ct}{1 + r \pm ct - \alpha t}\right).\end{aligned} \tag{2.20}$$

(2.20) implies (2.19), because we have

$$(1 + |\alpha|t)^{-1} \leq \frac{1 + r \pm ct}{1 + r \pm ct - \alpha t} \leq 1 + |\alpha|t$$

for $r \geq (|\alpha| + c)t$.

On the other hand, if $r < (|\alpha| + c)t$, it is easy to see

$$\int_0^t \left(1 + \left|\alpha\tau - |r \pm ct|\right|\right)^{-1} d\tau \leq C \log(2 + t + r) \leq C \log(2 + t).$$

This completes the proof. □

For $c > 0$, $a \geq 0$, $\rho \geq 1$ and $\mu \geq 0$, we define

$$K^{\pm}_{c,a,\rho,\mu}(t, r) = \int_0^t \langle \tau + |\lambda^{\pm}_c(\tau; t, r)|\rangle^{-\rho} \langle a\tau - |\lambda^{\pm}_c(\tau; t, r)|\rangle^{-(1+\mu)} d\tau.$$

Lemma 2.5. *Let $c > 0$.*

(i) *For $\mu \geq 0$ and $\rho \geq 1$, we have*

$$K^{+}_{c,c,\rho,\mu}(t, r) \leq C\Psi_\mu(t)\langle ct + r\rangle^{-\rho}, \tag{2.21}$$

$$K^{-}_{c,c,\rho,\mu}(t, r) \leq C\Psi_\mu(t)\langle ct - r\rangle^{-\rho} + C\Phi_{\rho-1}(ct, r)\langle ct - r\rangle^{-(1+\mu)} \tag{2.22}$$

for $(t, r) \in [0, \infty) \times [0, \infty)$, where $\Phi_{\rho-1}$ and Ψ_μ are from (2.2) *and* (2.3), *respectively.*

(ii) *Let $a \geq 0$, and suppose $a \neq c$. Then, for $\mu \geq 0$ and $\rho > 0$, we have*

$$K^{\pm}_{c,a,\rho,\mu}(t, r) \leq C\langle ct \pm r\rangle^{-\rho}\Psi_\mu(t) \tag{2.23}$$

for $(t, r) \in [0, \infty) \times [0, \infty)$, where Ψ_μ is given by (2.3).

Proof. First we note that $K^{\pm}_{c,a,\rho,\mu}(t, r)$ is bounded by

$$C \int_0^t (1 + c\tau + |\lambda^{\pm}_c(\tau; t, r)|)^{-\rho} \left(1 + \left|a\tau - |\lambda^{\pm}_c(\tau; t, r)|\right|\right)^{-(1+\mu)} d\tau.$$

Since $c\tau + \lambda_c^+(\tau; t, r) = ct + r$, and $c\tau - \lambda_c^+(\tau; t, r) = 2c\tau - (r + ct)$, Lemma 2.4 implies

$$K^+_{c,c,\rho,\mu}(t, r) \le C\langle ct + r\rangle^{-\rho} \int_0^t (1 + |2c\tau - (r + ct)|)^{-(1+\mu)}\, d\tau$$
$$\le C\langle ct + r\rangle^{-\rho}\Psi_\mu(t).$$

Suppose $r < ct$. Observing that we have

$$|\lambda_c^-(\tau; t, r)| = \begin{cases} c\tau - (ct - r) & \text{if} \quad \tau \ge \dfrac{ct - r}{c}, \\ -c\tau + ct - r & \text{if} \quad \tau < \dfrac{ct - r}{c}, \end{cases}$$

we get

$$K^-_{c,c,\rho,\mu}(t, r) \le C\langle ct - r\rangle^{-\rho} \int_0^{t\wedge\{(ct-r)/c\}} (1 + |2c\tau - (ct - r)|)^{-(1+\mu)}\, d\tau$$
$$+ C\langle ct - r\rangle^{-(1+\mu)} \int_{t\wedge\{(ct-r)/c\}}^t (1 + 2c\tau - (ct - r))^{-\rho}\, d\tau,$$

where $\alpha \wedge \beta = \min\{\alpha, \beta\}$. By Lemma 2.4, we see that the first term on the right-hand side of the above is bounded by $C\langle ct - r\rangle^{-\rho}\Psi_\mu(t)$. We also see that the second term is bounded by $C\langle ct - r\rangle^{-(1+\mu)}\Phi_{\rho-1}(ct, r)$, since we have

$$\int_{(ct-r)/c}^t (1 + 2c\tau - (ct - r))^{-1}\, d\tau = \frac{1}{2c} \log \frac{1 + ct + r}{1 + ct - r}$$

and

$$\int_{(ct-r)/c}^t (1 + 2c\tau - (ct - r))^{-\rho}\, d\tau \le \frac{1}{2c(\rho - 1)}(1 + ct - r)^{-\rho+1}$$

for $\rho > 1$, provided that $(ct - r)/c < t$.

Now suppose $r > ct$. Then we have

$$K^-_{c,c,\rho,\mu}(t, r) \le C\langle ct - r\rangle^{-(1+\mu)} \int_0^t (1 + 2c\tau + (r - ct))^{-\rho}\, d\tau,$$

and a similar argument to the above leads to

$$K^-_{c,c,\rho,\mu}(t, r) \le C\langle ct - r\rangle^{-(1+\mu)}\Phi_{\rho-1}(ct, r).$$

Finally we are going to prove (2.23). Let $a \geq 0$ and $a \neq c$. Since we have $c\tau + |\lambda_c^{\pm}(\tau; t, r)| \geq C|ct \pm r|$ for $\tau \geq 0$, $a\tau - \lambda_c^{+}(\tau; t, r) = (a + c)\tau - (r + ct)$, and

$$a\tau - |\lambda_c^{-}(\tau; t, r)| = \begin{cases} (a - c)\tau - (r - ct) & \text{if} \quad \tau \geq \dfrac{ct - r}{c}, \\ (a + c)\tau - (ct - r) & \text{if} \quad \tau < \dfrac{ct - r}{c}, \end{cases}$$

Lemma 2.4 implies

$$\begin{aligned} K_{c,a,\rho,\mu}^{\pm}(t, r) &\leq C\langle ct \pm r\rangle^{-\rho} \int_0^t \left(1 + \left|a\tau - |\lambda_c^{\pm}(\tau; t, r)|\right|\right)^{-(1+\mu)} d\tau \\ &\leq C\langle ct \pm r\rangle^{-\rho} \Psi_\mu(t). \end{aligned}$$

This completes the proof. □

Let $c_1, \ldots, c_m$ be given positive constants, and $c_0 = 0$ as before. For $i \in \{1, 2, \ldots, m\}$, we define

$$w_i(t, r) = \min_{\substack{0 \leq j \leq m \\ j \neq i}} \langle c_j t - r\rangle.$$

