5-COLORED KNOT DIAGRAM WITH FOUR COLORS

Dedicated to Professor Takao Matumoto on his sixtieth birthday

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Abstract

We study Fox 5-colorings for diagrams of 1- and 2-dimensional knots. We prove that any 5-colorable 1-knot has a non-trivially 5-colored diagram such that exactly four colors of five are assigned to the arcs of the diagram. Moreover, we prove that there is a 5-colorable 2-knot such that, for any non-trivially 5-colored diagram, all five colors are assigned to the sheets of the diagram.

1. Introduction

Let us observe the 5-colored diagrams of the knots $4_1$, $5_1$, and $7_4$ as shown in Fig. 1, where the pallet $\mathbb{Z}_5 = \{0, 1, \ldots, 4\}$ is used to provide a 5-coloring for each diagram. What is the common property of these 5-colorings?

Each 5-coloring in the figure uses exactly four colors $1, 2, 3, 4$ except $0$. Hence, it is natural to ask the question: Which 5-colorable knot has a 5-colored diagram with exactly four colors? The first aim of this note is to give the answer to this question as follows:

**Theorem 1.1.** Any 5-colorable knot has a non-trivially 5-colored diagram with exactly four colors.

Harary and Kauffman [5] study the minimal number of colors assigned to the arcs for all non-trivially $p$-colored diagrams of a $p$-colorable knot $K$, which is denoted by $C_p(K)$. Refer to [7] also. Theorem 1.1 implies that $C_5(K) = 4$ for any 5-colorable knot $K$. We remark that, if $p$ is a prime with $p > 3$, then any $p$-colorable knot $K$ satisfies $C_p(K) \geq 4$ (Lemma 2.1).

On the other hand, a $p$-coloring is also defined for a diagram of a 2-dimensional knot (a 2-sphere in $\mathbb{R}^4$), which satisfies the property that any non-trivial $p$-coloring needs at least four colors for $p > 3$. Hence, we can ask a similar question to the 1-dimensional knot case concerning the minimal number of colors for all non-trivial 5-colorings. The second aim of this note is to prove the following:

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Fig. 1. 5-colored diagrams of $4_1$, $5_1$, and $7_4$.

**Theorem 1.2.** There is a 2-knot whose any non-trivially 5-colored diagram needs all of the five colors.

For example, Theorem 1.2 holds for the 2-twist-spun figure-eight knot and (2, 5)-torus knot, which are both non-ribbon 2-knots. On the other hand, we have the following for the family of ribbon 2-knots:

**Proposition 1.3.** Any 5-colorable ribbon 2-knot has a non-trivially 5-colored diagram with exactly four colors.

### 2. 5-colored 1-knot diagrams

Throughout this section, a knot means a circle embedded in $\mathbb{R}^3$. Any knot diagram $D$ is regarded as a disjoint union of arcs obtained from the projected planar curves by cutting the lower paths at crossings. For an odd prime $p$, we consider an assignment of an element of $\mathbb{Z}_p = \{0, 1, \ldots, p - 1\}$ to each arc of $D$. It is called a $p$-coloring if $a + c = 2b$ in $\mathbb{Z}_p$ holds near each crossing, where the lower arcs are colored by $a$ and $c$ and the upper is colored by $b$. The color of the crossing is denoted by $[a \mid b \mid c]$. We say that a $p$-coloring is trivial if all arcs of $D$ have the same color, and otherwise non-trivial.

**Lemma 2.1.** If $p > 3$, then any non-trivial $p$-coloring for $D$ needs at least four colors of $0, 1, \ldots, p - 1$.

Proof. By definition, $D$ has a crossing with the color $[a \mid b \mid c]$ which does not satisfy $a = b = c$. Since $a + c = 2b$, we see that $a, b, c$ are mutually different. Hence, any non-trivial $p$-coloring needs at least three colors.

Assume that exactly three colors are assigned to the arcs of $D$. Then it is easy to see that $D$ has a pair of crossings whose colors are $[a \mid b \mid c]$ and $[a \mid c \mid b]$ for some mutually different $a, b, c \in \mathbb{Z}_p$. By the equations $a + c = 2b$ and $a + b = 2c$, we have $3(b - c) = 0$. This is impossible for $p > 3$ and $b \neq c$. \qed

Lemma 2.2. Any 5-colorable knot has a non-trivially 5-colored diagram $D$ with no crossing whose color is $[0 \mid 0 \mid 0]$.

