ON TUNNEL NUMBER ONE LINKS
WITH SURGERIES YIELDING THE 3-SPHERE

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Abstract

Gordon and Luecke showed that knots are determined by their complements. Therefore a non-trivial Dehn surgery on a non-trivial knot does not yield the 3-sphere. But the situation for links is different from that for knots. Berge constructed some examples of Dehn surgeries of 2-component links yielding the 3-sphere with interesting properties. By extending Berge’s example, we construct infinitely many examples of tunnel number one links in the 3-sphere, such that their components are non-trivial, and that non-trivial Dehn surgeries on them yield the 3-sphere.

1. Introduction and results

Let $S^3$ be the 3-sphere. By a Dehn surgery or Dehn filling yielding $S^3$ and a Heegaard diagram for $S^3$, we mean a Dehn surgery or Dehn filling yielding a 3-manifold homeomorphic to $S^3$ and a Heegaard diagram for a 3-manifold homeomorphic to $S^3$ respectively throughout this paper.

Gordon and Luecke [5] showed that knots are determined by their complements. In other words a non-trivial Dehn surgery on a non-trivial knot in $S^3$ does not yield $S^3$. But the situation for links is different from that for knots. In fact, there is a link in $S^3$ which admits a non-trivial Dehn surgery yielding $S^3$. Here a non-trivial Dehn surgery means a Dehn surgery along a non-meridional slopes. If a link has a trivial component or has a non-separating essential annulus in its exterior, we can easily see that the link admits infinitely many such surgeries. These are called trivial examples. Non-trivial examples of links with such surgeries have been constructed. Berge [1] gave some examples of tunnel number one links. Kawauchi [8], [9] showed that we can construct infinitely many examples of hyperbolic links of any number of components by using imitation theory. Teragaito [13] gave an example of an $n$-component link of which tunnel number is $n - 1$ for any $n \geq 2$. Classes of links without such surgeries are also known. See for example [10].

Let $L$ be a knot or link in a closed, orientable 3-manifold $N$, and let $M$ be the exterior of $L$ in $N$. $L$ is a tunnel number one link if $M$ is homeomorphic to a handle-
body $H$ of genus two with a single 2-handle attached to $H$ along a simple closed curve $C$ in $\partial H$. Note that $L$ is a knot if and only if $C$ is a non-separating curve in $\partial H$, and $L$ is a two component link if and only if $C$ is a separating curve in $\partial H$.

Berge [1] determined whether such $M$ can be embedded in $S^3$ or not, and found all such embeddings if they exist. He showed that if $L$ is a tunnel number one link in $S^3$, whose exterior $E(L)$ does not contain any non-separating essential annulus and any Dehn filling on one of the boundary components of $E(L)$ does not yield the solid torus, then $L$ has at most five non-trivial Dehn surgeries yielding $S^3$. Also, he described a Heegaard diagram for the exterior of a link with five non-trivial Dehn surgeries yielding $S^3$.

Let $M$ be a 3-manifold whose boundary components are $k$ tori $T_i$ ($i = 1, \ldots, k$), $\Lambda$ be a subset of $\{1, \ldots, k\}$ and $m_j, m'_j$ ($j \in \Lambda$) be essential simple closed curves in $T_j$. The Dehn filling of $M$ along $\bigcup_{j \in \Lambda} m_j$ is said to be equivalent to that of $M$ along $\bigcup_{j \in \Lambda} m'_j$ if $\bigcup_{j \in \Lambda} m_j$ is isotopic to $\bigcup_{j \in \Lambda} m'_j$ in $\partial M$. Two Dehn surgeries of a link $L$ in $S^3$ are said to be equivalent if their corresponding Dehn fillings of the exterior $E(L)$ of $L$ are equivalent.

**Theorem 1.1.** There is an infinite family of mutually distinct tunnel number one links $\{L_n\}_{n=1}^{\infty}$ in $S^3$ such that each $L_n$ has exactly five non-trivial Dehn surgeries yielding $S^3$ up to equivalence.

**Remark 1.1.** Since $L_n$ has only finite non-trivial Dehn surgeries yielding $S^3$, $L_n$ has the following properties.

1. $L_n$ has no trivial component.
2. The exterior $E(L_n)$ of $L_n$ does not contain any non-separating essential annulus.
3. Any Dehn filling on one of the boundary components of $E(L_n)$ does not yield the solid torus.

Because $L_n$ must have infinitely many non-trivial Dehn surgeries yielding $S^3$ up to equivalence if one of (1), (2) and (3) does not hold. All tunnel number one links whose exteriors contain a non-separating essential annulus are determined by [3], and all these links have a trivial component.

**Theorem 1.2.** There is an infinite family of mutually distinct pairs of tunnel number one links $\{L_n, L'_n\}_{n=1}^{\infty}$ in $S^3$ with the following properties.

1. $L_n$ has no trivial component.
2. $L'_n$ has a trivial component.
3. $E(L_n)$ is homeomorphic to $E(L'_n)$.

**Remark 1.2.** Berge [1] gave an example of a pair of distinct hyperbolic links without trivial component whose exteriors are homeomorphic to each other. The examples of Theorem 1.2 are entirely different from Berge’s one.
2. Basic facts and notions for proofs of Theorems

In this section we will prepare some basic facts and notations for proofs of Theorems 1.1 and 1.2.

Following [6] we will recall Heegaard diagrams for 3-manifolds. Let $H_n$ be a handlebody of genus $n$. A set of simple closed curves $u_1, u_2, \ldots, u_n \subset \partial H_n$ is a meridian system of $H_n$, if there are disks $D_1, D_2, \ldots, D_n \subset H_n$ such that $D_i \cap \partial H = \partial D_i = u_i$ (for each $i \in \{1, 2, \ldots, n\}$), $D_i \cap D_j = \emptyset$ (if $i \neq j$), and $\text{Cl}(H_n - \{\bigcup_{j=1}^{n} D_j\})$ is homeomorphic to the 3-ball $B^3$. Here $\text{Cl}(\cdot)$ means the closure and $N(\cdot)$ means regular neighborhood.

Let $H \cup_F H'$ be Heegaard splitting of a closed orientable 3-manifold $M$ and $[u_1, u_2, \ldots, u_n]$ (resp. $[u'_1, u'_2, \ldots, u'_n]$) be a meridian system of $H$ (resp. $H'$). We call $D = (F; [u_1, u_2, \ldots, u_n], [u'_1, u'_2, \ldots, u'_n])$ a Heegaard diagram of genus $n$. This definition is extended to the definition of Heegaard diagrams for non-closed compact orientable 3-manifolds ($H, H'$ are compression bodies), by choosing collections of core curves of 2-handles for each of compression bodies $H, H'$. A Heegaard diagram $D = (F; [u_1, u_2, \ldots, u_n], [u'_1, u'_2, \ldots, u'_n])$ is said to be normalized if $(\bigcup_{i=1}^{n} u_i) \cap (\bigcup_{j=1}^{n} u'_j)$ contains no isotopically removable point. When $(\bigcup_{i=1}^{n} u_i) \cup (\bigcup_{j=1}^{n} u'_j)$ is connected, $D$ is called a connected diagram. For normalized Heegaard diagram $D = (F; [u_1, u_2, \ldots, u_n], [u'_1, u'_2, \ldots, u'_n])$, a simple arc $w$ in $F$ is called a wave if $w$ satisfies the following conditions:

(1) there is a meridian $u \in [u_1, u_2, \ldots, u_n] \cup [u'_1, u'_2, \ldots, u'_n]$ satisfying $w \cap ((\bigcup_{i=1}^{n} u_i) \cup (\bigcup_{j=1}^{n} u'_j)) = w \cap u = \partial w$,
(2) a small neighborhood $N(\partial w; w)$ of $\partial w$ in $w$ is the same side of $u$, that is, the closure of one component of $N(u) - u$ contains $N(\partial w; w)$,
(3) each component of $u - \partial w$ intersects $\{u_1, u_2, \ldots, u_n\} \cup \{u'_1, u'_2, \ldots, u'_n\} - \{u\}$.

The wave $w$ is said to be associated with $u$ specifying the meridian which $w$ attaches. Note that any non-connected, normalized Heegaard diagram of genus two for $S^3$ is the standard one $D_0 = (F; [u_1, u_2], \{u'_1, u'_2\})$, where $D_0$ is normalized and satisfies $u_i \cap u'_j = \{\text{a point}\}$ if $i = j$ and $u_i \cap u'_j = \emptyset$ if $i \neq j$ for $i, j \in \{1, 2\}$.

**Theorem 2.1** (Homma, Ochiai and Takahashi [6]). Any connected normalized Heegaard diagram of genus two for $S^3$ has a wave.

For a survey of the proof, see [4].

