Cho, B., Kim, N.-M. and Park, Y.-K.
Osaka J. Math.
46 (2009), 479–502

ON THE COEFFICIENTS OF CERTAIN FAMILY OF MODULAR EQUATIONS

BUMKYU CHO, NAM MIN KIM and YOON KYUNG PARK

(Received February 5, 2008)

Abstract

The n-th modular equation for the elliptic modular function $j(z)$ has large coefficients even for small n, and those coefficients grow rapidly as $n \to \infty$. The growth of these coefficients was first obtained by Cohen ([5]). And, recently Cais and Conrad ([1], §7) considered this problem for the Hauptmodul $j_5(z)$ of the principal congruence group $\Gamma(5)$. They found that the ratio of logarithmic heights of n-th modular equations for $j(z)$ and $j_5(z)$ converges to 60 as $n \to \infty$, and observed that 60 is the group index $[\overline{\Gamma(1)} : \overline{\Gamma(5)}]$. In this paper we prove that their observation is true for Hauptmoduln of somewhat general Fuchsian groups of the first kind with genus zero.

1. Introduction

Let $\mathfrak{H} = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$ be the complex upper half plane and $j(z) = q^{-1} + 744 + 196884q + \cdots$ be the elliptic modular function on $SL_2(\mathbb{Z})$ with $z \in \mathfrak{H}$ and $q = e^{2\pi i z}$. Further, let $\Phi_n^j(X, Y) = 0$ be the n-th modular equation for $j(z)$ (see [6, 10, 11]). Then $\Phi_n^j(X, Y)$ is a polynomial with integral coefficients satisfying $\Phi_n^j(j(z), j(nz)) = 0$, and is irreducible as a polynomial in X over $\mathbb{C}(Y)$. Moreover it is known that $\Phi_p^j(X, Y)$ satisfies the Kronecker congruences, and $\Phi_n^j(X, Y)$ has large coefficients even for small n. For example,

$$\begin{aligned}
\Phi_3^j(X, Y) = {} & X(X + 2^{15} \cdot 3 \cdot 5^3)^3 + Y(Y + 2^{15} \cdot 3 \cdot 5^3)^3 - X^3Y^3 \\
& + 2^3 \cdot 3^2 \cdot 31X^2Y^2(X + Y) - 2^2 \cdot 3^3 \cdot 9907XY(X^2 + Y^2) \\
& + 2 \cdot 3^4 \cdot 13 \cdot 193 \cdot 6367X^2Y^2 + 2^{16} \cdot 3^5 \cdot 5^3 \cdot 17 \cdot 263XY(X + Y) \\
& - 2^{31} \cdot 5^6 \cdot 22973XY.
\end{aligned}$$

Note that the coefficients of $\Phi_n^j(X, Y)$ grow quite rapidly as $n \to \infty$, which was first estimated by Cohen ([5]) as follows.

For a nonzero polynomial $P(X_1, \dots, X_r) \in \mathbb{C}[X_1, \dots, X_r]$, let $h(P(X_1, \dots, X_r))$ be the *logarithmic height* of $P(X_1, \dots, X_r)$ defined by the logarithm of the maximum of

2000 Mathematics Subject Classification. 11F03, 11F11, 11P55.
This work was partially supported by the SRC Program of KOSEF Research Grant R11-2007-035-01001-0.

the absolute values of its coefficients. And, throughout this article we use $\mathcal{O}$-notation which has the following meaning; let f and g be complex valued functions defined on some set S and h be a real valued positive function defined on S. Then $f = g + \mathcal{O}(h)$ means that there exists an absolute positive constant A such that $|f - g| \le A \cdot h$ on S. With the aid of height and $\mathcal{O}$-notation Cohen showed that how rapidly $h(\Phi_n^j(X, Y))$ grows as $n \to \infty$, that is, for any positive integer n we have

$$h(\Phi_n^j(X, Y)) = 6\psi(n)\left\{\log n - 2\sum_{p|n}\frac{\log p}{p} + \mathcal{O}(1)\right\}, \tag{1.1}$$

where $\psi(n) = n\prod_{p|n}(1 + 1/p)$.

On the other hand Cais and Conrad recently considered the modular equations of the Hauptmodul $j_5(z) = q^{-1/5}(1 + q - q^3 + q^5 + \cdots)$ of $\Gamma(5)$. For a positive integer n with $(n, 5) = 1$ we let $\Phi_n^{j_5}(X, Y) = 0$ be the n-th modular equation for $j_5(z)$ defined as in [1, Definition 6.4]. Then $\Phi_n^{j_5}(X, Y)$ is a polynomial with integral coefficients satisfying $\Phi_n^{j_5}(j_5(z), j_5(nz)) = 0$, and is irreducible as a polynomial in X over $\mathbb{C}(Y)$. In addition, $\Phi_p^{j_5}(X, Y)$ also satisfies the Kronecker congruences ([1, Theorem 6.8]). But unlike the case of $\Phi_n^j(X, Y)$, $\Phi_n^{j_5}(X, Y)$ has much smaller coefficients, for example,

$$\Phi_3^{j_5}(X, Y) = X^4Y^3 + X^3 - 3X^2Y^2 - XY^4 - Y.$$

They indeed estimated the logarithmic height of $\Phi_n^{j_5}(X, Y)$, precisely, for any positive integer n with $(n, 5) = 1$

$$h(\Phi_n^{j_5}(X, Y)) = \frac{1}{10}\psi(n)\left\{\log n - 2\sum_{p|n}\frac{\log p}{p} + \mathcal{O}(1)\right\},$$

from which they derived by comparing with $h(\Phi_n^j(X, Y))$ that

$$\lim_{\substack{n\to\infty \\ (n,5)=1}}\frac{h(\Phi_n^j(X, Y))}{h(\Phi_n^{j_5}(X, Y))} = 60 = [\overline{\Gamma(1)} : \overline{\Gamma(5)}]$$

where $\overline{\Gamma(1)}$ and $\overline{\Gamma(5)}$ denote the images of $\Gamma(1)$ and $\Gamma(5)$ in $PSL_2(\mathbb{R})$. But Cais and Conrad did not explain why the ratio of logarithmic heights converges to the group index.

So it is natural and worthwhile to ask whether

$$\frac{h(\Phi_n^j(X, Y))}{h(\Phi_n^f(X, Y))} \to [\overline{\Gamma(1)} : \overline{\Gamma}]$$

as $n \to \infty$ with some conditions on n for a Hauptmodul $f(z)$ of arbitrary congruence subgroup Γ. In Theorem 2.1 (1) we shall prove that the answer is affirmative for classical congruence subgroups. We further consider a similar question about subgroups $\langle \Gamma_0(N), W_e \rangle_{e \in S}$ of $SL_2(\mathbb{R})$ which appear in "Monstrous Moonshine" phenomenon. And we will prove in Theorem 2.1 (2) that the ratio of logarithmic heights in this case is also related to a certain summand of group indices.

In what follows we fix an integer N, and define necessary congruence subgroups

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \,\middle|\, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \bmod N \right\},$$

$$\Gamma^0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \,\middle|\, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} * & 0 \\ * & * \end{pmatrix} \bmod N \right\},$$

$$\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \,\middle|\, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \bmod N \right\},$$

$$\Gamma^1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \,\middle|\, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix} \bmod N \right\},$$

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \,\middle|\, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod N \right\}.$$

2. Preliminaries and statements of the results

In this section we recall the definition of modular equations for Hauptmoduln of various subgroups of $SL_2(\mathbb{R})$.

For a Fuchsian group Γ of the first kind with genus zero, we define a Hauptmodul of Γ by an automorphic function $f(z)$ for Γ satisfying $A_0(\Gamma) = \mathbb{C}(f(z))$. Here by $A_0(\Gamma)$ we mean the field of all automorphic functions for Γ (see [11]). In this paper we fix that $\Gamma = \Gamma_1(N) \cap \Gamma_0(mN)$ for a positive integer m, and $f(z) = q^{-1} + \sum_{n=0}^{\infty} a_n q^n$ is a Hauptmodul of Γ with $a_n \in \mathbb{R}$ for all $n \geq 0$. While considering this Hauptmodul $f(z)$ of Γ, it is a necessary condition that the genus of Γ is zero, and as for the genus formula of Γ we refer to [9, Theorem 1.1].

For a positive integer n with $(n, mN) = 1$ we have the following disjoint coset decomposition

$$\Gamma \begin{pmatrix} 1 & 0 \\ 0 & n \end{pmatrix} \Gamma = \bigcup_{\substack{a>0 \\ ad=n}} \bigcup_{\substack{0 \leq b < d \\ (a,b,d)=1}} \Gamma \sigma_a \begin{pmatrix} a & b \\ 0 & d \end{pmatrix},$$

where $\sigma_a \in SL_2(\mathbb{Z})$ satisfies $\sigma_a \equiv \begin{pmatrix} a^{-1} & 0 \\ 0 & a \end{pmatrix} \bmod mN$. This can be proved by observing

$$\left| \Gamma \setminus \Gamma \begin{pmatrix} 1 & 0 \\ 0 & n \end{pmatrix} \Gamma \right| = n \prod_{p \mid n} \left(1 + \frac{1}{p} \right) = \psi(n)$$

and using [11, Proposition 3.36].