Proof. Assume that $D$ has a crossing of $[0 \mid 0 \mid 0]$. Then it is easy to see that $D$ has an adjacent pair of crossings $P$ and $Q$ such that $P$ is of $[0 \mid 0 \mid 0]$ and $Q$ is of $[a \mid 0 \mid 4a]$ or $[0 \mid a \mid 2a]$ for some $a \neq 0$. See the left or middle of Fig. 2. We deform the arc with the color $a$ near $Q$ which detours around $P$ passing over the arcs. Then the color of $P$ changes into $[2a \mid 2a \mid 2a]$, and new crossings are of $[0 \mid a \mid 2a]$. See the right of the figure. We repeat the deformation above if the obtained diagram still has a crossing of $[0 \mid 0 \mid 0]$. \hfill \Box

Lemma 2.3. Any 5-colorable knot has a non-trivially 5-colored diagram $D$ with no crossing whose color is $[* \mid 0 \mid *]$.

Proof. We may assume that $D$ has no crossing whose color is $[0 \mid 0 \mid 0]$ by Lemma 2.2. Assume that $D$ has a crossing of $[a \mid 0 \mid 4a]$ for some $a \neq 0$. Then we deform the arc with the color $a$ which detour around the crossing. See Fig. 3. Then the color of the original crossing changes into $[a \mid 2a \mid 3a]$, and new crossings are of $[0 \mid a \mid 2a]$ and $[3a \mid 2a \mid 4a]$. We repeat the deformation above if the obtained diagram still has a crossing of $[a \mid 0 \mid 4a]$ for some $a \neq 0$. \hfill \Box

Proof of Theorem 1.1. Let $D$ be a non-trivially 5-colored diagram of the knot. By Lemma 2.3, we may assume that the upper arc of any crossing of $D$ have a non-zero color. Hence, each arc with the color 0 connects a pair of crossings directly whose colors are $[0 \mid a \mid 2a]$ and $[0 \mid b \mid 2b]$ for some $a, b \neq 0$. According to $b =
Fig. 4. Eliminating an arc colored by 0.

Fig. 5. A 7-colored diagram of $5_2$ with four colors.

$a, 2a, 3a, 4a$, we deform the arc as shown in Fig. 4 so that the arc with 0 is eliminated. We repeat the deformation above if the obtained diagram still has an arc whose color is 0. □

Remark 2.4. (i) The argument as above can be easily applied to the families of 5-colored virtual knot diagrams [6] and virtual arc diagrams [9].
(ii) In the proof of Theorem 1.1, we eliminate the color 0 from a 5-colored diagram. By adding $a \neq 0$ to the color of each arc, we can easily eliminate the color $a$ instead of 0.

For the cases of 7- and 11-colorings, we have just several examples as follows:

Example 2.5. By Lemma 2.1, any non-trivial 7-coloring requires at least four colors assigned to the arcs of a diagram.

Consider the 7-colored diagram of the knot $5_2$ with five colors $0, 1, 2, 4, 6$ as shown in the left of Fig. 5. We deform a neighborhood of the arc with 6 as in the right, so that we can eliminate the color 6 without introducing new colors except 0, 1, 2, 4. Hence, we have $C_7(5_2) = 4$.

Similarly, consider the 7-colored diagram of the $(2, 7)$-torus knot $T_{2,7}$ with five colors $0, 1, 2, 3, 4$ as shown in the left of Fig. 6. We deform neighborhoods of the arcs with the color 3 as in the right, so that we obtain a diagram colored by $0, 1, 2, 4$. Hence, we have $C_7(T_{2,7}) = 4$. (Kauffman and Lopes [7] conjectured $C_p(T_{2,p}) = (p + 3)/2$.)
Fig. 6. A 7-colored diagram of $T_{2,7}$ with four colors.

Fig. 7. A 11-colored diagram of $6_2$ with five colors.

Fig. 8. A 11-colored diagram of $T_{2,11}$ with five colors.

**Question 2.6.** Does it hold $C_7(K) = 4$ for any 7-colorable knot $K$?

**Example 2.7.** Similarly to Lemma 2.1, it is easy to see that if $p > 7$, then any non-trivial $p$-coloring for a knot diagram needs at least five colors of $0, 1, \ldots, p - 1$ assigned to the arcs of a diagram, that is, $C_p(K) \geq 5$.

Consider the 11-colored diagram of the knot $6_2$ with six colors 0, 1, 2, 4, 7, 10 as shown in the left of Fig. 7. We deform a neighborhood of the arc with 10 as in the right, so that we obtain a diagram colored by 0, 1, 2, 4, 7. Hence, we have $C_{11}(6_2) = 5$.

Similarly, consider the 11-colored diagram of the $(2, 11)$-torus knot $T_{2,11}$ with seven colors 0, 1, 2, 3, 4, 5, 6 as shown in the left of Fig. 8. We deform neighborhoods of the arcs with 5 and 6 as shown in the right, so that we obtain a diagram colored by 0, 1, 2, 3, 6. Hence, we have $C_{11}(T_{2,11}) = 5$. 
QUESTION 2.8. Does it hold $C_{11}(K) = 5$ for any 11-colorable knot $K$?