Lemma 2.6. *Let $i \in \{1, \ldots, m\}$. Suppose $G \in C^1([0, \infty) \times [0, \infty))$, and let v be a classical solution to*

$$\Box_{c_i} v(t, x) = G(t, |x|) \quad \textit{for} \quad (t, x) \in (0, \infty) \times \mathbb{R}^3$$

with $v = \partial_t v = 0$ at $t = 0$. Set

$$\mathcal{D}[v](t, x) = \langle r\rangle \sum_{|\alpha|=1} |\partial_{t,r}^{\alpha} v(t, x)| + r \sum_{|\alpha|=2} |\partial_{t,r}^{\alpha} v(t, x)|,$$

where $r = |x|$, $\partial_r = \sum_{j=1}^{3}(x_j/|x|)\partial_j$, and $\partial_{t,r}^{\alpha}$ denotes $\partial_t^{\alpha_1}\partial_r^{\alpha_2}$ for a multi-index $\alpha = (\alpha_1, \alpha_2)$. We define $D_{+,c} = \partial_t + c\partial_r$ for $c > 0$, and

$$\mathcal{D}_i^{+}[v](t, x) = \langle r\rangle |D_{+,c_i} v(t, x)| + r \sum_{|\alpha|=1} |\partial_{t,r}^{\alpha} D_{+,c_i} v(t, x)|.$$

We also set

$$\mathcal{M}[G](t, r) = \langle r\rangle |G(t, r)| + r \sum_{|\alpha|=1} |\partial_{t,r}^{\alpha} G(t, r)|.$$

Then we have the following estimates:

(i) *For* $\rho \geq 1$ *and* $\mu \geq 0$, *we have*

$$(\Psi_\mu(t)\langle c_i t - r\rangle^{-\rho} + \Phi_{\rho-1}(c_i t, r)\langle c_i t - r\rangle^{-(1+\mu)})^{-1}\mathcal{D}[v](t, x) \leq C \sup_{(\tau,\lambda)\in\Theta_i^*(t,r)} \langle \tau + \lambda\rangle^\rho w(\tau, \lambda)^{1+\mu}\mathcal{M}[G](\tau, \lambda).$$

(ii) *For* $0 < \rho \leq 1$ *and* $\mu > 0$, *we have*

$$\langle c_i t - r\rangle^\rho \mathcal{D}[v](t, x) \leq C \sup_{(\tau,\lambda)\in\Theta_i^*(t,r)} \langle \tau + \lambda\rangle^\rho w_i(\tau, \lambda)^{1+\mu}\mathcal{M}[G](\tau, \lambda).$$

(iii) *For* $\rho \geq 1$ *and* $\mu \geq 0$, *we have*

$$\langle t + r\rangle \Phi_{\rho-1}(c_i t, r)^{-1}\mathcal{D}_i^+[v](t, x) \leq C\Psi_\mu(t) \sup_{(\tau,\lambda)\in\Theta_i^*(t,r)} \langle \tau + \lambda\rangle^\rho w(\tau, \lambda)^{1+\mu}\mathcal{M}[G](\tau, \lambda).$$

Proof. For $\rho > 0$, Lemma 2.1 implies

$$\begin{aligned}\langle t + r\rangle\langle c_i t - r\rangle^\rho |U_{c_i}^*[0, G(0, |\cdot|)](t, x)| &\leq C\|G(0, |\cdot|)\|_{2+\rho, c_i t, r} \\ &\leq C \sup_{(\tau,\lambda)\in\Theta_i^*(t,r)} \langle \tau + \lambda\rangle^\rho w(\tau, \lambda)\mathcal{M}[G](\tau, \lambda),\end{aligned} \tag{2.24}$$

since we have $\langle\lambda\rangle^{2+\rho} = \langle 0 + \lambda\rangle^\rho w(0, \lambda)\langle\lambda\rangle$. For $\rho > 0$, $\mu \geq 0$, and $0 \leq a \leq 3$, Lemma 2.2 leads to

$$\begin{aligned}&\langle t + r\rangle \Phi_{\rho-1}(c_i t, r)^{-1}|U_{c_i}[\partial_a \Box_{c_i} v](t, x)| \\ &\leq C\Psi_\mu(t) \sup_{(\tau,\lambda)\in\Theta_i^*(t,r)} \langle \tau + \lambda\rangle^\rho w(\tau, \lambda)^{1+\mu}\lambda|(\partial_a G)(\tau, \lambda)|.\end{aligned} \tag{2.25}$$

Since we have $\partial_a v = U_{c_i}[\partial_a \Box_{c_i} v] + \delta_{a0}U_{c_i}^*[0, G(0, |\cdot|)]$ with the Kronecker delta δ_{ab}, and $\langle t + r\rangle^{-1}\Phi_{\rho-1}(c_i t, r) \leq C\langle c_i t - r\rangle^{-\rho}$ for $\rho > 0$, from Lemma 2.2, (2.24) and (2.25) we get

$$\begin{aligned}&\langle c_i t - r\rangle^\rho(|v(t, x)| + |\partial_t v(t, x)| + |\partial_r v(t, x)|) \\ &\leq C\Psi_\mu(t) \sup_{(\tau,\lambda)\in\Theta_i^*(t,r)} \langle \tau + \lambda\rangle^\rho w(\tau, \lambda)^{1+\mu}\mathcal{M}[G](\tau, \lambda)\end{aligned} \tag{2.26}$$

for $\rho > 0$ and $\mu \geq 0$.

For $\rho > 0$, it is easy to see

$$\langle t + r\rangle^\rho r|G(t, r)| \leq C \sup_{(\tau,\lambda)\in\Theta_i^*(t,r)} \langle \tau + \lambda\rangle^\rho \mathcal{M}[G](\tau, \lambda). \tag{2.27}$$

Now we set

$$\tilde{\mathcal{D}}[v](t,x) = |\partial_t(rv(t,x))| + |\partial_r(rv(t,x))| + |\partial_r\partial_t(rv(t,x))| + |\partial_t^2(rv(t,x)) - rG(t,r)| + |\partial_r^2(rv(t,x))|.$$

Since we have

$$\mathcal{D}[v](t,x) \le C\left(\tilde{\mathcal{D}}[v](t,x) + \sum_{|\alpha|\le 1} |\partial_{t,r}^\alpha v(t,x)| + r|G(t,r)|\right),$$

in view of (2.26) and (2.27) we only have to prove (i) and (ii) with $\mathcal{D}[v]$ replaced by $\tilde{\mathcal{D}}[v]$. As we have mentioned before, we have $rv(t,x) = L_{c_i}[G](t,r)$. Therefore, from (2.14), (2.15) and (2.16) we get

$$\tilde{\mathcal{D}}[v](t,x) \le C \sum_{s=+,-} (|I_{c_i}^s[\check{G}](t,r)| + |I_{c_i}^s[\partial_r \check{G}](t,r)|), \tag{2.28}$$

and we find

$$\tilde{\mathcal{D}}[v](t,x) \le C \sum_{s=+,-} \int_0^t \mathcal{W}(\tau, |\lambda_{c_i}^s(\tau;t,r)|)^{-1}\,d\tau \times \sup_{(\tau,\lambda)\in\Theta_i^*(t,r)} \mathcal{W}(\tau,\lambda)\mathcal{M}[G](\tau,\lambda) \tag{2.29}$$

for any positive function $\mathcal{W} = \mathcal{W}(\tau,\lambda)$.