REMARK. Since $\sigma_a \Gamma \sigma_a^{-1} = \Gamma$ for any positive divisor a of n, we have $\mathbb{C}(f) = A_0(\Gamma) = A_0(\sigma_a^{-1}\Gamma\sigma_a) = \mathbb{C}(f \circ \sigma_a)$, and hence for given a we can define a rational function $P_a(T) \in \mathbb{C}(T)$ such that $f \circ \sigma_a = P_a(f)$. For positive divisors a, b of n we easily see that
(1) $a \equiv \pm 1 \bmod N \Leftrightarrow P_a(T) = T$, and $\overline{a} = \overline{b} \in (\mathbb{Z}/N\mathbb{Z})^\times/\{\pm 1\} \Leftrightarrow P_a(T) = P_b(T)$,
(2) $P_a(P_b(T)) = P_{ab}(T) = P_b(P_a(T))$.

If we let $P_a(T) = A(T)/B(T) \in \mathbb{C}(T)$ with $A(T), B(T) \in \mathbb{C}[T]$ and $(A(T), B(T)) = 1$, then $\deg A(T)$, $\deg B(T) \leq 1$ except when $\deg A(T) = \deg B(T) = 0$ because $\mathbb{C}(f \circ \sigma_a) = \mathbb{C}(f)$.

We now consider the following polynomial $\Psi_n^f(X, z)$ with the indeterminate X

$$\Psi_n^f(X, z) = \prod_{\substack{a>0 \\ ad=n}} \prod_{\substack{0 \leq b < d \\ (a,b,d)=1}} \left(X - f \circ \sigma_a \circ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}(z) \right).$$

Note that $\deg_X \Psi_n^f(X, z) = \psi(n)$. Since all the coefficients of $\Psi_n^f(X, z)$ are the elementary symmetric functions of the $f \circ \sigma_a \circ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$, they are invariant under Γ, i.e., $\Psi_n^f(X, z) \in \mathbb{C}(f(z))[X]$ and we may write $\Psi_n^f(X, f(z))$ instead of $\Psi_n^f(X, z)$. Then as in the usual argument of modular equations, we see that $\Psi_n^f(X, f(z))$ is irreducible over $\mathbb{C}(f(z))$. And we see from [8] that $f(z)^{r_n} \Psi_n^f(X, f(z)) \in \mathbb{C}[X, f(z)]$ for $r_n = -\sum_{s \in S_{1,\infty} \cap S_{2,0}} \mathrm{ord}_s f(z)$, where $S_{1,\infty}$ (respectively, $S_{2,0}$) is the set of all points of $\left(\Gamma \cap \begin{pmatrix} n & 0 \\ 0 & 1 \end{pmatrix}^{-1} \Gamma \begin{pmatrix} n & 0 \\ 0 & 1 \end{pmatrix}\right) \setminus \mathfrak{H}^*$ such that $f(z)$ (respectively, $f(nz)$) has poles (respectively, zeros) (see also [3, Theorem 3.3] or the proof of [4, Theorem 10]). Here we note that $r_n \leq -\sum_{s \in S_{1,\infty}} \mathrm{ord}_s f(z) = [\mathbb{C}(f(z), f(nz)) : \mathbb{C}(f(z))] \leq n \prod_{p|n}(1 + 1/p)$, because $\Psi_n^f(P_n(f(nz)), f(z)) = 0$.

Therefore for those Hauptmoduln $f(z)$ of Γ and integer n with $(n, mN) = 1$ we define the n-th modular equation $\Phi_n^f(X, Y) = Y^{r_n} \Psi_n^f(X, Y)$, namely

$$\Phi_n^f(X, f(z)) = f(z)^{r_n} \cdot \prod_{\substack{a>0 \\ ad=n}} \prod_{\substack{0 \leq b < d \\ (a,b,d)=1}} \left(X - f \circ \sigma_a \circ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}(z) \right).$$

Here we remark that if we confine ourselves to a Hauptmodul $f(z) = q^{-1} + \sum_{n=0}^{\infty} a_n q^n$ with $a_n \in \mathbb{Z}$, we could justify that $\Phi_n^f(X, Y) \in \mathbb{Z}[X, Y]$ and $\Phi_p^f(X, Y)$ satisfies the Kronecker congruences depending on $P_p(T)$ in the above remark. But we will not go further into this direction.

Next, unlike the case $\Gamma = \Gamma_1(N) \cap \Gamma_0(mN)$ we further consider a subgroup $\langle \Gamma_0(N), W_e \rangle_{e \in S}$ of $SL_2(\mathbb{R})$ which appears in "Monstrous Moonshine" phenomenon. For the details, we recommend the readers to refer [2].

Let $N > 1$ be an integer and e be a Hall divisor of N, that is, e is a positive divisor of N such that $(e, N/e) = 1$. For a Hall divisor e of N we define an Atkin-Lehner involution of $\Gamma_0(N)$ as a matrix with determinant 1 of the form

$$\begin{pmatrix} a\sqrt{e} & \dfrac{b}{\sqrt{e}} \\ c\dfrac{N}{\sqrt{e}} & d\sqrt{e} \end{pmatrix} \quad \text{where} \quad a, b, c, d \in \mathbb{Z}.$$

Let W_e be the set of all Atkin-Lehner involutions with a fixed Hall divisor e of N. Then these sets satisfy the following multiplication rule:

$$W_e W_f = W_f W_e = W_k \quad \text{where} \quad k = \frac{e}{(e,\ f)} \cdot \frac{f}{(e,\ f)}. \tag{2.1}$$

Notice that k is a Hall divisor of N if e and f are Hall divisors of N. Assume that S is a subset of the Hall divisors of N closed under the above multiplication rule. By $\langle \Gamma_0(N), W_e \rangle_{e\in S}$ we mean the subgroup of $SL_2(\mathbb{R})$ generated by all elements of $\Gamma_0(N)$ and W_e for all $e \in S$. If $\langle \Gamma_0(N), W_e \rangle_{e\in S}$ is of genus zero, then we can choose a Hauptmodul $f(z) = q^{-1} + \sum_{n=0}^{\infty} a_n q^n$ with $a_n \in \mathbb{Z}$. In [2] Chen and Yui defined, for a positive integer n prime to N, the n-th modular equation $\Phi_n^f(X, Y) = 0$ for which

$$\Phi_n^f(X,\ f(z)) = \prod_{\substack{a>0 \\ ad=n}} \prod_{\substack{0\le b<d \\ (a,b,d)=1}} \left(X - f \circ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} (z) \right).$$

And they proved that $\Phi_n^f(X, Y)$ is a polynomial with integral coefficients satisfying $\Phi_n^f(f(z),\ f(nz)) = 0$ and it is irreducible as a polynomial in X over $\mathbb{C}(Y)$. But, for the purpose of this article, it is enough to assume that $f(z)$ has only real Fourier coefficients, i.e., $a_n \in \mathbb{R}$ for all $n \geq 0$.

Now we are ready to state our main theorem.

Theorem 2.1. (1) *Let* $f(z) = q^{-1} + \sum_{n=0}^{\infty} a_n q^n$ *be a Hauptmodul of* $\Gamma = \Gamma_1(N) \cap \Gamma_0(mN)$ *with* $a_n \in \mathbb{R}$. *For a positive integer* n *with* $(n, mN) = 1$, *we get*

$$h(\Phi_n^f(X, Y)) = \frac{6\psi(n)}{[\overline{\Gamma(1)} : \overline{\Gamma}]} \left\{ \log n - 2 \sum_{p|n} \frac{\log p}{p} + \mathcal{O}(1) \right\}.$$

(2) *Let* $f(z) = q^{-1} + \sum_{n=0}^{\infty} a_n q^n$ *be a Hauptmodul of* $\langle \Gamma_0(N), W_e \rangle_{e\in S}$ *with* $a_n \in \mathbb{R}$. *For a positive integer* n *with* $(n, N) = 1$, *we have*

$$h(\Phi_n^f(X, Y)) = \sum_{e\in S} \frac{6\psi(n)}{[\overline{\Gamma(1)} : \overline{\Gamma_0(N/e)}]} \left\{ \log n - 2 \sum_{p|n} \frac{\log p}{p} + \mathcal{O}(1) \right\}.$$

Combining (1.1) and Theorem 2.1, we can readily achieve the following corollary.

Corollary 2.2. (1) *With the notations as in* Theorem 2.1 (1), *we obtain*

$$\lim_{\substack{n\to\infty\\(n,mN)=1}} \frac{h(\Phi_n^f(X,Y))}{h(\Phi_n^j(X,Y))} = \frac{1}{[\overline{\Gamma(1)}:\overline{\Gamma}]}.$$

(2) *With the notations as in* Theorem 2.1 (2), *we get*

$$\lim_{\substack{n\to\infty\\(n,N)=1}} \frac{h(\Phi_n^f(X,Y))}{h(\Phi_n^j(X,Y))} = \sum_{e\in S} \frac{1}{[\overline{\Gamma(1)}:\overline{\Gamma_0(N/e)}]}.$$

We conclude this section with some remarks. For an arbitrary intersection of classical congruence subgroups

$$\Gamma' = \Gamma_0(N_1)\cap\Gamma^0(N_2)\cap\Gamma_1(N_3)\cap\Gamma^1(N_4)\cap\Gamma(N_5),$$

we have $\alpha^{-1}\Gamma'\alpha = \Gamma_1(N)\cap\Gamma_0(mN)$ where $N = lcm(N_3, N_4, N_5)$ and

$$\alpha = \begin{pmatrix} lcm(N_2, N_4, N_5) & 0 \\ 0 & 1 \end{pmatrix}, \quad m = \frac{lcm(N_1, N_3, N_5)\, lcm(N_2, N_4, N_5)}{N}.$$

If $g(z) = q_h^{-1} + \sum_{n=0}^{\infty} a_n q_h^n$ is a Hauptmodul of Γ' with $h = lcm(N_2, N_4, N_5)$ and $q_h = e^{2\pi iz/h}$, then $f(z) := g\circ\alpha(z) = q^{-1} + \sum_{n=0}^{\infty} a_n q^n$ is a Hauptmodul of $\Gamma_1(N)\cap\Gamma_0(mN)$. Since the n-th modular equation $\Phi_n^g(X,Y)$ for $g(z)$ is, essentially, irreducible as a polynomial in X over $\mathbb{C}(Y)$ satisfying $\Phi_n^g(g(z), g(nz)) = 0$, we obtain $\Phi_n^f(X,Y) = \Phi_n^g(X,Y)$ by observing $\Phi_n^g(g(hz), g(hnz)) = 0$ and $f(z) = g(hz)$. Thus Theorem 2.1 (1) holds for any congruence subgroup of $\Gamma_0(N_1)$, $\Gamma^0(N_2)$, $\Gamma_1(N_3)$, $\Gamma^1(N_4)$, $\Gamma(N_5)$ or arbitrary intersection of them. For example, since

$$\begin{pmatrix} 5 & 0 \\ 0 & 1 \end{pmatrix}^{-1} \Gamma(5) \begin{pmatrix} 5 & 0 \\ 0 & 1 \end{pmatrix} = \Gamma_1(5)\cap\Gamma_0(25)$$

and $f(z) := j_5(5z)$ is a Hauptmodul of $\Gamma_1(5)\cap\Gamma_0(25)$ with the same n-th modular equation when $(n, 5) = 1$, we can recover the result of Cais and Conrad from Theorem 2.1 (1).

