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THE CAUCHY PROBLEM FOR FINITELY DEGENERATE HYPERBOLIC EQUATIONS WITH POLYNOMIAL COEFFICIENTS

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Abstract

We consider hyperbolic Cauchy problems with characteristics of variable multiplicity and coefficients of polynomial growth in the space variables; we focus on second order equations and admit finite order intersections between the characteristics. We obtain well posedness results in $\mathcal{S}(\mathbb{R}^n)$, $\mathcal{S}'(\mathbb{R}^n)$ by imposing suitable Levi conditions on the lower order terms. By an energy estimate in weighted Sobolev spaces we show that regularity and behavior at infinity of the solution are different from the ones of the data.

1. Introduction and main result

In this paper we deal with the Cauchy problem for hyperbolic equations with coefficients of polynomial growth in the space variables. A pioneering work on this topic is the book by Cordes [9], where strictly hyperbolic equations are considered. The author proves well posedness for the related Cauchy problem in $\mathcal{S}(\mathbb{R}^n)$, $\mathcal{S}'(\mathbb{R}^n)$ and in the scale of weighted Sobolev spaces H_{s_1,s_2}, $s_1, s_2 \in \mathbb{R}$, of all $u \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|u\|_{s_1,s_2} = \|\langle\xi\rangle^{s_1}\mathcal{F}_{x\to\xi}(\langle x\rangle^{s_2}u)\|_{L^2(\mathbb{R}^n)} < +\infty, \tag{1.1}$$

($\mathcal{F}$ denotes here the Fourier transform and $\langle a\rangle = (1+|a|^2)^{1/2}$ for $a \in \mathbb{R}^n$). The case $s_2 = 0$ corresponds to the standard Sobolev spaces. We recall that

$$\bigcap_{s_1,s_2\in\mathbb{R}} H_{s_1,s_2} = \mathcal{S}(\mathbb{R}^n), \qquad \bigcup_{s_1,s_2\in\mathbb{R}} H_{s_1,s_2} = \mathcal{S}'(\mathbb{R}^n). \tag{1.2}$$

Results in [9] have been extended to weakly hyperbolic equations with constant multiplicities by Coriasco [11], Coriasco and Rodino [12] by imposing Levi conditions on the lower order terms. This improvement was possible thanks to a suitable Fourier integral operator calculus developed in [10]. Analogous results in a Gevrey framework have been proved in [5] (see also the recent paper by Gourdin and Gramchev [14]). In this paper we admit variable multiplicity for the characteristics, focusing on second order operators.

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Local well posedness for the hyperbolic Cauchy problem with double characteristics has been intensively studied, see [15], [18], [19], [20] and the references therein.

In our paper we consider equations globally defined on $\mathbb{R}^n$ in the space variables and investigate global existence and uniqueness of a solution. In this setting, for uniformly bounded coefficients, some important results have been proved under an intermediate condition between effective hyperbolicity (cf. [21], [22]) and Levi conditions. Namely, Colombini, Ishida and Orrù [6] proved C^∞ well posedness of the Cauchy problem for the operator

$$P(t, D_t, D_x) = D_t^2 - a(t, D_x) + b(t, D_x) \tag{1.3}$$

with

$$a(t, \xi) = \sum_{i,j=1}^{n} a_{ij}(t)\xi_i\xi_j, \quad b(t, \xi) = \sum_{j=1}^{n} b_j(t)\xi_j,$$
$$a(t, \xi) \geq 0, \ \ t \in [0, T], \ \ |\xi| = 1,$$

by assuming the existence of an integer $k \geq 2$ such that

$$\begin{cases} \displaystyle\sum_{j=0}^{k} |\partial_t^j a(t, \xi)| \neq 0, \\ |b(t, \xi)| \leq C a^{\gamma}(t, \xi), \quad \gamma = \dfrac{1}{2} - \dfrac{1}{k}, \\ \forall t \in [0, T], \quad |\xi| = 1. \end{cases} \tag{1.4}$$