We use (2.29) with

$$\mathcal{W}(\tau,\lambda) = \langle \tau+\lambda\rangle^\rho w(\tau,\lambda)^{1+\mu} \quad \text{and} \quad \mathcal{W}(\tau,\lambda) = \langle \tau+\lambda\rangle^\rho w_i(\tau,\lambda)^{1+\mu}$$

to obtain (i) and (ii), respectively. Noting that we have

$$w(\tau,\lambda)^{-(1+\mu)} \le \sum_{0\le j\le m} \langle c_j\tau - \lambda\rangle^{-(1+\mu)},$$
$$w_i(\tau,\lambda)^{-(1+\mu)} \le \sum_{\substack{0\le j\le m\\ j\ne i}} \langle c_j\tau - \lambda\rangle^{-(1+\mu)}$$

for $\mu \ge 0$, and using Lemma 2.5 to estimate $\int_0^t \mathcal{W}(\tau, |\lambda_{c_i}^\pm(\tau;t,r)|)^{-1}\,d\tau$, we obtain (i) and (ii) with $\mathcal{D}[v]$ replaced by $\tilde{\mathcal{D}}[v]$.

Concerning (iii), our task is to estimate $\sum_{|\alpha|\le 1} |r\partial_{t,r}^\alpha D_{+,c_i} v(t,x)|$, because (2.24) and (2.25) imply the desired bound for $|D_{+,c_i} v(t,x)|$.

We have

$$r\partial_t^j D_{+,c_i} v(t,x) = \partial_t^j D_{+,c_i}(rv(t,x)) - c_i\partial_t^j v(t,x) \quad \text{for} \quad j = 0, 1,$$
$$r\partial_r D_{+,c_i} v(t,x) = \partial_r D_{+,c_i}(rv(t,x)) - D_{+,c_i} v(t,x) - c_i\partial_r v(t,x),$$

and by (2.14)–(2.16) we also have

$$\sum_{|\alpha|\le 1} |\partial_{t,r}^{\alpha} D_{+,c_i}(rv(t,x))| \le C(I_{c_i}^{+}[\check{G}] + I_{c_i}^{+}[\partial_r \check{G}]) + C|\check{G}(t,r)|.$$

Hence we obtain the desired estimate for $\sum_{|\alpha|\le 1} r|\partial_{t,r}^{\alpha} D_{+,c_i} v(t,x)|$ from Lemmas 2.2 and 2.5 together with (2.24), (2.25) and (2.27) (note that we have $\langle t+r\rangle \Phi_{\rho-1}^{-1}(t,r) \le \langle t+r\rangle^{\rho}$ for $\rho \ge 1$). This completes the proof. □

From the proof of Lemma 2.6, with using Corollary 2.3 in place of Lemma 2.2 and choosing $\mathcal{W}$ as in Corollary 2.3, we also have

Corollary 2.7. *Let v and G be as in* Lemma 2.6, *and*

$$\mathcal{W}(t,r)^{-1} = A_1 \langle t+r\rangle^{-1} w(t,r)^{-1} + A_2 \langle t+r\rangle^{-1} w(t,r)^{-2}$$

with some positive constants A_1 and A_2. Then we have

$$\begin{aligned}
&(\langle c_i t - r\rangle^{-1} + \Phi_0(c_i t, r)\langle c_i t - r\rangle^{-2})^{-1} \mathcal{D}[v](t,x) \\
&+ \langle t+r\rangle \Phi_0(c_i t, r)^{-1} \mathcal{D}_i^{+}[v](t,x) \\
&\le C(A_1 \log(2+t) + A_2) \sup_{(\tau,\lambda)\in \Theta_i^{*}(t,r)} \mathcal{W}(\tau,\lambda) \mathcal{M}[G](\tau,\lambda),
\end{aligned}$$

where $r = |x|$.

We conclude this section with a decay estimate for $\mathcal{D}_i^{+}[U_{c_i}^{*}]$.

Lemma 2.8. *Let $i \in \{1, \ldots, m\}$, $\kappa > 0$, and $v = U_{c_i}^{*}[\phi, \psi]$. Suppose that ϕ and ψ are radially symmetric functions. Then we have*

$$\begin{aligned}
&\langle t+|x|\rangle \langle c_i t - |x|\rangle^{\kappa} \mathcal{D}_i^{+}[v](t,x) \\
&\le C\left(\sum_{0\le k\le 2} \|\partial_r^k \phi\|_{2+\kappa, c_i t, |x|} + \sum_{0\le k\le 1} \|\partial_r^k \psi\|_{2+\kappa, c_i t, |x|} \right).
\end{aligned}$$

Proof. Since ϕ and ψ are radially symmetric, we see that v also is radially symmetric. Set $w(t,r) = rv(t,(|r|,0,0))$, $\check{\phi}(r) = r\phi(|r|,0,0)$, and $\check{\psi}(r) = r\psi(|r|,0,0)$. Then we get $(\partial_t^2 - c_i^2 \partial_r^2) w(t,r) = 0$ for $(t,r) \in [0,\infty) \times \mathbb{R}$, with $w = \check{\phi}$ and $\partial_t w = \check{\psi}$ at $t = 0$. It is easy to check that we have

$$\mathcal{D}_i^{+}[v](t,x) \le C \sum_{|\alpha|\le 1} |(D_{+,c_i} \partial_{t,r}^{\alpha} w)(t,|x|)| + \sum_{|\alpha|\le 1} |\partial_{t,r}^{\alpha} v(t,x)|.$$

By Lemma 2.1, we obtain

$$
\begin{aligned}
&\langle t+|x|\rangle\langle c_i t-|x|\rangle^{\kappa}\sum_{|\alpha|\le 1}|\partial_{t,r}^{\alpha}v(t,x)|\\
&\le C\left(\sum_{0\le k\le 2}\|\partial_r^k\phi\|_{2+\kappa,c_i t,r}+\sum_{0\le k\le 1}\|\partial_r^k\psi\|_{2+\kappa,c_i t,r}\right).
\end{aligned}
\tag{2.30}
$$

Since $(\partial_t - c_i\partial_r)(D_{+,c_i}\partial_{t,r}^{\alpha}w)(t,r)=0$, we get

$$
D_{+,c_i}\partial_{t,r}^{\alpha}w(t,r)=(D_{+,c_i}\partial_{t,r}^{\alpha}w)(0,c_i t+r).
\tag{2.31}
$$

Now it is easy to see

$$
\begin{aligned}
&\langle t+r\rangle^{1+\kappa}\sum_{|\alpha|\le 1}|(D_{+,c_i}\partial_{t,r}^{\alpha}w)(0,c_i t+r)|\\
&\le\sum_{|\alpha|\le 1}|\langle c_i t+r\rangle^{1+\kappa}(D_{+,c_i}\partial_{t,r}^{\alpha}w)(0,c_i t+r)|\\
&\le C\left(\sum_{0\le k\le 2}\|\partial_r^k\phi\|_{2+\kappa,c_i t,r}+\sum_{0\le k\le 1}\|\partial_r^k\psi\|_{2+\kappa,c_i t,r}\right).
\end{aligned}
\tag{2.32}
$$

(2.30), (2.31), and (2.32) imply the desired result. □

3. Proof of Theorem 1.1

For brevity, when $v=v(t,x)$ is radially symmetric, we write sometimes $v=v(t,x)$ and sometimes $v=v(t,r)$ with $r=|x|$ in the following. In other words, if there exists $\tilde v=\tilde v(t,r)$ such that $v(t,x)=\tilde v(t,|x|)$, we do not distinguish v from $\tilde v$.