If S contains all the Hall divisors of N, we write $\Gamma_0(N)+$ as the group $\langle\Gamma_0(N), W_e\rangle_{e\in S}$. In [2, Appendix 2] Chen and Yui calculated some modular equations for Hauptmoduln of $\Gamma_0(N)$ and $\Gamma_0(N)+$. For instance,

$$\begin{aligned}\Phi_2^{\Gamma_0(3)}(X,Y) = X^3 &+ (-Y^2 + 108)X^2 + (-153Y + 2268)X \\ &+ (Y^3 + 108Y^2 + 2268Y - 46224),\end{aligned}$$

$$\Phi_2^{\Gamma_0(3)+}(X, Y) = X^3 + (-Y^2 + 1566)X^2 + (17343Y + 741474)X$$
$$+ (Y^3 + 1566Y^2 + 7417474Y - 28166076),$$

where $\Phi_2^{\Gamma_0(3)}$ and $\Phi_2^{\Gamma_0(3)+}$ stand for the second modular equations of the (normalized) Hauptmoduln of $\Gamma_0(3)$ and $\Gamma_0(3)+$, respectively. We remark that Theorem 2.1 (2) also gives a reason why the logarithmic height of $\Phi_n^{\Gamma_0(3)}$ is smaller than that of $\Phi_n^{\Gamma_0(3)+}$ for not only $n = 2$ but also sufficiently large n.

3. Proof of Theorem 2.1

To prove Theorem 2.1 it is necessary to study the behavior of Hauptmodul at each cusp of $\Gamma = \Gamma_1(N) \cap \Gamma_0(mN)$ or $\langle \Gamma_0(N), W_e \rangle_{e \in S}$. In this section we recall some lemmas which give us useful informations about these cusps.

First lemma provides us a criterion to determine whether or not given two cusps are equivalent under Γ.

Lemma 3.1. *Let* $\Gamma = \Gamma_1(N) \cap \Gamma_0(mN)$ *and*

$$\Delta = \{\overline{\pm(1 + Nk)} \in (\mathbb{Z}/mN\mathbb{Z})^\times \mid k = 0, 1, \ldots, m - 1\}.$$

We assume that a, c, a' *and* c' *are integers such that* $(a, c) = (a', c') = 1$. *By* $\pm 1/0$ *we mean* ∞. *Then the cusp* a/c *is equivalent to* a'/c' *under* Γ *if and only if there exist* $x \in \Delta$ *and* $n \in \mathbb{Z}$ *such that*

$$\begin{pmatrix} a' \\ c' \end{pmatrix} \equiv \begin{pmatrix} xa + nc \\ x^{-1}c \end{pmatrix} \mod mN.$$

Proof. Suppose that a/c is equivalent to a'/c' under Γ, i.e., there exists $\gamma \in \Gamma$ such that $a'/c' = \gamma(a/c)$. Since a, c, a', c' are integers satisfying $(a, c) = (a', c') = 1$, we have $\begin{pmatrix} a' \\ c' \end{pmatrix} = \pm\gamma\begin{pmatrix} a \\ c \end{pmatrix}$. By putting $\gamma = \begin{pmatrix} x & n \\ z & w \end{pmatrix} \in \Gamma$ we have the desired assertion. Conversely suppose that there exist $x \in \Delta$ and $n \in \mathbb{Z}$ satisfying the above congruence in the hypothesis. Since the natural reduction map of $SL_2(\mathbb{Z})$ into $SL_2(\mathbb{Z}/mN\mathbb{Z})$ is surjective, let $\gamma \in SL_2(\mathbb{Z})$ be a preimage of $\begin{pmatrix} x & n \\ 0 & x^{-1} \end{pmatrix} \in SL_2(\mathbb{Z}/mN\mathbb{Z})$. Note that $\gamma \in \{\pm 1\} \cdot \Gamma$ and $\begin{pmatrix} a' \\ c' \end{pmatrix} \equiv \gamma \begin{pmatrix} a \\ c \end{pmatrix} \mod mN$. Now it is an elementary fact that if u, v, z, w are integers such that $(u, v) = (z, w) = 1$ and $\begin{pmatrix} u \\ v \end{pmatrix} \equiv \begin{pmatrix} z \\ w \end{pmatrix} \mod N$, then u/v and z/w are equivalent under $\Gamma(N)$ ([11, Lemma 1.42]). So in our case there exists $\gamma' \in \Gamma(mN)$ such that $a'/c' = \gamma'(\gamma(a/c))$. This completes the proof since $\Gamma(mN) \subset \Gamma$. □

Let $\phi(x)$ be the Euler function. Then it is worthy of remarking that

$$[\overline{\Gamma(1)} : \overline{\Gamma}] = [\overline{\Gamma(1)} : \overline{\Gamma_0(mN)}][\overline{\Gamma_0(mN)} : \overline{\Gamma}] = \frac{[\overline{\Gamma(1)} : \overline{\Gamma_0(mN)}]\phi(mN)}{|\Delta|}, \tag{3.1}$$

which will be used in the proof of Lemma 3.10 and Lemma 3.11. From the next two lemmas we can determine whether a given cusp is equivalent to the cusp infinity under $\langle \Gamma_0(N), W_e \rangle_{e\in S}$.

Lemma 3.2. *Let $S_{\Gamma_0(N)}$ be the set of pairs (c, a) satisfying*
(1) $(1, 0) \in S_{\Gamma_0(N)}$.
(2) $c > 1,\ c \mid N,\ 1 \le a < c,\ (c, a) = 1$.
(3) *If* $(c, a), (c, a_1) \in S_{\Gamma_0(N)}$ *and* $a_1 \equiv a \bmod (c, N/c)$ *then* $a = a_1$.
Then the set $\{a/c \mid (c, a) \in S_{\Gamma_0(N)}\}$ is a set of complete representatives of all inequivalent cusps of $\Gamma_0(N)$.

Proof. This lemma is indeed well-known ([7, Proposition 1.23]). For the reader's convenience we give an alternative proof. We first observe that the cardinality of $S_{\Gamma_0(N)}$ is $1+\sum_{c>1, c|N}\varphi((c, N/c))$ because the natural map $(\mathbb{Z}/c\mathbb{Z})^\times \to (\mathbb{Z}/(c, N/c)\mathbb{Z})^\times$ is surjective. Since the number of inequivalent cusps of $\Gamma_0(N)$ is $\sum_{d|N}\varphi((d, N/d))$ (see [11, Proposition 1.43]), it is enough to prove that arbitrary two distinct pairs $(c, a), (c', a') \in S_{\Gamma_0(N)}$ are inequivalent to each other. Suppose that they are equivalent under $\Gamma_0(N)$. By substituting $N = 1$, $m = N$, and $\Delta = (\mathbb{Z}/N\mathbb{Z})^\times$ in Lemma 3.1, we must have that $c = c'$ and $x \equiv 1 \bmod N/c$. Thus $a' \equiv xa + nc \bmod N$ with $x \equiv 1 \bmod N/c$ implies that $a' \equiv a \bmod (c, N/c)$. By hypothesis (3) we have $a' = a$. □

Lemma 3.3. *Let S be a subset of Hall divisors of N closed under the multiplication rule* (2.1). *Then the cusps*

$$\left\{ \frac{1}{N/e} \;\middle|\; e \in S \right\}$$

are all those equivalent under $\langle \Gamma_0(N), W_e \rangle_{e\in S}$ to ∞ among the set of representatives $\{a/c \mid (c, a) \in S_{\Gamma_0(N)}\}$ described in Lemma 3.2.

Proof. For given $e \in S$ there exist $b, d \in \mathbb{Z}$ satisfying $de - b(N/e) = 1$. Thus we have $W_e = \Gamma_0(N)\begin{pmatrix} \sqrt{e} & b/\sqrt{e} \\ N/\sqrt{e} & d\sqrt{e} \end{pmatrix}$. Since $\begin{pmatrix} \sqrt{e} & b/\sqrt{e} \\ N/\sqrt{e} & d\sqrt{e} \end{pmatrix}(\infty) = 1/(N/e)$, we have the assertion. □

Using the above lemmas we are able to prove Theorem 2.1 by adopting the idea of Cais and Conrad ([1]). For convenience, if $f(z) = q^{-1} + \sum_{n=0}^{\infty} a_n q^n$ with $a_n \in \mathbb{R}$ is a Hauptmodul of Γ (respectively, $\langle \Gamma_0(N), W_e \rangle_{e\in S}$), then we simply write "$f(z)$ is on Γ" (respectively, on $\langle \Gamma_0(N), W_e \rangle_{e\in S}$).

