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ASYMPTOTIC BEHAVIOR OF SOLUTIONS TO THE COMPRESSIBLE NAVIER-STOKES EQUATION IN A CYLINDRICAL DOMAIN

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Abstract

Asymptotic behavior of solutions to the compressible Navier-Stokes equation around a given constant state is investigated on a cylindrical domain in $\mathbf{R}^3$, under the no slip boundary condition for the velocity field. The L^2 decay estimate is established for the perturbation from the constant state. It is also shown that the time-asymptotic leading part of the perturbation is given by a function satisfying a 1 dimensional heat equation. The proof is based on an energy method and asymptotic analysis for the associated linearized semigroup.

1. Introduction

This paper studies the initial boundary value problem for the compressible Navier-Stokes equation in a cylindrical domain Ω:

$$\partial_t \rho + \operatorname{div}(\rho v) = 0, \tag{1.1}$$

$$\partial_t(\rho v) + \operatorname{div}(\rho v \otimes v) + \nabla P(\rho) = \mu \Delta v + (\mu + \mu')\nabla \operatorname{div} v, \tag{1.2}$$

$$v|_{\partial\Omega} = 0, \quad \rho|_{t=0} = \rho_0(x), \quad v|_{t=0} = v_0(x). \tag{1.3}$$

Here Ω is a cylindrical domain in $\mathbf{R}^3$ that is defined by

$$\Omega = \{x = (x', x_n);\, x' = (x_1, x_2) \in D,\ x_3 \in \mathbf{R}\},$$

where D is a bounded domain in $\mathbf{R}^2$ with smooth boundary; $\rho = \rho(x, t)$ and $v = (v^1(x, t), v^2(x, t), v^3(x, t))$ denote the unknown density and velocity at time $t \geq 0$ and position $x \in \Omega$, respectively; $P = P(\rho)$ is the pressure; μ and μ' are the viscosity coefficients that satisfy $\mu > 0$, $(2/3)\mu + \mu' \geq 0$.

Our main concern is the large time behavior of solutions to problem (1.1)–(1.3) when the initial value $\{\rho_0, v_0\}$ is sufficiently close to a given constant state $\{\rho_*, 0\}$, where ρ_* is a given positive number.

Matsumura and Nishida [15, 16] proved the global in time existence of solutions to the Cauchy problem for (1.1)–(1.2) on the whole space $\mathbf{R}^3$ around $(\rho_*, 0)$ and ob-

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tained the optimal L^2 decay rate of the perturbation $u(t) = \{\rho(t) - \rho_*, v(t)\}$. Kawashima, Matsumura and Nishida [11] then showed that the leading part of $u(t)$ is given by the solution of the linearized problem. (See [10] for the case of a general class of quasilinear hyperbolic-parabolic systems.) The solution of the linearized problem exhibits a hyperbolic-parabolic aspect of system (1.1)–(1.2), a typical property of system (1.1)–(1.2). Its asymptotically leading part in large time is given by the sum of two terms, one is given by the convolution of the heat kernel and the fundamental solution of the wave equation, which is the so-called *diffusion wave*, and the other is the solution of the heat equation. Hoff and Zumbrun [2, 3] showed that there appears some interesting interaction of hyperbolic and parabolic aspects of the system in the decay properties of L^p norms with $1 \leq p \leq \infty$. (See also [14].) Such an interaction phenomena also appears in the exterior domain problem [12, 13] and the half space problem [7, 8].

On the other hand, solutions on the infinite layer $\mathbf{R}^{n-1} \times (0, 1)$ behave in a manner different from the ones on the domains mentioned above. It was shown in [6] that the leading part of the solution on the infinite layer is given by a solution of an $n - 1$ dimensional heat equation and any hyperbolic feature does not appear in the leading part. This is due to the fact that the infinite layer has an infinite extent in $n - 1$ unbounded directions and the remaining one direction has a finite thickness. In this paper we will prove that an analogues result holds for solutions on the cylindrical domain Ω that has one unbounded direction x_3 and two dimensional bounded cross section D. We will show that under suitable assumptions on the initial value, $u(t) = \{\rho(t) - \rho_*, v(t)\}$ satisfies

$$\|u(t)\|_{L^2} = O(t^{-1/4}), \quad \|u(t) - u^{(0)}(t)\|_{L^2} = O(t^{-3/4} \log t) \tag{1.4}$$

as $t \to \infty$. Here $u^{(0)} = \{\phi^{(0)}(x_3, t), 0\}$ with $\phi^{(0)}(x_3, t)$ satisfying

$$\partial_t \phi^{(0)} - \kappa \partial_{x_3}^2 \phi^{(0)} = 0, \quad \phi^{(0)}|_{t=0} = \frac{1}{|D|} \int_D (\rho_0(x', x_3) - \rho_*)\, dx',$$

where κ is a positive constant and $|D|$ denotes the Lebesgue measure of D. We will also establish the decay estimate $\|\partial_x u(t)\|_{L^2} = O(t^{-3/4})$. As in the case of the infinite layer, the leading part of $u(t)$ is given by a solution of the 1 dimensional heat equation and no hyperbolic feature appears in the leading part. We also note that any effect from the nonlinearity does not appear in the leading part.

The proof of (1.4) is based on the H^3 energy estimate and the asymptotic analysis for the linearized semigroup. The H^3 energy estimate is obtained by the energy method in [17], which also gives the global solvability for the problem (1.1)–(1.3). To prove the asymptotic properties in (1.4), we analyze the linearized resolvent problem, which takes the form (after some transformation)

$$(\lambda + L)u = f. \tag{1.5}$$

Here $u = {}^T(\rho, v)$ (the superscript ^{T} stands for the transposition), and L is the operator with domain $D(L)$ defined by

$$L = \begin{pmatrix} 0 & \gamma \,\mathrm{div} \\ \gamma \nabla & -\nu \Delta I_3 - \tilde{\nu} \nabla \,\mathrm{div} \end{pmatrix}, \quad D(L) = H^1(\Omega) \times [H^2(\Omega) \cap H_0^1(\Omega)],$$

where I_3 denotes the 3×3 identity matrix, and ν, $\tilde{\nu}$ and γ are suitable positive constants. The resolvent problem will be considered through the Fourier transform in x_3 variable that leads to the problem of the form:

$$(\lambda + \hat{L}_\xi)\hat{u} = \hat{f}. \tag{1.6}$$

Here $\xi \in \mathbf{R}$ denotes the dual variable, $\hat{u} = \hat{u}(x', \xi)$ and $\hat{f} = \hat{f}(x', \xi)$ are functions in $x' \in D$, and $\hat{L}_\xi$ is the operator with ∂_{x_3} replaced by $i\xi$ in L. As in the case of the infinite layer [5], the spectrum of $-\hat{L}_\xi$ for $|\xi| \ll 1$ can be regarded as a perturbation from the one with $\xi = 0$, and we show that the spectrum near the origin is given by a simple eigenvalue $\lambda_0(\xi) = -\kappa\xi^2 + O(\xi^4)$ as $\xi \to 0$. On the other hand, as for $|\xi| \gg 1$, an explicit integral formula for $(\lambda + \hat{L}_\xi)^{-1}$ was used to obtain the L^p estimates in the case of the infinite layer. Such an explicit integral formula cannot be expected to be obtained in the case of the cylindrical domain Ω, and, so, as a first step of the analysis, we employ an energy method to obtain the L^2 estimates for $|\xi| \gg 1$.

This paper is organized as follows. In Section 2 we state our main results of this paper: asymptotic behavior of solutions of the linearized and nonlinear problems. In this paper we will give a proof only for the linearized problem, since the nonlinear problem can be treated in a similar argument to that given in [6], based on the linearized analysis and the energy method in [17]. In Section 3 we study the resolvent problem (1.6) for $|\xi| \gg 1$. Section 4 is devoted to the analysis of (1.6) for $|\xi| \ll 1$. We then investigate the asymptotic behavior of the linearized semigroup in Section 5.

2. Main result

We first introduce some notation which will be used throughout the paper. We denote by $L^2(\Omega)$ the usual Lebesgue space of all square summable functions on Ω and its norm is denoted by $\|\cdot\|_2$. Let l be a nonnegative integer. The symbol $H^l(\Omega)$ denotes the l-th order L^2 Sobolev space on Ω with norm $\|\cdot\|_{H^l}$. $C_0^l(\Omega)$ stands for the set of all C^l functions which have compact support in Ω. We denote by $H_0^1(\Omega)$ the completion of $C_0^1(\Omega)$ in $H^1(\Omega)$.

We simply denote by $L^2(\Omega)$ (resp., $H^l(\Omega)$) the set of all vector fields $v = (v^1, v^2, v^3)$ on Ω with $v^j \in L^2(\Omega)$ (resp., $H^l(\Omega)$), $j = 1, 2, 3$, and its norm is also denoted by $\|\cdot\|_2$ (resp., $\|\cdot\|_{H^l}$). We will frequently consider column vectors ${}^T(v^1, v^2, v^3)$, and, for simplicity, the set of all column vectors ${}^T(v^1, v^2, v^3)$ with $v^j \in L^2(\Omega)$ (resp., $H^l(\Omega)$), $j = 1, 2, 3$, is also denoted by $L^2(\Omega)$ (resp., $H^l(\Omega)$) and its norm is also denoted by $\|\cdot\|_2$ (resp., $\|\cdot\|_{H^l}$). Here and in what follows ${}^T\cdot$ stands for the transposition. For

$u = {}^T(\phi, v)$ with $\phi \in H^k(\Omega)$ and $v = (v^1, v^2, v^3) \in H^l(\Omega)$, we define $\|u\|_{H^k \times H^l}$ by $\|u\|_{H^k \times H^l} = \|\phi\|_{H^k} + \|v\|_{H^l}$. When $k = l$, we simply write $\|u\|_{H^k \times H^k} = \|u\|_{H^k}$.

Similarly, we define the function spaces on D, namely, $L^2(D)$ and $H^l(D)$; and their norms are denoted by $|\cdot|_2$ and $|\cdot|_{H^l}$, respectively.

We define $L^1_{x_3}(\mathbf{R}; L^2(D))$ by

$$L^1_{x_3}(\mathbf{R}; L^2(D)) = \{u = {}^T(\phi(x', x_3), v(x', x_3)); \||u|_2\|_{L^1_{x_3}} < \infty\},$$

where

$$\||u|_2\|_{L^1_{x_3}} = \int_{\mathbf{R}} |u(\,\cdot\,, x_3)|_2 \, dx_3 = \int_{\mathbf{R}} \left(\int_D |u(x', x_3)|^2 \, dx' \right)^{1/2} dx_3.$$

Similarly, we define $L^1_{x_3}(\mathbf{R}; H^1(D) \times L^2(D))$ and $\||u|_{H^1 \times L^2}\|_{L^1_{x_3}}$.

The inner product of $L^2(D)$ is denoted by

$$(f, g) = \int_D f(x')\overline{g(x')} \, dx', \quad f, g \in L^2(D).$$

Here $\overline{g}$ denotes the complex conjugate of g. Furthermore, we define $\langle \cdot, \cdot \rangle$ and $\langle \cdot \rangle$ by

$$\langle f, g \rangle = \frac{1}{|D|}(f, g) \quad \text{and} \quad \langle f \rangle = \langle f, 1 \rangle = \frac{1}{|D|} \int_D f(x') \, dx'$$

for $f, g \in L^2(D)$, respectively.

Partial derivatives of a function u in x, x', x_3 and t are denoted by $\partial_x u$, $\partial_{x'} u$, $\partial_{x_3} u$ and $\partial_t u$, respectively. We also write higher order partial derivatives of u in x as $\partial_x^k u = (\partial_x^\alpha u; |\alpha| = k)$.

We denote the $n \times n$ identity matrix by I_n. We define 4×4 diagonal matrices Q_0, $\tilde{Q}$ and Q' by

$$Q_0 = \mathrm{diag}(1, 0, 0, 0), \quad \tilde{Q} = \mathrm{diag}(0, 1, 1, 1), \quad Q' = \mathrm{diag}(0, 1, 1, 0).$$

We then have, for $u = {}^T(\phi, v) \in \mathbf{R}^4$, $v = (v^1, v^2, v^3)$,

$$Q_0 u = \begin{pmatrix} \phi \\ 0 \end{pmatrix}, \quad \tilde{Q} u = \begin{pmatrix} 0 \\ v \end{pmatrix}, \quad Q' u = \begin{pmatrix} 0 \\ v^1 \\ v^2 \\ 0 \end{pmatrix}.$$

For a function $f = f(x_3)$ $(x_3 \in \mathbf{R})$, we denote its Fourier transform by $\hat{f}$ or $\mathscr{F} f$:

$$\hat{f}(\xi) = (\mathscr{F} f)(\xi) = \int_{\mathbf{R}} f(x_3) e^{-i\xi x_3} \, dx_3 \quad (\xi \in \mathbf{R}).$$

The inverse Fourier transform is denoted by $\mathscr{F}^{-1}$:

$$(\mathscr{F}^{-1} f)(x_3) = (2\pi)^{-1} \int_{\mathbf{R}} f(\xi) e^{i\xi x_3}\, d\xi \quad (x_3 \in \mathbf{R}).$$

We denote the resolvent set of a closed operator A by $\rho(A)$ and the spectrum of A by $\sigma(A)$. For $\Lambda \in \mathbf{R}$ and $\theta \in (\pi/2, \pi)$ we will denote

$$\Sigma(\Lambda, \theta) = \{\lambda \in \mathbf{C};\, |\arg(\lambda - \Lambda)| \le \theta\}.$$

We next rewrite problem (1.1)–(1.3). We set $\phi = \rho - \rho_*$. Then problem (1.1)–(1.3) is reduced to finding $u = \{\phi, v\}$ that satisfies

$$\partial_t \phi + v \cdot \nabla \phi + \rho \operatorname{div} v = 0, \tag{2.1}$$

$$\rho(\partial_t v + v \cdot \nabla v) - \mu \Delta v - (\mu + \mu')\nabla \operatorname{div} v + P'(\rho)\nabla \phi = 0, \tag{2.2}$$

$$v|_{\partial\Omega} = 0; \quad u|_{t=0} = u_0, \tag{2.3}$$

where $\rho = \phi + \rho_*$ and

$$u_0 = \{\phi_0, v_0\}, \quad \phi_0 = \rho_0 - \rho_*.$$

Here (1.1) is used to obtain (2.2).

We first consider the linearized problem. Substituting $\rho = \phi + \rho_*$ in (2.1)–(2.3) and omitting the terms $O(|\phi|^2 + |v|^2)$, we have the linearized problem

$$\begin{aligned} &\partial_t \phi + \operatorname{div} v = 0, \\ &\partial_t v - \nu \Delta v - \tilde{\nu} \operatorname{div} \nabla v + p_1 \nabla \phi = 0, \\ &v|_{\partial\Omega} = 0, \quad \phi|_{t=0} = \phi_0, \quad v|_{t=0} = v_0, \end{aligned}$$

where $p_1 = P'(\rho_*)$. By transforming $\phi \mapsto \sqrt{\rho_*/p_1}\phi$, the problem is reduced to

$$\partial_t u + Lu = 0, \quad u|_{t=0} = u_0.$$

Here $u = {}^T(\phi, v)$, $u_0 = {}^T(\phi_0, v_0)$ and L is the operator defined in (1.5) with $\nu = \mu/\rho_*$, $\tilde{\nu} = (\mu + \mu')/\rho_*$ and $\gamma = \sqrt{p_1/\rho_*}$.

As for the linearized problem, we have the following result.

Theorem 2.1. *The operator $-L$ generates an analytic semigroup e^{-tL} on $H^1(\Omega) \times L^2(\Omega)$. Furthermore, if $u_0 = {}^T(\phi_0, v_0) \in (H^1(\Omega) \times L^2(\Omega)) \cap L^1(\Omega) \cap L^1_{x_3}(\mathbf{R}; H^1(D) \times L^2(D))$, then $e^{-tL}u_0$ is written as*:

$$e^{-tL}u_0 = \mathscr{U}_0(t)u_0 + \mathscr{U}_1(t)u_0 + \mathscr{R}(t)u_0,$$

where each term on the right has the following properties.
(i) $\mathscr{U}_0(t)u_0$ *has the form*

$$\mathscr{U}_0(t)u_0 = \begin{pmatrix} \phi^{(0)}(t) \\ 0 \end{pmatrix},$$

where $\phi^{(0)} = \phi^{(0)}(x_3, t)$ *satisfies the following heat equation*

$$\partial_t \phi^{(0)} - \kappa \partial_{x_3}^2 \phi^{(0)} = 0, \quad \phi^{(0)}|_{t=0} = \langle \phi_0 \rangle$$

with a positive constant κ. *Furthermore,* $\mathscr{U}_0(t)u_0$ *satisfies the estimates*

$$\|\partial_x^l \mathscr{U}_0(t)u_0\|_2 \leq Ct^{-1/4-l/2}\|Q_0u_0\|_{L^1}, \quad l = 0, 1, 2.$$

(ii) $\mathscr{U}_1(t)u_0$ *satisfies the estimates*

$$\|\mathscr{U}_1(t)u_0\|_{H^1} \leq Ct^{-3/4}\||\tilde{Q}u_0|_{H^1\times L^2}\|_{L^1_{x_3}},$$
$$\|\partial_x \mathscr{U}_1(t)\tilde{Q}u_0\|_2 \leq Ct^{-5/4}\||\tilde{Q}u_0|_2\|_{L^1_{x_3}},$$
$$\|\mathscr{U}_1(t)[\partial_x \tilde{Q}u_0]\|_2 \leq Ct^{-3/4}\||\tilde{Q}u_0|_2\|_{L^1_{x_3}} + Ct^{-5/4}\||\partial_x \tilde{Q}u_0|_2\|_{L^1_{x_3}}.$$

(iii) $\mathscr{R}(t)u_0$ *satisfies the estimate*

$$\|\mathscr{R}(t)u_0\|_{H^1} \leq Ce^{-c_0t}\|u_0\|_{H^1\times L^2}$$

for some positive constant c_0.