Notice that for $k = 2$ the operator P is effectively hyperbolic and the Levi condition is void (i.e. $\gamma = 0$); on the other hand, if $k = \infty$ no assumption on the degeneracy of the roots is required to get C^∞ well posedness, see [7]. As shown in [17], the bound γ in (1.4) is sharp. Colombini and Nishitani [8] allowed a dependence on x in the lower order terms of (1.3) and proved the same result of [6] but for the larger value $\gamma = 1/2 - 1/(2(k-1))$, cf. (1.4). Recently, the first author and Cicognani [3] dealt with the Cauchy problem for operators of the form

$$\begin{cases} P(t, x, D_t, D_x) = D_t^2 - a(t, x, D_x) + b(t, x, D_x) + c(t, x), \\ a(t, x, D_x) = \alpha(t) Q(x, D_x), \\ b(t, x, \xi) = \displaystyle\sum_{j=1}^{n} b_j(t, x)\xi_j, \end{cases} \tag{1.5}$$

$(t, x) \in [0, T] \times \mathbb{R}^n$, assuming that $\alpha(t) \geq 0$ and

$$Q(x, \xi) = \sum_{i,j=1}^{n} q_{ij}(x)\xi_i\xi_j \geq q_0|\xi|^2, \quad q_0 > 0. \tag{1.6}$$

They proved C^∞ well posedness by assuming the existence of an integer $k \geq 2$ such that

$$\begin{cases} \sum_{j=0}^{k} |\alpha^{(j)}(t)| \neq 0, \quad t \in [0, T], \\ |\partial_x^\beta b_j(t, x)| \leq C_\beta \alpha^\gamma(t), \quad \gamma = \dfrac{1}{2} - \dfrac{1}{k}, \\ t \in [0, T], \quad x \in \mathbb{R}^n, \quad j = 1, \dots, n, \quad \beta \in \mathbb{Z}_+^n. \end{cases} \tag{1.7}$$

The result in [3] comes out from $H^{\pm\infty}$ well posedness of the Cauchy problem for (1.5), obtained by means of an energy estimate in Sobolev spaces. Notice that condition (1.7) is clearly consistent with (1.4); this allowed to improve the result of [8] at least in the case $n = 1$. A more general Q in (1.5) depending also on t was considered in [4].

In the present paper we study the Cauchy problem

$$\begin{cases} P(t, x, D_t, D_x)u(t, x) = 0, \quad (t, x) \in [0, T] \times \mathbb{R}^n, \\ u(0, x) = f(x), \\ \partial_t u(0, x) = g(x) \end{cases} \tag{1.8}$$

for $P = D_t^2 - a(t)Q(t, x, D_x) + b(t, x, D_x) + c(t, x)$ and we assume that:

- $a \in C^\infty([0, T]; \mathbb{R})$, $a(t) \geq 0$ for $t \in [0, T]$, and there exists $k \geq 2$ such that

$$\sum_{j=0}^{k} |a^{(j)}(t)| \neq 0; \tag{1.9}$$

- $Q(t, x, D_x)$ is a pseudodifferential operator with symbol $Q(t, x, \xi)$ satisfying the estimate

$$\sup_{t \in [0,T]} |\partial_\xi^\alpha \partial_x^\beta Q(t, x, \xi)| \leq c_{\alpha\beta} \langle x \rangle^{\sigma - |\beta|} \langle \xi \rangle^{2 - |\alpha|} \quad \text{for some} \quad \sigma \in [0, 2] \tag{1.10}$$

for every $\alpha, \beta \in \mathbb{Z}_+^n$ and the global ellipticity condition

$$|Q(t, x, \xi)| \geq C \langle x \rangle^\sigma \langle \xi \rangle^2, \quad t \in [0, T], \ |x| + |\xi| \geq R, \tag{1.11}$$

for some positive constants C, R; concerning the regularity with respect to t we require that Q is in $C^1[0, T]$;