Lemma 3.4. Γ *and* $\langle \Gamma_0(N), W_e \rangle_{e\in S}$ *have no elliptic points on* $i\mathbb{R}_{>1}$.

Proof. If it ($t > 1$) is a fixed point of an elliptic element $\sigma \in SL_2(\mathbb{R})$, then the absolute value of the trace of σ, $|\mathrm{tr}(\sigma)|$, is less than 2. Moreover, if $\sigma \in SL_2(\mathbb{Z})$, we have $\sigma = \pm\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ which gives rise to a contradiction. If $\sigma \in SL_2(\mathbb{R}) \setminus SL_2(\mathbb{Z})$, then we may assume $\sigma = \pm\begin{pmatrix} a\sqrt{e} & b/\sqrt{e} \\ cN/\sqrt{e} & d\sqrt{e} \end{pmatrix}$ for $a, b, c, d \in \mathbb{Z}$ and a Hall divisor e of N. Since σ fixes it and $|\mathrm{tr}(\sigma)| < 2$, we have $a = d$; hence $a = 0$ and $\sigma = \pm\begin{pmatrix} 0 & b/\sqrt{e} \\ cN/\sqrt{e} & 0 \end{pmatrix}$. Since σ has determinant 1, we obtain $-bcN = e$ and so $bc(N/e) = -1$, that is, $b/c = -1$. On the other hand σ fixes it, so we have $b = -cNt$. Thus $Nt^2 = 1$, which is a contradiction. $\square$

Since $f(z) = q^{-1} + \cdots$ has real Fourier coefficients, $f(it)$ is real and $|f(it)| \to \infty$ as $t \to \infty$. Moreover $f'(z)$ is nonvanishing on $i\mathbb{R}_{>1}$ by Lemma 3.4, so we see that $f(it)$ is strictly increasing for $t \geq 1$. Thus we can choose real numbers $s > 1$ and $1 \leq t_0 \leq t_1$ such that $f(it_0) = s$, $f(it_1) = 2s$.

Lemma 3.5. *For $t_0 \leq t \leq t_1$, we have*

$$h(\Phi_n^f(X,\ f(it))) = \sum_{\substack{a>0 \\ ad=n}} S_d(t) + \mathcal{O}(\psi(n)),$$

where

$$S_d(t) = \begin{cases} \displaystyle\sum_{\substack{0\leq b<d \\ (a,b,d)=1}} \log\max\left\{1,\ \left|f \circ \sigma_a\left(\frac{ait+b}{d}\right)\right|\right\} & \text{if} \quad f(z) \quad \text{is on} \quad \Gamma, \\ \displaystyle\sum_{\substack{0\leq b<d \\ (a,b,d)=1}} \log\max\left\{1,\ \left|f\left(\frac{ait+b}{d}\right)\right|\right\} & \text{if} \quad f(z) \quad \text{is on} \quad \langle\Gamma_0(N),\ W_e\rangle_{e\in S}. \end{cases}$$

Here the implicit $\mathcal{O}$-constant depends only on f, t_0 and t_1.

Proof. It is well-known that the coefficients of a monic polynomial $P(x) = (x - w_1)\cdots(x - w_d)$ are laid in between $2^{-d}M$ and 2^dM where $M = \prod_{j=1}^d \max\{1, |w_j|\}$. Taking logarithm we see that

$$h(P) = \sum_{j=1}^{d} \log\max\{1, |w_j|\} + \mathcal{O}(d) \tag{3.2}$$

with an implicit absolute $\mathcal{O}$-constant which is independent of d and P.

If $f(z)$ is on Γ, then for $t_0 \leq t \leq t_1$

$$\Phi_n^f(X,\ f(it)) = f(z)^{r_n} \prod_{\substack{a>0\\ ad=n}} \prod_{\substack{0\leq b<d\\ (a,b,d)=1}} \left(X - (f \circ \sigma_a)\left(\frac{ait+b}{d}\right)\right).$$

Applying (3.2) we have

$$\begin{aligned} &h(\Phi_n^f(X,\ f(it))) \\ &= r_n \log f(it) + \sum_{\substack{a>0\\ ad=n}} \sum_{\substack{0\leq b<d\\ (a,b,d)=1}} \log\max\left\{1, \left| f \circ \sigma_a\left(\frac{ait+b}{d}\right)\right|\right\} + \mathcal{O}(\psi(n)). \end{aligned}$$

Since $0 \leq r_n \leq \psi(n)$ and $s = f(it_0) \leq f(it) \leq f(it_1) = 2s$, we get $r_n \log f(it) = \mathcal{O}(\psi(n))$ where the implicit $\mathcal{O}$-constant depends only on f, t_0 and t_1.

As for the case where $f(z)$ is on $\langle \Gamma_0(N), W_e\rangle_{e\in S}$, the same argument can be applied, and hence we omit the detailed proof. □

Next goal is to calculate each term in the summation $S_d(t)$. For this purpose we are in need of the following lemma.

Lemma 3.6. *For $z = \xi + i\eta \in \mathfrak{H}$, let $g(z) = a_{-1}q_h^{-1} + \sum_{n=0}^{\infty} a_n {q_h}^n$ with $q_h = e^{2\pi i z/h}$ for a positive integer h. We assume that if $a_{-1} = 0$ (respectively, $a_{-1} \neq 0$), then $g(z)$ (respectively, $q_h g(z)$) is absolutely convergent for $\eta > 0$. Then for $\eta \geq 1/2$, we have*

$$\log\max\{1, |g(z)|\} = \begin{cases} \mathcal{O}(1) & \text{if} \quad a_{-1} = 0, \\ \dfrac{2\pi i\eta}{h} + \mathcal{O}(1) & \text{if} \quad a_{-1} \neq 0. \end{cases}$$

Here the implicit $\mathcal{O}$-constants depend only $g(z)$.

Proof. Since $g(z+h) = g(z)$, we may assume that $-h/2 \leq \xi \leq h/2$. Suppose first that $a_{-1} = 0$. Since $|g(z)| \to |a_0|$ as $\eta \to \infty$, there is a real number $\eta_0 \geq 1/2$ such that for $\eta > \eta_0$, $|a_0|/2 \leq |g(z)| \leq |a_0| + 1$. Hence, for $\eta > \eta_0$ we derive $\log\max\{1, |g(z)|\} = \mathcal{O}(1)$. Here the implicit $\mathcal{O}$-constant depends only on a_0, that is g. For $1/2 \leq \eta \leq \eta_0$ we note that $\log\max\{1, |g(z)|\}$ is a continuous function on the set

$$\left\{\xi + i\eta \in \mathfrak{H} \;\middle|\; -\frac{h}{2} \leq \xi \leq \frac{h}{2} \text{ and } \frac{1}{2} \leq \eta \leq \eta_0\right\}$$

and hence is bounded on this set. Note that the upper bound depends only on g and is independent of the choice of η_0.

If $a_{-1} \neq 0$, $|q_h g(z)| \to |a_{-1}|$ as $\eta \to \infty$ so that we obtain the assertion by the same argument as above. □

Let M be a positive integer. Then it is more convenient to consider the displaced interval $I_M = [1/(M+1), (M+2)/(M+1))$ rather than the usual interval $[0, 1)$. Cohen proved in [5] that I_M can be expressed as

$$I_M = \bigcup_{k=1}^{M} \bigcup_{\substack{h=1 \\ (h,k)=1}}^{k} I_M\left(\frac{h}{k}\right),$$

which is a disjoint union of sets $I_M(h/k)$. Here each $I_M(h/k)$ is an interval of the form $[\rho_1^{(h/k)}, \rho_2^{(h/k)})$ containing h/k and

$$\frac{1}{2Mk} \le \frac{h}{k} - \rho_1^{(h/k)} \le \frac{1}{(M+1)k},$$
$$\frac{1}{2Mk} \le \rho_2^{(h/k)} - \frac{h}{k} \le \frac{1}{(M+1)k}.$$

For real numbers h, k and x, we put

$$g_{h,k}(x) = \frac{2\pi nt/d^2k^2}{(at/d)^2 + (x - h/k)^2},$$

which will be used for estimating the sum $S_d(t)$. Thus a, d and t are related to $S_d(t)$. Note that the width of the cusp $\sigma_a(\infty)$ is 1, because $f \circ \sigma_a = P_a(f)$ as remarked in §2. Also observe that $\langle \Gamma_0(N), W_e \rangle_{e \in S}$ contains $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Hence in any case we may reindex the sum in $S_d(t)$ via

$$b \mapsto \begin{cases} b & \text{if} \quad \dfrac{b}{d} \in \left[\dfrac{1}{N+1}, 1\right), \\ b + d & \text{if} \quad \dfrac{b}{d} \in \left[0, \dfrac{1}{N+1}\right). \end{cases}$$

Lemma 3.7. *Let f be on Γ.*
(1) *If $at/d \ge 1/2$, then we have*

$$\log \max\left\{1, \left| f \circ \sigma_a\left(\frac{ait + b}{d}\right)\right|\right\} = \begin{cases} \dfrac{2\pi nt}{d^2} + \mathcal{O}(1) & \textit{if} \quad \overline{a} \in \Delta, \\ \mathcal{O}(1) & \textit{otherwise.} \end{cases}$$

(2) *Put $M = [d/\sqrt{nt}]$. If $at/d \le 1$, then $M \ge 1$ and, for $b/d \in I_M(h/k)$, we get*

$$\log \max\left\{1, \left| f \circ \sigma_a\left(\frac{ait + b}{d}\right)\right|\right\}$$
$$= \begin{cases} g_{h,k}(b/d) + \mathcal{O}(1) & \textit{if} \quad k \equiv 0 \bmod mN \quad \textit{and} \quad \overline{h} \in \overline{a}\Delta, \\ \mathcal{O}(1) & \textit{otherwise.} \end{cases}$$

In both cases the implicit $\mathcal{O}$-constants depend only on f.