Suppose that all the assumptions in Theorem 1.1 are fulfilled. Let $(u^{(k)})_{1\le k\le 3}=((u_i^{(k)})_{1\le i\le m})_{1\le k\le 3}$ be a solution to

$$
\begin{aligned}
\Box_{c_i}u_i^{(1)}=&\,N_i(\partial u^{(1)},\partial u^{(1)})+r_i^{\mathrm{I}}(\partial u^{(1)},\partial u^{(1)})+r_i^{\mathrm{III}}(u^{(1)},\partial_t u^{(1)})\\
&+H_i(u^{(1)},\partial u^{(1)}),
\end{aligned}
\tag{3.1}
$$

$$
\begin{aligned}
\Box_{c_i}u_i^{(2)}=&\,2N_i(\partial u^{(1)},\partial u^{(2)+(3)})+2r_i^{\mathrm{I}}(\partial u^{(1)},\partial u^{(2)+(3)})\\
&+N_i(\partial u^{(2)+(3)},\partial u^{(2)+(3)})+r_i^{\mathrm{I}}(\partial u^{(2)+(3)},\partial u^{(2)+(3)})\\
&+r_i^{\mathrm{II}}(\partial u^{(1)+(2)+(3)},\partial u^{(1)+(2)+(3)})\\
&+r_i^{\mathrm{III}}(u^{(1)+(2)+(3)},\partial_t u^{(2)})\\
&+H_i(u^{(1)+(2)+(3)},\partial u^{(1)+(2)+(3)})-H_i(u^{(1)},\partial u^{(1)}),
\end{aligned}
\tag{3.2}
$$

$$
\Box_{c_i}u_i^{(3)}=r_i^{\mathrm{III}}(u^{(2)+(3)},\partial_t u^{(1)})+r_i^{\mathrm{III}}(u^{(1)+(2)+(3)},\partial_t u^{(3)})
\tag{3.3}
$$

for $1 \le i \le m$, with initial data

$$u^{(1)} = \varepsilon f, \quad \partial_t u^{(1)} = \varepsilon g, \quad u^{(2)} = \partial_t u^{(2)} = u^{(3)} = \partial_t u^{(3)} = 0 \tag{3.4}$$

at $t = 0$, where $u^{(j)+(k)} = u^{(j)} + u^{(k)}$ for $1 \le j, k \le 3$, and $u^{(1)+(2)+(3)} = u^{(1)} + u^{(2)} + u^{(3)}$. Since f and g are radially symmetric, and nonlinearity is rotationally invariant, we see that $u^{(k)}$ $(k = 1, 2, 3)$ are radial functions. Note that we have $(\nabla_x v) \cdot (\nabla_x w) = (\partial_r v)(\partial_r w)$ for radial functions v and w. Set $u = u^{(1)} + u^{(2)} + u^{(3)}$, and we find that u satisfies (1.1)–(1.2). Hence our task is to solve the Cauchy problem (3.1)–(3.3) with (3.4).

From the classical local existence theorem, the Cauchy problem (3.1)–(3.4) admits a unique solution $(u^{(k)})_{1\le k\le 3}$ for $0 \le t < T$ with some $T > 0$. Let T_ε be the supremum of such T.

We define

$$\begin{aligned} e_1[u^{(1)}](t, r) = \sum_{i=1}^{m} \{ &\langle t + r\rangle \langle c_i t - r\rangle (|u_i^{(1)}(t, r)| + \mathcal{D}_i^+[u_i^{(1)}](t, r)) \\ &+ \langle c_i t - r\rangle^2 \mathcal{D}[u_i^{(1)}](t, r)\}, \\ e_2[u^{(2)}](t, r) = \sum_{i=1}^{m} \{ &\langle t + r\rangle \Phi_0(c_i t, r)^{-1} (|u_i^{(2)}(t, r)| + \mathcal{D}_i^+[u_i^{(2)}](t, r)) \\ &+ \langle c_i t - r\rangle \mathcal{D}[u_i^{(2)}](t, r)\}, \\ e_3[u^{(3)}](t, r) = \sum_{i=1}^{m} \{ &\langle t + r\rangle \Phi_0(c_i t, r)^{-1} (|u_i^{(3)}(t, r)| + \mathcal{D}_i^+[u_i^{(3)}](t, r)) \\ &+ (\langle c_i t - r\rangle^{-1} + \langle c_i t - r\rangle^{-2} \Phi_0(c_i t, r))^{-1} \mathcal{D}[u_i^{(3)}](t, r)\}, \end{aligned}$$

where $\Phi_0(c_i t, r)$ is given by (2.2), and $\mathcal{D}[v](t, r)$ and $\mathcal{D}_i^+[v](t, r)$ are from Lemma 2.6. We also define

$$E_k(T) = \sup_{(t,r)\in[0,T)\times[0,\infty)} e_k[u^{(k)}](t, r) \quad (k = 1, 2, 3)$$

for $0 < T \le T_\varepsilon$, and $E_k(0) = \sup_{r\in[0,\infty)} e_k[u^{(k)}](0, r)$.

Proposition 3.1. *Assume* $0 < T < T_\varepsilon$, *and let* M_k $(k = 1, 2, 3)$ *be positive constants. Suppose that* ε *is a positive constant satisfying*

$$M_3\varepsilon^3 \le M_2\varepsilon^2 \le M_1\varepsilon \le 1$$

and $\varepsilon \le 1$. *Then there exist three positive constants* C_1, C_2 *and* C_3 (*which are in-*

dependent of T, T_ε, ε, *and* M_k ($k = 1, 2, 3$)) *such that*

$$E_k(T) \le M_k \varepsilon^k \quad (k = 1, 2, 3) \tag{3.5}$$

implies

$$E_1(T) \le C_1(\varepsilon + M_1^2 \varepsilon^2), \tag{3.6}$$

$$E_2(T) \le C_2(M_1^2\varepsilon^2 + (M_1^2 + M_2)M_2\varepsilon^4 \log(2+T) + M_3^2\varepsilon^6(\log(2+T))^2), \tag{3.7}$$

$$E_3(T) \le C_3(M_1M_2\varepsilon^3 + M_2M_3\varepsilon^5 \log(2+T)). \tag{3.8}$$

Before proving Proposition 3.1, we show that Theorem 1.1 follows from it.