- $b(t, x, D_x)$ is a pseudodifferential operator with symbol $b(t, x, \xi)$ such that

$$|\partial_\xi^\alpha \partial_x^\beta b(t, x, \xi)| \leq c_{\alpha\beta} a^\gamma(t) \langle x \rangle^{\sigma/2 - |\beta|} \langle \xi \rangle^{1 - |\alpha|}, \quad \gamma = \frac{1}{2} - \frac{1}{k}, \tag{1.12}$$

for some constant $c_{\alpha\beta} > 0$ and for every $(t, x, \xi) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^n$; moreover, b is continuous with respect to $t \in [0, T]$.

We are going to prove the following

Theorem 1.1. *Under the assumptions* (1.9), (1.10), (1.11), (1.12) *the Cauchy problem* (1.8) *is well posed in* $\mathcal{S}(\mathbb{R}^n)$, $\mathcal{S}'(\mathbb{R}^n)$. *Furthermore, there exist positive constants* δ_1, δ_2 *such that for every* $f \in H_{s_1,s_2}(\mathbb{R}^n)$, $g \in H_{s_1-1,s_2-1}(\mathbb{R}^n)$ *the solution* $u(t, x)$ *of* (1.8) *satisfies the energy estimate*:

$$\|u(t)\|^2_{s_1-\delta_1,s_2-\delta_2} + \|\partial_t u(t)\|^2_{s_1-\delta_1-1,s_2-\delta_2-1} \leq C(\|f\|^2_{s_1,s_2} + \|g\|^2_{s_1-1,s_2-1}) \tag{1.13}$$

for every $t \in [0, T]$.

Our results apply to operators with coefficients of polynomial growth in x. A typical model is the operator

$$L_0 = D_t^2 - a(t)\langle x\rangle^\sigma (1 - \triangle),$$

$\sigma \in (0, 2]$, with $a(t)$ satisfying the assumptions of Theorem 1.1. We underline that in the case $\sigma \neq 0$ the global ellipticity assumption (1.11) will be crucial in the proof of our results in order to reduce (1.8) to a suitable system of diagonal form, while the assumption (1.12) is instrumental to estimate the remainder term in (3.14).

As a novelty with respect to [3], [6], [8], the energy estimate (1.13) reveals a loss δ_2 in the second Sobolev index (the one related to the behavior at infinity). This phenomenon has been already observed in other degenerate hyperbolic problems with polynomial coefficients in x, cf. [1], [2]. As we shall see in Section 4, δ_2 depends on σ. This also allows to relate our result in the case $\sigma = 0$ with the one proved in [3]. We address the reader to Section 4, Remark 3, for a precise comparison.

2. Preliminaries

In this section we recall some basic facts about pseudodifferential operators of SG type that will be useful in the proofs of our results. We start by introducing the class of symbols we are dealing with.

DEFINITION 2.1. For any $m_1, m_2 \in \mathbb{R}$ we shall denote by $SG^{(m_1,m_2)}$ the space of all functions $a(x, \xi) \in C^\infty(\mathbb{R}^{2n})$ such that

$$\sup_{(x,\xi)\in\mathbb{R}^{2n}} \langle x\rangle^{-m_2+|\beta|}\langle\xi\rangle^{-m_1+|\alpha|}|D_\xi^\alpha D_x^\beta a(x, \xi)| < +\infty$$

for all $\alpha, \beta \in \mathbb{Z}^n_+$. We shall denote by $LG^{(m_1,m_2)}$ the space of all pseudodifferential operators of the form

$$Au(x) = a(x, D_x)u(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} a(x, \xi)\hat{u}(\xi)\, d\xi, \quad u \in \mathcal{S}(\mathbb{R}^n), \tag{2.1}$$

with $a \in SG^{(m_1,m_2)}$, where $\hat{u}$ denotes the Fourier transform of u. We shall write SG^0 for $SG^{(0,0)}$ and LG^0 for $LG^{(0,0)}$.