We next state our results on the nonlinear problem (2.1)–(2.3). We will look for the solution $u = \{\phi, v\} \in \bigcap_{j=0}^{[s/2]} C([0, \infty); H^{s-2j}(\Omega))$ for $s = 2, 3$. We therefore mention the compatibility condition for the initial value. By the boundary condition $v|_{\partial\Omega} = 0$ in (2.3), we have to require $v_0 \in H_0^1$ for $s = 2, 3$. In addition to this, we will require

$$v_0 \cdot \nabla v_0 + \frac{1}{\rho_0}\nabla P(\rho_0) - \frac{1}{\rho_0}\mu\Delta v_0 - \frac{1}{\rho_0}(\mu + \mu')\nabla \operatorname{div} v_0 \in H_0^1 \tag{2.4}$$

for $s = 3$, where $\rho_0 = \phi_0 + \rho_*$.

We first state the global in time existence of strong solutions.

Theorem 2.2. *Let* $s = 2, 3$. *Let* $P'(\rho_*) > 0$. *Assume that* $u_0 = \{\phi_0, v_0\} \in H^s(\Omega)$ *and* $v_0 \in H_0^1(\Omega)$. *Assume also that* u_0 *satisfies* (2.4) *when* $s = 3$. *Then there exists a positive number* $\varepsilon_0 > 0$ *such that if*

$$\|u_0\|_{H^s} \leq \varepsilon_0,$$

then there exists a unique solution $u(t) = \{\phi(t), v(t)\} \in \bigcap_{j=0}^{[s/2]} C([0,\infty); H^{s-2j}(\Omega))$ *of* (2.1)–(2.3). *Furthermore*, $u(t)$ *satisfies the following estimate*:

$$\|u(t)\|_{H^s}^2 + \int_0^t \|\partial_x v\|_{H^s}^2 + \|\partial_x \rho\|_{H^{s-1}}^2 \, d\tau \le C\|u_0\|_{H^s}^2$$

for all $t \ge 0$.

In addition to the assumptions for $s = 3$ of Theorem 2.2, if $u_0 \in L^1(\Omega) \cap L^1_{x_3}(\mathbf{R}; H^1(D) \times L^2(D))$, we have the following asymptotic behavior.

Theorem 2.3. *In addition to the assumptions for* $s = 3$ *of* Theorem 2.2, *assume also that* $u_0 \in L^1(\Omega) \cap L^1_{x_3}(\mathbf{R}; H^1(D) \times L^2(D))$. *Then there hold the following estimates*:
(i) $\|\partial_x^l u(t)\|_2 = O(t^{-1/4-l/2})$ $(l = 0, 1)$,
(ii) $\|u(t) - \mathscr{U}_0(t)u_0\|_2 = O(t^{-3/4}\log t)$
as $t \to \infty$, *provided that* $\|u_0\|_{H^3} + \|u_0\|_1 + \||u_0|_{H^1\times L^2}\|_{L^1_{x_3}}$ *is sufficiently small. Here* $\mathscr{U}(t)u_0$ *is the function given in* Theorem 2.1 (i).

REMARK. Since $\|\phi^{(0)}(t)\|_2 = O(t^{-1/4})$, the estimate (ii) of Theorem 2.2 shows that the asymptotic leading part of $u(t)$ is given by $\mathscr{U}_0(t)u_0$.

We omit the proof of Theorem 2.2 since it is proved by the energy method in the same way as given in [17]. Theorem 2.3 is proved by combining the estimates in Theorems 2.1 and 2.2. We also omit the proof of Theorem 2.3 since it is proved in a similar manner to the argument given in [6], which is based on the energy estimate and the linearized analysis. Therefore, in this paper we give a proof of Theorem 2.1 only.

3. Resolvent problem I

In this and next sections we consider the resolvent for the linearized problem, which leads to the asymptotic properties of the semigroup e^{-tL} in Theorem 2.1

We will first show that L is a sectorial operator on $H^1(\Omega) \times L^2(\Omega)$. We will then investigate the resolvent in detail by using the Fourier transform with respect to x_3 variable.

Let us consider the resolvent problem

$$(\lambda + L)u = f, \tag{3.1}$$

where $u = {}^T(\phi, v)$, $f = {}^T(f^0, g)$, and L is the operator with domain $D(L)$ defined by

$$L = \begin{pmatrix} 0 & \gamma \operatorname{div} \\ \gamma\nabla & -\nu\Delta I_3 - \tilde{\nu}\nabla \operatorname{div} \end{pmatrix}, \quad D(L) = H^1(\Omega) \times [H^2(\Omega) \cap H_0^1(\Omega)]$$

with $\nu = \mu/\rho_*$, $\tilde{\nu} = (\mu + \mu')/\rho_*$ and $\gamma = \sqrt{p_1/\rho_*}$.

The following proposition shows that $-L$ generates an analytic semigroup e^{-tL} on $H^1(\Omega) \times L^2(\Omega)$.

Proposition 3.1. *There exist constants $\Lambda_0 > 0$ and $\theta_0 \in (\pi/2, \pi)$ such that the following assertions hold: if $\lambda \in \Sigma(\Lambda_0, \theta_0)$, then for any $f = {}^T(f^0, g) \in H^1(\Omega) \times L^2(\Omega)$, there exists a unique solution $u = {}^T(\phi, v) \in D(L)$ of (3.1), and $u = (\lambda + L)^{-1} f$ satisfies*

$$|\lambda| \|(\lambda + L)^{-1} f\|_{H^1 \times L^2} + \sum_{k=1}^{2} |\lambda|^{1-2/k} \|\partial_x^k \tilde{Q}(\lambda + L)^{-1} f\|_2 \leq C\|f\|_{H^1 \times L^2}.$$

Proof. In this proof we denote by (f, g) the inner product of f and g in $L^2(\Omega)$. We first give a proof of the estimate for $u = (\lambda + L)^{-1} f$.

We write (3.1) as

$$\lambda\phi + \gamma \operatorname{div} v = f^0, \tag{3.2}$$

$$\lambda v - \nu\Delta v - \tilde{\nu}\nabla \operatorname{div} v + \gamma\nabla\phi = g, \quad v|_{\partial\Omega} = 0. \tag{3.3}$$

Assume that $\lambda \neq 0$. Then it follows from (3.2) that

$$\phi = \frac{1}{\lambda}\{f^0 - \gamma \operatorname{div} v\}. \tag{3.4}$$

Substituting (3.4) into (3.3), we have

$$\lambda v - \nu\Delta v - \tilde{\nu}\nabla \operatorname{div} v = F, \quad v|_{\partial\Omega} = 0, \tag{3.5}$$

where

$$F = g - \frac{\gamma}{\lambda}\nabla f^0 + \frac{\gamma^2}{\lambda}\nabla \operatorname{div} v.$$

Since $B = -\nu\Delta v - \tilde{\nu}\nabla \operatorname{div} v$ is strongly elliptic, there exist constants $\Lambda_0 > 0$ and $\theta_0 \in (\pi/2, \pi)$ such that if $\lambda \in \Sigma(\Lambda_0, \theta_0)$, then

$$\sum_{k=0}^{2} |\lambda|^{1-k/2} \|\partial_x^k v\|_2 \leq C\|F\|_2.$$

Since $\|F\|_2 \leq C\{\|f\|_{H^1 \times L^2} + \|\partial_x^2 v\|_2/|\lambda|\}$, taking Λ_0 larger if necessary, we obtain

$$\sum_{k=0}^{2} |\lambda|^{1-k/2} \|\partial_x^k v\|_2 \leq C\|f\|_{H^1 \times L^2}.$$

This, together with (3.4), gives

$$\|\phi\|_{H^1} \leq \frac{C}{|\lambda|}\{\|f^0\|_{H^1} + \|\text{div}\, v\|_{H^1}\} \leq \frac{C}{|\lambda|}\|f\|_{H^1\times L^2}$$

for $\lambda \in \Sigma(\Lambda_0, \theta_0)$. We thus obtain the desired estimate.

We next consider the existence of solutions. Let us assume $\lambda > 1$. We first look for a weak solution of (3.5) for $\lambda > 1$. Set $G = g - \gamma\nabla f^0/\lambda$ and consider the problem to find $v \in H_0^1(\Omega)$ satisfying

$$a(v, w) = (G, w) \quad (\forall w \in H_0^1(\Omega)). \tag{3.6}$$

Here

$$a(v, w) = \lambda(v, w) + \nu(\nabla v, \nabla w) + \left(\tilde{\nu} + \frac{\gamma^2}{\lambda}\right)(\text{div}\, v, \text{div}\, w).$$

It is easy to see that

$$|a(v, w)| \leq C\|v\|_{H^1}\|w\|_{H^1},$$
$$\text{Re}\, a(v, w) \geq \lambda\|v\|_2^2 + \nu\|\nabla v\|_2^2 + \left(\tilde{\nu} + \frac{\gamma^2}{\lambda}\right)\|\text{div}\, v\|_2^2 \geq c\|v\|_{H^1}^2$$

for some positive constants c and C. The Lax-Milgram theorem then implies that for any $G \in L^2(\Omega)$ there exists a unique solution $v \in H_0^1(\Omega)$ of (3.6). Since $B_\lambda = -\nu\Delta v - (\tilde{\nu} + \gamma^2/\lambda)\nabla\, \text{div}\, v$ is strongly elliptic for $\lambda > 1$, we see that $v \in H^2(\Omega)$. For this v we define ϕ by (3.4). Then $\phi \in H^1(\Omega)$, and, therefore, $u = {}^T(\phi, v)$ is a solution of (3.1) belonging to $D(L)$. The existence of solutions for other $\lambda \in \Sigma(\Lambda_0, \theta_0)$ follows from the estimate already obtained above and the standard perturbation argument. This completes the proof. □

Proposition 3.1 shows that $-L$ generates an analytic semigroup e^{-tL} on $H^1(\Omega) \times L^2(\Omega)$, which is represented as

$$e^{-tL} = \frac{1}{2\pi i}\int_{\Gamma_0} e^{\lambda t}(\lambda + L)^{-1}\, d\lambda,$$

where $\Gamma_0 = \{\lambda \in \mathbf{C}; |\arg(\lambda - \Lambda_0)| = \theta_0\}$ with Λ_0 and θ_0 given in Proposition 3.1.

To investigate the asymptotic behavior of e^{-tL} as $t \to \infty$, we consider the Fourier transform of the resolvent with respect to x_3 variable.

In what follows we denote

$$x = \begin{pmatrix} x' \\ x_3 \end{pmatrix}, \quad x' = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in D, \quad \nabla' = \begin{pmatrix} \partial_{x_1} \\ \partial_{x_2} \end{pmatrix}, \quad \triangle' = \partial_{x_1}^2 + \partial_{x_2}^2.$$

We also write

$$v = \begin{pmatrix} v' \\ v^3 \end{pmatrix}, \quad v' = \begin{pmatrix} v^1 \\ v^2 \end{pmatrix}, \quad g = \begin{pmatrix} g' \\ g^3 \end{pmatrix}, \quad g' = \begin{pmatrix} g^1 \\ g^2 \end{pmatrix}.$$

We take the Fourier transform of (3.2) and (3.3) in x_3 to obtain

$$\begin{cases} \lambda\hat{\phi} + \gamma\nabla' \cdot \hat{v}' + i\gamma\xi\hat{v}^3 = \hat{f}^0, \\ \lambda\hat{v}' - \nu\triangle'\hat{v}' + \nu\xi^2\hat{v}' - \tilde{\nu}\nabla'(\nabla' \cdot \hat{v}' + i\xi\hat{v}^3) + \gamma\nabla'\hat{\phi} = \hat{g}', \\ \lambda\hat{v}^3 - \nu\triangle'\hat{v}^3 + \nu\xi^2\hat{v}^3 - i\tilde{\nu}\xi(\nabla' \cdot \hat{v}' + i\xi\hat{v}^3) + i\gamma\xi\hat{\phi} = \hat{g}^3, \\ \hat{v}|_{\partial D} = 0. \end{cases} \tag{3.7}$$

For simplicity in notation we omit "ˆ" in (3.7), and so, the problem under consideration is written as

$$\lambda\phi + \gamma\nabla' \cdot v' + i\gamma\xi v^3 = f^0, \tag{3.8}$$

$$\lambda v' - \nu\triangle' v' + \nu\xi^2 v' - \tilde{\nu}\nabla'(\nabla' \cdot v' + i\xi v^3) + \gamma\nabla'\phi = g', \tag{3.9}$$

$$\lambda v^3 - \nu\triangle' v^3 + \nu\xi^2 v^3 - i\tilde{\nu}\xi(\nabla' \cdot v' + i\xi v^3) + i\gamma\xi\phi = g^3, \tag{3.10}$$

$$v|_{\partial D} = 0. \tag{3.11}$$

Here f^0, g', g^3 are given functions on D with values in $\mathbf{C}$ and ϕ, v', v^3 are unknown functions on D with values in $\mathbf{C}$. Problem (3.8)–(3.11) is also written as

$$\lambda u + \hat{L}_\xi u = f, \tag{3.12}$$

where $f = {}^T(f^0, g', g^3)$, $u = {}^T(\phi, v', v^3)$ and $\hat{L}_\xi$ is the operator on $H^1(D) \times L^2(D)$ with domain $D(\hat{L}_\xi)$ defined by

$$\hat{L}_\xi = \begin{pmatrix} 0 & \gamma^T\nabla' & i\gamma\xi \\ \gamma\nabla' & -\nu\triangle' I_2 + \nu\xi^2 I_2 - \tilde{\nu}\nabla'{}^T\nabla' & -i\tilde{\nu}\xi\nabla' \\ i\gamma\xi & -i\tilde{\nu}\xi^T\nabla' & -\nu\triangle' + (\nu + \tilde{\nu})\xi^2 \end{pmatrix},$$

$$D(\hat{L}_\xi) = H^1(D) \times [H^2(D) \times H_0^1(D)].$$

In the remaining of this section we investigate the Fourier transform of the resolvent $u = (\lambda + \hat{L}_\xi)^{-1} f$ for $|\xi| \geq r > 0$, where r is any fixed positive number. We will show that for any $r > 0$ there are numbers $\Lambda_1 > 0$ and $\theta_1 \in (\pi/2, \pi)$ such that $\Sigma(-\Lambda_1, \theta_1) \subset \rho(-\hat{L}_\xi)$ for $|\xi| \geq r$ and that $(\lambda + \hat{L}_\xi)^{-1}$ satisfies suitable estimates. The proof is given by an L^2-type energy method similar to that for the nonlinear problem given by Matsumura and Nishida [17]. There are several steps different from the one in [17], since the computations are done for any fixed ξ. Among them, Proposition 3.11 is one of the key steps.

In the following we denote by $u = (\lambda + \hat{L}_\xi)^{-1} f$ the solution of (3.12) belonging to $D(\hat{L}_\xi)$.

Proposition 3.2. *There holds the estimate*

$$(\mathrm{Re}\,\lambda + c|\mathrm{Im}\,\lambda|^2)|u|_2^2 + \nu|\xi|^2|v|_2^2 + \frac{\nu}{2}|\partial_{x'} v|_2^2 + \frac{\tilde{\nu}}{2}|\nabla' \cdot v' + i\xi v^3|_2^2$$
$$\leq \varepsilon|\phi|_2^2 + C_\varepsilon|f^0|_2^2 + C|g|_2^2$$

for any $\varepsilon \in (0, 1]$.

Proof. Taking the inner product of (3.12) with u and integrating by parts we have

$$\begin{aligned}&\lambda|u|_2^2 + \nu|\xi|^2|v|_2^2 + \nu|\partial_{x'} v|_2^2 + \tilde{\nu}|\nabla' \cdot v' + i\xi v^3|_2^2 \\ &+ \gamma(\nabla' \cdot v' + i\xi v^3, \phi) - \gamma(\phi, \nabla' \cdot v' + i\xi v^3) \\ &= (f, u).\end{aligned} \tag{3.13}$$

Since

$$\gamma(\nabla' \cdot v' + i\xi v^3, \phi) - \gamma(\phi, \nabla' \cdot v' + i\xi v^3) = 2i\gamma\,\mathrm{Im}(\nabla' \cdot v' + i\xi v^3, \phi),$$

we see from (3.13) that

$$\mathrm{Re}\,\lambda|u|_2^2 + \nu|\xi|^2|v|_2^2 + \nu|\partial_{x'} v|_2^2 + \tilde{\nu}|\nabla' \cdot v' + i\xi v^3|_2^2 = \mathrm{Re}(f, u), \tag{3.14}$$

$$\mathrm{Im}\,\lambda|u|_2^2 + 2\gamma\,\mathrm{Im}(\nabla' \cdot v' + i\xi v^3, \phi) = \mathrm{Im}(f, u). \tag{3.15}$$

By (3.15), we have

$$\begin{aligned}|\mathrm{Im}\,\lambda|^2|u|_2^4 &= |\mathrm{Im}(f, u) - 2\gamma\,\mathrm{Im}(\nabla' \cdot v' + i\xi v^3, \phi)|^2 \\ &\leq C\{|\nabla' \cdot v' + i\xi v^3|_2^2 + |f|_2^2\}|u|_2^2,\end{aligned}$$

and whence,

$$|\mathrm{Im}\,\lambda|^2|u|_2^2 \leq C\{|\nabla' \cdot v' + i\xi v^3|_2^2 + |f|_2^2\}. \tag{3.16}$$

It follows from (3.14) and (3.16) that for any $\eta > 0$ and $\varepsilon > 0$, there holds the estimate

$$\begin{aligned}&(\mathrm{Re}\,\lambda + c|\mathrm{Im}\,\lambda|^2)|u|_2^2 + \nu|\xi|^2|v|_2^2 + \nu|\partial_{x'} v|_2^2 + \frac{\tilde{\nu}}{2}|\nabla' \cdot v' + i\xi v^3|_2^2 \\ &\leq C\{|(f, u)| + |f|_2^2\} \\ &\leq C_\eta|g|_2^2 + \eta|v|_2^2 + C_\varepsilon|f^0|_2^2 + \varepsilon|\phi|_2^2.\end{aligned} \tag{3.17}$$

Since $|\partial_{x'} v|_2^2 \geq C|v|_2^2$ by Poincaré's inequality, we obtain the desired estimate by taking $\eta > 0$ suitably small in (3.17). This complete the proof. □

Proposition 3.3. *There holds the estimate*

$$\begin{aligned}&(\mathrm{Re}\,\lambda + c|\mathrm{Im}\,\lambda|^2)|u|_2^2 + (\mathrm{Re}\,\lambda + c|\mathrm{Im}\,\lambda| + c)\{|\xi|^2|v|_2^2 + |\partial_{x'} v|_2^2\} + c|\lambda|^2|v|_2^2\\ &\leq \varepsilon|\phi|_2^2 + C_\varepsilon|f^0|_2^2 + C|g|_2^2\end{aligned}$$

for any $\varepsilon \in (0, 1]$.