Proof of Theorem 1.1. Set

$$\begin{aligned} &M_1 = \max\left\{\frac{2E_1(0)}{\varepsilon}, 4C_1\right\}, \quad M_2 = \max\left\{\frac{2E_2(0)}{\varepsilon^2}, 6C_2M_1^2\right\}, \\ &M_3 = \max\left\{\frac{2E_3(0)}{\varepsilon^3}, 4C_3M_1M_2\right\}. \end{aligned} \tag{3.9}$$

Choose a positive constant ε_1 (≤ 1) which is small enough to satisfy

$$M_1\varepsilon_1 \le \frac{1}{4C_1} \quad \text{and} \quad M_3\varepsilon_1^3 \le M_2\varepsilon_1^2 \le M_1\varepsilon_1 \le 1. \tag{3.10}$$

Let $0 < \varepsilon \le \varepsilon_1$, and assume $\varepsilon^2 \log(2 + T_\varepsilon) \le C_4$, where

$$C_4 = \min\left\{\frac{1}{6C_2(M_1^2 + M_2)}, \sqrt{\frac{M_2}{6C_2M_3^2}}, \frac{1}{4C_3M_2}\right\}. \tag{3.11}$$

Define

$$T = \sup\{S \in [0, T_\varepsilon]; E_k(S) \le M_k\varepsilon^k \ (k = 1, 2, 3)\}.$$

Since (3.9) implies $E_k(0) \le M_k\varepsilon^k/2$ ($k = 1, 2, 3$), the continuity of E_k implies $E_k(S) \le M_k\varepsilon^k$ ($k = 1, 2, 3$) for some $S \in (0, T_\varepsilon]$. Hence T is positive.

Now suppose $T < T_\varepsilon$. Then, Proposition 3.1, (3.9), (3.10), and (3.11) lead to $E_k(T) \le M_k\varepsilon^k/2$ ($k = 1, 2, 3$), which contradict the definition of T because E_k ($k = 1, 2, 3$) are continuous functions. Hence we conclude $T = T_\varepsilon$, and $E_k(T_\varepsilon) \le M_k\varepsilon^k$ for $1 \le k \le 3$.

From these *a priori* estimates, we see that $\sum_{|\alpha|\le 1}\sum_{k=1}^3 |\partial_{t,r}^\alpha u^{(k)}(t, r)|$ is bounded for $(t, r) \in [0, T_\varepsilon) \times [0, \infty)$. Then, from the system (3.1)–(3.3), it is easy to show that $\sum_{|\alpha|=2}\sum_{k=1}^3 |\partial_{t,r}^\alpha u^{(k)}(t, r)|$ is also bounded for $(t, r) \in [0, T_\varepsilon) \times [0, \infty)$. Now the classical local existence theorem assures that we can extend the solution $(u^{(k)})_{1\le k\le 3}$ beyond

T_ε. This contradicts the definition of T_ε. Accordingly we find $\varepsilon^2 \log(2+T_\varepsilon) > C_4$ for $0 < \varepsilon \le \varepsilon_1$, which immediately implies $T_\varepsilon \ge \exp(C_5\varepsilon^{-2})$ for $\varepsilon \le \varepsilon_0$ with appropriately chosen positive constants C_5 and ε_0. This completes the proof. □

Now we are going to prove Proposition 3.1.

Proof of Proposition 3.1. Let ϕ and ψ are radially symmetric C^2-functions. First we observe that we have

$$\mathcal{M}[\phi(\partial_{t,r}^{\gamma}\psi)](t,r) \le C(|\phi(t,r)| + \langle r\rangle^{-1}\mathcal{D}[\phi](t,r))\mathcal{D}[\psi](t,r), \tag{3.12}$$

$$\mathcal{M}[(\partial_{t,r}^{\beta}\phi)(\partial_{t,r}^{\gamma}\psi)](t,r) \le C\langle r\rangle^{-1}\mathcal{D}[\phi](t,r)\mathcal{D}[\psi](t,r) \tag{3.13}$$

for $|\beta| = |\gamma| = 1$, where $\mathcal{M}$ is defined as in Lemma 2.6. Since

$$2Q_0(\phi,\psi;c_i) = (\partial_t\phi - c_i\partial_r\phi)(D_{+,c_i}\psi) + (\partial_t\psi - c_i\partial_r\psi)(D_{+,c_i}\phi)$$

for any radially symmetric functions ϕ and ψ, we also obtain

$$\begin{aligned}&\mathcal{M}[Q_0(\phi,\psi;c_i)](t,r)\\ &\le C\langle r\rangle^{-1}(\mathcal{D}[\phi](t,r)\mathcal{D}_i^{+}[\psi](t,r) + \mathcal{D}[\psi](t,r)\mathcal{D}_i^{+}[\phi](t,r)).\end{aligned} \tag{3.14}$$

For $0 \le j \le m$, we define

$$\Lambda_j = \{(t,r) \in [0,\infty)\times[0,\infty);\, |c_jt - r| < \delta t\}$$

with some small $\delta > 0$, where $c_0 = 0$ as before. Note that $\langle c_jt - r\rangle$ is equivalent to $\langle t+r\rangle$ outside Λ_j. If δ is chosen sufficiently small, then there is no intersection between Λ_j and Λ_k for $j \ne k$. Hence we have

$$\langle c_jt - r\rangle^{-1}\langle c_kt - r\rangle^{-1} \le C\langle t+r\rangle^{-1}w(t,r)^{-1} \tag{3.15}$$

for $j \ne k$. Moreover, if we have $j \ne i$ and $k \ne i$ in addition, we get

$$\langle c_jt - r\rangle^{-1}\langle c_kt - r\rangle^{-1} \le C\langle t+r\rangle^{-1}w_i(t,r)^{-1}. \tag{3.16}$$

Similarly, for $\kappa \ge 0$ and $j \ne k$, we have

$$\langle c_jt - r\rangle^{-\kappa}\Phi_0(c_kt,r) \le C(\langle c_jt - r\rangle^{-\kappa} + \langle t+r\rangle^{-\kappa}\Phi_0(c_kt,r)), \tag{3.17}$$

$$\Phi_0(c_jt,r)\Phi_0(c_kt,r) \le C(\Phi_0(c_jt,r) + \Phi_0(c_kt,r)). \tag{3.18}$$

From (3.15), we especially have $\langle r\rangle\langle c_jt - r\rangle^{-1} \le C\langle t+r\rangle^{-1}$ for $1 \le j \le m$, and we obtain

$$|u_j^{(1)}(t,r)| + \langle r\rangle^{-1}\mathcal{D}[u_j^{(1)}](t,r) \le C\langle t+r\rangle^{-1}\langle c_jt - r\rangle^{-1}M_1\varepsilon, \tag{3.19}$$

$$|u_j^{(k)}(t,r)| + \langle r\rangle^{-1}\mathcal{D}[u_j^{(k)}](t,r) \le C\langle t+r\rangle^{-1}\Phi_0(c_jt,r)M_k\varepsilon^k \tag{3.20}$$

for $k = 2, 3$. Having these estimates in mind, we are going to evaluate each nonlinearity.