Birman and Hilden [2] showed that every 3-manifold with a Heegaard splitting of genus two is two-sheeted cyclic branched cover of $S^3$ branched over a knot or link in $S^3$, see Takahashi [12] for alternative proof. By a solution of the Smith conjecture [11], we obtain the following well known theorem.

**Theorem 2.2.** Let $N$ be a closed, connected, simply connected 3-manifold with a Heegaard splitting of genus two. Then $N$ is homeomorphic to $S^3$. 
By loop theorem and Schoenflies theorem, we obtain the following well known theorem.

**Theorem 2.3.** Let $M$ be a 3-manifold homeomorphic to the exterior of a knot. Then $M$ is homeomorphic to the solid torus if and only if the fundamental group $\pi_1(M)$ is isomorphic to the infinite cyclic group $\mathbb{Z}$.

The following is the Dehn filling version theorem of Gordon–Luecke [5].

**Theorem 2.4.** Let $M$ be a 3-manifold homeomorphic to the exterior of a non-trivial knot in $S^3$. Then the Dehn filling of $M$ yielding $S^3$ is unique up to equivalence.

## 3. Proofs of Theorems

In this section, we will prove Theorems 1.1 and 1.2 by using Heegaard diagrams.

### 3.1. Definitions, Key Lemma and Basic Lemma

Let $M$ be a handlebody $H$ of genus two with a single 2-handle attached to $H$ along a separating simple closed curve $C$ in $\partial H$, and $\{u_1, u_2\}$ be a meridian system of $H$. Then $D = (\partial H; [u_1, u_2], C)$ is a Heegaard diagram for $M$. Some definitions in the section 2 for a Heegaard diagram for a closed orientable 3-manifold can be extended to that for such a Heegaard diagram $D$. The Heegaard diagram $D = (\partial H; [u_1, u_2], C)$ is said to be normalized if $(u_2 \cup u_1) \cap C$ contains no isotopically removable point. For normalized Heegaard diagram $D = (\partial H; [u_1, u_2], C)$, a simple arc $w$ in $\partial H$ is called a wave associated with $C$ if $w$ satisfies the following conditions:

1. $w \cap (u_1 \cup u_2 \cup C) = w \cap C = \partial w$,
2. a small neighborhood $N(\partial w; w)$ of $\partial w$ in $w$ is the same side of $C$, that is, the closure of one component of $N(C) - C$ contains $N(\partial w; w)$,
3. each component of $C - \partial w$ intersects $u_1 \cup u_2$.

For Heegaard diagram $D = (\partial H; [u_1, u_2], C)$ of genus two, by cutting $\partial H$ open along $u_1$ and $u_2$, we obtain the 2-sphere with four disks (we name these $A$, $a$, $B$, and $b$), where disks $A$, $a$ are obtained by cutting $\partial H$ open along $u_1$ and disks $B$, $b$ are obtained by cutting $\partial H$ open along $u_2$). Throughout this section, we consider such diagrams.

**Key Lemma 3.1.1.** Let $M$ be a handlebody $H$ of genus two with a single 2-handle attached to $H$ along a separating simple closed curve $C$ in $\partial H$. Let $D = (\partial H; [u_1, u_2], C)$ be a Heegaard diagram for $M$ where $\{u_1, u_2\}$ is a meridian system of $H$. Suppose that $D$ is of the type as shown in Fig. 1 below, where each arc represents a family of arcs parallel to it, the labels $c$, $d$, $e$, $f$ for arcs indicate the numbers of arcs in each family respectively, and $w_i (i \in \{1, 2, 3\}, x \in [l, r])$ is a wave associated with $C$ in $D$. Let $m_{ix} (i \in \{1, 2, 3\}, x \in [l, r])$ be the simple closed curves $w_{ix} \cup \alpha_{ix}$ where $\alpha_{ix}$ is a component of $C - w_{ix}$. If $c, d, e, f \geq 1$, then for any Dehn filling of $M$ yielding $S^3$ (if it exists),
one of the two simple closed curves $m_l, m_r$ in $\partial H - C$ corresponding to this Dehn filling coincides with one of $m_{l1}, m_{l2}, m_{l3}, m_{r1}, m_{r2}$ and $m_{r3}$ up to isotopy on the closure of one component of $\partial H - C$.

Proof. Our proof is based on the idea of Berge [1]. We may assume that (1) $(m_l \cup m_r) \cap (u_1 \cup u_2) = \emptyset$. The triplet $D' = (\partial H; [u_1, u_2], [m_l, m_r])$ is a Heegaard diagram for $S^3$. Suppose that $D'$ is the standard Heegaard diagram for $S^3$. Then, without loss of generality, we may assume (2) $u_1 \cap m_l = \{a \text{ point}\}, u_2 \cap m_r = \{a \text{ point}\}, u_1 \cap m_r = \emptyset$, $u_2 \cap m_l = \emptyset$. By (2), $(m_l \cup m_r) \cap C = \emptyset$ and Fig. 1, $C$ must contain $c$ simple closed curves parallel to $m_l$ and $d$ simple closed curves parallel to $m_r$. This is contradicts connectivity of $C$ because of $c + d \geq 2$. Therefore $D'$ is not the standard Heegaard diagram for $S^3$ and so $D'$ is connected. Then, by Theorem 2.1, $D'$ has a wave $w$ associated with $m_x$ ($x = l$ or $r$) or $u_j$ ($j = 1 \text{ or } 2$). We may assume that (3) $w \cap C$ has no isotopically removable point by isotopy if necessary.

CASE 1. Suppose that $w$ is a wave associated with $m_x$ ($x = l$ or $r$). Then we have $w \cap C \neq \emptyset$ because, if $w \cap C = \emptyset$, $w$ must be an arc as shown in Fig. 2 of $D'$ obtained by cutting $\partial H$ along $m_l$ and $m_r$, where one of the two components of $m_x - \partial w$ does not intersect $u_1 \cup u_2$, and so is not a wave associated with $m_x$ ($x = l$ or $r$). By $\partial w \subset m_x \subset \partial H - C$, $\partial w$ is contained in one of two components of $\partial H - C$. 

Fig. 1.
and so, by $w \cap C \neq \emptyset$, $w$ contains a subarc $w_C$ such that (4) $w_C \cap C = \partial w_C$, $w_C \cap (u_1 \cup u_2 \cup m_1 \cup m_r) = \emptyset$, and $w_C - \partial w_C \subset F'_y$ where $F'_y$ is the component of $\partial H - C$ containing $m_y$ ($y \neq x, y = l$ or $r$). Then, by (3) and (4), $w_C$ is a wave associated with $C$. Let $\beta_{w_C}$ be any one of the two components $\beta_1, \beta_2$ of $C - \partial w_C$ and $F_y$ be the closure of $F'_y$. (Note that two simple closed curves $w_C \cup \beta_1$ and $w_C \cup \beta_2$ are isotopic in $F_y$.) Then by (3), $w_C \cup \beta_{w_C}$ is an essential simple closed curve in $F_y$ and so by $(w_C \cup \beta_{w_C}) \cap m_y \subset (w \cup C) \cap m_y = (w \cap m_y) \cup (C \cap m_y) = \emptyset$, $w_C \cup \beta_{w_C}$ is isotopic to $m_y$ in $F_y$. By (3), $w_C$ is isotopic (in $F_y$) to one wave $w_{ij}$ ($i \in \{1, 2, 3\}$) keeping $(w_C - \partial w_C) \cap C = \emptyset$ and $\partial w_C$ in $C$, and so $w_C \cup \beta_{w_C}$ is isotopic to $m_{ij}$ in $\partial H$. Therefore, we have $m_y = m_{ij}$ up to isotopy in $F_y$.