Proof. We compute the inner products $((3.9), \lambda v')$ and $((3.10), \lambda v^3)$, and then add the resulting identities to have

$$\begin{aligned}&|\lambda|^2|v|_2^2 + \nu\overline{\lambda}(|\xi|^2|v|_2^2 + |\partial_{x'} v|_2^2)\\ &+ \tilde{\nu}\overline{\lambda}|\nabla' \cdot v' + i\xi v^3|_2^2 - \gamma\overline{\lambda}(\phi, \nabla' \cdot v' + i\xi v^3)\\ &= \overline{\lambda}(g, v).\end{aligned} \tag{3.18}$$

Assume that $\lambda \neq 0$. It then follows from (3.8) that

$$\phi = -\frac{\gamma}{\lambda}\{\nabla' \cdot v' + i\xi v^3\} + \frac{1}{\lambda} f^0.$$

We thus obtain

$$\gamma\overline{\lambda}(\phi, \nabla' \cdot v' + i\xi v^3) = -\gamma^2\frac{(\overline{\lambda})^2}{|\lambda|^2}|\nabla' \cdot v' + i\xi v^3|_2^2 + \gamma\frac{(\overline{\lambda})^2}{|\lambda|^2}(f^0, \nabla' \cdot v' + i\xi v^3).$$

Substituting this into (3.18), we have

$$\begin{aligned}&|\lambda|^2|v|_2^2 + \overline{\lambda}\{\nu|\xi|^2|v|_2^2 + |\partial_{x'} v|_2^2 + \tilde{\nu}|\nabla' \cdot v' + i\xi v^3|_2^2\}\\ &= -\gamma^2\frac{(\overline{\lambda})^2}{|\lambda|^2}|\nabla' \cdot v' + i\xi v^3|_2^2 + \gamma\frac{(\overline{\lambda})^2}{|\lambda|^2}(f^0, \nabla' \cdot v' + i\xi v^3) + \overline{\lambda}(g, v).\end{aligned}$$

It then follows that

$$\begin{aligned}&|\lambda|^2|v|_2^2 + \mathrm{Re}\,\lambda\{\nu|\xi|^2|v|_2^2 + \nu|\partial_{x'} v|_2^2 + \tilde{\nu}|\nabla' \cdot v' + i\xi v^3|_2^2\}\\ &= \mathrm{Re}\left\{-\gamma^2\frac{(\overline{\lambda})^2}{|\lambda|^2}|\nabla' \cdot v' + i\xi v^3|_2^2 + \gamma\frac{(\overline{\lambda})^2}{|\lambda|^2}(f^0, \nabla' \cdot v' + i\xi v^3) + \overline{\lambda}(g, v)\right\},\end{aligned} \tag{3.19}$$

$$
\begin{aligned}
&- \operatorname{Im}\lambda\{\nu|\xi|^2|v|_2^2 + \nu|\partial_{x'}v|_2^2 + \tilde{\nu}|\nabla'\cdot v' + i\xi v^3|_2^2\} \\
&= \operatorname{Im}\left\{-\gamma^2\frac{(\bar{\lambda})^2}{|\lambda|^2}|\nabla'\cdot v' + i\xi v^3|_2^2 + \gamma\frac{(\bar{\lambda})^2}{|\lambda|^2}(f^0, \nabla'\cdot v' + i\xi v^3) + \bar{\lambda}(g, v)\right\}.
\end{aligned} \tag{3.20}
$$

Since

$$
\begin{aligned}
&\left|-\gamma^2\frac{(\bar{\lambda})^2}{|\lambda|^2}|\nabla'\cdot v' + i\xi v^3|_2^2 + \gamma\frac{(\bar{\lambda})^2}{|\lambda|^2}(f^0, \nabla'\cdot v' + i\xi v^3) + \bar{\lambda}(g, v)\right| \\
&\le \gamma^2|\nabla'\cdot v' + i\xi v^3|_2^2 + \gamma|f^0|_2|\nabla'\cdot v' + i\xi v^3|_2 + |\lambda||g|_2|v|_2 \\
&\le C\{|\xi|^2|v|_2^2 + |\partial_{x'}v|_2^2 + |f|_2^2\} + \frac{1}{4}|\lambda|^2|v|_2^2,
\end{aligned}
$$

we deduce from (3.19) and (3.20) that

$$
\begin{aligned}
&(\operatorname{Re}\lambda + c|\operatorname{Im}\lambda| + c)\{\nu|\xi|^2|v|_2^2 + \nu|\partial_{x'}v|_2^2 + \tilde{\nu}|\nabla'\cdot v' + i\xi v^3|_2^2\} + c|\lambda|^2|v|_2^2 \\
&\le C\{|\xi|^2|v|_2^2 + |\partial_{x'}v|_2^2 + |f|_2^2\}.
\end{aligned}
$$

This, together with Proposition 3.2, yields the desired estimate. In case $\lambda = 0$, the desired estimate is nothing but the one obtained in Proposition 3.2. This completes the proof. □

We next establish the estimates for higher order derivatives near the boundary ∂D. For this purpose, we introduce a local curvilinear coordinate system. Let $\bar{x}' \in \partial D$. Since ∂D is smooth, there are an open neighborhood $\mathcal{O}$ of $\bar{x}'$, a ball B of $\mathbf{R}^2$ with center 0, and a smooth map $\Phi = {}^T(\Phi_1, \Phi_2): \mathcal{O} \to B$ with the following properties.

$$
\det(\nabla_{x'}\Phi) \neq 0 \quad \text{on } \overline{\mathcal{O}}, \ \Phi \text{ and } \Phi^{-1} \text{ are } C^\infty \text{ maps.} \tag{3.21}
$$

$$
\begin{aligned}
&\Phi(\bar{x}') = 0, \quad \Phi(D \cap \mathcal{O}) = \{y' = {}^T(y_1, y_2) \in B; y_1 > 0\}, \\
&\Phi(\partial D \cap \mathcal{O}) = \{y' = {}^T(y_1, y_2) \in B; y_1 = 0\}.
\end{aligned} \tag{3.22}
$$

By the implicit function theorem we may assume that there is a smooth function ψ on an open interval ω such that $\bar{x}' = {}^T(\psi(\bar{y}_2), \bar{y}_2)$ and $x' \in \partial D \cap \mathcal{O}$ is represented as $x' = {}^T(\psi(y_2), y_2)$ $(y_2 \in \omega)$ by taking $\mathcal{O}$ smaller if necessary. Set

$$
\begin{cases}
a_1(y_2) = \dfrac{\nabla_{x'}\Phi_1(x')}{|\nabla_{x'}\Phi_1(x')|} \quad (x' = {}^T(\psi(y_2), y_2)), \\[2ex]
a_2(y_2) = \dfrac{\tilde{a}_2(y_2)}{|\tilde{a}_2(y_2)|}, \quad \tilde{a}_2(y_2) = {}^T(\dot{\psi}(y_2), y_2),
\end{cases} \tag{3.23}
$$

where $\dot{\psi} = d\psi/dy_2$. Then $a_1(y_2)$ and $a_2(y_2)$ are the unit inner normal vector and a unit tangent vector at $x' = {}^T(\psi(y_2), y_2) \in \partial D$, respectively. Note that by the orthonormality of $\{a_1(y_2), a_2(y_2)\}$ there holds the relation

$$\begin{pmatrix} \dot{a}_1(y_2) \\ \dot{a}_2(y_2) \end{pmatrix} = \begin{pmatrix} 0 & k(y_2) \\ -k(y_2) & 0 \end{pmatrix} \begin{pmatrix} a_1(y_2) \\ a_2(y_2) \end{pmatrix}$$

for some $k(y_2)$. The tubular neighborhood theorem then implies that there exists a positive number such that $x' \in D \cap \mathcal{O}$ is represented as

$$x' = y_1 a_1(y_2) + \begin{pmatrix} \psi(y_2) \\ y_2 \end{pmatrix} \quad (y' = {}^T(y_1, y_2) \in \tilde{\mathcal{O}},\ y_1 > 0) \tag{3.24}$$

for some open neighborhood $\tilde{\mathcal{O}}$ of $\overline{y}' = {}^T(0, \overline{y}_2)$ by changing $\mathcal{O}$ suitably if necessary. It then follows that

$$\frac{\partial x'}{\partial y'} = (a_1(y_2),\ J(y_1, y_2)a_2(y_2)),$$

where $J(y_1, y_2) = |\tilde{a}_2(y_2)| + k(y_2)y_1$. We may assume that $J = J(y_1, y_2) > 0$ by changing $\mathcal{O}$ suitably if necessary. We thus obtain

$$\nabla_{x'} = A(y_1, y_2)\nabla_{y'} = a_1(y_2)\partial_{y_1} + \frac{1}{J(y_1, y_2)} a_2(y_2)\partial_{y_2},$$

and, by using the orthonormality,

$$\nabla_{y'} = (A(y_1, y_2))^{-1}\nabla_{x'} = a_1(y_2)\partial_{x_1} + J(y_1, y_2)a_2(y_2)\partial_{x_2}.$$

We write

$$(A(y_1, y_2))^{-1} = \begin{pmatrix} a^{11}(x') & a^{12}(x') \\ a^{21}(x') & a^{22}(x') \end{pmatrix}.$$

Then $a^{jk}(x')$ is smooth and

$$\partial_{y_j} = a^{j1}(x')\partial_{x_1} + a^{j2}(x')\partial_{x_2} \quad (j = 1, 2).$$

We note that ∂_{y_1} is the inward normal derivative at $x' = {}^T(\psi(y_2), y_2) \in \partial D$ and ∂_{y_2} is the tangential derivative at $x' = {}^T(\psi(y_2), y_2) \in \partial D$. In what follows we denote the normal and tangential derivatives by ∂_n and ∂, respectively, i.e.,

$$\partial_n = \partial_{y_1} = a^{11}(x')\partial_{x_1} + a^{12}(x')\partial_{x_2},$$
$$\partial = \partial_{y_2} = a^{21}(x')\partial_{x_1} + a^{22}(x')\partial_{x_2}.$$

If $v \in H^2(D)$, then $v|_{\partial D} = 0$ implies that $\partial^k v|_{\partial D\cap\mathcal{O}} = 0$ $(k = 0, 1)$. We also note that

$$\partial^k v = \sum_{|\alpha|=0}^{k} a^\alpha(x')\partial_{x'}^\alpha v$$

with some smooth $a^\alpha(x')$.

In the following we will denote by $[A, B]$ the commutator of A and B, i.e., $[A, B] = AB - BA$.

We fix a function $\chi \in C_0^\infty(\mathcal{O})$.

Lemma 3.4. *There hold the following estimates.*

(i) $|(\chi[\partial, \partial_{x'}]v, \chi\partial v)| \le C|\chi\partial_{x'}v|_2^2$.

(ii) $|(\chi[\partial, \partial_{x'}^2]v, \chi\partial v)| \le \eta|\chi\partial_{x'}\partial v|_2^2 + C_\eta|\partial_{x'}v|^2_{L^2(D\cap\mathcal{O})}$ *for all* $\eta > 0$ *and* $v \in H^2(D)$ *with* $\partial v|_{\partial D\cap\mathcal{O}} = 0$.

Proof. The estimate (i) follows from a direct computation. As for (ii), we have

$$\begin{aligned}[\partial, \partial_{x'}^2]v &= -\sum_{k=1}^{2}\partial_{x'}^2 a^{2k}(x')\partial_{x_k}v - 2\sum_{k=1}^{2}\partial_{x'}a^{2k}(x')\partial_{x_k}\partial_{x'}v\\ &= \sum_{k=1}^{2}\partial_{x'}^2 a^{2k}(x')\partial_{x_k}v - 2\sum_{k=1}^{2}\partial_{x'}(\partial_{x'}a^{2k}(x')\partial_{x_k}v)\\ &\equiv I_1 + I_2.\end{aligned}$$

As for I_1, we easily see $|(\chi I_1, \chi\partial v)| \le C|\chi\partial_{x'}v|_2^2$. As for I_2, by integrating by parts, we have

$$\begin{aligned}&|(\chi I_2, \chi\partial v)|\\ &\le C\sum_{k=1}^{2}|(\chi\partial_{x'}a^{2k}(x')\partial_{x_k}v, \chi\partial_{x'}\partial v) - 2(\partial_{x'}\chi\partial_{x'}a^{2k}(x')\partial_{x_k}v, \chi\partial v)|\\ &\le \eta|\chi\partial_{x'}\partial v|_2^2 + C_\eta|\partial_{x'}v|^2_{L^2(D\cap\mathcal{O})}.\end{aligned}$$

This completes the proof. □

We derive the estimate for ∂u similar to that in Proposition 3.2.

Proposition 3.5. *There holds the estimate*

$$\begin{aligned}&(\mathrm{Re}\,\lambda + c\varepsilon|\mathrm{Im}\,\lambda|)|\chi\partial u|_2^2 + c\{|\xi|^2|\chi\partial v|_2^2 + |\chi\partial_{x'}\partial v|_2^2\}\\ &\le \varepsilon|\chi\partial\phi|_2^2 + C_\varepsilon\{|\chi\partial f^0|_2^2 + |\partial_{x'}v|_2^2\} + C|\chi g|_2^2\end{aligned}$$

for any $\varepsilon \in (0, 1]$.

Proof. Applying ∂ to (3.8)–(3.11), we have

$$\lambda(\partial\phi) + \gamma\nabla'\cdot\partial v' + i\gamma\xi\partial v^3 = F^0, \tag{3.25}$$

$$\lambda(\partial v') - \nu\triangle'(\partial v') + \nu\xi^2\partial v' - \tilde{\nu}\nabla'(\nabla'\cdot\partial v' + i\xi\partial v^3) + \gamma\nabla'(\partial\phi) = G', \tag{3.26}$$

$$\lambda(\partial v^3) - \nu\triangle'\partial v^3 + \nu\xi^2\partial v^3 - i\tilde{\nu}\xi(\nabla'\cdot\partial v' + i\xi\partial v^3) + i\gamma\xi\partial\phi = G^3, \tag{3.27}$$

$$\partial v|_{\partial D\cap\mathcal{O}} = 0. \tag{3.28}$$

Here

$$\begin{aligned} F^0 &= \partial f^0 - \gamma[\partial, \nabla'\cdot]v', \\ G' &= \partial g' + \nu[\partial, \triangle']v' + \tilde{\nu}[\partial, \nabla'\nabla'\cdot]v' + i\tilde{\nu}\xi[\partial, \nabla']v^3 - \gamma[\partial, \nabla']\phi, \\ G^3 &= \partial g^3 + \nu[\partial, \triangle']v^3 + i\tilde{\nu}\xi[\partial, \nabla'\cdot]v'. \end{aligned}$$

In the following we set $F = {}^T(F^0, G', G^3)$, $G = {}^T(G', G^3)$.