First we estimate nonlinear terms contained in (3.1). We have

$$\begin{aligned}\mathcal{M}[N_i(\partial u^{(1)}, \partial u^{(1)})](t, r) &\le C\sum_j \langle r\rangle^{-1}\langle t+r\rangle^{-1}\langle c_j t - r\rangle^{-3} M_1^2\varepsilon^2\\ &\le C\langle t+r\rangle^{-2} w(t, r)^{-3} M_1^2\varepsilon^2,\\ \mathcal{M}[r_i^{\mathrm{I}}(\partial u^{(1)}, \partial u^{(1)})](t, r) &\le C\sum_{j\ne k}\langle r\rangle^{-1}\langle c_j t - r\rangle^{-2}\langle c_k t - r\rangle^{-2} M_1^2\varepsilon^2\\ &\le C\langle t+r\rangle^{-3} w(t, r)^{-2} M_1^2\varepsilon^2,\\ \mathcal{M}[r_i^{\mathrm{III}}(u^{(1)}, \partial_t u^{(1)})](t, r) &\le C\sum_{j\ne k}\langle t+r\rangle^{-1}\langle c_j t - r\rangle^{-1}\langle c_k t - r\rangle^{-2} M_1^2\varepsilon^2\\ &\le C\langle t+r\rangle^{-2} w(t, r)^{-2} M_1^2\varepsilon^2.\end{aligned}$$

On the other hand, since H_i is a rotationally invariant function of cubic order, we have

$$\begin{aligned}\mathcal{M}[H_i(u^{(1)}, \partial u^{(1)})](t, r) &\le C\{\langle r\rangle|u^{(1)}|^3 + (|u^{(1)}|^2 + |\partial_{t,r}u^{(1)}|^2)\mathcal{D}[u^{(1)}]\}\\ &\le C\langle t+r\rangle^{-2} w(t, r)^{-3} M_1^3\varepsilon^3.\end{aligned}$$

Summing up, we get

$$\mathcal{M}[\Box_{c_i} u_i^{(1)}](t, r) \le C\langle t+r\rangle^{-2} w(t, r)^{-2} M_1^2\varepsilon^2. \tag{3.21}$$

Hence Lemmas 2.1, 2.8, 2.2, and Lemma 2.6-(i), (iii) with $(\rho, \mu) = (2, 1)$ lead to

$$E_1(T) \le C_1(\varepsilon + M_1^2\varepsilon^2). \tag{3.22}$$

Next we turn our attention to (3.2). Let ν be a positive and small constant. Then we have

$$\Phi_0(c_i t, r) \le C\langle t+r\rangle^{\nu}\langle c_i t - r\rangle^{-\nu}. \tag{3.23}$$

Using this inequality, we start with

$$\begin{aligned}&\mathcal{M}[N_i(\partial u^{(2)+(3)}, \partial u^{(2)+(3)})](t, r)\\ &\le C\sum_j \langle r\rangle^{-1}\langle t+r\rangle^{-1}\langle c_j t - r\rangle^{-1}\Phi_0(c_j t, r)^2 (M_2\varepsilon^2 + M_3\varepsilon^3)^2\\ &\le C\langle t+r\rangle^{-2+2\nu} w(t, r)^{-1-2\nu} M_2^2\varepsilon^4,\\ &\mathcal{M}[r_i^{\mathrm{I}}(\partial u^{(2)+(3)}, \partial u^{(2)+(3)})](t, r)\\ &\le C\sum_{j\ne k}\langle r\rangle^{-1}\langle c_j t - r\rangle^{-1}\langle c_k t - r\rangle^{-1}\Phi_0(c_j t, r)\Phi_0(c_k t, r)(M_2\varepsilon^2 + M_3\varepsilon^3)^2\\ &\le C\langle t+r\rangle^{-2+\nu} w(t, r)^{-1-\nu} M_2^2\varepsilon^4.\end{aligned}$$

Since $\partial u^{(1)}$ enjoys a better estimate than $\partial u^{(2)+(3)}$, it is easy to obtain

$$\mathcal{M}[N_i(\partial u^{(1)}, \partial u^{(2)+(3)})](t,r) \le C\langle t+r\rangle^{-2+\nu} w(t,r)^{-2-\nu} M_1 M_2 \varepsilon^3,$$
$$\mathcal{M}[r_i^{\mathrm{I}}(\partial u^{(1)}, \partial u^{(2)+(3)})](t,r) \le C\langle t+r\rangle^{-2+\nu} w(t,r)^{-2-\nu} M_1 M_2 \varepsilon^3.$$

Now we are proceeding to rather delicate parts. For simplicity of exposition, we set $u = u^{(1)+(2)+(3)}$. Then we have

$$\begin{aligned}
&\mathcal{M}[r_i^{\mathrm{II}}(\partial u, \partial u)](t,r)\\
&\le C\sum_{j\ne i}\langle r\rangle^{-1}\{\langle c_j t - r\rangle^{-4} M_1^2\varepsilon^2 + \langle c_j t - r\rangle^{-2}(M_2^2\varepsilon^4 + M_3^2\varepsilon^6)\\
&\qquad\qquad + \langle c_j t - r\rangle^{-4}\Phi_0(c_j t, r)^2 M_3^2\varepsilon^6\}\\
&\le C\langle t+r\rangle^{-1} w_i(t,r)^{-2}(M_1^2\varepsilon^2 + M_3^2\varepsilon^6\{\log(2+T)\}^2)
\end{aligned}$$

for $0 \le t < T$. Here we have used $\Phi_0(c_j t, r) \le C\log(2+t)$. We also get

$$\begin{aligned}
&\mathcal{M}[r_i^{\mathrm{III}}(u^{(1)}, \partial_t u^{(2)})](t,r)\\
&\le C\sum_{j\ne k}\langle t+r\rangle^{-1}\langle c_j t - r\rangle^{-1}\langle c_k t - r\rangle^{-1} M_1 M_2\varepsilon^3\\
&\le C\langle t+r\rangle^{-2} w(t,r)^{-1} M_1 M_2 \varepsilon^3 \le C\langle t+r\rangle^{-2+\nu} w(t,r)^{-1-\nu} M_1 M_2\varepsilon^3,\\
&\mathcal{M}[r_i^{\mathrm{III}}(u^{(2)+(3)}, \partial_t u^{(2)})](t,r)\\
&\le C\sum_{j\ne k}\langle t+r\rangle^{-1}\Phi_0(c_j t, r)\langle c_k t - r\rangle^{-1}(M_2\varepsilon^2 + M_3\varepsilon^3)M_2\varepsilon^2\\
&\le C(\langle t+r\rangle^{-1} w(t,r)^{-1} + \langle t+r\rangle^{-2+\nu} w(t,r)^{-\nu})M_2^2\varepsilon^4\\
&\le C\langle t+r\rangle^{-1} w(t,r)^{-1} M_2^2\varepsilon^4.
\end{aligned}$$

Setting $\tilde{H}_i = H_i(u, \partial u) - H_i(u^{(1)}, \partial u^{(1)})$, we obtain

$$\begin{aligned}
\mathcal{M}[\tilde{H}_i](t,r) &\le C\left\{\sum_{k=1}^{3}(|u^{(k)}| + \langle r\rangle^{-1}\mathcal{D}[u^{(k)}])\right\}^2 \sum_{k=2}^{3}(\langle r\rangle|u^{(k)}| + \mathcal{D}[u^{(k)}])\\
&\le C\langle t+r\rangle^{-2+3\nu} w(t,r)^{-3\nu} M_1^2 M_2\varepsilon^4\\
&\le C\langle t+r\rangle^{-1} w(t,r)^{-1} M_1^2 M_2 \varepsilon^4.
\end{aligned}$$

Now we set

$$G_{i,1} = r_i^{\mathrm{II}}(\partial u, \partial u), \quad G_{i,2} = r_i^{\mathrm{III}}(u^{(2)+(3)}, \partial_t u^{(2)}) + \tilde{H}_i,$$
$$G_{i,3} = \Box_{c_i} u_i^{(2)} - G_{i,1} - G_{i,2}.$$