A detailed calculus for the class defined above can be found in many papers, see for example [9], [13], [23], [24]. Here we limit ourselves to remind some basic results. For every $a \in SG^{(m_1,m_2)}$, the operator (2.1) is linear and continuous from $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}(\mathbb{R}^n)$ and extends to a continuous map from $\mathcal{S}'(\mathbb{R}^n)$ to itself. Moreover, for any s_1, s_2, A maps continuously H_{s_1,s_2} into $H_{s_1-m_1,s_2-m_2}$. We also recall that

$$\bigcap_{m_1,m_2\in\mathbb{R}} SG^{(m_1,m_2)} = \mathcal{S}(\mathbb{R}^{2n}).$$

Operators with symbol in $\mathcal{S}(\mathbb{R}^{2n})$ map continuously $\mathcal{S}'(\mathbb{R}^n)$ into $\mathcal{S}(\mathbb{R}^n)$. They are called *regularizing* and their class will be denoted in the following by $\mathcal{K}$.

Theorem 2.1. *Let $A = a(x, D_x) \in LG^{(m_1,m_2)}$, $B = b(x, D_x) \in LG^{(m_1',m_2')}$. Then there exists $c(x,\xi) \in SG^{(m_1+m_1',m_2+m_2')}$ such that $AB = c(x, D_x) + R_1$ where $R_1 \in \mathcal{K}$ and*

$$c(x, \xi) \sim \sum_{\alpha} \frac{1}{\alpha!} \partial_\xi^\alpha a(x, \xi) D_x^\alpha b(x, \xi). \tag{2.2}$$

Similarly, the commutator $[A, B] = d(x, D_x) + R_2$, where $d \in SG^{(m_1+m_1'-1,m_2+m_2'-1)}$ has the following asymptotic expansion

$$d(x, \xi) \sim \sum_{\alpha\neq 0} \frac{1}{\alpha!} (\partial_\xi^\alpha a(x, \xi) D_x^\alpha b(x, \xi) - \partial_\xi^\alpha b(x, \xi) D_x^\alpha a(x, \xi)) \tag{2.3}$$

and $R_2 \in \mathcal{K}$.

DEFINITION 2.2. A symbol $a \in SG^{(m_1,m_2)}$ is said to be md-elliptic (or SG-elliptic) if there exist positive constants C, R such that

$$|a(x, \xi)| \geq C\langle x\rangle^{m_2}\langle\xi\rangle^{m_1} \quad \text{for} \quad |x| + |\xi| \geq R. \tag{2.4}$$

Theorem 2.2. *Let $A = a(x, D_x) \in LG^{(m_1,m_2)}$. Then there exist $E_i \in LG^{(-m_1,-m_2)}$, $i = 1, 2$ such that*

$$E_1 A = I + R_1, \quad AE_2 = I + R_2$$

for some $R_i \in \mathcal{K}$ if and only if a is md-elliptic (I denotes here the identity operator).

The operators E_1, E_2 in the theorem above are called *left* (*respectively right*) *parametrix* of A.

3. Reduction to a system

In this section we factorize the operator P and reduce the equation $Pu = 0$ in (1.8) to a first order system of diagonal form; in Section 4 we shall derive an energy estimate for that system.

First of all we separate the characteristic roots of P defining

$$\tilde{\lambda}(t, x, \xi) = \sqrt{a(t) + \langle x\rangle_h^{-\sigma}\langle\xi\rangle^{-2}}\sqrt{Q(t, x, \xi)}, \tag{3.1}$$

where we denote $\langle x\rangle_h = (h^2 + |x|^2)^{1/2}$, with $h \geq 1$ to be chosen later on. We remind that $\langle x\rangle_h \in SG^{(0,1)}$ and that for every $\beta \in \mathbb{Z}_+^n$, we have $|\partial_x^\beta \langle x\rangle_h| \leq C_\beta \langle x\rangle_h^{1-|\beta|}$ for some positive constant C_β independent of h. Now by Theorem 2.1 and by (3.1) we get