We compute the inner products $(\chi(3.25), \chi\partial\phi)$, $(\chi(3.26), \chi\partial v')$ and $(\chi(3.27), \chi\partial v^3)$, and add the resulting identities, as in the proof of Proposition 3.2, to obtain

$$\begin{aligned} &\lambda|\chi\partial u|_2^2 + \nu|\xi|^2|\chi\partial v|_2^2 + \nu|\chi\partial_{x'}\partial v|_2^2 \\ &+ \tilde{\nu}|\chi(\nabla'\cdot\partial v' + i\xi\partial v^3)|_2^2 + 2i\gamma\,\mathrm{Im}(\chi\nabla'\cdot\partial v' + i\xi\partial v^3, \chi\partial\phi) \\ &= (\chi F, \chi\partial u) + \gamma(\partial\phi, \nabla'(\chi^2)\partial v') - \nu(\nabla'\partial v, \nabla'(\chi)^2\partial v) \\ &\quad - \tilde{\nu}(\nabla'\cdot\partial v', +i\xi\partial v^3, \nabla'(\chi)^2\partial v'). \end{aligned}$$

By Young's inequality, we have

$$|\mathrm{Im}(\chi(\nabla'\cdot\partial v' + i\xi\partial v^3), \chi\partial\phi)| \le \frac{\varepsilon}{2}|\chi\partial\phi|_2^2 + \frac{C}{\varepsilon}\{|\chi\nabla'(\partial v)|_2^2 + \xi^2|\chi\partial v|_2^2\}$$

for any $\varepsilon > 0$. Using Lemma 3.4 and Young's inequality, we obtain

$$|\mathrm{Im}(\chi F^0, \partial u)| \le \frac{\varepsilon}{2}|\chi\partial\phi|_2^2 + \frac{C}{\varepsilon}\{|\chi\partial f^0|_2^2 + |\partial_{x'}v|_{L^2(D\cap\mathcal{O})}^2\}$$

for any $\varepsilon > 0$. Furthermore, by integration by parts, we have

$$\begin{aligned} &|\mathrm{Im}\{(\chi G, \chi\partial v) - \nu(\nabla'\partial v, \nabla'(\chi^2)\partial v) - \tilde{\nu}(\nabla'\cdot\partial v' + \xi\partial v^3, \nabla'(\chi^2)\partial v')\}| \\ &\le \eta\{|\chi\nabla'(\partial v)|_2^2 + \xi^2|\chi\partial v|_2^2\} + C_\eta\{|\chi g|_2^2 + |\partial_{x'}v|_{L^2(D\cap\mathcal{O})}^2\} \end{aligned}$$

for any $\eta > 0$. It then follows that

$$
\begin{aligned}
&|\mathrm{Im}\,\lambda||\chi\partial u|_2^2 \\
&\le \varepsilon|\chi\partial\phi|_2^2 + \eta\{|\chi\nabla'(\partial v)|_2^2 + \xi^2|\chi\partial v|_2^2\} \\
&\quad + \frac{C}{\varepsilon}\{|\chi\nabla'(\partial v)|_2^2 + \xi^2|\chi\partial v|_2^2 + |\nabla'\partial f^0|_2^2 + |\partial_{x'}v|^2_{L^2(D\cap\mathcal{O})}\} \\
&\quad + C_\eta\{|\chi g|_2^2 + |\partial_{x'}v|^2_{L^2(D\cap\mathcal{O})}\}.
\end{aligned}
\tag{3.29}
$$

Similarly, we have

$$
\begin{aligned}
&\mathrm{Re}\,\lambda|\chi\partial u|_2^2 + \frac{\nu}{2}\{|\chi\nabla'(\partial v)|_2^2 + \xi^2|\chi\partial v|_2^2\} \\
&\le \varepsilon|\chi\partial\phi|_2^2 + C_\varepsilon\{|\chi\partial f^0|_2^2 + |\partial_{x'}v|^2_{L^2(D\cap\mathcal{O})}\} \\
&\quad + C|\chi g|_2^2 + C_\eta|\partial_{x'}v|^2_{L^2(D\cap\mathcal{O})} + \eta\{|\chi\nabla'(\partial v)|_2^2 + \xi^2|\chi\partial v|_2^2\}.
\end{aligned}
\tag{3.30}
$$

For $0 < \varepsilon \le 1$, adding (3.29) $\times\, \varepsilon\nu/(4C)$ and (3.30), we obtain the desired estimate by taking $\eta > 0$ suitably small. This completes the proof. □

We next estimate the normal derivative of ϕ.

Proposition 3.6. *There holds the estimate*

$$
\begin{aligned}
&(\mathrm{Re}\,\lambda + c|\mathrm{Im}\,\lambda|^2 + c)|\chi\partial_n\phi|_2^2 + c\left\{\left|\lambda + \frac{\gamma^2}{\nu+\tilde{\nu}}\right|^2|\chi\partial_n\phi|_2^2 + |\chi\partial_n(\nabla'\cdot v')|_2^2\right\} \\
&\le C\{|\chi\partial_{x'}f^0|_2^2 + |\chi g|_2^2 + |\lambda|^2|\chi v|_2^2 + \xi^2|\chi\partial_{x'}v|_2^2 \\
&\quad + \xi^4|\chi v|_2^2 + |\chi\partial_{x'}\partial v|_2^2 + |\chi\partial_{x'}v|_2^2\}.
\end{aligned}
$$

Proof. We set $\tilde{\phi}(y') = \phi(x')$ with $x' \in D\cap\mathcal{O}$ and $y' \in \tilde{\mathcal{O}}\cap\{y_1 > 0\}$ given in (3.24). Then our aim here is to estimate $\partial_{y_1}\tilde{\phi}$ on $\tilde{\mathcal{O}}\cap\{y_1 > 0\}$.

Let us derive an useful identity for $\partial_{y_1}\tilde{\phi}$. We transform $v'(x')$ into $\tilde{v}'(y')$ as $v'(x') = E'(y')\tilde{v}'(y')$, where $E'(y')$ is an orthogonal matrix defined by $E'(y') = (a_1(y_2), a_2(y_2))$ with $a_1(y_2)$ and $a_2(y_2)$ given in (3.23). We also define $\tilde{v}^3(y')$ by $\tilde{v}^3(y') = v^3(x')$ with y' and x' as above. We will derive the equations for $\tilde{\phi}(y')$ and ${}^T(\tilde{v}'(y'), \tilde{v}^3(y'))$.

For a moment, we denote by $\phi(x)$ and $v(x) = {}^T(v^1(x), v^2(x), v^3(x))$ $(x \in \Omega)$ the functions satisfying the original problem (3.2)–(3.3).

We make a transformation of the vector field $v(x)$. We transform $v(x)$ as $v(x) = E(y)\tilde{v}(y)$, where $x = {}^T(x', x_3)$ and $y = {}^T(y', y_3)$ with $x' \in D\cap\mathcal{O}$ and $y' \in \tilde{\mathcal{O}}\cap\{y_1 > 0\}$ as above and $y_3 = x_3 \in \mathbf{R}$, and $E(y)$ is an orthogonal matrix defined by $E(y) = (e_1(y_2), e_2(y_2), e_3)$ with $e_j(y_2) = {}^T(a_j(y_2), 0)$ $(j = 1, 2)$ and $e_3 = {}^T(0, 0, 1)$. We also define $\tilde{\phi}(y)$ by $\tilde{\phi}(y) = \phi(x)$ with x and y as above. Under these transformations, problem (3.2)–(3.3) is transformed into the following one on $\tilde{\mathcal{O}} \cap \{y_1 > 0\}$:

$$\begin{cases} \lambda\tilde{\phi} + \gamma \operatorname{div}_y \tilde{v} = \tilde{f}^0, \\ \lambda\tilde{v} + \nu \operatorname{rot}_y \operatorname{rot}_y \tilde{v} - (\nu + \tilde{\nu})\nabla_y \operatorname{div}_y \tilde{v} + \gamma\nabla_y\tilde{\phi} = \tilde{g}, \\ \tilde{v}|_{\tilde{\mathcal{O}}\cap\{y_1=0\}} = 0. \end{cases} \tag{3.31}$$

Here $f^0(x) = \tilde{f}^0(y)$ and $g(x) = E(y)\tilde{g}(y)$ with x, y and $E(y)$ as above, and ∇_y, div_y and rot_y denote the gradient, divergence and rotation in the curvilinear coordinates y which are written as

$$\nabla_y\tilde{\phi} = e_1\partial_{y_1}\tilde{\phi} + \frac{1}{J}e_2\partial_{y_2}\tilde{\phi} + e_3\partial_{y_3}\tilde{\phi},$$
$$\operatorname{div} \tilde{v} = \frac{1}{J}(\partial_{y_1}(J\tilde{v}^1) + \partial_{y_2}\tilde{v}^2 + \partial_{y_3}(J\tilde{v}^3)),$$
$$\operatorname{rot}_y \tilde{v} = (\operatorname{rot}_y \tilde{v})^1 e_1 + (\operatorname{rot}_y \tilde{v})^2 e_2 + (\operatorname{rot}_y \tilde{v})^3 e_3,$$

where $(\operatorname{rot}_y \tilde{v})^i$ is defined by

$$(\operatorname{rot}_y \tilde{v})^1 = \frac{1}{J}(\partial_{y_2}\tilde{v}^3 - \partial_{y_3}(J\tilde{v}^2)), \quad (\operatorname{rot}_y \tilde{v})^2 = \partial_{y_3}\tilde{v}^1 - \partial_{y_1}\tilde{v}^3,$$
$$(\operatorname{rot}_y \tilde{v})^3 = \frac{1}{J}(\partial_{y_1}(J\tilde{v}^2) - \partial_{y_2}\tilde{v}^1),$$
$$(\operatorname{rot}_y \operatorname{rot}_y \tilde{v})^1 = \frac{1}{J}\{\partial_{y_2}(\operatorname{rot}_y \tilde{v})^3 - \partial_{y_3}(J(\operatorname{rot}_y \tilde{v})^2)\},$$
$$(\operatorname{rot}_y \operatorname{rot}_y \tilde{v})^2 = \partial_{y_3}(\operatorname{rot}_y \tilde{v})^1 - \partial_{y_1}(\operatorname{rot}_y \tilde{v})^3,$$
$$(\operatorname{rot}_y \operatorname{rot}_y \tilde{v})^3 = \frac{1}{J}\{\partial_{y_1}(J(\operatorname{rot}_y \tilde{v})^2) - \partial_{y_2}(\operatorname{rot}_y \tilde{v})^1\}.$$

To obtain (3.31), we used $\Delta v = -\operatorname{rot}\operatorname{rot} v + \nabla \operatorname{div} v$.

We now take the Fourier transform of (3.31) in y_3. Then in the resulting equations we replace the Fourier transforms $\mathscr{F}\tilde{\phi}$ and ${}^T(\mathscr{F}\tilde{v}', \mathscr{F}\tilde{v}^3)$ by $\tilde{\phi}(y')$ and ${}^T(\tilde{v}'(y'), v^3(y'))$ to obtain the equations for $\tilde{\phi}(y')$ and ${}^T(\tilde{v}'(y'), v^3(y'))$:

$$\lambda\tilde{\phi} + \gamma\mathscr{F}(\operatorname{div}_y \tilde{v}) = \tilde{f}^0, \tag{3.32}$$

$$\lambda\tilde{v}^1 + \nu\mathscr{F}(\operatorname{rot}_y \operatorname{rot}_y \tilde{v})^1 - (\nu + \tilde{\nu})\mathscr{F}(\nabla_y \operatorname{div}_y \tilde{v})^1 + \gamma\partial_{y_1}\tilde{\phi} = \tilde{g}^1, \tag{3.33}$$
$$\lambda\tilde{v}^2 + \nu\mathscr{F}(\operatorname{rot}_y \operatorname{rot}_y \tilde{v})^2 - (\nu + \tilde{\nu})\mathscr{F}(\nabla_y \operatorname{div}_y \tilde{v})^2 + \frac{\gamma}{J}\partial_{y_2}\tilde{\phi} = \tilde{g}^2,$$
$$\lambda\tilde{v}^3 + \nu\mathscr{F}(\operatorname{rot}_y \operatorname{rot}_y \tilde{v})^3 - (\nu + \tilde{\nu})\mathscr{F}(\nabla_y \operatorname{div}_y \tilde{v})^3 + i\gamma\xi\tilde{\phi} = \tilde{g}^3.$$

Here $\mathscr{F}(\operatorname{div}_y \tilde{v})$, $\mathscr{F}(\operatorname{rot}_y \operatorname{rot}_y \tilde{v})^1, \dots,$ stand for the functions with ∂_{y_3} replaced by $i\xi$ in the functions $\operatorname{div}_y \tilde{v}$, $(\operatorname{rot}_y \operatorname{rot}_y \tilde{v})^1, \dots,$ respectively. These equations are the desired equations for $\tilde{\phi}(y')$ and ${}^T(\tilde{v}'(y'), v^3(y'))$.

Since equation (3.32) is written as

$$\lambda \tilde{v}^1 + \nu \mathscr{F}(\mathrm{rot}_y\ \mathrm{rot}_y\ \tilde{v})^1 - (\nu + \tilde{\nu})\partial_{y_1}\mathscr{F}(\mathrm{div}_y\ \tilde{v}) + \gamma \partial_{y_1}\tilde{\phi} = \tilde{g}^1, \tag{3.34}$$

we add ∂_{y_1}(3.32) and $(\gamma/(\nu + \tilde{\nu})) \times$ (3.34) to obtain

$$\left(\lambda + \frac{\gamma^2}{\nu + \tilde{\nu}}\right)\partial_{y_1}\tilde{\phi} = \partial_{y_1} \tilde{f}^0 + h. \tag{3.35}$$

Here

$$h = \frac{\gamma}{\nu + \tilde{\nu}}\{\tilde{g}^1 - \lambda \tilde{v}^1 - \nu \mathscr{F}(\mathrm{rot}_y\ \mathrm{rot}_y\ \tilde{v})^1\}. \tag{3.36}$$

Therefore, considering $\int_{\tilde{\mathcal{O}}\cap\{y_1>0\}} \tilde{\chi} \times (3.35) \times \tilde{\chi}\,\overline{\partial_{y_1}\tilde{\phi}}\, J\, dy'$ with $\tilde{\chi}(y') = \chi(x')$, we see that

$$\left(\lambda + \frac{\gamma^2}{\nu + \tilde{\nu}}\right)|\tilde{\chi}\,\partial_{y_1}\tilde{\phi}|_2^2 = (\tilde{\chi}\,\partial_{y_1} \tilde{f}^0,\ \tilde{\chi}\,\partial_{y_1}\tilde{\phi}) - (\tilde{\chi} h,\ \tilde{\chi}\,\partial_{y_1}\tilde{\phi}).$$

This implies that

$$\begin{aligned} &\left|\lambda + \frac{\gamma^2}{\nu + \tilde{\nu}}\right| |\tilde{\chi}\,\partial_{y_1}\tilde{\phi}|_2 \\ &\le C\{|\tilde{\chi}\,\partial_{y_1} \tilde{f}^0|_2 + |\tilde{\chi}\tilde{g}|_2 + |\lambda|\,|\tilde{\chi}\tilde{v}|_2 \\ &\qquad + |\xi|\,|\tilde{\chi}\,\partial_{y_1}\tilde{v}|_2 + \xi^2|\tilde{\chi}\tilde{v}|_2 + |\tilde{\chi}\,\partial_{y_2}((\mathrm{rot}_y\ \tilde{v})^3)|_2\} \end{aligned} \tag{3.37}$$

and

$$\begin{aligned} &\left(\mathrm{Re}\,\lambda + |\mathrm{Im}\,\lambda|^2 + \frac{\gamma^2}{2(\nu + \tilde{\nu})}\right)|\tilde{\chi}\,\partial_{y_1}\tilde{\phi}|_2^2 \\ &\le C\{|\tilde{\chi}\,\partial_{y_1} \tilde{f}^0|_2^2 + |\tilde{\chi}\tilde{g}|_2^2 + |\lambda|^2|\tilde{\chi}\tilde{v}|_2^2 \\ &\qquad + \xi^2|\tilde{\chi}\,\partial_{y_1}\tilde{v}|_2^2 + \xi^4|\tilde{\chi}\tilde{v}|_2^2 + |\tilde{\chi}\,\partial_{y_2}((\mathrm{rot}_y\ \tilde{v})^3)|_2^2\}. \end{aligned} \tag{3.38}$$

Since $\mathscr{F}(\mathrm{div}_y\ \tilde{v}) = (\mathrm{div}_{y'}\ \tilde{v}') + i\gamma\xi\tilde{v}^3$, we see from (3.32) that

$$\gamma \mathscr{F}(\mathrm{div}_{y'}\ \tilde{v}') = \tilde{f}^0 - \lambda\tilde{\phi} - i\gamma\xi\tilde{v}^3.$$

We thus obtain

$$\begin{aligned} &|\tilde{\chi}\,\partial_{y_1}\mathscr{F}(\mathrm{div}_{y'}\ \tilde{v}')|_2^2 \\ &\le C\left\{|\tilde{\chi}\,\partial_{y_1} \tilde{f}^0|_2^2 + \left|\lambda + \frac{\gamma^2}{\nu + \tilde{\nu}}\right|^2 |\tilde{\chi}\,\partial_{y_1}\tilde{\phi}|_2^2 + |\tilde{\chi}\,\partial_{y_1}\tilde{\phi}|_2^2 + \xi^2|\tilde{\chi}\,\partial_{y_1}\tilde{v}^3|_2^2\right\}. \end{aligned} \tag{3.39}$$

The desired estimate follows from (3.37), (3.38) and (3.39) by inverting to the original coordinates x' and noting $\partial_{y_2} = \partial$. This completes the proof. □

We next derive the estimate for the derivative of $\nabla' \cdot v'$.

Proposition 3.7. *There holds the estimate*

$$\begin{aligned}
&(\mathrm{Re}\,\lambda + c\varepsilon|\mathrm{Im}\,\lambda|)|\chi\,\partial u|_2^2 + (\mathrm{Re}\,\lambda + c|\mathrm{Im}\,\lambda|^2 + c)|\chi\,\partial_n\phi|_2^2 \\
&+ c\left\{\xi^2|\chi\,\partial v|_2^2 + |\chi\,\partial_{x'}\partial v|_2^2 + \left|\lambda + \frac{\gamma^2}{\nu+\tilde{\nu}}\right|^2|\chi\,\partial_n\phi|_2^2 + |\chi\,\partial_{x'}(\nabla'\cdot v')|_2^2\right\} \\
&\le \varepsilon|\chi\,\partial\phi|_2^2 + C_\varepsilon\{|\chi\,\partial_{x'} f^0|_2^2 + |\partial_{x'}v|_2^2\} \\
&\quad + C\{|\lambda|^2|\chi v|_2^2 + \xi^2|\chi\,\partial_{x'}v|_2^2 + \xi^4|\chi v|_2^2 + |\chi g|_2^2\}
\end{aligned}$$

for any $\varepsilon \in (0, 1]$.