Since we have shown

$$\mathcal{M}[G_{i,3}](t, r) \leq C\langle t + r\rangle^{-2+2\nu} w(t, r)^{-1-\nu} M_1 M_2 \varepsilon^3,$$

from Lemma 2.2 and Lemma 2.6-(i), (iii) with $(\rho, \mu) = (2 - 2\nu, \nu)$ (note that we may assume $2 - 2\nu > 1$), we get

$$e_2[U_{c_i}[G_{i,3}]](t, r) \leq C M_1 M_2 \varepsilon^3 \leq C M_1^2 \varepsilon^2. \tag{3.24}$$

Also these lemmas with $(\rho, \mu) = (1, 0)$ yield

$$e_2[U_{c_i}[G_{i,2}]](t, r) \leq C(M_1^2 + M_2) M_2 \varepsilon^4 \log(2 + t). \tag{3.25}$$

On the other hand, by Lemmas 2.2 and 2.6-(ii), (iii) with $(\rho, \mu) = (1, 1)$, we get

$$e_2[U_{c_i}[G_{i,1}]](t, r) \leq C(M_1^2 \varepsilon^2 + M_3^2 \varepsilon^6 \{\log(2 + T)\}^2). \tag{3.26}$$

Now (3.24), (3.25) and (3.26) imply

$$E_2(T) \leq C_2(M_1^2 \varepsilon^2 + (M_1^2 + M_2) M_2 \varepsilon^4 \log(2 + T) + M_3^2 \varepsilon^6 \{\log(2 + T)\}^2) \tag{3.27}$$

with an appropriate constant C_2.

Finally we consider (3.3). We get

$$\begin{aligned}
&\mathcal{M}[r_i^{\mathrm{III}}(u^{(2)+(3)}, \partial_t u^{(1)})] \\
&\leq C \sum_{j \neq k} \langle t + r\rangle^{-1} \Phi_0(c_j t, r) \langle c_k t - r\rangle^{-2} M_1 \varepsilon (M_2 \varepsilon^2 + M_3 \varepsilon^3) \\
&\leq C(\langle t + r\rangle^{-1} w(t, r)^{-2} + \langle t + r\rangle^{-3+\nu} w(t, r)^{-\nu}) M_1 M_2 \varepsilon^3 \\
&\leq C\langle t + r\rangle^{-1} w(t, r)^{-2} M_1 M_2 \varepsilon^3,
\end{aligned}$$

where ν (> 0) is a small constant. We also obtain

$$\begin{aligned}
\mathcal{M}[r_i^{\mathrm{III}}(u^{(1)}, \partial_t u^{(3)})] &\leq C \sum_{j \neq k} \langle t + r\rangle^{-1} \langle c_j t - r\rangle^{-1} \langle c_k t - r\rangle^{-1} \\
&\qquad \times (1 + \langle c_k t - r\rangle^{-1} \Phi_0(c_k t, r)) M_1 M_3 \varepsilon^4 \\
&\leq C(\langle t + r\rangle^{-2} w(t, r)^{-1} + \langle t + r\rangle^{-2+\nu} w(t, r)^{-2-\nu}) M_1 M_3 \varepsilon^4 \\
&\leq C\langle t + r\rangle^{-1} w(t, r)^{-2} M_1 M_3 \varepsilon^4.
\end{aligned}$$

From Lemma 2.2 and Lemma 2.6-(i), (iii) with $(\rho, \mu) = (1, 1)$, we obtain

$$e_3[U_{c_i}[r_i^{\mathrm{III}}(u^{(2)+(3)}, \partial_t u^{(1)}) + r_i^{\mathrm{III}}(u^{(1)}, \partial_t u^{(3)})]](t, r) \leq C M_1 M_2 \varepsilon^3. \tag{3.28}$$

On the other hand, for $j \neq k$ and small ν (> 0), we have

$$
\begin{aligned}
&\Phi_0(c_j t, r)\langle c_k t - r\rangle^{-1}(1 + \langle c_k t - r\rangle^{-1}\Phi_0(c_k t, r)) \\
&\le C(\langle t + r\rangle^{-1}\Phi_0(c_j t, r) + w(t, r)^{-1} + w(t, r)^{-2}\Phi_0(c_k t, r)) \\
&\le C(\langle t + r\rangle^{-1+\nu} w(t, r)^{-\nu} + w(t, r)^{-1} + w(t, r)^{-2}\log(2 + T)) \\
&\le C(w(t, r)^{-1} + w(t, r)^{-2}\log(2 + T)).
\end{aligned}
$$

Hence we get

$$
\begin{aligned}
&\mathcal{M}[r_i^{\mathrm{III}}(u^{(2)+(3)}, \partial_t u^{(3)})](t, r) \\
&\le C\sum_{j\neq k}\langle t + r\rangle^{-1}\Phi_0(c_j t, r)\langle c_k t - r\rangle^{-1} \\
&\qquad \times (1 + \langle c_k t - r\rangle^{-1}\Phi_0(c_k t, r))(M_2\varepsilon^2 + M_3\varepsilon^3)M_3\varepsilon^3 \\
&\le C\langle t + r\rangle^{-1}(w(t, r)^{-1} + w(t, r)^{-2}\log(2 + T))M_2 M_3\varepsilon^5.
\end{aligned}
$$

Therefore, by Corollaries 2.3 and 2.7, we obtain

$$
e_3[U_i[r_i^{\mathrm{III}}(u^{(2)+(3)}, \partial_t u^{(3)})]](t, r) \le C M_2 M_3 \varepsilon^5 \log(2 + T). \tag{3.29}
$$

Finally (3.28) and (3.29) imply

$$
E_3(T) \le C_3\{M_1 M_2\varepsilon^3 + M_2 M_3\varepsilon^5\log(2 + T)\}. \tag{3.30}
$$

This completes the proof. □

4. Proof of Theorem 1.2

Suppose that all the assumptions in Theorem 1.2 are fulfilled. Let $u = (u_1, u_2)$ be a solution to (1.1)–(1.2) (with (1.19) and (1.20)) for $0 \le t < T_\varepsilon$. Fix κ satisfying $1/2 < \kappa < 1$. We put

$$
\begin{aligned}
e_1^*[u_1](t, r) &= \langle t + r\rangle^{\kappa}(|u_1(t, r)| + \mathcal{D}_1^+[u_1](t, r)) \\
&\quad + \langle c_1 t - r\rangle^{\kappa}\mathcal{D}[u_1](t, r), \\
e_2^*[u_2](t, r) &= \langle t + r\rangle\Phi_0(c_2 t, r)^{-1}(|u_2(t, r)| + \mathcal{D}_2^+[u_2](t, r)) \\
&\quad + \langle c_2 t - r\rangle\mathcal{D}[u_2](t, r),
\end{aligned}
$$

and

$$
E(T) = \sup_{(t,r)\in[0,T)\times[0,\infty)} (e_1^*[u_1](t, r) + e_2^*[u_2](t, r)) \tag{4.1}
$$

for $0 < T \le T_\varepsilon$, with $E(0) = \sup_{r\in[0,\infty)}(e_1^*[u_1](0, r) + e_2^*[u_2](0, r))$.

Similarly to the proof of Theorem 1.1, what we need for the proof of Theorem 1.2 is the following.