**Case 2.** Suppose that $w$ is a wave associated with $u_j$ ($j = 1$ or $2$). We may consider Fig. 1 a graph in a 2-sphere $\Sigma$. For a wave $w$ in $\Sigma$, there is a simple closed curve $u \in \{\partial A, \partial a, \partial B, \partial b\}$ satisfying (5) $w \cap (\partial A \cup \partial a \cup \partial B \cup \partial b) = w \cap u = \partial w$. Let $\bar{u}$ be a component of $u - \partial w$. Then the simple closed curve $w \cup \bar{u}$ in $\Sigma$ separates $u' \cup u'' \cup u'''$ into $u' \cup u''$ and $u'''$ where $\{u, u'\} = \{\partial A, \partial a\}$ or $\{\partial B, \partial b\}$, and $\{u'', u''\} = \{\partial A, \partial a, \partial B, \partial b\} - \{u, u'\}$. Since the number of subarcs of $C$ in Fig. 1 connecting $u' \cup u''$ and $u'''$ is one of the integers $d + f$, $d + e$, $c + f$, $c + e \geq 2$. Then $w$ contains a subarc $w_C$ such that (6) $w_C \cap C = \partial w_C$ and $w_C \cap (u_1 \cup u_2 \cup m_1 \cup m_r) = \emptyset$. By (3) and (6), $w_C$ is a wave associated with $C$. Choose $m_y$ ($y = l$ or $r$) such that both $m_y$ and $w_C - \partial w_C$ are contained in the same component $F'_y$ of $\partial H - C$. Let $F_y$ be the closure of $F'_y$. By (3) and (6), $w_C$ is isotopic (in $F_y$) to one wave $w_{ij}$ ($i \in \{1, 2, 3\}$) keeping $(w_C - \partial w_C) \cap C = \emptyset$ and $\partial w_C$ in $C$. By the same argument in Case 1, we have $m_y = m_{ij}$ up to isotopy in $F_y$. 

\[ \square \]
Let \( \alpha \) and \( \beta \) be \( p \)-based closed curves in \( H - \partial H \) as shown in Fig. 3, where \( p \) is base point. Let \([\alpha]_p\) (resp. \([\beta]_p\)) be the \( p \)-base homotopy class of \( \alpha \) (resp. \( \beta \)) in \( H \). The fundamental group \( \pi_1(H) = \pi_1(H, p) \) of \( H \) is the free group generated by \([\alpha]_p \) and \([\beta]_p \). Put \( A = [\alpha]_p, B = [\beta]_p \) and \( a = A^{-1}, b = B^{-1} \) in order to get a word expression for an element of \( \pi_1(H) \) easily. For any (oriented) closed curve \( \gamma \) in \( H \), its free homotopy class \([\gamma] \) in \( H \) contains a \( p \)-based closed curve \( \gamma_p \in [\gamma] \). And \([\gamma_p]_p \in \pi_1(H) = \langle A, B \rangle \) is represented by a word \( W(\gamma) \) defined for a closed curve \( \gamma \) is unique up to conjugation in \( \pi_1(H) = \langle A, B \rangle \). If an oriented simple closed curve \( \gamma \) in \( \partial H \) has finite transversal intersections with \( u_1 \cup u_2 \) and a starting point \( q \in \gamma - (u_1 \cup u_2) \) is given, then we can obtain a word \( W(\gamma) \) uniquely by reading the intersection \( \gamma \cap (u_1 \cup u_2) \) along \( \gamma \) starting from \( q \). This is a well-known algorithm to get \( W(\gamma) \).

Let \( W_i \ (i = 1, \ldots, m) \) be a word in the alphabets \( A_j \ (j = 1, \ldots, n) \) and \( e \) be the unit element of the free group \( \langle A_1, \ldots, A_n \rangle \). Let \( N(W_1, \ldots, W_m) \) be the smallest normal subgroup of \( \langle A_1, \ldots, A_n \rangle \) containing \( \langle W_1, \ldots, W_m \rangle \). The factor group \( \langle A_1, \ldots, A_n \rangle / N(W_1, \ldots, W_m) \) is denoted by \( \langle A_1 \cdots A_n \mid W_1 = e, \ldots, W_m = e \rangle \). Note that \( \langle A_1 \cdots A_n \mid W_1 = e, \ldots, W_m = e \rangle \equiv \langle A_1 \cdots A_n \mid W_1' = e, \ldots, W_m' = e \rangle \) holds if \( W_i \) is conjugate to \( W_i' \) in \( \langle A_1, \ldots, A_n \rangle \) for each \( i = 1, \ldots, m \). For two groups \( G_1 \) and \( G_2, G_1 \equiv G_2 \) means that \( G_1 \) is isomorphic to \( G_2 \). By van Kampen’s theorem, the next lemma holds.

**Basic Lemma 3.1.2.** Let \( M \) and \( C \) be the same ones in Key Lemma 3.1.1. Let \( m_1 \) (resp. \( m_r \)) be an essential simple closed curve in one component (resp. the other component) of \( \partial H - C \). Let \( M(m_1) \) (resp. \( M(m_r) \)) be a 3-manifold obtained by Dehn filling of \( M \) along \( m_1 \) (resp. \( m_r \)) as a meridian and \( M(m_1, m_r) \) be the one along \( m_1 \) and \( m_r \) as meridians. Then the followings hold.
(1) \( \pi_1(M) \equiv \langle A, B \mid W(C) = e \rangle \).
(2) \( \pi_1(M(m_i)) \equiv \langle A, B \mid W(m_i) = e \rangle \), \( \pi_1(M(m_r)) \equiv \langle A, B \mid W(m_r) = e \rangle \).
(3) \( \pi_1(M(m_i, m_r)) \equiv \langle A, B \mid W(m_i) = e, W(m_r) = e \rangle \).

**Remark 3.1.** Since
\[
W_1^{-1}W(m_i)W_2W_1^{-1}W(m_r)^{-1}W_2 = W(C)
\]
holds for certain words \( W_1, W_2, W_1', W_2' \in \langle A, B \rangle = \pi_1(H) \), \( W(m_i) = e \) (resp. \( W(m_r) = e \)) implies \( W(C) = e \) and,
\[
N(W(C), W(m_i)) = N(W(m_i)), \quad N(W(C), W(m_r)) = N(W(m_r))
\]
and
\[
N(W(C), W(m_i), W(m_r)) = N(W(m_i), W(m_r))
\]
hold. And so,
\[
\langle A, B \mid W(C) = e, W(m_i) = e \rangle \equiv \langle A, B \mid W(m_i) = e \rangle,
\]
\[
\langle A, B \mid W(C) = e, W(m_r) = e \rangle \equiv \langle A, B \mid W(m_r) = e \rangle
\]
and
\[
\langle A, B \mid W(C) = e, W(m_i) = e, W(m_r) = e \rangle \equiv \langle A, B \mid W(m_i) = e, W(m_r) = e \rangle
\]
hold.

**3.2. Proof of Theorem 1.1.** Let \( D_n \) be a Heegaard diagram \( (\partial H; \{u_1, u_2\}, C_n) \) shown by Fig. 4 below where \( n \) is a positive integer. Note that \( D_n \) is a special case of \( D \) in Key Lemma 3.1.1. Throughout this subsection, we assume \( D = D_n \), \( M_n, C_n \) mean \( M, C \) in Key Lemma 3.1.1 respectively and \( w_{i,x}, m_{i,x} (i \in \{1, 2, 3\}, x \in \{l, r\}) \) mean the ones in Key Lemma 3.1.1 in the case of \( D = D_n \) respectively.

**Remark 3.2.** A Heegaard diagram \( D_1 \) is an example given by Berge [1].

**Lemma 3.2.1.** If \( C_n, m_{1l}, m_{2l}, m_{3l}, m_{1r}, m_{2r} \text{ and } m_{3r} \) are oriented as shown in Fig. 4 respectively, then there exists a starting point on each of these simple closed curves respectively such that the following equations hold.

(1)
\[
W(C_n) = B(abAB)^{(n-1)}a(BAba)^{(n-1)}BA(BAbA)^{(n-1)}BB(AbaB)^{(n-1)}AA(BabA)^{(n-1)}
\]
\[
\times B(AbaB)^{(n-1)}Ab(ABab)^{(n-1)}A(babA)^{(n-1)}ba(bABa)^{(n-1)}bb(aBAb)^{(n-1)}
\]
\[
\times aa(bABa)^{(n-1)}b(aBAb)^{(n-1)}a.
\]
Fig. 4.

(2) $W(m_{1l}) = b(ABab)^{(n-1)}A(bABa)^{(n-1)}ba(bABa)^{(n-1)}bb(aBAb)^{(n-1)}a$.

(3) $W(m_{2l}) = a(bABa)^{(n-1)}b(aBAb)^{(n-1)}aB(abAB)^{(n-1)}a(BAba)^{(n-1)}B$.

(4) $W(m_{3l}) = A(BabA)^{(n-1)}BB(AbaB)^{(n-1)}AA(BabA)^{(n-1)}B(AbaB)^{(n-1)}A$.

(5) $W(m_{4l}) = (BabA)^{(n-1)}AB(BabA)^{(n-1)}BB(AbaB)^{(n-1)}AA(BabA)^{(n-1)}B$.

(6) $W(m_{5l}) = b(aBAb)^{(n-1)}ab(bABa)^{(n-1)}b(aBAb)^{(n-1)}ab(bABa)^{(n-1)}b$.

(7) $W(m_{6l}) = (AbaB)^{(n-1)}Ab(ABab)^{(n-1)}A(baBA)^{(n-1)}ba(bABa)^{(n-1)}b$.