3. 5-colored 2-knot diagrams

Throughout this section, a 2-knot means a 2-dimensional sphere embedded in $\mathbb{R}^4$ smoothly. A diagram of a 2-knot $K$ is a projection image $\pi(K)$ under a projection $p: \mathbb{R}^4 \to \mathbb{R}^3$ equipped with crossing information. Refer to [3] for more details.

Any 2-knot diagram is regarded as a disjoint union of compact, connected surfaces, each of which is called a sheet. For an odd prime $p$, an assignment of an element of $\mathbb{Z}_p$ to each sheet of the diagram is called a $p$-coloring if $a + c = 2b$ in $\mathbb{Z}_p$ holds near each double point, where the lower sheets are colored by $a$ and $c$ and the upper is colored by $b$.

Let $D$ be a $p$-colored diagram of a 2-knot. Consider a triple point of $D$, where the top sheet is colored by $a$, the middle sheets are colored by $b_1$ and $b_2$, and the bottom sheets on both sides of the middle sheet with $b_i$ are colored by $c_{i1}$ and $c_{i2}$ ($i = 1, 2$). We may assume that the bottom sheets colored by $c_{1j}$ and $c_{2j}$ are adjacent along the top sheet ($j = 1, 2$). See Fig. 9. We say that a triple point is degenerated with respect to the $p$-coloring if $a = b_i$ or $b_i = c_{ij}$ holds for some $i, j \in \{1, 2\}$, and otherwise non-degenerated. Hence, a triple point is non-degenerated if and only if $a \neq b_i \neq c_{ij}$ holds for any $i, j \in \{1, 2\}$. The notion of non-degeneracy was used in [10].

Lemma 3.1. For a non-degenerated triple point with the colors as above, we have the following.

(i) It holds that $c_{11} \neq c_{12}$ and $c_{21} \neq c_{22}$.
(ii) It holds that $c_{11} \neq c_{22}$ and $c_{12} \neq c_{21}$.
(iii) It holds that $c_{11} \neq c_{21}$ or $c_{12} \neq c_{22}$.

Proof. We first remark that, since $b_1 + b_2 = 2a$ and $a \neq b_1, b_2$, it holds that $b_1 \neq b_2$.

(i) Since $c_{11} + c_{12} = 2b_1$ and $b_i \neq c_{ij}$ for $i, j = 1, 2$, we see that $b_1, c_{i1}, c_{i2}$ are mutually different.

(ii) Assume that $c_{11} = c_{22}$ (the case $c_{12} \neq c_{21}$ is similarly proved). Then it holds that $c_{12} = 2a - c_{22} = 2a - c_{11} = c_{21}$, and hence, $b_1 = (c_{11} + c_{12})/2 = (c_{22} + c_{21})/2 = b_2$. This contradicts to $b_1 \neq b_2$.

(iii) Assume that $c_{11} = c_{21}$ and $c_{12} = c_{22}$. Then it holds that $b_1 = (c_{11} + c_{12})/2 = (c_{21} + c_{22})/2 = b_2$, which contradicts to $b_1 \neq b_2$. □

Lemma 3.2. Let $p$ be a prime with $p > 3$. If a $p$-colored diagram of a 2-knot has a non-degenerated triple point, then the $p$-coloring needs at least five colors assigned to the sheets of the diagram.

Proof. It is sufficient to prove that there are at least five different colors in the set $\{a, b_i, c_{ij} \mid i, j = 1, 2\}$ near a non-degenerated triple point. By Lemma 3.1, we have
two cases with respect to the colors of the bottom sheets (by changing the indices if necessary):

(i) \(c_{11}, c_{12}, c_{21}, \) and \(c_{22}\) are four different colors.

(ii) \(c_{11} = c_{21}, c_{12}, \) and \(c_{22}\) are three different colors.

For the case (i), since \(a/b = c_{ij}\) holds for any \(i, j = 1, 2\), we have five different colors \(a, c_{11}, c_{12}, c_{21}, \) and \(c_{22}\).

Consider the case (ii). Since \(c_{11} = c_{21} = a\), each of the triplets \([a, b_1, b_2], [a, c_{12}, c_{22}], [a, b_1, c_{12}], \) and \([a, b_2, c_{22}]\)
consists of mutually different colors. Hence, to prove the lemma, it is sufficient to prove \(b_1 \neq c_{22}\) and \(b_2 \neq c_{12}\). Assume that \(b_1 = c_{22}\) (the case \(b_2 = c_{12}\) is similarly proved).