$$\begin{aligned}
&(D_t - \tilde{\lambda}(t, x, D_x))(D_t + \tilde{\lambda}(t, x, D_x)) \\
&= D_t^2 - \tilde{\lambda}(t, x, D_x)\tilde{\lambda}(t, x, D_x) + (D_t\tilde{\lambda})(t, x, D_x) \\
&= D_t^2 - a(t)Q(t, x, D_x) - \mathrm{op}\left(\sum_{j=1}^n \partial_{\xi_j}\tilde{\lambda}(t, x, \xi)D_{x_j}\tilde{\lambda}(t, x, \xi)\right) \\
&\quad + \mathrm{op}\left(-\frac{ia'(t)\sqrt{Q(t, x, \xi)}}{2\sqrt{a(t) + \langle x\rangle_h^{-\sigma}\langle\xi\rangle^{-2}}} + \frac{\sqrt{a(t) + \langle x\rangle_h^{-\sigma}\langle\xi\rangle^{-2}}D_tQ(t, x, \xi)}{2\sqrt{Q(t, x, \xi)}}\right) \\
&\quad + r_0(t, x, D_x),
\end{aligned} \tag{3.2}$$

for some $r_0(t, x, \xi) \in C([0, T], SG^0)$. Developing further in the right-hand side of (3.2) we obtain

$$\begin{aligned}
&(D_t - \tilde{\lambda}(t, x, D_x))(D_t + \tilde{\lambda}(t, x, D_x)) \\
&= D_t^2 - a(t)Q(t, x, D_x) - s_1(t, x, D_x) - s_2(t, x, D_x) + r_1(t, x, D_x)
\end{aligned} \tag{3.3}$$

where $r_1 \in C([0, T], LG^0)$,

$$s_1(t, x, \xi) = \frac{ia'(t)\sqrt{Q(t, x, \xi)}}{2\sqrt{a(t) + \langle x\rangle_h^{-\sigma}\langle\xi\rangle^{-2}}} \tag{3.4}$$

and

$$\begin{aligned}
s_2(t, x, \xi) = {}& \frac{\sqrt{a(t) + \langle x\rangle_h^{-\sigma}\langle\xi\rangle^{-2}}D_tQ(t, x, \xi)}{2\sqrt{Q(t, x, \xi)}} \\
&+ \frac{a(t)}{4Q(t, x, \xi)}\sum_{j=1}^n \partial_{\xi_j}Q(t, x, \xi)D_{x_j}Q(t, x, \xi).
\end{aligned} \tag{3.5}$$

Let now

$$\omega(t, x, \xi) = \sqrt{1 + a(t)\langle x\rangle_h^{\sigma}\langle\xi\rangle^2} \in C([0, T], SG^{(1,\sigma/2)}). \tag{3.6}$$

We observe that the symbol $1/\omega(t,x,\xi)$ has order $(0,0)$ since $a(t)$ may vanish at some points. Nevertheless, we have that

$$\frac{a(t)}{\omega(t,\,x,\,\xi)} \in C([0,\,T],\,SG^{(-1,-\sigma/2)}).$$

Moreover, the operator $\omega(t,\,x,\,D_x)$ is invertible. Namely, we have the following result.

Proposition 3.1. *Let $\omega(t,x,\xi)$ be defined by* (3.6). *There exists $h_0 \geq 1$ such that for every $h \geq h_0$, the operator $\omega(t,x,D_x)$ is invertible. The inverse operator is given by*

$$(\omega(t,\,x,\,D_x))^{-1} = \mathrm{op}\left(\frac{1}{\omega(t,\,x,\,\xi)}\right) \circ S(t,\,x,\,D_x), \quad S = I + S_{-1},$$

I the identity operator, for some $S_{-1} \in C([0,\,T],\,LG^{(-1,-1)})$.