Proof. Let $c_i$, $d_i$, $e_i$, $f_i$, $g_i$ and $h_i$ be subarcs of $C_n$ respectively as shown in Fig. 4. Then $C_n$ can be represented by connecting these subarcs as

$$
c_1 \prod_{i=0}^{n-2} (d_{12i+2}e_{12i+6}e_{12i+8}f_{12i+12}) d_{12n-10} \prod_{i=0}^{n-2} (c_{12(n-i)-9}e_{12(n-i)-11}d_{12(n-i)-15}f_{12(n-i)-17}),
$$

$$
c_3 e_1 \prod_{i=0}^{n-2} (f_{12i+2}d_{12i+4}e_{12i+8}e_{12i+10}) f_{12n-10} h_2 \prod_{i=0}^{n-2} (e_{12(n-i)-9}d_{12(n-i)-13}f_{12(n-i)-15}e_{12(n-i)-19}),
$$

$$
e_3 g_1 \prod_{i=0}^{n-2} (f_{12i+4}d_{12i+6}e_{12i+10}e_{12i+12}) f_{12n-8} \prod_{i=0}^{n-2} (e_{12(n-i)-7}d_{12(n-i)-11}f_{12(n-i)-13}e_{12(n-i)-17}),
$$
Each of five groups \( \mathcal{B}_1 \) defined by 

\[
\prod_{i=0}^{n-2} (e_{12i+2}f_{12i+6}d_{12i+8}e_{12i+12})c_{12n-10} 
\prod_{i=0}^{n-2} (d_{12(n-i)-9}f_{12(n-i)-11}c_{12(n-i)-15}e_{12(n-i)-17}),
\]

\[
d_3f_1 \prod_{i=0}^{n-2} (e_{12i+2}c_{12i+4}f_{12i+8}d_{12i+10})e_{12n-10}h_1 
\prod_{i=0}^{n-2} (f_{12(n-i)-9}c_{12(n-i)-13}e_{12(n-i)-15}d_{12(n-i)-19}),
\]

\[
f_{32} \prod_{i=0}^{n-2} (e_{12i+4}c_{12i+6}f_{12i+10}d_{12i+12})e_{12n-8} 
\prod_{i=0}^{n-2} (f_{12(n-i)-7}c_{12(n-i)-11}e_{12(n-i)-13}d_{12(n-i)-17}) f_5.
\]

Take a starting point on \( c_1 - a_{c_1} \) for \( C_n \) and a starting point on \( u_{1x} - \partial u_{1x} \) (\( i \in \{1, 2, 3\} \), \( x \in \{l, r\} \)) for \( m_{1x} \) respectively.

For two words \( W, W' \in \{A, B\} = \pi_1(H) \), \( W \equiv W' \) means that \( W \) is conjugate to \( W' \) in \( \{A, B\} = \pi_1(H) \).

**Lemma 3.2.2.** Let \( \varphi : \{A, B\} \to \{A, B\} \) (resp. \( \varphi' : \{A, B\} \to \{A, B\} \)) be an isomorphism defined by \( \varphi(A) = B \) and \( \varphi(B) = A \) (resp. \( \varphi'(A) = A \) and \( \varphi'(B) = ab \)) and \( \varphi'' : \{A, B\} \to \{A, B\} \) be the composed isomorphism \( \varphi' \circ \varphi \) (and so \( \varphi''(A) = ab \) and \( \varphi''(B) = A \)). Then the followings hold.

1. \( \varphi(W(m_{1l})) \equiv W(m_{2r}), \quad \varphi(W(m_{2l})) \equiv W(m_{3r}), \quad \varphi(W(m_{3l})) \equiv W(m_{1r}), \quad \varphi(W(m_{2r}))(W(m_{3r}))^2 \equiv W(m_{1l})(W(m_{2l}))^2, \quad \varphi(W(m_{1r}))(W(m_{2r}))^2 \equiv W(m_{3l})(W(m_{1l}))^2, \quad \varphi(W(m_{3r}))(W(m_{1r}))^2 \equiv W(m_{2l})(W(m_{3l}))^2. \)

2. \( \varphi(W(m_{1l})) \equiv W(m_{1r}), \quad \varphi'(W(m_{2l}))(W(m_{3l}))^2 \equiv W(m_{2l})(W(m_{3l}))^2. \)

3. \( \varphi''(W(m_{1l})) \equiv W(m_{2l}), \quad \varphi''(W(m_{2l}))(W(m_{3l}))^2 \equiv W(m_{1l})(W(m_{2l}))^2. \)

And so the followings hold.

4. \( \{A, B \mid W(m_{1x}) = e\} \equiv \{A, B \mid W(m_{1x}) = e\} \) for any \( i \in \{1, 2, 3\} \) and any \( x \in \{l, r\} \).

5. Each of five groups \( \{A, B \mid W(m_{2r}) = e, \ W(m_{1r})(W(m_{2r}))^2 = e\}, \quad \{A, B \mid W(m_{3r}) = e, \ W(m_{3r})(W(m_{1r}))^2 = e\}, \quad \{A, B \mid W(m_{2r}) = e, \ W(m_{3r})(W(m_{2r}))^2 = e\}, \quad \{A, B \mid W(m_{3r}) = e, \ W(m_{3r})(W(m_{1r}))^2 = e\} \) is isomorphic to \( \{A, B \mid W(m_{1l}) = e, \ W(m_{2l})(W(m_{3l}))^2 = e\} \).

**Proof.** We define words \( W_{1l}, W_{2l}, W_{3l}, W_{1r}, W_{2r}, W_{3r} \in \{A, B\} \) as follows.

\[
W_{1l} = b(ABab)^{(n-1)}A(babA)^{(n-1)}b,
\]

\[
W_{2l} = a(bAbA)^{(n-1)}b(aBAb)^{(n-1)}a,
\]

\[
W_{3l} = A(BabA)^{(n-1)}BB(AbaB)^{(n-1)}A,
\]

\[
W_{1r} = (BAba)^{(n-1)}BA(BabA)^{(n-1)}B,
\]

\[
W_{2r} = b(aBAb)^{(n-1)}aa(bAba)^{(n-1)}b,
\]

\[
W_{3r} = (AbaB)^{(n-1)}Ab(ABab)^{(n-1)}A.
\]
Then we can check the followings.

\(1\)
\[
\varphi(W(m_{1l})) = W_{2r}^{-1}W(m_{2r})W_{2r},
\]
\[
\varphi(W(m_{2l})) = W_{3r}^{-1}W(m_{3r})W_{3r},
\]
\[
\varphi(W(m_{3l})) = W_{1r}^{-1}W(m_{1r})W_{1r},
\]
\[
\varphi(W(m_{2r})(W(m_{3r}))^2) = W_{1l}^{-1}W(m_{1l})(W(m_{2l}))^2W_{1l},
\]
\[
\varphi(W(m_{1r})(W(m_{2r}))^2) = W_{3l}^{-1}W(m_{3l})(W(m_{1l}))^2W_{3l},
\]
\[
\varphi(W(m_{3r})(W(m_{1r}))^2) = W_{2l}^{-1}W(m_{2l})(W(m_{3l}))^2W_{2l}.
\]

\(2\)
\[
\varphi'(W(m_{1l})) = W_{1r}^{-1}W(m_{1r})W_{1r},
\]
\[
\varphi'(W(m_{2r})(W(m_{3r}))^2) = W_{2l}^{-1}W(m_{2l})(W(m_{3l}))^2W_{2l}.
\]

\(3\)
\[
\varphi''(W(m_{1l})) = aW(m_{2l})A,
\]
\[
\varphi''(W(m_{2r})(W(m_{3r}))^2) = aW(m_{1r})(W(m_{2r}))^2A.
\]

\[\square\]

**Lemma 3.2.3.** If \(m_{ix} (i \in \{1, 2, 3\}, x \in \{l, r\})\) is oriented as shown in Fig. 4, then the followings hold.

1) For each \(m_{ix} (i \in \{1, 2, 3\}, x \in \{l, r\})\), there exists an oriented simple closed curve \(\tilde{m}_{ix}\) in the component \(F'_{x}\) (\(x \in \{l, r\}\)) of \(\partial H - C_n\) intersecting \(m_{ix}\) such that \(W(\tilde{m}_{ix}) = W(m_{ix})\) and \(\tilde{m}_{ix}\) is isotopic to \(m_{ix}\).

2) For each word \(W(m_{ix})(W(m_{jx}))^2\) ((\(i, j\)) \( \in \{(1, 2), (2, 3), (3, 1)\}\), \(x \in \{l, r\}\)), there exists an oriented simple closed curve \(\tilde{m}_{ijx}\) in the component of \(\partial H - C_n\) intersecting \(m_{ix} \cup m_{jx}\) such that \(W(\tilde{m}_{ijx}) = W(m_{ix})(W(m_{jx}))^2\).