Proof. (1) By Lemma 3.1, $\sigma_a(\infty)$ is equivalent to ∞ under Γ if and only if $\overline{a} \in \Delta$. Using this, Lemma 3.6 gives us the assertion.

(2) Since $(h,k)=1$, we can find $\gamma_{h,k} := \begin{pmatrix} v & u \\ -k & h \end{pmatrix} \in SL_2(\mathbb{Z})$. By routine calculation we see that

$$\operatorname{Im}\left(\gamma_{h,k}\left(\frac{ait+b}{d}\right)\right) = \frac{nt/d^2k^2}{(at/d)^2+(b/d-h/k)^2} = \frac{1}{2\pi} g_{h,k}\left(\frac{b}{d}\right).$$

Since $b/d \in I_M(h/k) = [\rho_1^{(h/k)}, \rho_2^{(h/k)})$, we obtain

$$\left|\frac{b}{d} - \frac{h}{k}\right| \le \frac{1}{(M+1)k} \le \frac{\sqrt{nt}}{dk}.$$

Moreover, we achieve

$$\frac{at}{d} = \frac{nt}{d^2} \le \frac{\sqrt{nt}}{dk}$$

which implies that

$$\operatorname{Im}\left(\gamma_{h,k}\left(\frac{ait+b}{d}\right)\right) \ge \frac{1}{2}.$$

By Lemma 3.1, $\sigma_a(\gamma_{h,k}^{-1}(\infty))$ is equivalent to ∞ under Γ if and only if $k \equiv 0 \bmod mN$ and $\overline{h} \in \overline{a}\Delta$. Taking $g(z) = f \circ \sigma_a \circ \gamma_{h,k}^{-1}(z)$ in Lemma 3.6, we have the assertion. More precisely, if $k \equiv 0 \bmod mN$ and $\overline{h} \in \overline{a}\Delta$, then

$$\begin{aligned}\left| f \circ \sigma_a\left(\frac{ait+b}{d}\right)\right| &= \left| f \circ \sigma_a \circ \gamma_{h,k}^{-1}\left(\gamma_{h,k}\left(\frac{ait+b}{d}\right)\right)\right| \\ &= 2\pi \operatorname{Im}\left(\gamma_{h,k}\left(\frac{ait+b}{d}\right)\right) = g_{h,k}\left(\frac{b}{d}\right).\end{aligned}$$

Other case corresponds to the holomorphic one in Lemma 3.6. Therefore we prove the lemma. □

Lemma 3.8. *Let f be on $\langle \Gamma_0(N), W_e\rangle_{e\in S}$.*
(1) *If $at/d \ge 1/2$, then we have*

$$\log \max\left\{1, \left| f\left(\frac{ait+b}{d}\right)\right|\right\} = \frac{2\pi at}{d} + \mathcal{O}(1).$$

(2) *Put* $M = [d/\sqrt{nt}]$. *If* $at/d \leq 1$, *then* $M \geq 1$ *and, for* $b/d \in I_M(h/k)$, *we establish*

$$\log\max\left\{1, \left|f\left(\frac{ait+b}{d}\right)\right|\right\}$$
$$= \begin{cases} g_{h,k}\left(\dfrac{b}{d}\right) + \mathcal{O}(1) & \textit{for} \quad e \in S, \quad k \equiv 0 \bmod N/e \quad \textit{and} \quad \overline{h} \in (\mathbb{Z}/(N/e)\mathbb{Z})^{\times}, \\ \mathcal{O}(1) & \textit{otherwise.} \end{cases}$$

In both cases the implicit $\mathcal{O}$*-constants depend only on* f.

Proof. Since the first assertion can be proved in a similar way to Lemma 3.7, we only prove (2). The fact that $\gamma_{h,k}^{-1}(\infty)$ is equivalent to ∞ under $\langle\Gamma_0(N), W_e\rangle_{e\in S}$ yields by Lemma 3.7 that h/k is equivalent to $1/(N/e)$ under $\Gamma_0(N)$ for some Hall divisor $e \in S$ exactly. In other words, by Lemma 3.1 there are $\overline{x} \in (\mathbb{Z}/N\mathbb{Z})^{\times}$, $n \in \mathbb{Z}$ such that $h \equiv x^{-1} + n \cdot (N/e) \bmod N$ and $k \equiv x \cdot (N/e) \bmod N$. This is equivalent to $h \equiv x^{-1} \bmod (N/e)$ and $k \equiv 0 \bmod (N/e)$, because N/e is also a Hall divisor. Thus we have the conclusion. □

Now, we calculate $S_d(t)$ more precisely in Lemma 3.10 and 3.10. To this end we need the following lemma in advance.

Lemma 3.9. *Let* k, j *and* a *be positive integers satisfying* $j \mid k$ *and* $(j, a) = 1$. *We further let* ζ *be a primitive* k*-th root of unity and let*

$$c_k'(l) = \sum_{\substack{h\in(\mathbb{Z}/k\mathbb{Z})^{\times} \\ h\equiv a \bmod j}} \zeta^{hl} \quad \textit{for} \quad l \in \mathbb{Z}.$$

Then

$$|c_k'(l)| \leq j \cdot (k, l) \quad \textit{for any} \quad l \in \mathbb{Z}. \tag{3.3}$$

Proof. Using a primitive j-th root of unity $\zeta^{k/j}$ we may rewrite the sum as

$$c_k'(l) = \frac{1}{j} \sum_{i\in\mathbb{Z}/j\mathbb{Z}} \zeta^{-kia/j} \sum_{h\in(\mathbb{Z}/k\mathbb{Z})^{\times}} \zeta^{(l+ik/j)h}.$$

Let $\mu(x)$ be the Möbius function. Since the Ramanujan's sum satisfies

$$\sum_{h\in(\mathbb{Z}/k\mathbb{Z})^{\times}} \zeta^{hx} = \mu\left(\frac{k}{(k,x)}\right) \cdot \phi(k) \Big/ \phi\left(\frac{k}{(k,x)}\right)$$

for $x \in \mathbb{Z}$ and $\phi(xy) \le x\phi(y)$ for any $x, y \in \mathbb{Z}_{>0}$, we have

$$\left|\sum_{h\in(\mathbb{Z}/k\mathbb{Z})^\times} \zeta^{(l+ik/j)h}\right| \le \frac{\phi(k)}{\phi(k/(k, l+ik/j))} \le \left(k, l+\frac{ik}{j}\right) = \left(k, l+\frac{ik}{j}, jl\right)$$
$$\le (k, jl) \le j\cdot\left(\frac{k}{j}, l\right) \le j\cdot(k, l),$$

which implies $|c_k'(l)| \le j\cdot(k, l)$. □

Here we remark that Cais and Conrad dealt with the case of a rational prime j dividing k in [1, Lemma D.3], but it seems to be not true. Indeed, we can find a counterexample when $k = p = 3$, $a = m = 1$ with the notations as in there. So we correct it and prove the expanded version. It doesn't crucially matter, however, to the results because we need just its boundedness.

Lemma 3.10. *Let f be on Γ.*
(1) *If $d < \sqrt{nt}$, then $S_d(t) = \mathcal{O}(n/d)$. Here the implicit $\mathcal{O}$-constant depends only upon f, t_0 and t_1.*
(2) *If $d \ge \sqrt{nt}$, then*

$$S_d(t) = \frac{1}{[\overline{\Gamma(1)} : \overline{\Gamma}]}\cdot\frac{6d}{(a, d)}\phi((a, d))\log\left(\frac{d^2}{n}\right) + \mathcal{O}\left(\sigma_1\left(\frac{d}{(a, d)}\right)\right) + \mathcal{O}\left(\frac{d\sigma_1((a, d))}{(a, d)}\right),$$

where $\phi(x)$ is the Euler function and $\sigma_1(x)$ is the sum of positive divisors of x. Here the implicit $\mathcal{O}$-constant depends only upon Γ, f, t_0 and t_1.

Proof. (1) Since the number of elements in $\{b \mid 0 \le b < d,\ (a, b, d) = 1\}$ is $d\phi((a, d))/(a, d)$, by Lemma 3.6 and the fact that $\phi((a, d))/(a, d) \le 1$ we have

$$|S_d(t)| \le \sum_{\substack{0\le b<d\\(a,b,d)=1}} \log\max\left\{1, \left|f\circ\sigma_a\left(\frac{ait+b}{d}\right)\right|\right\}$$
$$= \begin{cases} \dfrac{d\phi((a, d))}{(a, d)}\dfrac{2\pi nt}{d^2} + \mathcal{O}\left(\dfrac{d\phi((a, d))}{(a, d)}\right) & \text{if } \ \overline{a} \in \Delta, \\ \mathcal{O}\left(\dfrac{d\phi((a, d))}{(a, d)}\right) & \text{otherwise} \end{cases}$$
$$\le \begin{cases} \dfrac{2\pi nt}{d} + C\cdot d & \text{if } \ \overline{a} \in \Delta \\ C'd & \text{otherwise.} \end{cases}$$

Using the fact that $d < nt/d \le nt_1/d$ we conclude the first assertion.