Proof. By Lemma 3.4 we have

$$\begin{aligned}
|\chi\,\partial(\nabla'\cdot v')|_2^2 &\le C\{|\chi(\nabla'\cdot\partial v')|_2^2 + |\chi[\partial, \nabla'\cdot]v'|_2^2\} \\
&\le C\{|\chi(\nabla'\cdot\partial v')|_2^2 + |\chi\,\partial_{x'}v|_2^2\}.
\end{aligned}$$

This, together with Propositions 3.5 and 3.6, implies that

$$\begin{aligned}
&(\mathrm{Re}\,\lambda + c\varepsilon|\,\mathrm{Im}\,\lambda|)|\chi\,\partial u|_2^2 + (\mathrm{Re}\,\lambda + c|\,\mathrm{Im}\,\lambda|^2 + c)|\chi\,\partial_n\phi|_2^2 \\
&+ c\left\{\xi^2|\chi\,\partial v|_2^2 + |\chi\,\partial_{x'}\partial v|_2^2 + \left|\lambda + \frac{\gamma^2}{\nu+\tilde{\nu}}\right|^2|\chi\,\partial_n\phi|_2^2\right. \\
&\qquad \left. + |\chi\,\partial(\nabla'\cdot v')|_2^2 + |\chi\,\partial_n(\nabla'\cdot v')|_2^2\right\} \\
&\le \varepsilon|\chi\,\partial\phi|_2^2 + C_\varepsilon\{|\chi\,\partial_{x'} f^0|_2^2 + |\partial_{x'}v|_{L^2(D\cap\mathcal{O})}^2\} \\
&\quad + C\{|\lambda|^2|\chi v|_2^2 + \xi^2|\chi\,\partial_{x'}v|_2^2 + \xi^4|\chi v|_2^2 + |\chi g|_2^2\}.
\end{aligned}$$

Since

$$|\chi\,\partial_{x'}(\nabla'\cdot v')|_2^2 \le C\{|\chi\,\partial(\nabla'\cdot v')|_2^2 + |\chi\,\partial_n(\nabla'\cdot v')|_2^2\},$$

we have the desired estimate. This completes the proof. □

We next derive the interior estimate for the derivative of ϕ. We fix a function $\chi_0 \in C_0^\infty(D)$.

Proposition 3.8. *There holds the estimate*

$$\left(\left|\lambda + \frac{\gamma^2}{\nu+\tilde{\nu}}\right|^2 + \mathrm{Re}\,\lambda + |\mathrm{Im}\,\lambda|^2 + \frac{\gamma^2}{2(\nu+\tilde{\nu})}\right)\{|\chi_0\partial_{x'}\phi|_2^2 + \xi^2|\chi_0\phi|_2^2\}$$
$$\leq C\{|\partial_{x'} f^0|_2^2 + |g|_2^2 + \xi^2|f^0|_2^2 + |\lambda|^2|v|_2^2 + |\partial_{x'} v|_2^2 + \xi^2|v|_2^2\}.$$

Proof. We compute

$$(\chi_0\nabla'(3.8),\ \chi_0\nabla'\phi) + (\chi_0 i\xi(3.8),\ \chi_0 i\xi\phi)$$
$$+ \frac{\gamma}{\nu+\tilde{\nu}}\{(\chi_0(3.9),\ \chi_0\nabla'\phi) + (\chi_0(3.10),\ \chi_0 i\xi\phi)\}.$$

By integration by parts, we have

$$\begin{aligned}
&(\chi_0(\Delta' v' - \xi^2 v'),\ \chi_0\nabla'\phi) + (\chi_0(\Delta' v^3 - \xi^2 v^3),\ \chi_0 i\xi\phi)\\
&= (\chi_0\nabla'(\nabla'\cdot v' + i\xi v^3),\ \chi_0\nabla'\phi) + (\chi_0 i\xi(\nabla'\cdot v' + i\xi v^3),\ \chi_0 i\xi\phi)\\
&\quad - ({}^T\nabla'(\chi_0^2)\nabla' v',\ \nabla'\phi) + (\nabla' v'\nabla'(\chi_0^2),\ \nabla'\phi) - (\nabla'(\chi_0^2)\cdot\nabla' v^3,\ i\xi\phi)\\
&\quad + (\nabla'(\chi_0^2)\cdot i\xi v',\ i\xi\phi),
\end{aligned}$$

where $\nabla' v'$ is the 2×2 matrix $(\partial_k v^j)$. Noting this fact, we see that the term

$$\gamma\{(\chi_0\nabla'(\nabla'\cdot v' + i\xi v^3),\ \chi_0\nabla'\phi) + (\chi_0 i\xi(\nabla'\cdot v' + i\xi v^3),\ \chi_0 i\xi\phi)\}$$

vanishes. We thus obtain

$$\left(\lambda + \frac{\gamma^2}{\nu+\tilde{\nu}}\right)\{|\chi_0\nabla'\phi|_2^2 + |\chi_0 i\xi\phi|_2^2\} = F.$$

Here

$$\begin{aligned}
F &= (\chi_0\nabla' f^0,\ \chi_0\nabla'\phi) + (\chi_0 i\xi f^0,\ \chi_0 i\xi\phi) + \frac{\gamma}{\nu+\tilde{\nu}}\{(\chi_0 g,\ \chi_0\nabla'\phi) + (\chi_0 g^3,\ \chi_0 i\xi\phi)\}\\
&\quad - \frac{\gamma}{\nu+\tilde{\nu}}\{\lambda(\chi_0 v',\ \chi_0\nabla'\phi) + \lambda(\chi_0 v^3,\ \chi_0 i\xi\phi) + ({}^T\nabla'(\chi_0^2)\nabla' v',\ \nabla'\phi)\\
&\qquad - (\nabla' v'\nabla'(\chi_0^2),\ \nabla'\phi) + (\nabla'(\chi_0^2)\cdot\nabla' v^3,\ i\xi\phi) - (\nabla'(\chi_0^2)\cdot i\xi v',\ i\xi\phi)\}.
\end{aligned}$$

Since F is estimated as

$$\begin{aligned}
|F| &\leq C\{|\chi_0\nabla'\phi|_2 + |\chi_0 i\xi\phi|_2\}\\
&\qquad \times\{|\partial_{x'} f^0|_2 + |\xi||f^0|_2 + |g|_2 + |\lambda||v|_2 + |\partial_{x'} v|_2 + |\xi||v|_2\}\\
&\leq \frac{\gamma^2}{8(\nu+\tilde{\nu})}\{|\chi_0\nabla'\phi|_2^2 + |\chi_0 i\xi\phi|_2^2\}\\
&\qquad + \{|\partial_{x'} f^0|_2^2 + \xi^2|f^0|_2^2 + |g|_2^2 + |\lambda|^2|v|_2^2 + |\partial_{x'} v|_2^2 + \xi^2|v|_2^2\},
\end{aligned}$$

we obtain the desired estimate. This completes the proof. □

We next derive the interior estimate for the derivative of $\nabla' \cdot v'$.

Proposition 3.9. *For* λ *satisfying* $\mathrm{Re}\,\lambda + |\mathrm{Im}\,\lambda| + \gamma^2/(4(\nu+\tilde{\nu})) \geq 0$, *there holds the estimate*

$$|\chi_0 \partial_{x'}(\nabla' \cdot v')|_2^2 \leq C\{|\partial_{x'} f^0|_2^2 + \xi^2 |f^0|_2^2 + |\lambda|^2 |v|_2^2 + |\partial_{x'} v|_2^2 + \xi^4 |v|_2^2 + \xi^2 |\partial_{x'} v|_2^2\}.$$

Proof. Since $\gamma > 0$, we see from (3.8) that

$$\nabla' \cdot v' = \frac{1}{\gamma}\{f^0 - \lambda\phi - i\gamma\xi v^3\}.$$

It follows from Proposition 3.8 that if $\mathrm{Re}\,\lambda + |\mathrm{Im}\,\lambda| + \gamma^2/(4(\nu+\tilde{\nu})) \geq 0$, then

$$\begin{aligned}
&|\chi_0 \partial_{x'}(\nabla' \cdot v')|_2^2 \\
&\leq C\{|\partial_{x'} f^0|_2^2 + |\lambda|^2 |\chi_0 \partial_{x'}\phi|_2^2 + \xi^2 |\chi_0 \partial_{x'} v^3|_2^2\} \\
&\leq C\left\{|\partial_{x'} f^0|_2^2 + \left|\lambda + \frac{\gamma^2}{(\nu+\tilde{\nu})}\right|^2 |\chi_0 \partial_{x'}\phi|_2^2 + \left(\frac{\gamma^2}{(\nu+\tilde{\nu})}\right)^2 |\chi_0 \partial_{x'}\phi|_2^2 + \xi^2 |\partial_{x'} v|_2^2\right\} \\
&\leq C\{|\partial_{x'} f^0|_2^2 + \xi^2 |f^0|_2^2 + |g|_2^2 + |\lambda|^2 |v|_2^2 + |\partial_{x'} v|_2^2 + \xi^2 |v|_2^2 + \xi^2 |\partial_{x'} v|_2^2\}.
\end{aligned}$$

This completes the proof. □

Proposition 3.10. *Let* λ *satisfy* $\mathrm{Re}\,\lambda + |\mathrm{Im}\,\lambda| + \gamma^2/(4(\nu+\tilde{\nu})) \geq 0$. *Then there holds the estimate*

$$\begin{aligned}
&(\mathrm{Re}\,\lambda + c|\mathrm{Im}\,\lambda|^2)|u|_2^2 + (\mathrm{Re}\,\lambda + c|\mathrm{Im}\,\lambda| + c)\{\xi^2 |v|_2^2 + |\partial_{x'} u|_2^2\} \\
&+ c\left\{|\lambda|^2 |v|_2^2 + \left|\lambda + \frac{\gamma^2}{\nu+\tilde{\nu}}\right|^2 |\partial_{x'}\phi|_2^2 + |\partial_{x'}^2 v'|_2^2\right\} \\
&\leq \varepsilon|\phi|_2 + C_\varepsilon |f^0|_2 + C\{|\partial_{x'} f^0|_2^2 + \xi^2 |f^0|_2^2 + |g|_2^2 + \xi^4 |v|_2^2 + \xi^2 |\partial_{x'} v|_2^2\}
\end{aligned}$$

for any $\varepsilon \in (0, 1]$.

Proof. We see from (3.9) that

$$\begin{cases} -\nu\triangle' v' + \gamma\nabla'\phi = g' - \{\lambda v' + \nu\xi^2 v' - \tilde{\nu}\nabla'(\nabla' \cdot v') - i\tilde{\nu}\xi\nabla' v^3\}, \\ v'|_{\partial D} = 0. \end{cases}$$

Applying the regularity estimates for the Stokes equations on bounded domains (e.g., [1]), we have

$$\begin{aligned}
&|\partial_{x'}^2 v'|_2^2 + |\partial_{x'}\phi|_2^2 \\
&\leq C\{|g'|_2^2 + |\nabla' \cdot v'|_{H^1}^2 + |\lambda|^2 |v|_2^2 + \xi^4 |v|_2^2 + \xi^2 |\partial_{x'} v|_2^2 + |\partial_{x'} v|_2^2\}.
\end{aligned}$$

This, together with Proposition 3.3, implies that

$$
\begin{aligned}
&(\operatorname{Re}\lambda + c|\operatorname{Im}\lambda|^2)|u|_2^2 + (\operatorname{Re}\lambda + c|\operatorname{Im}\lambda| + c)\{\xi^2|v|_2^2 + |\partial_{x'}v|_2^2\}\\
&+ c\{|\partial_{x'}^2 v'|_2^2 + |\partial_{x'}\phi|_2^2 + |\lambda|^2|v|_2^2\}\\
&\le \varepsilon|\phi|_2^2 + C_\varepsilon|f^0|_2^2 + C\{|g|_2^2 + \xi^4|v|_2^2 + \xi^2|\partial_{x'}v|_2^2 + |\nabla'\cdot v'|_{H^1}^2\}.
\end{aligned}
\tag{3.40}
$$

Let us estimate $|\nabla'\cdot v'|_{H^1}$ on the right of (3.40). We take an open covering $\{\mathcal{O}_m\}_{m=0}^N$ of D, a partition of unity $\{\chi_m\}_{m=0}^N$ subordinate to $\{\mathcal{O}_m\}_{m=0}^N$, and C^∞ maps $\{\Phi_m\}_{m=1}^N$ with the following properties.

(i) $\overline{\mathcal{O}_0} \subset D$, $D \cap \mathcal{O}_m \neq \emptyset$ $(m = 1, \dots, N)$.

(ii) $\sum_{m=0}^N \chi_m \equiv 1$ on D, $\chi_m \in C_0^\infty(\mathcal{O}_m)$ $(m = 0, 1, \dots, N)$.

(iii) For each $m = 1, \dots, N$, $\mathcal{O}_m$ and Φ_m have the properties as those of $\mathcal{O}$ and Φ stated in (3.21) and (3.22) so that there exists a local curvilinear coordinate system on $\mathcal{O}_m$ such as $y' = {}^T(y_1, y_2) \in \tilde{\mathcal{O}}$ given in (3.24).

Note that the estimate in Proposition 3.7 holds for $\mathcal{O} = \mathcal{O}_m$ and $\chi = \chi_m$ $(m = 1, \dots, N)$ with constants c and C uniformly in $m = 1, \dots, N$.

Combining Propositions 3.7–3.9 with (3.40), we see that if $\operatorname{Re}\lambda + |\operatorname{Im}\lambda| + \gamma^2/(4(\nu + \tilde{\nu})) \ge 0$, then

$$
\begin{aligned}
&(\operatorname{Re}\lambda + c|\operatorname{Im}\lambda|^2)|u|_2^2 + (\operatorname{Re}\lambda + c|\operatorname{Im}\lambda| + c)\{\xi^2|v|_2^2 + |\partial_{x'}u|_2^2\}\\
&+ c\left\{\left|\lambda + \frac{\gamma^2}{\nu + \tilde{\nu}}\right|^2 |\partial_{x'}\phi|_2^2 + |\partial_{x'}^2 v'|_2^2 + |\lambda|^2|v|_2^2\right\}\\
&\le \varepsilon_1|\partial_{x'}\phi|_2^2 + \varepsilon|\phi|_2^2 + C_\varepsilon|f^0|_2^2 + C_{\varepsilon_1}\{|\partial_{x'}f^0|_2^2 + |\partial_{x'}v|_2^2\}\\
&\quad + C\{\xi^2|f^0|_2^2 + |g|_2^2 + \xi^4|v|_2^2 + \xi^2|\partial_{x'}v|_2^2\}.
\end{aligned}
$$

Taking $\varepsilon_1 > 0$ sufficiently small and estimating $|\partial_{x'}v|_2^2$ by Proposition 3.3, we obtain the desired estimate. This completes the proof. □

The following proposition is a key step to obtain a dissipative estimate for $|\phi|_2$. We make use of an orthogonal decomposition of ϕ. We decompose ϕ as

$$
\phi = \phi_0 + \phi_1, \quad \phi_0 = \langle\phi\rangle = \frac{1}{|D|}\int_D \phi(x')\,dx'.
$$

As for this decomposition, the following relations hold:

$$
|\phi|_2^2 = |\phi_0|_2^2 + |\phi_1|_2^2, \quad |\phi_1|_2 \le C|\partial_{x'}\phi_1|_2 = C|\partial_{x'}\phi|_2.
$$

Here, the latter inequality follows from Poincaré's inequality, since ϕ_1 satisfies $\int_D \phi_1(x')\,dx' = 0$.

Proposition 3.11. *Let $r > 0$. Then there are positive constants $C_1 = C_1(r)$ and $C_2 = C_2(r)$ such that the following estimates hold uniformly for $|\xi| \geq r$.*

$$(\operatorname{Re}\lambda + |\operatorname{Im}\lambda|^2)|\phi|_2^2 + C_1|\phi_0|_2^2 \leq C_2\{|\lambda|^2|v|_2^2 + \xi^2|v|_2^2 + |\partial_{x'}v|_2^2 + |\partial_{x'}\phi|_2^2 + |f|_2^2\}.$$

Proof. We define an operator A with domain $D(A)$ by $A\varphi = -\nu\triangle'\varphi$ for $\varphi \in D(A) = H^2(D) \cap H_0^1(D)$. By (3.10), we have

$$v^3 = -(\xi^2 + A)^{-1}\{\lambda v^3 - i\tilde{\nu}\xi(\nabla' \cdot v' + i\xi v^3) + i\gamma\xi\phi - g^3\}.$$

Substituting this into (3.8), we arrive at

$$\lambda\phi + \gamma^2\xi^2(\xi^2 + A)^{-1}\phi = h. \tag{3.41}$$

Here

$$h = -\gamma\nabla' \cdot v' + f^0 + i\gamma\xi(\xi^2 + A)^{-1}\{\lambda v^3 - i\tilde{\nu}\xi(\nabla' \cdot v' + i\xi v^3) - g^3\}.$$

As for A, it is well-known that the following inequalities hold:

$$(A\varphi, \varphi) \geq C|\varphi|_2^2 \quad (\forall\varphi \in D(A)), \tag{3.42}$$

$$|(\mu + A)^{-1}h|_2 \leq \frac{C}{\mu + 1}|h|_2 \quad (\forall\mu \geq 0), \tag{3.43}$$

$$|(\mu + A)^{-1/2}h|_2^2 = ((\mu + A)^{-1}h, h) \leq \frac{C}{\mu + 1}|h|_2^2 \quad (\forall\mu \geq 0). \tag{3.44}$$

Taking the inner product of (3.41) with ϕ we have

$$\lambda|\phi|_2^2 + \gamma^2\xi^2|(\xi^2 + A)^{-1/2}\phi|_2^2 = (h, \phi). \tag{3.45}$$

By (3.45) we obtain

$$|(h, \phi)| \leq C\{|\lambda||v|_2 + |\xi||v|_2 + |\partial_{x'}v|_2 + |f|_2\}|\phi|_2. \tag{3.46}$$

Using (3.44) we see that

$$\begin{aligned}
\xi^2|(\xi^2 + A)^{-1/2}\phi|_2^2 &= \xi^2\{|(\xi^2 + A)^{-1/2}\phi_0|_2^2 + |(\xi^2 + A)^{-1/2}\phi_1|_2^2 \\
&\qquad + 2\operatorname{Re}((\xi^2 + A)^{-1/2}\phi_0, (\xi^2 + A)^{-1/2}\phi_1)\} \\
&\geq \frac{1}{2}\xi^2|(\xi^2 + A)^{-1/2}\phi_0|_2^2 - C|\phi_1|_2^2 \\
&\geq \frac{1}{2}\xi^2|(\xi^2 + A)^{-1/2}\phi_0|_2^2 - C|\partial_{x'}\phi_1|_2^2.
\end{aligned} \tag{3.47}$$

We now apply the following fact: for any $r > 0$ there exists a positive constant $C(r)$ such that

$$\mu|(\mu + A)^{-1/2} \cdot 1|_2^2 \geq C(r) \quad (\forall \mu \geq r^2). \tag{3.48}$$

We will give a proof of (3.48) later.