Proof. Let \(F_x\) be the closure of \(F'_{x}\) and \(F_{jx}\) (\(j \in \{1, 2, \ldots, 24n - 15\}\)) be the closure of each component of \(F_{x} - (u_1 \cup u_2)\) such that \(F'_{1l} \supset e_1 \cup g_2 \cup d_1, F_{2l} \supset c_1 \cup f_1 \cup g_1, F_{1r} \supset c_{12n-9} \cup h_1 \cup e_{12n-7}\), and \(F_{2r} \supset f_{12n-7} \cup d_{12n-9} \cup h_2\). Note that \(F_{jx} \cap (u_1 \cup u_2)\) has three arc-components for \(j \in \{1, 2\}\), \(x \in \{l, r\}\) or two arc-components for other case. Then \(\bigcup_{j=3}^{24n-15} F_{jx}\) consists of three bands connecting \(F_{1x}\) and \(F_{2x}\) for each \(x \in \{l, r\}\). By \(F_x = \bigcup_{j=1}^{24n-15} F_{jx}\) and the subarc-expression of \(C_n\) in the proof of Lemma 3.2.1, \(F_x\) can be shown as in Fig. 5 up to homeomorphism. Note that i) \(\partial F_x = C_n\) and ii) all subarcs of \(u_1 \cup u_2\) in \(F_x\) are in three bands obtained by connecting \(F_{jx}\) (\(j \in \{3, \ldots, 24n - 15\}\)) along two subarcs or one subarc of \(u_1 \cup u_2\).

1) By deforming \(m_{ix}\) isotopically, we obtain a simple closed curve \(\tilde{m}_{ix}\) in \(F_x - \partial F_x = F_x - C_n\) such that \(\tilde{m}_{ix} \cup F_{jx}\) and \(m_{ix} \cup F_{jx}\) are both empty or parallel (two) arcs

...
Fig. 5.

in $F_{jx}$ for any $j \in \{1, 2, \ldots, 24n - 15\}$. Suppose that $\tilde{m}_{lx}$ has an orientation induced by that of $m_{lx}$ and a starting point on an open arc $\tilde{m}_{lx} \cap (F_{1x} - \partial F_{1x})$. Then we have $W(\tilde{m}_{lx}) = W(m_{lx})$. (See Fig. 6 for the case of $i = 1$, $x = l$.)

2) Since waves $w_{1x}, w_{2x}, w_{3x}$ ($x \in [l, r]$) are mutually disjoint and each component of $C_n - \partial w_{lx}$ contains one point of $\partial w_{jx}$ for any $i \in \{1, 2, 3\}$ and any $j \in \{1, 2, 3\} - \{i\}$, we may assume that any two simple closed curves of $\tilde{m}_{1x}, \tilde{m}_{2x}$ and $\tilde{m}_{3x}$ ($x \in [l, r]$) in (1) intersect transversely at one point in $F_{1x} - \partial F_{1x}$. For $(i, j) \in \{(1, 2), (2, 3), (3, 1)\}$ and $x \in [l, r]$, let $\tilde{m}_{ijx}$ be an oriented simple closed curve obtained from $\tilde{m}_{ix}$ with the orientation in (1) by applying the Dehn twist in $F_{i}$ along $\tilde{m}_{jx}$ twice. Suppose that a starting point of $\tilde{m}_{ijx}$ is the initial point of the oriented subarc $\tilde{m}_{ijx} \cap m_{ix}$ of $\tilde{m}_{ix}$. Then we have $W(\tilde{m}_{ijx}) = W(m_{ix})(W(m_{jx}))^2$. (See Fig. 7 for the case of $(i, j) = (1, 2)$ and $x = l$.)

\[\text{Lemma 3.2.4.} \quad 1) \quad \text{Each of the six fundamental groups} \]

\[
\pi_1(M_n(\tilde{m}_{1l}, \tilde{m}_{233r}), \quad \pi_1(M_n(\tilde{m}_{2l}, \tilde{m}_{122r}), \quad \pi_1(M_n(\tilde{m}_{3l}, \tilde{m}_{311r}), \quad \pi_1(M_n(\tilde{m}_{233l}, \tilde{m}_{1r})), \quad \pi_1(M_n(\tilde{m}_{122l}, \tilde{m}_{2r})), \quad \pi_1(M_n(\tilde{m}_{311l}, \tilde{m}_{3r}))
\]

\] is trivial.
2) Each $\pi_1(M_n(\tilde{m}_{ix}))$ ($i \in \{1, 2, 3\}$, $x \in \{l, r\}$) is not isomorphic to the infinite cycle group $\mathbb{Z}$.

Proof. 1) The presentations of the groups can be simplified by using “mutual substitutions” defined by Kaneto, see Definition 1 and Theorem 2 in [7]. Here we demonstrate how mutual substitutions can be applied.

\[
\pi_1(M_n(\tilde{m}_{1l}, \tilde{m}_{2l})) = \langle A, B \mid W(\tilde{m}_{1l}) = e, W(\tilde{m}_{233r}) = e \rangle = \langle A, B \mid W(m_{1l}) = e, W(m_{2r})(W(m_{3r}))^2 = e \rangle
\]

\[
\equiv \langle A, B \mid b(ABab)^{(n-1)}A(baBA)^{(n-1)}ba(bABA)(a^{-1})bb(abABA)(n-1)a = e, \\
\quad b(abAB)^{(n-1)}aa(bABA)(a^{-1})b(abABA)(n-1)b(abABA)(n-1)b = e \\
\quad b(Abab)^{(n-1)}A(baBA)(n-1)ba(abABA)(a^{-1})b = e \\
\quad b(babaBA)(n-1)ba(abABA)(n-1)b = e \rangle
\]

\[
\equiv \langle A, B \mid b = e, \quad b(abABA)(n-1)ba(bABA)(n-1)b = e \rangle
\]

By Basic Lemma 3.1.2 and Lemma 3.2.2 (5), we obtain the conclusion 1) of Lemma 3.2.4.

2) For a natural number $N$, let $\xi_N$ be an $N$-th primitive root of unity, and put

\[
\alpha_N = \begin{pmatrix} \xi_N & 0 \\ 0 & \xi_N^{-1} \end{pmatrix}
\]

and

\[
\beta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
\]

Let $\rho: \langle A, B \rangle \to GL(2, \mathbb{C})$ be a homomorphism defined by $\rho(A) = \alpha_N$ and $\rho(B) = \beta$. 
Since \(\rho(W(\bar{m}_{1})) = \rho(W(m_{1})) = \alpha_{N}^{-8n+5} \begin{pmatrix} \xi_{N}^{-8n+5} & 0 \\ 0 & \xi_{N}^{-8n+5} \end{pmatrix}\), by putting \(N = 8n - 5\), \(\rho(W(\bar{m}_{1})) = e\), and so \(\rho\) keeps the relation \(W(\bar{m}_{1}) = e\). Then we obtain the induced homomorphism \(\tilde{\rho}: (A, B \mid W(\bar{m}_{1}) = e) = \langle A, B \rangle / N(W(\bar{m}_{1})) \to GL(2, \mathbb{C})\), so that \(\tilde{\rho}(A) = \alpha_{8n-5}, \tilde{\rho}(B) = \beta\), here \(AN(W(\bar{m}_{1})), BN(W(\bar{m}_{1})) \in \langle A, B \rangle / N(W(\bar{m}_{1}))\) are denoted by \(A, B\) respectively for convenience. Since two elements \(\alpha_{8n-5}\) and \(\beta\) in \(GL(2, \mathbb{C})\) are non-commutative, \(\langle A, B \mid W(\bar{m}_{1}) = e \rangle\) is not isomorphic to \(Z\). By Lemma 3.2.2, each group \(\langle A, B \mid W(\bar{m}_{i}) = e \rangle \) \((i \in \{1, 2, 3\}, \ x \in \{l, r\})\) is not isomorphic to \(Z\). By Basic Lemma 3.1.2, each group \(\pi_{1}(M_{n}(m_{i})) \) \((i \in \{1, 2, 3\}, \ x \in \{l, r\})\) is not isomorphic to \(Z\).