(2) Note that the assumption $d \geq \sqrt{nt}$ implies $at/d \leq 1$. Put $M = [d/\sqrt{nt}] \geq 1$. Then we have by Lemma 3.7

$$\begin{aligned}
S_d(t) &= \sum_{k=1}^{M} \sum_{\substack{h=1 \\ (h,k)=1}}^{k} \sum_{\substack{b/d \in I_M(h/k) \\ 0 \leq b < d \\ (a,b,d)=1}} \log \max\left\{1, \left| f \circ \sigma_a\left(\frac{ait+b}{d}\right)\right|\right\} \\
&= \sum_{\substack{1 \leq h \leq k \leq M \\ (h,k)=1 \\ k \equiv 0 \bmod mN \\ \overline{h} \in \overline{a}\Delta}} \sum_{\substack{b/d \in I_M(h/k) \\ 0 \leq b < d \\ (a,b,d)=1}} \left(g_{h,k}\left(\frac{b}{d}\right) + \mathcal{O}(1)\right) + \sum_{\substack{1 \leq h \leq k \leq M \\ (h,k)=1 \\ \text{otherwise}}} \sum_{\substack{b/d \in I_M(h/k) \\ 0 \leq b < d \\ (a,b,d)=1}} \mathcal{O}(1) \\
&= \sum_{\substack{1 \leq h \leq k \leq M \\ (h,k)=1 \\ k \equiv 0 \bmod mN \\ \overline{h} \in \overline{a}\Delta}} \sum_{\substack{b/d \in I_M(h/k) \\ 0 \leq b < d \\ (a,b,d)=1}} g_{h,k}\left(\frac{b}{d}\right) + \mathcal{O}(d).
\end{aligned}$$

Since the total number for error terms $\mathcal{O}(1)$ is less than d and so $\mathcal{O}(d)$ lies outside of the summation, we can get the last expression in the above summation.

Meanwhile, we see from [5, Lemma 6] that

$$\sum_{\substack{b/d \in I_M(h/k) \\ 0 \leq b < d,\ (a,b,d)=1}} g_{h,k}\left(\frac{b}{d}\right) = k^{-2} \sum_{f|(a,d)} \mu(f) F_f\left(\frac{dh}{fk}\right) + \mathcal{O}\left(\frac{\sqrt{n}\sigma_1((a,d))}{k(a,d)}\right),$$

where $F_f(\theta) = (2\pi^2 d/f) \sum_{v \in \mathbb{Z}} e^{-2\pi|v|nt/df} e^{2\pi i v\theta}$ and $\mu(x)$ is the Möbius function.

Since we have as in [1]

$$\begin{aligned}
C\frac{\sqrt{n}\sigma_1((a,d))}{(a,d)} \sum_{\substack{1 \leq h \leq k \leq M \\ (h,k)=1}} \frac{1}{k} &= C\frac{\sqrt{n}\sigma_1((a,d))}{(a,d)} \sum_{1 \leq k \leq M} \frac{\phi(k)}{k} \leq C \cdot M \frac{\sqrt{n}\sigma_1((a,d))}{(q,d)} \\
&\leq C\frac{\sqrt{n}\sigma_1((a,d))}{(a,d)} \frac{d}{\sqrt{nt}} \leq C\frac{d\sigma_1((a,d))}{(a,d)},
\end{aligned} \tag{3.4}$$

we establish that

$$\begin{aligned}
S_d(t) &= \sum_{\substack{1 \leq h \leq k \leq M \\ (h,k)=1 \\ k \equiv 0 \bmod mN \\ \overline{h} \in \overline{a}\Delta}} \left\{k^{-2} \sum_{f|(a,d)} \mu(f) F_f\left(\frac{dh}{fk}\right) + \mathcal{O}\left(\frac{\sqrt{n}\sigma_1((a,d))}{k(a,d)}\right)\right\} + \mathcal{O}(d) \\
&= \sum_{\substack{1 \leq h \leq k \leq M \\ (h,k)=1 \\ k \equiv 0 \bmod mN \\ \overline{h} \in \overline{a}\Delta}} \left\{k^{-2} \sum_{f|(a,d)} \mu(f) F_f\left(\frac{dh}{fk}\right)\right\} + \mathcal{O}\left(\frac{d\sigma_1((a,d))}{(a,d)}\right)
\end{aligned}$$

$$= \sum_{f|(a,d)} \mu(f) \sum_{\substack{1\le h\le k\le M \\ (h,k)=1 \\ k\equiv 0 \bmod mN \\ \overline{h}\in\overline{a}\Delta}} k^{-2} F_f\left(\frac{dh}{fk}\right) + \mathcal{O}\left(\frac{d\sigma_1((a,d))}{(a,d)}\right).$$

We now consider the sum

$$\sum_{\substack{1\le h\le k\le M \\ (h,k)=1 \\ k\equiv 0 \bmod mN \\ \overline{h}\in\overline{a}\Delta}} k^{-2} F_f\left(\frac{dh}{fk}\right) = \frac{2\pi^2 d}{f} \sum_{v\in\mathbb{Z}} C_M\left(\frac{dv}{f}\right) e^{-2\pi|v|nt/df}, \tag{3.5}$$

where

$$C_M(l) = \sum_{\substack{1\le k\le M \\ k\equiv 0 \bmod mN}} k^{-2} c_k(l)$$

and

$$c_k(l) = \sum_{\substack{1\le h\le k \\ (h,k)=1 \\ \overline{h}\in\overline{a}\Delta}} e^{2\pi i hl/k} \quad \text{for any} \quad l\in\mathbb{Z}.$$

We have to calculate $C_M(l)$ and $c_k(l)$ to know the upper bound of the sum of (3.5). By Lemma 3.9 we know that $|c_k(l)| \le |\Delta| mN(k,l)$ for $l\in\mathbb{Z}-\{0\}$. So when $l\neq 0$, we have

$$|C_M(l)| \le |\Delta| mN \sum_{k=1}^{\infty} k^{-2}(k,l) \le |\Delta| mN \sum_{d|l} d \sum_{j=1}^{\infty} \frac{1}{j^2 d^2}$$
$$= |\Delta| mN \frac{\pi^2}{6} \frac{1}{|l|} \sum_{d|l} \frac{|l|}{d} = |\Delta| mN \frac{\pi^2}{6} \frac{\sigma_1(|l|)}{|l|};$$

hence

$$|C_M(l)| = \mathcal{O}\left(\frac{\sigma_1(|l|)}{|l|}\right)$$

for $l\neq 0$, where the implicit $\mathcal{O}$-constant depends only on Γ. In case of $l=0$ we consider the natural surjective homomorphism $\pi : (\mathbb{Z}/k\mathbb{Z})^\times \to (\mathbb{Z}/mN\mathbb{Z})^\times$ which gives us

$$c_k(0) = |\pi^{-1}(\Delta)| = |\Delta|\,|\ker\pi| = |\Delta| \frac{\phi(k)}{\phi(mN)}.$$

Hence by [1, Lemma D.1] and (3.1) we obtain

$$\begin{aligned}
C_M(0) &= \sum_{\substack{1\le k\le M\\ k\equiv 0 \bmod mN}} k^{-2}\frac{|\Delta|}{\phi(mN)}\phi(k)\\
&= \frac{6}{\pi^2}\frac{|\Delta|}{\phi(mN)[\Gamma(1):\Gamma_0(mN)]}\log M + \mathcal{O}(1)\\
&= \frac{6}{\pi^2[\overline{\Gamma(1)}:\overline{\Gamma}]}\log M + \mathcal{O}(1),
\end{aligned}$$

where the implicit $\mathcal{O}$-constant is absolute, i.e., it is independent of Γ and M.

Therefore we get

$$\begin{aligned}
\sum_{\substack{1\le h\le k\le M\\ (h,k)=1\\ h\equiv 0 \bmod mN\\ \overline{h}\in\overline{a}\Delta}} k^{-2}F_f\left(\frac{dh}{fk}\right) &= \frac{12d}{f[\overline{\Gamma(1)}:\overline{\Gamma}]}\log M + \mathcal{O}\left(\frac{d}{f}\right)\\
&\quad + \mathcal{O}\left(\sum_{v\in\mathbb{Z}-\{0\}}\frac{\sigma_1(d|v|/f)}{|v|}e^{-2\pi|v|nt/df}\right),
\end{aligned}$$

where the implicit $\mathcal{O}$-constants depend only on Γ and t. Since $f \mid (a,d)$ and $(a,d)\mid a = n/d$, we have $df \le n$; hence $1\le t_0\le t$ implies that

$$e^{-2\pi(|v|-1)nt/df} \le e^{-2\pi(|v|-1)t} \le e^{-2\pi(|v|-1)}.$$

By putting

$$C_1 = \sum_{v\in\mathbb{Z}-\{0\}}\frac{\sigma_1(|v|)}{|v|}e^{-2\pi(|v|-1)}$$

and using the fact

$$\sigma_1\left(\frac{d}{f}|v|\right) \le \sigma_1\left(\frac{d}{f}\right)\sigma_1(|v|),$$

we obtain

$$\sum_{v\in\mathbb{Z}-\{0\}}\frac{\sigma_1(d|v|/f)}{|v|}e^{-2\pi|v|nt/df} \le C_1\sigma_1\left(\frac{d}{f}\right)e^{-2\pi nt/df} \le C_1\sigma_1\left(\frac{d}{f}\right)e^{-2\pi n/df}.$$

Thus we deduce

$$\mathcal{O}\left(\sum_{v\in\mathbb{Z}-\{0\}}\frac{\sigma_1(d|v|/f)}{|v|}e^{-2\pi|v|nt/df}\right) = \mathcal{O}\left(\sigma_1\left(\frac{d}{f}\right)e^{-2\pi n/df}\right),$$

where the implicit $\mathcal{O}$-constant depends only on Γ.