Proposition 4.1. *Assume $0 < T < T_\varepsilon$, and let M_0 be a positive constant. Suppose that ε is a positive constant satisfying $M_0\varepsilon \le 1$ and $\varepsilon \le 1$. Then there exists a positive constant C_0, which is independent of T, T_ε, ε, and M_0, such that $E(T) \le M_0\varepsilon$ implies*

$$E(T) \le C_0(\varepsilon + M_0^2\varepsilon^2). \tag{4.2}$$

From Proposition 4.1, following a similar argument in the proof of Theorem 1.1, we see that $E(T)$ stays bounded as far as the solution u exists for $0 \le t < T$, and the local existence theorem implies Theorem 1.2.

Proof of Proposition 4.1. Lemmas 2.1 and 2.8 yield

$$\sum_{i=1}^{2} e_i^*[U_{c_i}^*[\varepsilon f_i, \varepsilon g_i]](t, r) \le C\varepsilon. \tag{4.3}$$

As for the nonlinearities, firstly we have

$$\begin{aligned}\mathcal{M}[Q_0(u_1, u_1; c_1)](t, r) &\le C\langle r\rangle^{-1}\langle t + r\rangle^{-\kappa}\langle c_1t - r\rangle^{-\kappa}M_0^2\varepsilon^2\\ &\le C\langle t + r\rangle^{-2\kappa}w(t, r)^{-1}M_0^2\varepsilon^2\\ &\le C\langle t + r\rangle^{-1-\nu}w(t, r)^{-2\kappa+\nu}M_0^2\varepsilon^2\end{aligned}$$

for $\nu > 0$ satisfying $2\kappa - \nu > 1$. Similarly we get

$$\begin{aligned}\mathcal{M}[Q_0(u_2, u_2; c_2)](t, r) &\le C\langle r\rangle^{-1}\langle t + r\rangle^{-1}\langle c_2t - r\rangle^{-1}\Phi_0(c_2t, r)M_0^2\varepsilon^2\\ &\le C\langle t + r\rangle^{-2+\nu}w(t, r)^{-1-\nu}M_0^2\varepsilon^2\\ &\le C\langle t + r\rangle^{-1-\nu}w(t, r)^{-2+\nu}M_0^2\varepsilon^2\end{aligned}$$

for small $\nu > 0$. Thus we obtain

$$\mathcal{M}[N_i(\partial u, \partial u)](t, r) \le C\langle t + r\rangle^{-1-\nu}w(t, r)^{-2\kappa+\nu}M_0^2\varepsilon^2. \tag{4.4}$$

On the other hand, we have

$$\begin{aligned}\mathcal{M}[r_i^{\mathrm{I}}(\partial u, \partial u)](t, r) &\le C\langle r\rangle^{-1}\langle c_1t - r\rangle^{-\kappa}\langle c_2t - r\rangle^{-1}M_0^2\varepsilon^2\\ &\le C\langle t + r\rangle^{-1-\kappa}w(t, r)^{-1}M_0^2\varepsilon^2\\ &\le C\langle t + r\rangle^{-1-\kappa/2}w(t, r)^{-1-\kappa/2}M_0^2\varepsilon^2.\end{aligned}$$

Summing up, from Lemma 2.2, (i) and (iii) in Lemma 2.6, we obtain

$$e_i^*[U_{c_i}[N_i(\partial u,\, \partial u) + r_i^{\mathrm{I}}(\partial u,\, \partial u)]](t,\, r) \le C M_0^2 \varepsilon^2 \tag{4.5}$$

for $i = 1, 2$.

Now we are going to estimate the main parts. For r_1^{IV}, we have

$$\begin{aligned}\mathcal{M}[r_1^{\mathrm{IV}}(u,\, \partial u)](t,\, r) &\le C\langle t + r\rangle^{-1} \Phi_0(c_2 t,\, r)\langle c_2 t - r\rangle^{-1} M_0^2 \varepsilon^2 \\ &\le C\langle t + r\rangle^{-1+\nu} w_1(t,\, r)^{-1-\nu} M_0^2 \varepsilon^2\end{aligned}$$

for small $\nu > 0$. By (ii) in Lemma 2.6 with $(\rho,\, \mu) = (1 - \nu,\, \nu)$, we get

$$\langle c_1 t - r\rangle^{\kappa} \mathcal{D}[U_{c_1}[r_1^{\mathrm{IV}}]](t,\, r) \le C M_0^2 \varepsilon^2, \tag{4.6}$$

provided that ν is small enough to satisfy $0 < \kappa \le 1 - \nu$.

On the other hand, since

$$\langle t + r\rangle^{-1+\nu} w_1(t,\, r)^{-1-\nu} \le C\langle t + r\rangle^{1-\kappa} \langle t + r\rangle^{-2+\kappa+\nu} w_1(t,\, r)^{-1-\nu}$$

and $\langle \tau + \lambda\rangle \le \langle t + r\rangle$ for $(\tau,\, \lambda) \in \Theta_i^*(t,\, r)$, we get

$$\sup_{(\tau,\lambda)\in\Theta_i^*(t,r)} \langle \tau + \lambda\rangle^{2-\kappa-\nu} w(\tau,\, \lambda)^{1+\nu} \mathcal{M}[r_1^{\mathrm{IV}}](\tau,\, \lambda) \le C M_0^2 \varepsilon^2 \langle t + r\rangle^{1-\kappa}.$$

Now, since we may assume $2 - \kappa - \nu > 1$, Lemmas 2.2 and 2.6–(iii) with $(\rho,\, \mu) = (2 - \kappa - \nu,\, \nu)$ lead to

$$|U_{c_1}[r_1^{\mathrm{IV}}](t,\, r)| + \mathcal{D}_1^+[U_{c_1}[r_1^{\mathrm{IV}}]](t,\, r) \le C\langle t + r\rangle^{-\kappa} M_0^2 \varepsilon^2. \tag{4.7}$$

From (4.6) and (4.7), we obtain

$$e_1^*[U_{c_1}[r_1^{\mathrm{IV}}(u,\, \partial u)]](t,\, r) \le C M_0^2 \varepsilon^2. \tag{4.8}$$

Similarly we also obtain

$$e_1^*[U_{c_1}[r_1^{\mathrm{II}}(\partial u,\, \partial u)]](t,\, r) \le C M_0^2 \varepsilon^2. \tag{4.9}$$

Since we have $2\kappa > 1$, we get

$$\begin{aligned}\mathcal{M}[r_2^{\mathrm{II}}(\partial u,\, \partial u)](t,\, r) &\le C\langle r\rangle^{-1} \langle c_1 t - r\rangle^{-2\kappa} M_0^2 \varepsilon^2 \\ &\le C\langle t + r\rangle^{-1} w_2(t,\, r)^{-2\kappa} M_0^2 \varepsilon^2.\end{aligned}$$

Hence, from Lemma 2.2, (ii) and (iii) in Lemma 2.6, we obtain

$$e_2^*[U_{c_2}[r_2^{\mathrm{II}}(\partial u,\, \partial u)]](t,\, r) \le C_0 M_0^2 \varepsilon^2. \tag{4.10}$$

Finally, (4.2) follows from (4.3), (4.5), (4.8), (4.9) and (4.10). This completes the proof. □

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