**Lemma 3.2.5.** If \(n \neq n'\), then \(\pi_{1}(M_{n})\) is not isomorphic to \(\pi_{1}(M_{n'})\).

Proof. Let \(\xi_{N}, \alpha_{N}, \beta\) and \(\rho\) be same ones in the proof of Lemma 3.2.4. Since \(\rho(W(C_{n})) = \alpha_{N}^{6n-10} = \left(\begin{array}{cc} \xi_{N}^{16n-10} & 0 \\ 0 & \xi_{N}^{-16n+10} \end{array}\right)\), by putting \(N = 16n - 10\), \(\rho(W(C_{n})) = e\), and so \(\rho\) keeps the relation \(W(C_{n}) = e\). Then we obtain the induced homomorphism \(\tilde{\rho}: (A, B \mid W(C_{n}) = e) = \langle A, B \rangle / N(W(C_{n})) \to GL(2, \mathbb{C})\). Let \(G_{N}\) be the subgroup of \(GL(2, \mathbb{C})\) generated by \(\alpha_{N}\) and \(\beta\). If \(\pi_{1}(M_{n})\) is isomorphic to \(\pi_{1}(M_{n'})\) for \(n' \leq n\), by Basic Lemma 3.1.2, there is a surjective homomorphism \(\tau: \langle A, B \mid W(C_{n'}) = e \rangle = \langle A, B \rangle / N(W(C_{n'})) \to G_{16n-10}\). Since \(\tau\) is surjective, two elements \(\tau(A), \tau(B)\) are generators of \(G_{16n-10}\). Any element of \(G_{N}\) is represented by \(\alpha_{N}^{k}\) or \(\alpha_{N}^{k}\beta\) for some integer \(k\), because of \(\beta^{2} = \left(\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right)\) and \(\beta\alpha_{N}^{k} = \left(\begin{array}{cc} 0 & \xi_{N}^{-k} \\ \xi_{N}^{k} & 0 \end{array}\right) = \alpha_{N}^{-k}\beta\). Hence, any pair of two generators of \(G_{N}\) is represented by \([\alpha_{N}, \alpha_{N}^{l}]\) or \([\alpha_{N}^{k}, \alpha_{N}^{l}]/\beta, \alpha_{N}^{l}],[\alpha_{N}^{k}, \alpha_{N}^{l}]/\beta\), where \(k, l\) are integers, \(k\) and \(N\) are relatively prime. Note that \(\xi_{N}^{k}\) is also an \(N\)-th primitive root of unity if and only if \(k\) and \(N\) are relatively prime. Then there are following four cases for \(\tau(A)\) and \(\tau(B)\).

1. If \(\tau(A) = \alpha_{16n-10}^{k}\) and \(\tau(B) = \alpha_{16n-10}^{l}\beta\), then \(\tau(W(C_{n})) = \alpha_{16n-10}^{(16n-10)k}\).
2. If \(\tau(A) = \alpha_{16n-10}^{k}\) and \(\tau(B) = \alpha_{16n-10}^{l}\), then \(\tau(W(C_{n})) = \alpha_{16n-10}^{(16n-10)k}\).
3. If \(\tau(A) = \alpha_{16n-10}^{k+l}\) and \(\tau(B) = \alpha_{16n-10}^{k}\beta\), then \(\tau(W(C_{n})) = \alpha_{16n-10}^{(16n-10)k}\).
4. If \(\tau(A) = \alpha_{16n-10}^{k+l}\) and \(\tau(B) = \alpha_{16n-10}^{k}\beta\), then \(\tau(W(C_{n})) = \alpha_{16n-10}^{(16n-10)k}\).

Here \(k\) and \(16n - 10\) are relatively prime. On the other hand, since \(W(C_{n'})\) represents a unit element of \(\langle A, B \mid W(C_{n'}) = e \rangle = \langle A, B \rangle / N(W(C_{n'}))\), \(\tau(W(C_{n})) = \left(\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right)\) holds. In each case of (1), (2), (3) and (4), \(n = n'\) holds, because \(\xi_{16n-10}^{k}\) is a \((16n - 10)\)-th primitive root of unity and \(16n' - 10 \leq 16n - 10\).

Here recall the definition of equivalence for Dehn fillings of a 3-manifold \(M\) with \(\partial M\) consisting of two tori. Two Dehn fillings of \(M\) yielding \(M(m_{1}, m_{2})\) and \(M(m'_{1}, m'_{2})\) respectively are said to be equivalent if \(m_{1} \cup m_{2}\) is isotopic to \(m'_{1} \cup m'_{2}\) in \(\partial M\).
**Proposition 3.2.1.** 1) The Dehn fillings of $M_n$ yielding $S^3$ are exactly the six ones yielding $M_n(\bar{m}_{1l}, \bar{m}_{233r})$, $M_n(\bar{m}_{2l}, \bar{m}_{122r})$, $M_n(\bar{m}_{3l}, \bar{m}_{311r})$, $M_n(\bar{m}_{233l}, \bar{m}_{1r})$, $M_n(\bar{m}_{122l}, \bar{m}_{2r})$ and $M_n(\bar{m}_{311l}, \bar{m}_{3r})$ respectively up to equivalence.  
2) If $n \neq n'$, then $M_n$ is not homeomorphic to $M_{n'}$. 

Proof. 1) By Lemma 3.2.4 1) and Theorem 2.2, each of Dehn fillings $M_n(\bar{m}_{1l}, \bar{m}_{233r})$, $M_n(\bar{m}_{2l}, \bar{m}_{122r})$, $M_n(\bar{m}_{3l}, \bar{m}_{311r})$, $M_n(\bar{m}_{233l}, \bar{m}_{1r})$, $M_n(\bar{m}_{122l}, \bar{m}_{2r})$ and $M_n(\bar{m}_{311l}, \bar{m}_{3r})$ of $M_n$ is homeomorphic to $S^3$. We assume that a Dehn filling $M_n(m_1, m_r)$ of $M_n$ yielding $S^3$. By Key Lemma 3.1.1, one of two simple closed curve $m_1$, $m_r$ in $\partial H - C_n$ coincides with one of $m_{1l}$, $m_{2l}$, $m_{3l}$, $m_{1r}$, $m_{2r}$, and $m_{3r}$ up to isotopy on $\partial M_n$. Recall that $\bar{m}_{ij}$ $(i \in \{1, 2, 3\}, x \in \{l, r\})$ is isotopic to $m_{ix}$ in $\partial M_n$. By Lemma 3.2.4 2) and Theorem 2.3, each of Dehn fillings $M_n(\bar{m}_{1l}), M_n(\bar{m}_{2l}), M_n(\bar{m}_{3l}), M_n(\bar{m}_{1r}), M_n(\bar{m}_{2r})$ and $M_n(\bar{m}_{3r})$ of $M_n$ is homeomorphic to the exterior of a non-trivial knot, because the exterior of the trivial knot is homeomorphic to the solid torus. Then, by Theorem 2.4, $m_l \cup m_r$ is isotopic to one of $\bar{m}_{1l} \cup \bar{m}_{233r}$, $\bar{m}_{2l} \cup \bar{m}_{122r}$, $\bar{m}_{3l} \cup \bar{m}_{311r}$, $\bar{m}_{233l} \cup \bar{m}_{1r}$, $\bar{m}_{122l} \cup \bar{m}_{2r}$ and $\bar{m}_{311l} \cup \bar{m}_{3r}$ on $\partial M_n$. 

2) If $n \neq n'$, then, by Lemma 3.2.5, $M_n$ is not homeomorphic to $M_{n'}$. 

By Proposition 3.2.1, there exists a homeomorphism $h_{n'}: M_n(\bar{m}_{1l}, \bar{m}_{233r}) \to S^3$. The closure of $M_n(\bar{m}_{1l}, \bar{m}_{233r})$ consists of two solid tori $N_1$, $N_r$ such that $\partial N_l \supset \bar{m}_{1l}$ and $\partial N_r \supset \bar{m}_{233r}$. Then there are two homeomorphisms $h_{n'}: D^2 \times S^1 \to N_i$ $(x \in \{l, r\})$. Let $\bar{K}_{n_l}$ $(x \in \{l, r\})$ be the simple closed curve $h_{nl}(\emptyset \times S^1)$ where $\emptyset$ is the center of unit disk $D^2$. Let $\bar{K}_{n_r}$ $(x \in \{l, r\})$ be the knot $h_{n}(\overline{K}_{n_l})$ in $S^3$ and $L_n$ be the link $K_{nl} \cup K_{nr}$ in $S^3$. By the definitions of $M_n$ and $M_n(\bar{m}_{1l}, \bar{m}_{233r})$, the link $\bar{K}_{nl} \cup \bar{K}_{nr}$ in $M_n(\bar{m}_{1l}, \bar{m}_{233r})$ is tunnel number one, and so the link $K_{nl} \cup K_{nr}$ in $S^3$ is tunnel number one. 

In order to complete the proof of Theorem 1.1, we will show the next proposition.

**Proposition 3.2.2.** 1) Each tunnel number one link $L_n$ in $S^3$ has exactly five nontrivial Dehn surgeries yielding $S^3$ up to equivalence. 
2) Two links $L, L'$ are said to be equivalent if there is a homeomorphism $h: S^3 \to S^3$ satisfying $h(L) = L'$. If $n \neq n'$, then $L_n$ is not equivalent to $L_{n'}$. 

Proof. By the definition of a link $L_n$, the exterior $E(L_n)$ of $L_n$ is homeomorphic to $M_n$. Then we obtain 1) of Proposition 3.2.2 from 1) of Proposition 3.2.1. If $n \neq n'$, by 2) of Proposition 3.2.1, $E(L_n)$ is not homeomorphic to $E(L_{n'})$. Then $L_n$ is not equivalent to $L_{n'}$. 