It follows from (3.48) that if $|\xi| \geq r$, then

$$\begin{aligned} \xi^2|(\xi^2 + A)^{-1/2}\phi_0|_2^2 &= \xi^2|\phi_0|^2|(\xi^2 + A)^{-1/2} \cdot 1|_2^2 \\ &\geq C(r)|\phi_0|_2^2. \end{aligned} \tag{3.49}$$

Here we note that ϕ_0 is a constant. From (3.45)–(3.47) and (3.49) we see that

$$\begin{aligned} &(\mathrm{Re}\,\lambda + |\mathrm{Im}\,\lambda|^2)|\phi|_2^2 + C(r)\gamma^2|\phi_0|_2^2 \\ &\leq \delta|\phi_0|_2^2 + C|\partial_{x'}\phi|_2^2 + C_\delta\{|\lambda|^2|v|_2^2 + |\partial_{x'}v|_2^2 + \xi^2|v|_2^2 + |f|_2^2\} \end{aligned}$$

for any $\delta > 0$. Taking $\delta > 0$ as $\delta < C(r)\gamma^2/2$, we obtain the desired estimate.

We finally prove (3.48). By (3.44), we have

$$\begin{aligned} \mu|(\mu + A)^{-1/2} \cdot 1|_2^2 &= \mu((\mu + A)^{-1} \cdot 1, 1) \\ &= ((1 + \mu^{-1}A)^{-1} \cdot 1, 1). \end{aligned}$$

Since A is sectorial, we have

$$((1 + \mu^{-1}A)^{-1} \cdot 1, 1) \to (1, 1) = |D| \quad (\mu \to \infty),$$

and, therefore, there exists a positive number R such that

$$\mu|(\mu + A)^{-1/2} \cdot 1|_2^2 \geq \frac{1}{2}|D|, \quad \forall \mu \geq R. \tag{3.50}$$

Since $|(\mu + A)^{-1/2} \cdot 1|_2^2$ is continuous in $\mu \geq 0$, and, furthermore, since

$$|(\mu + A)^{-1/2} \cdot 1|_2^2 > 0, \quad \forall \mu \geq 0,$$

we see that there exists a positive number $\tilde{C}(R)$ such that

$$|(\mu + A)^{-1/2} \cdot 1|_2^2 \geq \tilde{C}(R), \quad 0 \leq \forall \mu \leq R. \tag{3.51}$$

Combining (3.50) and (3.51) we obtain (3.48). This completes the proof. □

Proposition 3.12. *There holds the estimate*

$$(\mathrm{Re}\,\lambda + c\varepsilon|\mathrm{Im}\,\lambda|)\xi^2|u|_2^2 + c\{\xi^4|v|_2^2 + \xi^2|\partial_{x'}v|_2^2 + \xi^2|\nabla'\cdot v' + i\xi v^3|_2^2\}$$
$$\leq \varepsilon\xi^2|\phi|_2^2 + C_\varepsilon\xi^2|f^0|_2^2 + C|g|_2^2$$

for any $\varepsilon \in (0, 1]$.

Proof. We see from (3.15) that

$$|\mathrm{Im}\,\lambda|\xi^2|u|_2^2 \leq \varepsilon\xi^2|\phi|_2^2 + \frac{C}{\varepsilon}\{\xi^4|v|_2^2 + \xi^2|\partial_{x'}v|_2^2 + \xi^2|f^0|_2^2\} + C_\eta|g|_2^2 + \eta\xi^4|v|_2^2 \tag{3.52}$$

for any $\eta > 0$ and $\varepsilon > 0$. We also have

$$\xi^2|\mathrm{Re}(f, u)| \leq C_\eta|g|_2^2 + \eta|\xi|^4|v|_2^2 + \frac{C}{\varepsilon}\xi^2|f^0|_2^2 + \varepsilon\xi^2|\phi|_2^2 \tag{3.53}$$

for any $\eta > 0$ and $\varepsilon > 0$. Combining (3.14), (3.52) and (3.53), and taking $\eta > 0$ suitably small, we obtain the desired estimate. This completes the proof. □

Proposition 3.13. *Let* $r > 0$. *Then there exists a positive constant* $C_1 = C_1(r)$ *such that If* $|\xi| \geq r$ *and* $\mathrm{Re}\,\lambda + c|\mathrm{Im}\,\lambda| + C_1(r) \geq 0$, *then*

$$(\mathrm{Re}\,\lambda + c|\,\mathrm{Im}\,\lambda|^2 + C_1(r))|u|_2^2 + (\mathrm{Re}\,\lambda + c|\,\mathrm{Im}\,\lambda| + c)\{\xi^2|v|_2^2 + |\partial_{x'}u|_2^2\}$$
$$+ c\left\{\left|\lambda + \frac{\gamma^2}{\nu + \tilde{\nu}}\right|^2|\partial_{x'}\phi|_2^2 + |\partial_{x'}^2 v'|_2^2 + |\lambda|^2|v|_2^2\right\}$$
$$\leq C\{|f^0|_{H^1}^2 + \xi^2|f^0|_2^2 + |g|_2^2 + \xi^4|v|_2^2 + \xi^2|\partial_{x'}v|_2^2\}.$$

Proof. By Propositions 3.10 and 3.11, we have

$$\begin{aligned}(3.54)\quad &(\mathrm{Re}\,\lambda + c|\mathrm{Im}\,\lambda|^2)|u|_2^2 + (\mathrm{Re}\,\lambda + c|\mathrm{Im}\,\lambda|^2)|\phi|_2^2 + \tilde{C}_1(r)|\phi_0|_2^2\\ &+ (\mathrm{Re}\,\lambda + c|\mathrm{Im}\,\lambda| + c)\{\xi^2|v|_2^2 + |\partial_{x'}u|_2\}\\ &+ c\left\{\left|\lambda + \frac{\gamma^2}{\nu + \tilde{\nu}}\right|^2|\partial_{x'}\phi|_2^2 + |\partial_{x'}^2 v'|_2^2 + |\lambda|^2|v|_2^2\right\}\\ &\leq \varepsilon|\phi|_2 + C_\varepsilon|f^0|_2^2 + C\{|\partial_{x'}f^0|_2 + \xi^2|f^0|_2^2 + |g|_2^2 + \xi^4|v|_2^2 + \xi^2|\partial_{x'}v|_2^2\}.\end{aligned}$$

Since $|\partial_{x'}\phi|_2^2 = |\partial_{x'}\phi_1|_2^2 \geq C|\phi_1|_2^2$ by Poincaré inequality, the left-hand side of (3.54) is bounded from below by

$$(\mathrm{Re}\,\lambda + c|\mathrm{Im}\,\lambda|^2)|u|_2^2 + (\mathrm{Re}\,\lambda + c|\mathrm{Im}\,\lambda|^2 + C_1(r))|\phi|_2^2$$
$$+ (\mathrm{Re}\,\lambda + c|\mathrm{Im}\,\lambda| + c)\{\xi^2|v|_2^2 + |\partial_{x'}u|_2^2\}$$
$$+ c\left\{\left|\lambda + \frac{\gamma^2}{\nu + \tilde{\nu}}\right|^2|\partial_{x'}\phi|_2^2 + |\partial_{x'}^2 v'|_2^2 + |\lambda|^2|v|_2^2\right\}.$$

The desired estimate now follows by taking ε suitably small. This completes the proof. □

We now deduce the following two propositions on $(\lambda + \hat{L}_\xi)^{-1}$.

Proposition 3.14. *Let* $0 < r < \infty$. *Then there exist constants* $\Lambda_1 > 0$ *and* $\theta_1 \in (\pi/2, \pi)$ *such that for any* ξ *with* $|\xi| \geq r$ *problem* (3.12) *has a unique solution* $u \in H^1(D) \times [H^2(D) \cap H^1_0(D)]$ *for any* $f \in H^1(D) \times L^2(D)$, *provided that* $\lambda \in \Sigma(-\Lambda_1, \theta_1)$. *Furthermore,* $u = (\lambda + \hat{L}_\xi)^{-1} f$ *satisfies the estimate*

$$
\begin{aligned}
&(\operatorname{Re}\lambda + c|\operatorname{Im}\lambda|^2 + c)|(\lambda + \hat{L}_\xi)^{-1} f|_2^2 \\
&+ (\operatorname{Re}\lambda + c|\operatorname{Im}\lambda| + c)\{\xi^2 |\tilde{Q}(\lambda + \hat{L}_\xi)^{-1} f|_2^2 + |\partial_{x'}(\lambda + \hat{L}_\xi)^{-1} f|_2^2\} \\
&+ c\left\{\left|\lambda + \frac{\gamma^2}{\nu + \tilde{\nu}}\right| |\partial_{x'} Q_0(\lambda + \hat{L}_\xi)^{-1} f|_2^2 \right. \\
&\qquad \left. + |\partial_{x'}^2 Q'(\lambda + \hat{L}_\xi)^{-1} f|_2^2 + |\lambda|^2 |\tilde{Q}(\lambda + \hat{L}_\xi)^{-1} f|_2^2 \right\} \\
&\leq C(1 + \xi^4)\{|f^0|_{H^1}^2 + \xi^2 |f^0|_2^2\}.
\end{aligned}
$$

Here c *and* C *are some constants depending on* r.

Proof. Proposition 3.14 follows from Propositions 3.12 and 3.13. We omit the details. □

Proposition 3.15. *Let* $0 < r < \infty$ *and let* $\Lambda_1 > 0$ *and* $\theta_1 \in (\pi/2, \pi)$ *be the numbers given in* Proposition 3.14. *Then there holds the estimate*

$$
\begin{aligned}
&|(\lambda + \hat{L}_\xi)^{-1} f|_{H^1}^2 + \xi^2 |(\lambda + \hat{L}_\xi)^{-1} f|_2^2 + \sum_{k+l=2} \xi^{2k} |\partial_{x'}^l \tilde{Q}(\lambda + \hat{L}_\xi)^{-1} f|_2^2 \\
&\leq C\{|f|_{H^1 \times L^2}^2 + \xi^2 |f^0|_2^2\}
\end{aligned}
$$

uniformly for $|\xi| \geq r$ *and* $\lambda \in \Sigma(-\Lambda_1, \theta_1) \cap \{\lambda; |\lambda| \geq \Lambda_1/2\}$. *Here* C *is a positive constant depending on* r.

Proof. Let $u = {}^T(\phi, v) = (\lambda + \hat{L}_\xi)^{-1} f$. By Propositions 3.12 and 3.13, there exists a constant $C_1 = C_1(r) > 0$ such that if $|\xi| \geq r$ and $\operatorname{Re}\lambda + c|\operatorname{Im}\lambda| + C_1(r) \geq 0$, then the following estimate holds:

$$
\begin{aligned}
&(\operatorname{Re}\lambda + c|\operatorname{Im}\lambda|^2 + C_1(r))|u|_2^2 + (\operatorname{Re}\lambda + c|\operatorname{Im}\lambda| + c)\{\xi^2|v|_2^2 + |\partial_{x'} u|_2^2\} \\
&+ c\left\{|\lambda|^2 |v|_2^2 + \left|\lambda + \frac{\gamma^2}{\nu + \tilde{\nu}}\right|^2 |\partial_{x'}\phi|_2^2 + |\partial_{x'}^2 v'|_2^2 + \xi^4 |v|_2^2 + \xi^2 |\partial_{x'} v|_2^2\right\} \\
&\leq \varepsilon \xi^2 |\phi|_2^2 + C\{|f|_{H^1 \times L^2}^2 + \xi^2 |f^0|_2^2\}.
\end{aligned}
\tag{3.55}
$$

By (3.8), we have

$$
|\lambda|^2 \xi^2 |\phi|_2 \leq C\{\xi^2 |f^0|_2^2 + \xi^2 |\nabla' \cdot v'|_2^2 + \xi^4 |v^3|_2^2\}. \tag{3.56}
$$

It follows from (3.55) and (3.56) that if $\lambda \neq 0$, then

$$
\begin{aligned}
&(\operatorname{Re}\lambda + c|\operatorname{Im}\lambda|^2 + C_1(r))|u|_2^2 + (\operatorname{Re}\lambda + c|\operatorname{Im}\lambda| + c)\{\xi^2|v|_2^2 + |\partial_{x'}u|_2^2\} \\
&+ c\left\{|\lambda|^2|v|_2^2 + \left|\lambda + \frac{\gamma^2}{\nu+\tilde{\nu}}\right|^2 |\partial_{x'}\phi|_2^2 + |\partial_{x'}^2 v'|_2^2 + \xi^2|\partial_{x'}v|_2^2 + \xi^4|\partial_{x'}v|_2^2\right\} \\
&\leq \varepsilon\frac{C}{|\lambda|^2}\{\xi^2|\nabla'\cdot v'|_2^2 + \xi^4|v^3|_2^2\} + C_\varepsilon\left\{|f|_{H^1\times L^2}^2 + \left(1+\frac{1}{|\lambda|^2}\right)\xi^2|f^0|_2^2\right\}.
\end{aligned}
$$

Since $|\lambda| \geq \Lambda_1/2$ and $\lambda \in \Sigma(-\Lambda_1, \theta_1)$, taking ε suitably small, we have

$$
\begin{aligned}
&|u|_2^2 + \xi^2|v|_2^2 + |\partial_{x'}u|_2^2 + \xi^4|v|_2^2 + \xi^2|\partial_{x'}v|_2^2 + |\partial_{x'}^2 v'|_2^2 + |\lambda|^2|v|_2^2 \\
&\leq C\{|f|_{H^1\times L^2}^2 + \xi^2|f^0|_2^2\}.
\end{aligned} \tag{3.57}
$$

It follows from (3.56) and (3.57) that

$$\xi^2|\phi|_2^2 \leq C\{|f|_{H^1\times L^2}^2 + \xi^2|f^0|_2^2\}.$$

We finally consider the estimate for $|\partial_{x'}^2 v^3|_2$. By (3.10), v^3 satisfies the elliptic problem

$$-\nu\triangle' v^3 = -\{\lambda v^3 + \nu\xi^2 v^3 - i\tilde{\nu}\xi(\nabla'\cdot v' + i\xi v^3) + i\gamma\xi\phi - g^3\}, \quad v^3|_{\partial D} = 0,$$

so, the regularity theory for the elliptic problem gives

$$
\begin{aligned}
|\partial_{x'}^2 v^3|_2^2 &\leq C\{|\lambda|^2|v|_2^2 + \xi^4|v|_2^2 + \xi^2|\partial_{x'}v|_2^2 + \xi^2|\phi|_2^2 + |g|_2^2\} \\
&\leq C\{|f|_{H^1\times L^2}^2 + \xi^2|f^0|_2^2\},
\end{aligned}
$$

which is the desired estimate. This completes the proof. □

4. Resolvent problem II

In this section we investigate $(\lambda + \hat{L}_\xi)^{-1}$ for $|\xi| \ll 1$. We will show that if $|\xi| \ll 1$, then $\rho(-\hat{L}_0) \supset \{\lambda \neq 0, \operatorname{Re}\lambda + C_3|\operatorname{Im}\lambda| + C_4 \geq 0\}$ and $\sigma(-\hat{L}_\xi) \cap \{|\lambda| \leq C_4/2\} = \{\lambda_0(\xi)\}$ for some $C_3, C_4 > 0$, where $\lambda_0(\xi)$ is a simple eigenvalue of $-\hat{L}_\xi$, which satisfies

$$\lambda_0(\xi) = -\frac{a_1\gamma}{\nu}\xi^2 + O(\xi^4) \quad (\xi \to 0)$$

with some positive constant a_1.

We set $\xi = 0$ in (3.8)–(3.11) to obtain

$$
\begin{cases}
\lambda\phi + \gamma\nabla'\cdot v' = f^0, \\
\lambda v' - \nu\triangle' v' - \tilde{\nu}\nabla'(\nabla'\cdot v') + \gamma\nabla'\phi = g', \\
\lambda v^3 - \nu\triangle' v^3 = g^3, \\
v|_{\partial D} = 0.
\end{cases} \tag{4.1}
$$

Let $\phi = \phi_0 + \phi_1$ be the orthogonal decomposition of ϕ defined in Proposition 3.11. Similarly we decompose f^0 as

$$f^0 = f^0_0 + f^0_1, \quad f^0_0 = \langle f^0 \rangle = \frac{1}{|D|} \int_D f \, dx', \quad f^0_1 = f^0 - f^0_0.$$

It then follows that (4.1) is rewritten as

$$\lambda \phi_0 = f^0_0, \tag{4.2}$$

$$\lambda \phi_1 + \gamma \nabla' \cdot v' = f^0_1, \tag{4.3}$$

$$\lambda v' - \nu \triangle' v' - \tilde{\nu} \nabla' (\nabla' \cdot v') + \gamma \nabla \phi_1 = g', \quad v'|_{\partial D} = 0, \tag{4.4}$$

$$\lambda v^3 - \nu \triangle' v^3 = g^3, \quad v^3|_{\partial D} = 0. \tag{4.5}$$

We consider the solvability of each of (4.2), (4.3)–(4.4), and (4.5).