Since $M = [\sqrt{d^2/(nt)}]$ and $1 \le t_0 \le t \le t_1$, we see that $\log M = (1/2)\log(d^2/n) + \mathcal{O}(1)$ where the implicit $\mathcal{O}$-constant depends only on t_0 and t_1.

Consequently, we have

$$\sum_{\substack{1\le h\le k\le M\\(h,k)=1\\k\equiv 0 \bmod mN\\ \overline{h}\in\overline{a}\Delta}} k^{-2}F_f\left(\frac{dh}{fk}\right) = \frac{6d}{f[\overline{\Gamma(1)}:\overline{\Gamma}]}\log\left(\frac{d^2}{n}\right) + \mathcal{O}\left(\frac{d}{f}\right) + \mathcal{O}\left(\sigma_1\left(\frac{d}{f}\right)e^{-2\pi n/df}\right)$$

where the implicit $\mathcal{O}$-constants depend only on Γ, t_0 and t_1. By substituting this for the sum of $S_d(t)$ we obtain

$$\begin{aligned} S_d(t) = \sum_{f\mid(a,d)} \mu(f)\left(\frac{6d}{f[\overline{\Gamma(1)}:\overline{\Gamma}]}\log\left(\frac{d^2}{n}\right) + \mathcal{O}\left(\frac{d}{f}\right) + \mathcal{O}\left(\sigma_1\left(\frac{d}{f}\right)e^{-2\pi n/df}\right)\right) \\ + \mathcal{O}\left(\frac{d\sigma_1((a,d))}{(a,d)}\right), \end{aligned}$$

where the implicit $\mathcal{O}$-constants depend only on Γ, f, t_0 and t_1. Since

$$\sum_{f\mid(a,d)}\left|\mu(f)\frac{d}{f}\right| \le \sum_{f\mid(a,d)}\frac{d}{f} = \frac{d\sigma_1((a,d))}{(a,d)},$$

the first error term contributes $\mathcal{O}(d\sigma_1((a,d))/(a,d))$.

Similarly, since $\sigma_1(df/(a,d)) \le \sigma_1(d/(a,d))\sigma_1(f)$ and $e^{-2\pi nf/d(a,d)} \le e^{-2\pi f}$, we derive

$$\begin{aligned} \sum_{f\mid(a,d)}\left|\mu(f)\sigma_1\left(\frac{d}{f}\right)e^{-2\pi n/df}\right| \le \sum_{f\mid(a,d)}\sigma_1\left(\frac{d}{f}\right)e^{-2\pi n/df} = \sum_{f\mid(a,d)}\sigma_1\left(\frac{df}{(a,d)}\right)e^{-2\pi nf/d(a,d)} \\ \le \sigma_1\left(\frac{d}{(a,d)}\right)\sum_{f\mid(a,d)}\sigma_1(f)e^{-2\pi f}, \end{aligned}$$

and so the second error term contributes $\mathcal{O}(\sigma_1(d/(a,d)))$. From the fact $\phi((a,d)) = \sum_{f\mid(a,d)}\mu(f)(a,d)/f$ we finally obtain

$$S_d(t) = \frac{6}{[\overline{\Gamma(1)}:\overline{\Gamma}]}\frac{d}{(a,d)}\phi((a,d))\log\left(\frac{d^2}{n}\right) + \mathcal{O}\left(\sigma_1\left(\frac{d}{(a,d)}\right)\right) + \mathcal{O}\left(\frac{d\sigma_1((a,d))}{(a,d)}\right),$$

where the implicit $\mathcal{O}$-constants depend only on Γ, f, t_0 and t_1. This completes the proof. $\square$

Lemma 3.11. *Let f be on $\langle \Gamma_0(N), W_e \rangle_{e \in S}$.*
(1) *If $d < \sqrt{nt}$, then $S_d = \mathcal{O}(n/d)$. Here the implicit $\mathcal{O}$-constant depends only upon f, t_0 and t_1.*
(2) *If $d \geq \sqrt{nt}$, then*

$$S_d = \sum_{e \in S} \frac{1}{[\Gamma(1) : \Gamma_0(N/e)]} \frac{6d}{(a, d)} \phi((a, d)) \log\left(\frac{d^2}{n}\right) + \mathcal{O}\left(\sigma_1\left(\frac{d}{(a, d)}\right)\right) + \mathcal{O}\left(\frac{d\sigma_1((a, d))}{(a, d)}\right).$$

Here the implicit $\mathcal{O}$-constant depends only upon $\langle \Gamma_0(N), W_e \rangle_{e \in S}$, f, t_0 and t_1.

Proof. It is possible for us to prove (1) with similar arguments as in Lemma 3.10, so we omit the detail. We only give a proof of (2). By using Lemma 3.8 we have

$$\begin{aligned} S_d(t) &= \sum_{k=1}^{M} \sum_{\substack{h=1 \\ (h,k)=1}}^{k} \sum_{\substack{b/d \in I_M(h/k) \\ 0 \leq b < d \\ (a,b,d)=1}} \log \max\left\{1, \left| f\left(\frac{ait + b}{d}\right)\right|\right\} \\ &= \sum_{e \in S} \sum_{\substack{1 \leq h \leq k \leq M \\ (h,k)=1 \\ k \equiv 0 \bmod N/e \\ \overline{h} \in (\mathbb{Z}/(N/e)\mathbb{Z})^\times}} \sum_{\substack{b/d \in I_M(h/k) \\ 0 \leq b < d \\ (a,b,d)=1}} g_{h,k}\left(\frac{b}{d}\right) + \mathcal{O}(d \cdot s), \end{aligned}$$

where $M = [d/\sqrt{nt}]$ and s is the number of Hall divisors in S. From [5, Lemma 6] and (3.4) we can see that

$$\begin{aligned} S_d(t) &= \sum_{e \in S} \sum_{\substack{1 \leq h \leq k \leq M \\ (h,k)=1 \\ k \equiv 0 \bmod N/e \\ \overline{h} \in (\mathbb{Z}/(N/e)\mathbb{Z})^\times}} \left\{ k^{-2} \sum_{f | (a,d)} \mu(f) F_f\left(\frac{dh}{fk}\right) + \mathcal{O}\left(\frac{\sqrt{n}\sigma_1((a, d))}{k(a, d)}\right)\right\} + \mathcal{O}(d \cdot s) \\ &= \sum_{e \in S} \sum_{\substack{1 \leq h \leq k \leq M \\ (h,k)=1 \\ k \equiv 0 \bmod N/e \\ \overline{h} \in (\mathbb{Z}/(N/e)\mathbb{Z})^\times}} \left\{ k^{-2} \sum_{f | (a,d)} \mu(f) F_f\left(\frac{dh}{fk}\right) + \mathcal{O}\left(\frac{\sqrt{n}\sigma_1((a, d))}{k(a, d)}\right)\right\} + \mathcal{O}(d \cdot s) \\ &= \sum_{e \in S} \sum_{f | (a,d)} \mu(f) \sum_{\substack{1 \leq h \leq k \leq M \\ (h,k)=1 \\ k \equiv 0 \bmod N/e \\ \overline{h} \in (\mathbb{Z}/(N/e)\mathbb{Z})^\times}} k^{-2} F_f\left(\frac{dh}{fk}\right) + \mathcal{O}\left(\frac{d\sigma_1((a, d))}{(a, d)} \cdot s\right). \end{aligned}$$

As we did in (3.5) we change the inner summand as

$$\sum_{\substack{1\le h\le k\le M\\(h,k)=1\\k\equiv 0 \bmod N/e\\ \overline{h}\in(\mathbb{Z}/(N/e)\mathbb{Z})^\times}} k^{-2}F_f\left(\frac{dh}{fk}\right) = \frac{2\pi^2 d}{f}\sum_{v\in\mathbb{Z}} C_M\left(\frac{dv}{f}\right)e^{-2\pi|v|nt/df},$$

where

$$C_M(l) = \sum_{\substack{1\le k\le M\\k\equiv 0 \bmod N/e}} k^{-2}c_k(l)$$

and

$$c_k(l) = \sum_{\substack{1\le h\le k\le M\\(h,k)=1\\ \overline{h}\in(\mathbb{Z}/(N/e)\mathbb{Z})^\times}} e^{2\pi i hl/k} \quad \text{for any} \quad l\in\mathbb{Z}.$$

Then, by Lemma 3.9 we know that $|c_k(l)| \le \phi(N/e)\cdot(N/e)\cdot(k,l)$ for $l\in\mathbb{Z}-\{0\}$. So when $l\neq 0$, we have

$$|C_M(l)| = \mathcal{O}\left(\frac{\sigma_1(|l|)}{|l|}\right),$$

where the implicit $\mathcal{O}$-constant depends only on $\langle\Gamma_0(N), W_e\rangle_{e\in S}$. When $l=0$, we obtain $c_k(0)=\phi(k)$. Hence it follows from [1, Lemma D.1] that

$$\begin{aligned} C_M(0) &= \sum_{\substack{1\le k\le M\\k\equiv 0 \bmod N/e}} k^{-2}\phi(k)\\ &= \frac{6}{\pi^2}\frac{1}{[\overline{\Gamma(1)}:\overline{\Gamma_0(N/e)}]}\log M + \mathcal{O}(1), \end{aligned}$$

where the implicit $\mathcal{O}$-constant is absolute, namely it is independent of $\langle\Gamma_0(N), W_e\rangle_{e\in S}$ and M.