Since \(b_1 + b_2 = 2a\) and \(c_{12} + c_{22} = c_{12} + b_1 = 2a\), we have \(b_2 = c_{12}\). Hence, it holds that

\[c_{11} + c_{12} = a + b_2 = 2b_1\]

and

\[c_{21} + c_{22} = a + b_1 = 2b_2,\]

which induces \(3(b_1 - b_2) = 0\). This is impossible for \(p > 3\) and \(b_1 \neq b_2\).

Let \(D\) be a diagram of a 2-knot \(K\), and \(\gamma\) a (possibly trivial) \(p\)-coloring for \(D\).

By using the Mochizuki’s 3-cocycle [8] of the dihedral quandle of order \(p\), we can define a weight \(W_p(t, \gamma) \in \mathbb{Z}_p\) for a triple point \(t\) of \(D\) in an appropriate manner. Take the sum \(W_p(\gamma) = \sum W_p(t, \gamma)\) for all triple points of \(D\). The cocycle invariant of the 2-knot \(K\) is defined by

\[\Phi_p(K) = \{W_p(\gamma) \mid \gamma; \text{ any } p\text{-coloring for } D\}\]

as a multi-set [2]. The weight \(W_p(t, \gamma)\) has the property that, if \(t\) is degenerated with respect to \(\gamma\), then it holds that \(W_p(t, \gamma) = 0\). In particular, if \(\gamma\) is a trivial \(p\)-coloring, then any \(t\) is degenerated, and hence, we have \(W_p(\gamma) = 0\). In other words, if a \(p\)-coloring
γ satisfies $W_p(γ) \neq 0$, then there is a triple point $t$ with $W_p(t, γ) \neq 0$ which implies that $t$ is non-degenerated with respect to $γ$.

Proof of Theorem 1.2. Let $K$ be the the 2-twist-spun figure-eight knot, and $D$ a diagram of $K$. The cocycle invariant of $K$ is calculated in [4] such that

$$\Phi_5(K) = \{0, \ldots, 0 \ (5 \ \text{times}), \ 2, \ldots, 2 \ (10 \ \text{times}), \ 3, \ldots, 3 \ (10 \ \text{times})\}.$$

The number of 5-colorings for $D$ is 25 which includes 5 trivial ones. Hence, for any non-trivial 5-coloring $γ$, it holds that $W_5(γ) = 2$ or 3. Since $W_5(γ) \neq 0$, $D$ has a non-degenerated triple point with respect to $γ$. The proof is completed by Lemma 3.2.

Remark 3.3. (i) Let $K_{2n}$ be the 2$n$-twist-spun figure-eight knot. The cocycle invariant of $K_{2n}$ is given by $\Phi_5(K_{2n}) = n \cdot \Phi_5(K_2) = \{0, \ldots, 2n, \ldots, 3n, \ldots \}$ (cf. [1]). Hence, if $n$ is not divisible by 5, then $K_{2n}$ has the same property as in Theorem 1.2.

(ii) Since the cocycle invariant of the 2-twist-spun (2, 5)-torus knot $K$ is

$$\Phi_5(K) = \{0, \ldots, 0 \ (5 \ \text{times}), \ 1, \ldots, 1 \ (10 \ \text{times}), \ 4, \ldots, 4 \ (10 \ \text{times})\},$$

$K$ has the same property as in Theorem 1.2.

A ribbon 2-knot is obtained by adding 1-handles to a trivial 2-link [11]. It is known that any ribbon 2-knot is presented by a virtual arc diagram [9]. Given an oriented virtual arc diagram $A$, we construct a diagram $D$ of a ribbon 2-knot $\text{Tube}(A)$. In Fig. 10, we shows a part of $D$ corresponding to a classical crossing of $A$. Moreover, it is easy to see that there is a one-to-one correspondence between the set of the 5-colorings for $A$ and that for $D$.

Proof of Proposition 1.3. Let $K$ be a 5-colorable ribbon 2-knot. We may assume that $K = \text{Tube}(A)$ for some virtual arc diagram $A$. Since $K$ is 5-colorable, so is $A$. As mentioned in Remark 2.4 (i), we may assume that $A$ has a non-trivial 5-coloring with exactly four colors 1, 2, 3, 4.

Consider the 5-colored diagram $D$ of $K = \text{Tube}(A)$ corresponding to $A$. By the assumption for $A$, if $D$ has a sheet colored by 0, then the sheet is the small one colored.
by $2a - b (= 0)$ in Fig. 10. In a neighborhood of the sheet with 0, we deform the sheet with $2a (= b)$ as shown in Fig. 11 so that the color 0 is eliminated. The deformation is similar to the one in the most left of Fig. 4. This completes the proof.

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