As for (4.2), if $\lambda \neq 0$, then (4.2) has a unique solution $\phi_0 = (1/\lambda) f^0_0$. We also see that $\lambda = 0$ is a simple eigenvalue with eigenfunction $\phi_0 = 1$.

As for (4.5), it is well-known that there exists a sequence $\{\lambda_j\}_{j=1}^{\infty}$ ($\lambda_j < 0$, $|\lambda_1| < |\lambda_2| \leq |\lambda_3| \leq \cdots \to \infty$) that has the following properties. Each λ_j is a semi-simple eigenvalue and for any $\lambda \notin \{\lambda_j\}_{j=1}^{\infty}$ (4.5) has a unique solution $v^3 \in H^2(D) \cap H^1_0(D)$. Furthermore, if $|\arg(\lambda - (1/2)\lambda_1)| \leq \pi - \varepsilon$ ($\varepsilon > 0$), then there holds the estimate

$$|\lambda| \, |v^3|_2 + |\lambda|^{1/2} |\partial_{x'} v^3|_2 + |\partial^2_{x'} v^3|_2 \leq C_\varepsilon |g^3|_2.$$

As for the solvability of (4.3)–(4.4), we have the following result.

Proposition 4.1. *There are positive constants C_3 and C_4 such that If* $\mathrm{Re}\,\lambda + C_3 |\mathrm{Im}\,\lambda| + C_4 \geq 0$, *then for any* ${}^T(f^0_1, g') \in H^1(D) \times L^2(D)$ *with* $\int_D f^0_1 \, dx' = 0$ *there exists a unique solution* ${}^T(\phi_1, v') \in H^1(D) \times [H^2(D) \cap H^1_0(D)]$ *of* (4.3)–(4.4) *with* $\int_D \phi_1 \, dx' = 0$. *Furthermore, there holds the estimate*

$$\begin{aligned}
&(\mathrm{Re}\,\lambda + C_3 |\mathrm{Im}\,\lambda|^2 + C_4)\{|\phi_1|_2^2 + |v'|_2^2 + |\partial_{x'} \phi_1|_2^2\} \\
&+ (\mathrm{Re}\,\lambda + C_3 |\mathrm{Im}\,\lambda| + C_4)\{|\partial_{x'} \phi_1|_2^2 + |\partial_{x'} v'|_2\} \\
&+ c\{|\partial^2_{x'} v'|_2 + |\lambda|^2 |v'|_2^2\} + c \left| \lambda + \frac{\gamma}{\nu + \tilde{\nu}} \right|^2 \{|\phi_1|_2^2 + |\partial_{x'} \phi_1|_2^2\} \\
&\leq C\{|f^1_0|^2_{H^1} + |g'|_2^2\}.
\end{aligned}$$

Proof. The existence of solution can be proved as in the proof of Proposition 3.1. It is not difficult to see that the estimate in Proposition 3.10 also holds for $\xi = 0$ and

$\phi = \phi_1$. We thus have

$$
\begin{aligned}
&(\mathrm{Re}\,\lambda + C_3|\mathrm{Im}\,\lambda|^2)\{|\phi_1|_2^2 + |v'|_2^2\} \\
&+ (\mathrm{Re}\,\lambda + C_3|\mathrm{Im}\,\lambda| + C_4)\{|\partial_{x'}\phi_1|_2^2 + |\partial_{x'}v'|_2^2\} \\
&+ c\left\{|\partial_{x'}^2 v'|_2 + |\lambda|^2|v'|_2^2 + \left|\lambda + \frac{\gamma}{\nu+\tilde{\nu}}\right|^2|\partial_{x'}\phi_1|_2^2\right\} \\
&\le \varepsilon^2|\phi_1|_2^2 + C_\varepsilon|f_0^1|_{H^1}^2 + |g'|_2^2.
\end{aligned}
$$

Since $\int_D \phi_1\,dx' = 0$ and $v'|_{\partial D} = 0$, the Poincaré inequality gives

$$
|\partial_{x'}\phi_1|_2 \ge C|\phi_1|_2, \quad |\partial_{x'}v'|_2 \ge C|v'|_2.
$$

Taking $\varepsilon > 0$ suitably small, we obtain the desired estimate. This completes the proof. □

In what follows we represent $\hat{L}_\xi$ as

$$
\hat{L}_\xi = \hat{L}_0 + \xi\hat{L}^{(1)} + \xi^2\hat{L}^{(2)}.
$$

Here $\hat{L}_0$ is the operator with domain $D(\hat{L}_0)$ defined by

$$
\hat{L}_0 = \begin{pmatrix} 0 & \gamma^T\nabla' & 0 \\ \gamma\nabla' & -\nu\triangle' I_2 - \tilde{\nu}\nabla'^T\nabla' & 0 \\ 0 & 0 & -\nu\triangle' \end{pmatrix},
$$

$$
D(\hat{L}_0) = H^1(D) \times [H^2(D) \cap H_0^1(D)],
$$

and

$$
\hat{L}^{(1)} = \begin{pmatrix} 0 & 0 & i\gamma \\ 0 & 0 & -i\tilde{\nu}\nabla' \\ i\gamma & -i\tilde{\nu}^T\nabla' & 0 \end{pmatrix}, \quad \hat{L}^{(2)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \nu I_2 & 0 \\ 0 & 0 & \nu+\tilde{\nu} \end{pmatrix}.
$$

From the above observations on (4.2)–(4.5), we deduce the following results on $(\lambda + \hat{L}_0)^{-1}$.

Proposition 4.2. (i) *There are positive constants C_3 and C_4 such that*

$$
\Sigma_1 \equiv \{\lambda \ne 0,\ \mathrm{Re}\,\lambda + C_3|\mathrm{Im}\,\lambda| + C_4 \ge 0\} \subset \rho(-\hat{L}_0).
$$

Furthermore, if $\lambda \in \Sigma_1$, then

$$
|(\lambda + \hat{L}_0)^{-1}f|_{H^1\times L^2} \le \frac{C}{|\lambda|+1}\{|f_0^1|_{H^1} + |g|_2\} + \frac{C}{|\lambda|}|f_0^0|_2,
$$

$$
|\partial_{x'}^l\tilde{Q}(\lambda + \hat{L}_0)^{-1}f|_2 \le \frac{C}{(|\lambda|+1)^{1-l/2}}\{|f_0^1|_{H^1} + |g|_2\} \quad (l = 1, 2).
$$

(ii) $\lambda = 0$ *is a simple eigenvalue of* $-\hat{L}_0$, *and the associated eigenprojection* $\hat{P}_0$ *is given by*

$$\hat{P}_0 u = \begin{pmatrix} \langle\phi\rangle \\ 0 \end{pmatrix} \quad \textit{for} \quad u = \begin{pmatrix} \phi \\ v' \end{pmatrix}.$$

We next investigate the resolvent set $\rho(-\hat{L}_\xi)$ and the spectrum $(\lambda + \hat{L}_\xi)^{-1}$ for $|\xi| \ll 1$.

Proposition 4.3. *There exists a positive number* r_1 *such that if* $|\xi| \leq r_1$, *then*

$$\Sigma_1 \cap \{|\lambda| \geq C_4/2\} \subset \rho(-\hat{L}_\xi).$$

Furthermore, if $\lambda \in \Sigma_1 \cap \{|\lambda| \geq C_4/2\}$, *then*

$$|(\lambda + \hat{L}_\xi)^{-1} f|_{H^1 \times L^2} \leq \frac{C}{|\lambda| + 1} |f|_{H^1 \times L^2},$$
$$|\partial_{x'}^l \tilde{Q}(\lambda + \hat{L}_\xi)^{-1} f|_2 \leq \frac{C}{(|\lambda| + 1)^{1-l/2}} |f|_{H^1 \times L^2} \quad (l = 1, 2).$$

Proof. We have $\hat{L}^{(1)} u = {}^T(i\gamma v^3, -\tilde{\nu}\nabla' v^3, i\gamma\phi - i\tilde{\nu}\nabla' \cdot v')$ for $u = {}^T(\phi, v)$. Setting $u = (\lambda + \hat{L}_0)^{-1} f$ and noting that $|\lambda| \geq C_4/2$, we see from Proposition 4.2 that

$$|\hat{L}^{(1)} u|_2 \leq C\{|\phi|_2 + |v|_{H^1}\} \leq \frac{C}{(|\lambda| + 1)^{1/2}} |f|_{H^1 \times L^2},$$
$$|\partial_{x'} Q_0 \hat{L}^{(1)} u|_2 \leq C|\partial_{x'} v^3|_2 \leq \frac{C}{(|\lambda| + 1)^{1/2}} |f|_{H^1 \times L^2}.$$

Since $\hat{L}^{(2)} u = {}^T(0, \nu v', (\nu + \tilde{\nu})v^3)$, we similarly obtain by Proposition 4.2

$$|\hat{L}^{(2)} u|_2 \leq C|v|_2 \leq \frac{C}{|\lambda| + 1} |f|_{H^1 \times L^2}, \quad \partial_{x'} Q_0 \hat{L}^{(2)} u = 0.$$

Therefore, there exists a positive number r_1 such that

$$|(\xi \hat{L}^{(1)} + \xi^2 \hat{L}^{(2)})(\lambda + \hat{L}_0)^{-1} f|_{H^1 \times L^2} \leq \frac{1}{2} |f|_{H^1 \times L^2} \quad (\forall |\xi| \leq r_1).$$

This implies that $\Sigma_1 \cap \{|\lambda| \geq C_4/2\} \subset \rho(-\hat{L}_\xi)$, and we have the Neumann series expansion

$$(4.6) \qquad (\lambda + \hat{L}_\xi)^{-1} = (\lambda + \hat{L}_0)^{-1} \sum_{N=0}^{\infty} (-1)^N [(\xi \hat{L}^{(1)} + \xi^2 \hat{L}^{(2)})(\lambda + \hat{L}_0)^{-1}]^N,$$

and $(\lambda+\hat{L}_\xi)^{-1}$ is estimated as

$$|(\lambda+\hat{L}_\xi)^{-1}|_{H^1\times L^2}\le\frac{C}{|\lambda|+1}\sum_{N=0}^{\infty}\left(\frac{1}{2}\right)^N|f|_{H^1\times L^2}\le\frac{C}{|\lambda|+1}|f|_{H^1\times L^2}.$$

Similarly we find that

$$\begin{aligned}
&|\partial_{x'}\tilde{Q}(\lambda+\hat{L}_\xi)^{-1}f|_2\\
&=\left|\partial_{x'}\tilde{Q}(\lambda+\hat{L}_0)^{-1}\sum_{N=0}^{\infty}(-1)^N[(\xi\hat{L}^{(1)}+\xi^2\hat{L}^{(2)})(\lambda+\hat{L}_0)^{-1}]^Nf\right|_2\\
&\le\frac{C}{(|\lambda|+1)^{1/2}}\sum_{N=0}^{\infty}|[(\xi\hat{L}^{(1)}+\xi^2\hat{L}^{(2)})(\lambda+\hat{L}_0)^{-1}]^Nf|_{H^1\times L^2}\\
&\le\frac{C}{(|\lambda|+1)^{1/2}}\sum_{N=0}^{\infty}\left(\frac{1}{2}\right)^N|f|_{H^1\times L^2}\le\frac{C}{(|\lambda|+1)^{1/2}}|f|_{H^1\times L^2},\\
&|\partial_{x'}^2\tilde{Q}(\lambda+\hat{L}_\xi)^{-1}f|_2\\
&=\left|\partial_{x'}^2\tilde{Q}(\lambda+\hat{L}_0)^{-1}\sum_{N=0}^{\infty}(-1)^N[(\xi\hat{L}^{(1)}+\xi^2\hat{L}^{(2)})(\lambda+\hat{L}_0)^{-1}]^Nf\right|_2\\
&\le C\sum_{N=0}^{\infty}|[(\xi\hat{L}^{(1)}+\xi^2\hat{L}^{(2)})(\lambda+\hat{L}_0)^{-1}]^Nf|_{H^1\times L^2}\\
&\le C|f|_{H^1\times L^2}.
\end{aligned}$$

This completes the proof. □

REMARK. It is easy to see that Propositions 4.1–4.3 are also valid for the adjoint problem $(\lambda+\hat{L}_\xi^*)w=f$, since $\hat{L}_\xi^*$ has the following form:

$$\hat{L}_\xi^*=\hat{L}_0^*+\xi\hat{L}^{(1)*}+\xi^2\hat{L}^{(2)*},\quad D(\hat{L}_\xi^*)=H^1(D)\times[H^2(D)\cap H_0^1(D)],$$

$$\hat{L}_0^*=\begin{pmatrix}0 & -\gamma^T\nabla' & 0\\ -\gamma\nabla' & -\nu\triangle' I_2-\tilde{\nu}\nabla'^T\nabla' & 0\\ 0 & 0 & -\nu\triangle'\end{pmatrix},$$

$$\hat{L}^{(1)*}=\begin{pmatrix}0 & 0 & -i\gamma\\ 0 & 0 & -i\tilde{\nu}\nabla'\\ -i\gamma & -i\tilde{\nu}^T\nabla' & 0\end{pmatrix},\quad \hat{L}^{(2)*}=\begin{pmatrix}0 & 0 & 0\\ 0 & \nu I_2 & 0\\ 0 & 0 & \nu+\tilde{\nu}\end{pmatrix}.$$

As for the spectrum of $\sigma(-\hat{L}_\xi)$, we have the following result.

Proposition 4.4. *There exists a positive number r_2 such that if $|\xi| \le r_2$, then there holds*

$$\sigma(-\hat{L}_\xi) \cap \{|\lambda| \le C_4/2\} = \{\lambda_0(\xi)\}.$$

Here C_4 is the constant given in Proposition 4.2, *and $\lambda_0(\xi)$ is a simple eigenvalue of $-\hat{L}_\xi$, which satisfies*

$$\lambda_0(\xi) = -\frac{a_1\gamma}{\nu}\xi^2 + O(\xi^4) \quad (\xi \to 0)$$

for some positive constant a_1. Furthermore, the associated eigenprojection $\hat{P}_0(\xi)$ takes the form

$$\hat{P}_0(\xi) = \hat{P}_0 + \xi\hat{P}_1 + \hat{P}_2(\xi).$$

Here the right members have the following properties:

$$\hat{P}_0 u = \hat{P}_0\begin{pmatrix}\phi\\ v\end{pmatrix} = \begin{pmatrix}\langle\phi\rangle\\ 0\end{pmatrix},$$

$$|\hat{P}_1 u|_{H^1\times L^2} + \sum_{l=1}^{2} |\partial_{x'}^l \tilde{Q}\hat{P}_1 u|_2 \le C|u|_{H^1\times L^2},$$

$$\partial_{x'}\hat{P}_1\tilde{Q}u = 0,$$

$$|\hat{P}_2(\xi)u|_{H^1\times L^2} + \sum_{l=1}^{2} |\partial_{x'}^l \tilde{Q}\hat{P}_2(\xi)u|_2 \le C\xi^2|u|_{H^1\times L^2}.$$

Proof. By Proposition 4.3 we see that if $|\lambda| = C_4/2$, then $\lambda \in \rho(-\hat{L}_\xi)$ for $|\xi| \le r_1$. In particular,

$$\hat{P}_0(\xi) = \frac{1}{2\pi i}\int_{|\lambda|=C_4/2} (\lambda + \hat{L}_\xi)^{-1}\, d\lambda$$

is the eigenprojection for the eigenvalues lying inside the circle $|\lambda| = C_4/2$. The continuity of $(\lambda + \hat{L}_\xi)^{-1}$ in (λ, ξ) then implies that

$$\dim R(\hat{P}_0(\xi)) = \dim R(\hat{P}_0) = 1.$$

Therefore, $\sigma(-\hat{L}_\xi) \cap \{|\lambda| \le C_4/2\}$ consists of only one point, say, $\{\lambda_0(\xi)\}$, and $\lambda_0(\xi)$ is a simple eigenvalue. Furthermore, it follows from (4.6) that

$$(\lambda + \hat{L}_\xi)^{-1} = (\lambda + \hat{L}_0)^{-1} + \xi\hat{\mathcal{R}}^{(1)}(\lambda) + \hat{\mathcal{R}}^{(2)}(\lambda, \xi).$$

Here

$$\hat{\mathcal{R}}^{(1)}(\lambda) = -(\lambda + \hat{L}_0)^{-1}\hat{L}^{(1)}(\lambda + \hat{L}_0)^{-1},$$

$$\hat{\mathcal{R}}^{(2)}(\lambda, \xi) = -\xi^2(\lambda + \hat{L}_0)^{-1}\hat{L}^{(2)}(\lambda + \hat{L}_0)^{-1} + (\lambda + \hat{L}_0)^{-1}\sum_{N=2}^{\infty}(-1)^N[(\xi\hat{L}^{(1)} + \xi^2\hat{L}^{(2)})(\lambda + \hat{L}_0)^{-1}]^N.$$

We thus deduce that $\hat{P}_0(\xi)$ is written as

$$\hat{P}_0(\xi) = \hat{P}_0 + \xi\hat{P}_1 + \hat{P}_2(\xi),$$

where

$$\hat{P}_1 = \frac{1}{2\pi i}\int_{|\lambda|=C_4/2}\hat{\mathcal{R}}^{(1)}(\lambda)\,d\lambda = -\hat{S}\hat{L}^{(1)}\hat{P}_0 - \hat{P}_0\hat{L}^{(1)}\hat{S}$$

with

$$\hat{S} = [(I - \hat{P}_0)\hat{L}_0(I - \hat{P}_0)]^{-1},$$

and

$$\hat{P}_2(\xi) = \frac{1}{2\pi i}\int_{|\lambda|=\frac{C_4}{2}}\hat{\mathcal{R}}^{(2)}(\lambda, \xi)\,d\lambda.$$

Applying Proposition 4.3 we see that

$$|\hat{P}_2(\xi)u|_{H^1\times L^2} + \sum_{l=1}^{2}|\partial_{x'}^l\tilde{Q}\hat{P}_2(\xi)u|_2 \le C\xi^2|u|_{H^1\times L^2}.$$

Since $\hat{P}_0\tilde{Q}u = 0$ and $\partial_{x'}\hat{P}_0 u = 0$, we have

$$\partial_{x'}\hat{P}_1\tilde{Q}u = -\partial_{x'}[\hat{P}_0\hat{L}^{(1)}\hat{S}\tilde{Q}u] = 0.$$

We next prove the asymptotic formula for $\lambda_0(\xi)$ as $\xi \to 0$. By the analytic perturbation theory ([9]), $\lambda_0(\xi)$ is written as

$$\lambda_0(\xi) = \lambda^{(0)} + \xi\lambda^{(1)} + \xi^2\lambda^{(2)} + \xi^3\lambda^{(3)} + O(\xi^4) \quad (\xi \to 0).$$

Here $\lambda^{(0)} = 0$. Furthermore, we have $\lambda^{(1)} = \lambda^{(3)} = 0$. This follows from a symmetry. In fact, it holds $\hat{L}_{\pm\xi} = T_{\pm}^{-1}\hat{L}_{\pm\xi}T_{\pm}$ for $T_{\pm} = \mathrm{diag}(1, 1, 1, \pm 1)$, which implies that $\lambda_0(\xi) = \lambda_0(-\xi)$, $\lambda_0(\xi) \in \mathbf{R}$, since $\lambda_0(\xi)$ is simple. We thus see that $\lambda^{(1)} = \lambda^{(3)} = 0$.