Theorem 1.1 follows from Proposition 3.2.2.

### 3.3. Proof of Theorem 1.2.

Let $D_n$ be a Heegaard diagram $(\partial H; \{u_1, u_2\}, C_n)$ shown by Fig. 8 below where $n$ is a positive integer. Note that $D_n$ is a special case of $D$ in Key Lemma 3.1.1. Throughout this subsection, we assume $D = D_n$. $M_n, C_n$ mean $M, C$ in Key Lemma 3.1.1 respectively and $u_{ij}, m_{ix} (i \in \{1, 2, 3\}, x \in \{l, r\})$ mean the ones in Key Lemma 3.1.1 in the case of $D = D_n$ respectively.
Lemma 3.3.1. If $C_n$, $m_{1l}$, $m_{2l}$, $m_{3l}$, $m_{1r}$, $m_{2r}$ and $m_{3r}$ are oriented as shown in Fig. 8 respectively, then there exists a starting point on each of these simple closed curves respectively such that the following equations hold.

1. $W(C_n) = BABBAA(babA)^n babbba (BABa)^n$.
2. $W(m_{1l}) = babba$.
3. $W(m_{2l}) = a (BABA)^n B$.
4. $W(m_{3l}) = A B B A A(babA)^n$.
5. $W(m_{1r}) = (BABA)^n BAB B$.
6. $W(m_{2r}) = b a a$.
7. $W(m_{3r}) = A A (babA)^n bab$.

Proof. Let $c_i$, $d_i$, $e_i$, $f_i$, $g_i$ and $h_i$ be subarcs of $C_n$ respectively as shown in Fig. 8. Then $C_n$ can be represented by connecting theses subarcs as

$$c_1 e_1 f_{2n+2} h_2 e_{2n+3} g_1 \prod_{i=0}^{n-1} (d_{2(n-i)+1} f_{2(n-i)+1} e_{2i+2} e_{2i+2}),$$

$$d_1 f_1 e_{2n+2} h_1 f_{2n+3} g_2 \prod_{i=0}^{n-1} (c_{2(n-i)+1} e_{2(n-i)+1} f_{2i+2} d_{2i+2}).$$
Take a starting point on \( c_1 - d_1 \) for \( C_n \) and a starting point on \( w_{ix} - \partial w_{ix} \) (\( i \in [1, 2, 3] \), \( x \in [l, r] \)) for \( m_{ix} \) respectively.

**Lemma 3.3.2.** If \( m_{ix} \) (\( i \in [1, 2, 3] \), \( x \in [l, r] \)) is oriented as shown in Fig. 8, then the followings hold.

1) For each \( m_{ix} \) (\( i \in [1, 2, 3] \), \( x \in [l, r] \)), there exists an oriented simple closed curve \( \tilde{m}_{ix} \) in the component \( F'_x \) (\( x \in [l, r] \)) of \( \partial H - C_n \) intersecting \( m_{ix} \) such that \( W(\tilde{m}_{ix}) = W(\bar{m}_{ix}) \). \( \tilde{m}_{ix} \) is isotopic to \( m_{ix} \).

2) There exists an oriented simple closed curve \( \tilde{m}_{2l+1} \) in the component of \( \partial H - C_n \) intersecting \( m_{2l} \cup m_{1l} \) such that \( W(\tilde{m}_{2l+1}) = W(m_{2l})(W(m_{1l}))^{a+1} \).

Lemma 3.3.2 can be proved by same argument in the proof of Lemma 3.2.3.

**Lemma 3.3.3.** 1) Each of the two fundamental groups \( \pi_1(M_n(\tilde{m}_{2l}, \tilde{m}_{3r})) \) and \( \pi_1(M_n(\tilde{m}_{2l+1}, \tilde{m}_{2r})) \) is trivial.

2) Each of the two fundamental groups \( \pi_1(M_n(\tilde{m}_{2l})) \) and \( \pi_1(M_n(\tilde{m}_{3r})) \) is not isomorphic to the infinite cycle group \( \mathbb{Z} \).

3) The fundamental group \( \pi_1(M_n(\tilde{m}_{2r})) \) is isomorphic to the infinite cycle group \( \mathbb{Z} \).

**Proof.** 1) We will check them by using mutual substitutions.

\[
\pi_1(M_n(\tilde{m}_{2l}, \tilde{m}_{3r}))
\equiv \langle A, B \mid W(\tilde{m}_{2l}) = e, W(\tilde{m}_{3r}) = e \rangle
\equiv \langle A, B \mid W(m_{2l}) = e, W(m_{3r}) = e \rangle
\equiv \langle A, B \mid a(BABa)^n B = e, A(A(babA)^n bab = e \rangle
\equiv \langle A, B \mid A(babA)^n b = e, A(A(babA)^n bab = e \rangle
\equiv \langle A, B \mid (babA)^n b = e, b = e \rangle
\equiv \langle A, B \mid A = e, b = e \rangle = \{ e \}.
\]

\[
\pi_1(M_n(\tilde{m}_{2l+1}, \tilde{m}_{2r}))
\equiv \langle A, B \mid W(\tilde{m}_{2l+1}) = e, W(\tilde{m}_{2r}) = e \rangle
\equiv \langle A, B \mid W(m_{2l})(W(m_{1l}))^{a+1} = e, W(m_{2r}) = e \rangle
\equiv \langle A, B \mid a(BABa)^n abba(babba)^n = e, baa = e \rangle
\equiv \langle A, B \mid (babA)^n aabba(babba)^n = e, aab = e \rangle
\equiv \langle A, B \mid (babA)^{(n-1)} aabba(babba)^{(n-1)} = e, aab = e \rangle
\equiv \langle A, B \mid aabba = e, aab = e \rangle
\equiv \langle A, B \mid ba = e, aab = e \rangle \equiv \langle A, B \mid ba = e, a = e \rangle
\equiv \langle A, B \mid b = e, a = e \rangle = \{ e \}.
\]
2) Let $\xi_N, \alpha_N, \beta$ and $\rho$ be same ones in the proof of Lemma 3.2.4 and $\sigma : \langle A, B \rangle \rightarrow GL(2, \mathbb{C})$ be a homomorphism defined by $\sigma(A) = \alpha_N \beta, \sigma(B) = \beta$. Since $\sigma(W(m_2)) = \sigma(W(m_{21})) = \alpha_N^{2n+1}$ and $\rho(W(m_{3})) = \rho(W(m_{3})) = \alpha_N^{2n+3}$, by putting $N' = 2n + 1, N = 2n + 3, \sigma(W(m_{21})) = e, \rho(W(m_{3})) = e$, and so $\sigma$ (resp. $\rho$) keeps the relation $W(m_{21}) = e$ (resp. $W(m_{3}) = e$). Then we obtain the induced homomorphisms $\tilde{\sigma} : \langle A, B \mid W(m_{21}) = e \rangle = \langle A, B \rangle / N(W(m_{21})) \rightarrow GL(2, \mathbb{C})$ and $\tilde{\rho} : \langle A, B \mid W(m_{3}) = e \rangle = \langle A, B \rangle / N(W(m_{3})) \rightarrow GL(2, \mathbb{C})$. Since two elements $\alpha_{2n+1} \beta$ and $\beta$ (resp. $\alpha_{2n+3} \beta$ and $\beta$) in $GL(2, \mathbb{C})$ are non-commutative, $\langle A, B \mid W(m_{21}) = e \rangle$ (resp. $\langle A, B \mid W(m_{3}) = e \rangle$) is not isomorphic to $\mathbb{Z}$. By Basic Lemma 3.1.2, $\pi_1(M_n(m_{21}))$ (resp. $\pi_1(M_n(m_{3}))$) is not isomorphic to $\mathbb{Z}$.

3) By changing generators $A$ and $B$ of free group $\langle A, B \rangle$ into $A$ and $B' := W(m_{21}) = W(m_{3}) = baa$, we can check the following.

$$\pi_1(M_n(m_{21})) \cong \langle A, B \mid W(m_{21}) = e \rangle \equiv \langle A, B' \mid B' = e \rangle \equiv \langle A \mid - \rangle \equiv \mathbb{Z}. \quad \square$$

**Lemma 3.3.4.** If $n \neq n'$, then $\pi_1(M_n) \text{ is not isomorphic to } \pi_1(M_{n'})$.

Proof. Let $\xi_N, \alpha_N, \beta$ and $\rho$ be same ones in the proof of Lemma 3.2.4 and $G_N$ be same one in the proof of Lemma 3.2.5. Since $\rho(W(C_n)) = \alpha_N^{-4n+6} = \begin{pmatrix} \xi_N^{4n+6} & 0 \\ 0 & \xi_N^{-4n+6} \end{pmatrix}$, by putting $N = 4n + 6, \rho(W(C_n)) = e$, and so $\rho$ keeps the relation $W(C_n) = e$. Then we obtain the induced homomorphism $\tilde{\rho} : \langle A, B \mid W(C_n) = e \rangle = \langle A, B \rangle / N(W(C_n)) \rightarrow GL(2, \mathbb{C})$. If $\pi_1(M_n)$ is isomorphic to $\pi_1(M_{n'})$ for $n' \leq n$, by Basic Lemma 3.1.2, there is a surjective homomorphism $\tau : \langle A, B \mid W(C_{n'}) = e \rangle = \langle A, B \rangle / N(W(C_{n'})) \rightarrow G_{4n+6}$. Since $\tau$ is surjective, two elements $\tau(A), \tau(B)$ are generators of $G_{4n+6}$. By same argument in the proof of Lemma 3.2.5, there are following four cases for $\tau(A)$ and $\tau(B)$.