Therefore we get

$$\begin{aligned} &\sum_{\substack{1\le h\le k\le M\\(h,k)=1\\k\equiv 0 \bmod N/e\\ \overline{h}\in(\mathbb{Z}/(N/e)\mathbb{Z})^\times}} k^{-2}F_f\left(\frac{dh}{fk}\right)\\ &= \frac{12d}{f[\overline{\Gamma(1)}:\overline{\Gamma_0(N/e)}]}\log M + \mathcal{O}\left(\frac{d}{f}\right) + \mathcal{O}\left(\sum_{v\in\mathbb{Z}-\{0\}}\frac{\sigma_1(d|v|/f)}{|v|}e^{-2\pi|v|nt/df}\right), \end{aligned}$$

where the implicit $\mathcal{O}$-constants depend only on $\langle\Gamma_0(N), W_e\rangle_{e\in S}$ and t. Applying the same estimates as in the proof of Lemma 3.10 we have

$$\sum_{\substack{1\le h\le k\le M\\(h,k)=1\\k\equiv 0 \bmod N/e\\ \overline{h}\in(\mathbb{Z}/(N/e)\mathbb{Z})^\times}} k^{-2}F_f\left(\frac{dh}{fk}\right)$$
$$= \frac{6d}{f[\overline{\Gamma(1)}:\overline{\Gamma_0(N/e)}]}\log\left(\frac{d^2}{n}\right)+\mathcal{O}\left(\frac{d}{f}\right)+\mathcal{O}\left(\sigma_1\left(\frac{d}{f}\right)e^{-2\pi n/df}\right),$$

where the implicit $\mathcal{O}$-constants depend only on $\langle\Gamma_0(N), W_e\rangle_{e\in S}$, t_0 and t_1. By plugging this into the sum of $S_d(t)$ we achieve

$$S_d(t)=\sum_{e\in S}\sum_{f|(a,d)}\mu(f)\left(\frac{6d}{f[\overline{\Gamma(1)}:\overline{\Gamma_0(N/e)}]}\log\left(\frac{d^2}{n}\right)+\mathcal{O}\left(\frac{d}{f}\right)+\mathcal{O}\left(\sigma_1\left(\frac{d}{f}\right)e^{-2\pi n/df}\right)\right)$$
$$+\mathcal{O}\left(\frac{d\sigma_1((a,d))}{(a,d)}\cdot s\right),$$

where the implicit $\mathcal{O}$-constants depend only on $\langle\Gamma_0(N), W_e\rangle_{e\in S}$, f, t_0 and t_1. Thus, in like manner as in the proof of Lemma 3.10 we finally conclude

$$S_d(t)=\sum_{e\in S}\frac{6}{[\overline{\Gamma(1)}:\overline{\Gamma_0(N/e)}]}\frac{d}{(a,\,d)}\phi((a,\,d))\log\left(\frac{d^2}{n}\right)$$
$$+\mathcal{O}\left(\sigma_1\left(\frac{d}{(a,\,d)}\right)\cdot s\right)+\mathcal{O}\left(\frac{d\sigma_1((a,\,d))}{(a,\,d)}\cdot s\right),$$

where the implicit $\mathcal{O}$-constants depend only on $\langle\Gamma_0(N), W_e\rangle_{e\in S}$, f, t_0 and t_1. The number s depends only on the group $\langle\Gamma_0(N), W_e\rangle_{e\in S}$, and so we have the assertion. □

Lemma 3.12. *For* $1\le t_0\le t\le t_1$,
(1) *if* $f(z)$ *is on* Γ, *then we have*

$$h(\Phi_n^f(X,\ f(it)))=\frac{6\psi(n)}{[\overline{\Gamma(1)}:\overline{\Gamma}]}\left(\log n-2\sum_{p|n}\frac{\log p}{p}+\mathcal{O}(1)\right),$$

(2) *if* $f(z)$ *is on* $\langle\Gamma_0(N), W_e\rangle_{e\in S}$, *then we achieve*

$$h(\Phi_n^f(X,\ f(it)))=\sum_{e\in S}\frac{6\psi(n)}{[\overline{\Gamma(1)}:\overline{\Gamma_0(N/e)}]}\left(\log n-2\sum_{p|n}\frac{\log p}{p}+\mathcal{O}(1)\right).$$

In case (1) (*resp.,* (2)) *the implicit* $\mathcal{O}$*-constant depends only on* Γ (*resp.,* $\langle\Gamma_0(N), W_e\rangle_{e\in S}$), f, t_0 *and* t_1.

Proof. In each case we are able to use the same method. Thus we put down only the first case. From Lemma 3.10 we know that

$$h(\Phi_n^f(X,\ f(it))) = \sum_{\substack{a>0\\ ad=n}} S_d(t) + \mathcal{O}(\psi(n)) = H_1 + H_2 + \mathcal{O}(\psi(n)),$$

where

$$H_1 = \sum_{\substack{a>0,\ ad=n\\ d<\sqrt{nt}}} S_d(t) = \mathcal{O}(\psi(n))$$

and

$$H_2 = \sum_{\substack{a>0,\ ad=n\\ d\geq\sqrt{nt}}} S_d(t) = \frac{6\psi(n)}{[\overline{\Gamma(1)} : \overline{\Gamma}]}\left(\log n - 2\sum_{p|n} \frac{\log p}{p} + \mathcal{O}(1)\right)$$

by means of Cohen's results in [5, §4]. □

Lemma 3.13. *Let $P(X) \in \mathbb{C}[X]$ be any nonzero polynomial of degree $\leq D$. Then for any $\theta > 0$, there exists an absolute constant $c_\theta > 0$, depending only on θ, such that*

$$\left|(h(P(X)) - \log \sup_{\theta\leq x\leq 2\theta} |P(x)|\right| \leq c_\theta D.$$

Proof. We refer to [1] or [5]. □

Now we are ready to prove our main theorem.

Proof of Theorem 2.1. To avoid troublesome, we define $h(0) = -\infty$. Let $D = \psi(n)$ and we write

$$\Phi_n^f(X, Y) = P_0(Y)X^D + P_1(Y)X^{D-1} + \cdots + P_D(Y)$$

with $P_j(Y) \in \mathbb{C}[Y]$ and $P_0(Y) \neq 0$. Certainly, $h(\Phi_n^f(X,Y)) = \max_{0\leq j\leq D} h(P_j(Y))$. Since $\deg P_j(Y) \leq D$, Lemma 3.13 yields that

$$\begin{aligned} h(\Phi_n^f(X, Y)) &= \max_{0\leq j\leq D} \log \sup_{s\leq y\leq 2s} |P_j(y)| + \mathcal{O}(D) \\ &= \sup_{s\leq y\leq 2s} \max_{0\leq j\leq D} \log |P_j(y)| + \mathcal{O}(D) \end{aligned}$$

where the implicit $\mathcal{O}$-constant depends only on s. Since $\max_{0\leq j\leq D} \log|P_j(y)| = h(\Phi_n^f(X, y))$, we obtain

$$h(\Phi_n^f(X, Y)) = \sup_{s\leq y\leq 2s} h(\Phi_n^f(X, y)) + \mathcal{O}(D).$$

Here we note that the interval $[t_0, t_1]$ corresponds bijectively to the interval $[s, 2s]$, and so we have

$$h(\Phi_n^f(X, Y)) = \sup_{t_0 \le t \le t_1} h(\Phi_n^f(X, f(it))) + \mathcal{O}(D).$$

Therefore, we get the conclusion by Lemma 3.12. □

ACKNOWLEDGEMENT. The authors would like to thank their advisor Professor Ja Kyung Koo for his guidance to this problem and helpful comments.

References

[1] B. Cais and B. Conrad: *Modular curves and Ramanujan's continued fraction*, J. Reine Angew. Math. **597** (2006), 27–104.
[2] I. Chen and N. Yui: *Singular values of Thompson series*; in Groups, Difference Sets, and the Monster (Columbus, OH, 1993), Ohio State Univ. Math. Res. Inst. Publ. **4**, de Gruyter, Berlin, 1996, 255–326.
[3] B. Cho, N.M. Kim and J.K. Koo: *Affine models of the modular curves $X(p)$ and its application*, submitted.
[4] B. Cho, J.K. Koo and Y.K. Park: *Arithmetic of the Ramanujan-Göllnitz-Gordon continued fraction*, submitted.
[5] P. Cohen: *On the coefficients of the transformation polynomials for the elliptic modular function*, Math. Proc. Cambridge Philos. Soc. **95** (1984), 389–402.
[6] D.A. Cox: Primes of the form $x^2 + ny^2$, Wiley, New York, 1989.
[7] K. Harada: Moonshine of Finite Groups, Lecture Notes of The Ohio State University, 1987.
[8] N. Ishida and N. Ishii: *The equations for modular function fields of principal congruence subgroups of prime level*, Manuscripta Math. **90** (1996), 271–285.
[9] D. Jeon and C.H. Kim: *On the arithmetic of certain modular curves*, preprint.
[10] S. Lang: Elliptic Functions, second edition, Springer, New York, 1987.
[11] G. Shimura: Introduction to the Arithmetic Theory of Automorphic Functions, Princeton University Press, 1971.

Bumkyu Cho
Department of Mathematical Sciences
Korea Advanced Institute of Science and Technology
Guseong-dong, Yuseong-gu
Daejeon, 305–701
Korea
e-mail: bam@math.kaist.ac.kr

Nam Min Kim
Department of Mathematical Sciences
Korea Advanced Institute of Science and Technology
Guseong-dong, Yuseong-gu
Daejeon, 305–701
Korea
e-mail: nmkim@kaist.ac.kr

Yoon Kyung Park
Department of Mathematical Sciences
Korea Advanced Institute of Science and Technology
Guseong-dong, Yuseong-gu
Daejeon, 305–701
Korea
e-mail: ykpark@math.kaist.ac.kr