Let us next compute $\lambda^{(2)}$. Since $\sigma(-\hat{L}_0^*) = \sigma(-\hat{L}_0)$ and $\lambda = 0$ is a simple eigenvalue with eigenfunction $u^{(0)} = {}^T(1, 0)$, we see that

$$\lambda^{(2)} = -\langle\hat{L}^{(2)}u^{(0)}, u^{(0)}\rangle + \langle\hat{L}^{(1)}\hat{S}\hat{L}^{(1)}u^{(0)}, u^{(0)}\rangle.$$

Since $\hat{L}^{(2)}u^{(0)} = 0$, we have $\langle \hat{L}^{(2)}u^{(0)}, u^{(0)}\rangle = 0$. A direct computation shows $\hat{L}^{(1)}u^{(0)} = {}^{T}(0, 0, i\gamma)$, from which we have $\hat{S}\hat{L}^{(1)}u^{(0)} = {}^{T}(0, 0, (i\gamma/\nu)(-\triangle')^{-1}\cdot 1)$, and, therefore,

$$\hat{L}^{(1)}\hat{S}\hat{L}^{(1)}u^{(0)} = \begin{pmatrix} -\dfrac{\gamma^2}{\nu}(-\triangle')^{-1}\cdot 1 \\ \dfrac{\tilde{\nu}\gamma}{\nu}(-\triangle')^{-1}\cdot 1 \\ 0 \end{pmatrix}.$$

We thus conclude that

$$\lambda^{(2)} = \langle \hat{L}^{(1)}\hat{S}\hat{L}^{(1)}u^{(0)}, u^{(0)}\rangle = -\frac{a_1\gamma^2}{\nu}.$$

Here $a_1 = (1/|D|)\int_D(-\triangle')^{-1}\cdot 1\,dx' > 0$. As a result we obtain

$$\lambda_0(\xi) = -\frac{a_1\gamma^2}{\nu}\xi^2 + O(\xi^4).$$

This completes the proof. □

5. Proof of Theorem 2.1: asymptotic behavior of e^{-tL}

In this section we prove Theorem 2.1.

Proof of Theorem 2.1. Let $\chi_1(\xi) \in C_0^\infty(\mathbf{R})$ be a smooth cut-off function satisfying $0 \le \chi_1 \le 1$, $\chi_1(\xi) = 1$ for $|\xi| \le r_2/2$ and $\chi_1(\xi) = 1$ for $|\xi| > r_2/2$. Here r_2 is the positive number given in Proposition 4.4.

We set $\chi_\infty = 1 - \chi_1$. We then decompose e^{-tL} as

$$e^{-tL} = U_1(t) + U_\infty(t).$$

Here

$$U_1(t) = \mathscr{F}^{-1}[\chi_1(\xi)e^{-t\hat{L}_\xi}],$$

$$U_\infty(t) = \mathscr{F}^{-1}[\chi_\infty(\xi)e^{-t\hat{L}_\xi}],$$

$$e^{-t\hat{L}_\xi} = \frac{1}{2\pi i}\int_{\Gamma_0} e^{\lambda t}(\lambda + \hat{L}_\xi)^{-1}\,d\lambda,$$

where $\Gamma_0 = \{\lambda = \Lambda_0 + \eta e^{\pm\theta_0}, \eta \ge 0\}$ with some $\Lambda_0 > 0$ and $\theta_0 \in (\pi/2, \pi)$ that are taken in such a way that $\Gamma_0 \subset \rho(-\hat{L}_\xi)$ for all $\xi \in \mathbf{R}$.

We first estimate $U_\infty(t)$. By Proposition 3.14, we see that if $|\xi| \geq r_2/2$, then $\Sigma(-\Lambda_1, \theta_1) \subset \rho(-\hat{L}_\xi)$ for some $\Lambda_1 > 0$ and $\theta_1 \in (\pi/2, \pi)$, and, furthermore, $|(\lambda + \hat{L}_\xi)^{-1} u_0|_{H^1 \times L^2} \leq C_\xi |u_0|_{H^1 \times L^2}$ uniformly in $\lambda \in \Sigma(-\Lambda_1, \theta_1)$. (Here C_ξ depends on ξ.) We can thus deform the contour Γ_0 into $\Gamma_1 \equiv \{\lambda; |\arg(\lambda + \Lambda_1)| = \theta_1\}$ to obtain

$$\hat{U}_\infty(t)\hat{u}_0 = \frac{1}{2\pi i} \int_{\Gamma_1} \chi_\infty e^{\lambda t} (\lambda + \hat{L}_\xi)^{-1} \hat{u}_0 \, d\lambda.$$

Furthermore, Proposition 3.15 implies that there exists a positive number C such that

$$(5.1) \qquad |(\lambda + \hat{L}_\xi)^{-1} \hat{u}_0|_{H^1} + |\xi| |(\lambda + \hat{L}_\xi)^{-1} \hat{u}_0|_2 \leq C\{|\hat{u}_0|_{H^1 \times L^2} + |\xi| |Q_0 \hat{u}_0|_2\}$$

for all $|\xi| \geq r_2/2$ and $\lambda \in \Gamma_1$.

It then follows from (5.1) that

$$\begin{aligned} \|U_\infty(t)u_0\|_{H^1} &\leq C\{\| |\hat{U}_\infty(t)\hat{u}_0|_{H^1} \|_{L^2_\xi} + \| |\xi \hat{U}_\infty(t)\hat{u}_0|_2 \|_{L^2_\xi}\} \\ &\leq C \int_0^\infty e^{(-\Lambda_1 + \eta \cos\theta_1)t} \{\| |\hat{u}_0|_{H^1 \times L^2} \|_2 + \| |\xi| |Q_0 \hat{u}_0|_2 \|_2\} \, d\eta \\ &\leq C_{\theta_1} e^{-\Lambda_1 t} \|u_0\|_{H^1 \times L^2} \end{aligned}$$

for all $t \geq 1$.

We next consider $U_1(t)$. By Propositions 4.3 and 4.4, there are constants $\Lambda_2 > 0$ and $\theta_2 \in (\pi/2, \pi)$ such that if $|\xi| \leq r_2$, then

$$\{\lambda; |\arg(\lambda + \Lambda_2)| \leq \theta_2\} \cap \{\lambda; |\lambda| \geq \Lambda_2/2\} \subset \rho(-\hat{L}_\xi)$$

and

$$\{\lambda; |\lambda| \leq \Lambda_2/2\} \cap \sigma(-\hat{L}_\xi) = \{\lambda_0(\xi)\}.$$

We deform the contour Γ_0 into $\Gamma_2 \equiv \{\lambda; |\arg(\lambda + \Lambda_2)| = \theta_2\}$ to obtain, with the aid of the residue theorem,

$$\hat{U}_1(t) = \chi_1(\xi) e^{\lambda_0(\xi)t} \hat{P}_0(\xi) + \frac{1}{2\pi i} \int_{\Gamma_2} e^{\lambda t} \chi_1(\xi) (\lambda + \hat{L}_\xi)^{-1} \, d\lambda.$$

We write $\hat{U}_1(t)$ as

$$\hat{U}_1(t) = \hat{\mathscr{U}}_0(t) + \sum_{j=1}^{5} \hat{U}_1^j(t).$$

Here

$$\begin{gathered} \hat{\mathscr{U}}_0(t) = e^{-\kappa\xi^2 t} \hat{P}_0, \quad \hat{U}_1^{(1)}(t) = \chi_\infty(\xi) e^{-\kappa\xi^2 t} \hat{P}_0, \quad \hat{U}_1^{(2)}(t) = \chi_1(\xi) \xi e^{-\kappa\xi^2 t} \hat{P}_1, \\ \hat{U}_1^{(3)}(t) = \chi_1(\xi) e^{-\kappa\xi^2 t} \hat{P}_2(\xi), \quad \hat{U}_1^{(4)}(t) = \chi_1(\xi)(e^{\lambda_0(\xi)t} - e^{-\kappa\xi^2 t}) \hat{P}_0, \\ \hat{U}_1^{(5)}(t) = \frac{1}{2\pi i} \int_{\Gamma_1} e^{\lambda t} \chi_1(\xi) (\lambda + \hat{L}_\xi)^{-1} \, d\lambda. \end{gathered}$$

Here $\kappa = a_1\gamma^2/\nu$. Furthermore, we set

$$\mathscr{U}_0(t)u_0 = \mathscr{F}^{-1}[\hat{\mathscr{U}}_0(t)\hat{u}_0], \quad U_1^{(j)}(t)u_0 = \mathscr{F}^{-1}[\hat{U}_1^{(j)}(t)\hat{u}_0].$$

$U_1^{(5)}(t)$ can be estimated as $U_\infty(t)$, and we have

$$\|U_1^{(5)}(t)u_0\|_{H^1} \le Ce^{-\Lambda_2 t}\|u_0\|_{H^1\times L^2}.$$

It is easy to see that $\mathscr{U}_0(t)u_0$ is the function given in Theorem 2.1 (i) and satisfies the heat equation, and, thus, it satisfies the estimate $\|\partial_{x_3}^l \mathscr{U}_0(t)u_0\|_2 \le Ct^{-1/4-l/2}\|Q_0u_0\|_1$. Since $\partial_{x'}\hat{P}_0u_0 = 0$, we have $\partial_{x'}\mathscr{U}_0(t)u_0 = 0$.

Let us estimate $U_1^{(2)}(t)$. For $l = 0, 1$, we see from Proposition 4.4 that

$$\begin{aligned}\|\partial_{x'}^l U_1^{(2)}(t)u_0\|_2 &\le C\left(\int_0^\infty \xi^2 e^{-2\kappa\xi^2 t}|\hat{u}_0|_{H^1\times L^2}^2\,d\xi\right)^{1/2}\\ &\le C\left(\int_0^\infty \xi^2 e^{-2\kappa\xi^2 t}\,d\xi\right)^{1/2}\sup_\xi|\hat{u}_0|_{H^1\times L^2}\\ &\le Ct^{-3/4}\||u_0|_{H^1\times L^2}\|_{L^1_{x_3}},\end{aligned}$$

and

$$\begin{aligned}\|\partial_{x_3}U_1^{(2)}(t)u_0\|_2 &\le C\left(\int_0^\infty \xi^4 e^{-2\kappa\xi^2 t}|\hat{u}_0|_2^2\,d\xi\right)^{1/2}\\ &\le Ct^{-5/4}\||u_0|_2\|_{L^1_{x_3}}.\end{aligned}\tag{5.2}$$

Similarly we can estimate $U_1^{(3)}(t)$ to obtain

$$\|U_1^{(3)}(t)u_0\|_{H^1} \le Ct^{-5/4}\||u_0|_{H^1\times L^2}\|_{L^1_{x_3}}.\tag{5.3}$$

As for $U_1^{(4)}(t)$, since $\lambda_0(\xi) = -\kappa\xi^2 + \lambda^{(4)}(\xi)$ with $\lambda^{(4)}(\xi) = O(\xi^4)$, taking r_2 smaller if necessary, we have

$$|\lambda^{(4)}(\xi)| \le C\xi^4 \le \frac{\kappa}{4}\xi^2, \quad |\xi| \le r_2.$$

This implies that

$$\begin{aligned}|e^{\lambda_0(\xi)t} - e^{-\kappa\xi^2 t}| &= \left|\lambda^{(4)}(\xi)te^{-\kappa\xi^2 t}\int_0^1 e^{\theta\lambda^{(4)}(\xi)t}\,d\theta\right|\\ &\le C\xi^2 e^{(-(\kappa/2)\xi^2+|\lambda^{(4)}(\xi)|)t}\\ &\le C\xi^2 e^{-(\kappa/4)\xi^2 t}.\end{aligned}$$

Therefore, as in the estimate of $U_1^{(2)}(t)$, we obtain

$$\|U_1^{(4)}(t)u_0\|_{H^1} \le Ct^{-5/4}\| \, |u_0|_{H^1\times L^2}\|_{L^1_{x_3}}. \tag{5.4}$$

We now set

$$\mathscr{U}_1(t) = U_1^{(1)}(t) + U_1^{(2)}(t) + U_1^{(3)}(t) + U_1^{(4)}(t)$$

and

$$\mathscr{R}(t) = U_1^{(5)}(t) + U_\infty(t).$$

Then we obtain the desired estimates in Theorem 2.1 (ii) and (iii) for $\|\mathscr{U}_1(t)u_0\|_{H^1}$ and $\|\mathscr{R}(t)u_0\|_{H^1}$.

We next consider $\|\partial_x \mathscr{U}_1(t)\tilde{Q}u_0\|_2$. Since $\partial_x U_1^{(1)}(t)\tilde{Q}u_0 = 0$ and $\partial_x U_1^{(2)}(t)\tilde{Q}u_0 = 0$, we see from (5.2)–(5.4) that

$$\|\partial_x \mathscr{U}_1(t)\tilde{Q}u_0\|_2 \le Ct^{-4/5}\| \, |\tilde{Q}u_0|_2\|_{L^1_{x_3}}.$$

We finally estimate $\|\mathscr{U}_1(t)[\partial_x \tilde{Q}u_0]\|_2$. We here estimate only $U_1^{(2)}(t)[\partial_x \tilde{Q}u_0]$. In view of the above argument, it is not difficult to see that the other terms can be bounded by $Ct^{-5/4}\| \, |\partial_x \tilde{Q}u_0|_2\|_{L^1_{x_3}}$.

Let $\Psi = {}^T(\Phi, V) \in C_0^\infty(D)$. Since $\hat{P}_1 = -\hat{S}\hat{L}^{(1)}\hat{P}_0 - \hat{P}_0\hat{L}^{(1)}\hat{S}$ and $\hat{P}_0\tilde{Q} = 0$, we have

$$(\hat{P}_1(\partial_{x'}\tilde{Q}u_0), \Psi) = -(\hat{P}_0\hat{L}^{(1)}\hat{S}(\partial_{x'}\tilde{Q}u_0), \Psi) = -(\partial_{x'}\tilde{Q}u_0, \hat{S}^*\hat{L}^{(1)*}\hat{P}_0\Psi).$$

Here $\hat{S}^* = [(I - \hat{P}_0)\hat{L}_0^*(I - \hat{P}_0)]^{-1}$. Since $\hat{L}^{(1)*}\hat{P}_0\Psi = -i\gamma\, {}^T(0, 0, \langle\Phi\rangle)$, we have

$$\hat{S}^*\hat{L}^{(1)*}\hat{P}_0\Psi = -\frac{i\gamma}{\nu}\begin{pmatrix} 0 \\ 0 \\ (-\triangle')^{-1}\langle\Phi\rangle \end{pmatrix}. \tag{5.5}$$

Since $\hat{S}^*\hat{L}^{(1)*}\hat{P}_0\Psi|_{\partial D} = 0$, integrating by parts, we see from (5.5) that

$$\begin{aligned} |(\hat{P}_1(\partial_{x'}\tilde{Q}u_0), \Psi)| &\le |(\tilde{Q}u_0, \partial_{x'}\hat{S}^*\hat{L}^{(1)*}\hat{P}_0\Psi)| \\ &\le |\tilde{Q}u_0|_2 |\partial_{x'}\hat{S}^*\hat{L}^{(1)*}\hat{P}_0\Psi|_2 \\ &\le C|\tilde{Q}u_0|_2|\Psi|_2. \end{aligned}$$

By duality we have $|\hat{P}_1(\partial_{x'}\tilde{Q}u_0)|_2 \le C|\tilde{Q}u_0|_2$, from which we obtain

$$\|U_1^{(2)}(t)[\partial_{x'}u_0]\|_2 \le Ct^{-3/4}\| \, |\tilde{Q}u_0|_2\|_{L^1_{x_3}}.$$

Since $U_1^{(2)}(t)[\partial_{x_3}\tilde{Q}u_0] = \partial_{x_3}U_1^{(2)}(t)[\tilde{Q}u_0]$, we have

$$\|U_1^{(2)}(t)[\partial_{x_3}\tilde{Q}u_0]\|_2 \leq Ct^{-5/4}\|\,|\tilde{Q}u_0|_2\|_{L^1_{x_3}}.$$

This completes the proof. □

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