1) If $\tau(A) = \alpha_{4n+6}^k$ and $\tau(B) = \alpha_{4n+6}^{l}$, then $\tau(W(C_{n'})) = \alpha_{4n+6}^{k+l}$. 

2) If $\tau(A) = \alpha_{4n+6}^{k} \beta$ and $\tau(B) = \alpha_{4n+6}^{l} \beta$, then $\tau(W(C_{n'})) = \alpha_{4n+6}^{k+l}$. 

3) If $\tau(A) = \alpha_{4n+6}^{k} \beta$ and $\tau(B) = \alpha_{4n+6}^{l} \beta$, then $\tau(W(C_{n'})) = \alpha_{4n+6}^{k+l}$. 

4) If $\tau(A) = \alpha_{4n+6}^{k} \beta$ and $\tau(B) = \alpha_{4n+6}^{l} \beta$, then $\tau(W(C_{n'})) = \alpha_{4n+6}^{k+l}$. 

Here $k$ and $4n + 6$ are relatively prime. On the other hand, since $W(C_{n'})$ represents a unit element of $\langle A, B \mid W(C_{n'}) = e \rangle = \langle A, B \rangle / N(W(C_{n'})), \tau(W(C_{n'})) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ holds. In the case (1), $n = n'$ holds, because $\xi_{4n+6}^k$ is a $(4n + 6)$-th primitive root of unity and $4n + 6 \leq 4n + 6$. The other cases (2), (3) and (4) do not happen, because $\xi_{4n+6}^k$ is a $(4n + 6)$-th primitive root of unity and $2, 4n + 2 < 4n + 6$. \square

**Proposition 3.3.1.** 1) Each of the two Dehn fillings $M_n(m_{21}, m_{3})$ and $M_n(m_{21} + 1, \bar{m}_{3})$ is homeomorphic to $S^3$.

2) Each of the two Dehn fillings $M_n(m_{21})$ and $M_n(m_{3})$ is not homeomorphic to the solid torus.

3) A Dehn filling $M_n(m_{21})$ is homeomorphic to the solid torus.

4) If $n \neq n'$, then $M_n$ is not homeomorphic to $M_{n'}$. 


Proof. 1) By Lemma 3.3.3 1), each of the fundamental groups \( \pi_1(M_n(\bar{m}_{2l}, \bar{m}_{3r})) \) and \( \pi_1(M_n(\bar{m}_{2l+1}, \bar{m}_{2r})) \) is trivial. Then, by Theorem 2.4, each of \( M_n(\bar{m}_{2l}, \bar{m}_{3r}) \) and \( M_n(\bar{m}_{2l+1}, \bar{m}_{2r}) \) is homeomorphic to \( S^3 \).

2) By Lemma 3.3.3 2) and Theorem 2.3, each of \( M_n(\bar{m}_{2l}) \) and \( M_n(\bar{m}_{3r}) \) is not homeomorphic to the solid torus.

3) A Dehn filling \( M_n(\bar{m}_{2l}) \) is a submanifold of a Dehn filling of \( M_n(\bar{m}_{2l+1}, \bar{m}_{2r}) \). By 1), \( M_n(\bar{m}_{2l+1}, \bar{m}_{2r}) \) is homeomorphic to \( S^3 \). By Lemma 3.3.3 3), \( \pi_1(M_n(\bar{m}_{2l})) \) is isomorphic to \( Z \). Then, by Theorem 2.3, \( M_n(\bar{m}_{2l}) \) is homeomorphic to the solid torus.

4) If \( n \neq n' \), then, by Lemma 3.3.4, \( M_n \) is not homeomorphic to \( M_{n'} \). \( \square \)

By Proposition 3.3.1, there exists two homeomorphisms \( h_n: M_n(\bar{m}_{2l}, \bar{m}_{3r}) \rightarrow S^3 \) and \( h'_n: M_n(\bar{m}_{2l+1}, \bar{m}_{2r}) \rightarrow S^3 \). The closure of \( M_n(\bar{m}_{2l}, \bar{m}_{3r}) - M_n \) (resp. \( M_n(\bar{m}_{2l+1}, \bar{m}_{2r}) - M_n \)) consists of two solid tori \( N_l, N_r \) (resp. \( N'_l, N'_r \)) such that \( \partial N_l \supset \bar{m}_{2l} \) and \( \partial N_r \supset \bar{m}_{3r} \) (resp. \( \partial N'_l \supset \bar{m}_{2l+1} \) and \( \partial N'_r \supset \bar{m}_{2r} \)). Then there are four homeomorphisms \( h_{nx}: D^2 \times S^1 \rightarrow N_x (x \in [l, r]) \) and \( h'_{nx}: D^2 \times S^1 \rightarrow N'_x (x \in [l, r]) \). Let \( \tilde{K}_{nx} \) (resp. \( K'_{nx} \)) \( (x \in [l, r]) \) be the simple closed curve \( h_{nx}(\mathbf{0} \times S^1) \) (resp. \( h'_{nx}(\mathbf{0} \times S^1) \)) where \( \mathbf{0} \) is the center of unit disk \( D^2 \). Let \( K_{nx} \) (resp. \( K'_{nx} \)) \( (x \in [l, r]) \) be the knot \( h_n(\tilde{K}_{nx}) \) (resp. \( h_n(\tilde{K}'_{nx}) \)) in \( S^3 \) and \( L_n \) (resp. \( L'_n \)) be the link \( K_{nx} \cup K_{nr} \) (resp. \( K'_{nx} \cup K'_{nr} \)) in \( S^3 \). By the definitions of \( M_n, M_n(\bar{m}_{2l}, \bar{m}_{3r}) \) and \( M_n(\bar{m}_{2l+1}, \bar{m}_{2r}) \), each of the two links \( \tilde{K}_{nl} \cup \tilde{K}_{nr} \) in \( M_{n}(\bar{m}_{2l}, \bar{m}_{3r}) \) and \( \tilde{K}'_{nl} \cup \tilde{K}'_{nr} \) in \( M_{n}(\bar{m}_{2l+1}, \bar{m}_{2r}) \) is tunnel number one, and so each of the two links \( K_{nl} \cup K_{nr} \) and \( K'_{nl} \cup K'_{nr} \) in \( S^3 \) is tunnel number one.

In order to complete the proof of Theorem 1.2 we will show the next proposition.

**Proposition 3.3.2.**

1) \( L_n \) has no trivial component.

2) \( L'_n \) has a trivial component.

3) \( E(L_n) \) is homeomorphic to \( E(L'_n) \).

4) If \( n \neq n' \), then \( L_n \) is not equivalent to \( L_{n'} \). \( \square \)

Proof. 1) By the definition of \( L_n \), \( E(K_l) \) (resp. \( E(K_r) \)) is homeomorphic to \( M_n(\bar{m}_{3r}) \) (resp. \( M_n(\bar{m}_{2l}) \)). By Proposition 3.3.1, each of \( E(K_l), E(K_r) \) is not homeomorphic to the solid torus. Hence each of \( K_l, K_r \) is not a trivial knot.

2) By the definition of \( L'_n \), \( E(K'_l) \) is homeomorphic to \( M_n(\bar{m}_{2r}) \). By Proposition 3.3.1, \( E(K_l) \) is homeomorphic to the solid torus. Hence \( K_l \) is a trivial knot.

3) By the definition of \( L_n \) and \( L'_n \), each of the exteriors \( E(L_n), E(L'_n) \) is homeomorphic to \( M_n \). Hence \( E(L_n) \) is homeomorphic to \( E(L'_n) \).

4) If \( n \neq n' \), then, by Proposition 3.3.1, \( M_n \) is not homeomorphic to \( M_{n'} \), and so \( E(L_n) \) is not homeomorphic to \( E(L_{n'}) \). Hence \( L_n \) is not equivalent to \( L_{n'} \). \( \square \)

Theorem 1.2 follows from Proposition 3.3.2.

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