Proof. By Theorem 2.1 we have

$$\omega(t,\,x,\,D_x) \circ \mathrm{op}\left(\frac{1}{\omega(t,\,x,\,\xi)}\right) = I + R(t,\,x,\,D_x),$$

where $R(t,\,x,\,\xi)$ has principal part

$$r(t,\,x,\,\xi) = -\left(\frac{a(t)\langle x\rangle_h^{\sigma/2}\langle\xi\rangle}{2(1+a(t)\langle x\rangle_h^{\sigma}\langle\xi\rangle^2)}\right)^2 \sum_{j=1}^{n} \partial_{\xi_j}\langle\xi\rangle^2 D_{x_j}\langle x\rangle_h^{\sigma}.$$

We notice that $r \in C([0,\,T];\,SG^{(-1,-1)})$; moreover, for every $\alpha,\,\beta \in \mathbb{Z}_+^n$ we have

$$|D_\xi^\alpha D_x^\beta r(t,\,x,\,\xi)| \leq C_{\alpha\beta}\langle x\rangle_h^{-1-|\beta|}\langle\xi\rangle^{-1-|\alpha|},$$

for some $C_{\alpha\beta} > 0$ independent of h. Choosing $h \geq 1$ sufficiently large, we obtain that R is a bounded operator $L^2 \to L^2$ with norm

$$\|R\|_{\mathcal{L}(L^2(\mathbb{R}^n))} < 1,$$

cf. [9], Corollary 1.3 p. 102. Then, $I+R$ is invertible by Neumann series, and the inverse

$$S = I + \sum_{j=1}^{\infty} R^j(t,\,x,\,D_x)$$

belongs to $C([0,\,T],\,LG^0)$. Thus, $\mathrm{op}(1/\omega(t,\,x,\,\xi)) \circ S$ is the inverse of $\omega(t,\,x,\,D_x)$. □

We can now define

$$\begin{cases} u_0 = \omega(t, x, D_x)u, \\ u_1 = (D_t + \tilde{\lambda}(t, x, D_x))u. \end{cases} \tag{3.7}$$

Then by (3.3), (3.7) we get

$$\begin{aligned}(D_t + \tilde{\lambda}(t, x, D_x))u_0 = \omega(t, x, D_x)u_1 + (D_t\omega)(t, x, D_x)(\omega(t, x, D_x))^{-1}u_0 \\ + [\tilde{\lambda}(t, x, D_x), \omega(t, x, D_x)](\omega(t, x, D_x))^{-1}u_0\end{aligned}$$

and

$$\begin{aligned}&(D_t - \tilde{\lambda}(t, x, D_x))u_1 \\ &= -b(t, x, D_x)(\omega(t, x, D_x))^{-1}u_0 - c(t, x)(\omega(t, x, D_x))^{-1}u_0 \\ &\quad - s_1(t, x, D_x)(\omega(t, x, D_x))^{-1}u_0 - s_2(t, x, D_x)(\omega(t, x, D_x))^{-1}u_0 \\ &\quad + r_1(t, x, D_x)(\omega(t, x, D_x))^{-1}u_0.\end{aligned}$$

Taking into account (2.3) we obtain that

$$\begin{aligned}&[\tilde{\lambda}(t, x, D_x), \omega(t, x, D_x)](\omega(t, x, D_x))^{-1} \\ &= \mathrm{op}\left(\sum_{j=1}^{n} \frac{\partial_{\xi_j}\tilde{\lambda} D_{x_j}\omega - \partial_{\xi_j}\omega D_{x_j}\tilde{\lambda}}{\omega}\right) + r_2(t, x, D)\end{aligned}$$

with $r_2 \in C([0, T], SG^0)$. Since

$$\frac{D_{x_j}\omega(t, x, \xi)}{\omega(t, x, \xi)} = \frac{a(t)D_{x_j}\langle x\rangle^{\sigma}\langle\xi\rangle^2}{2\omega^2(t, x, \xi)} \in C([0, T]; SG^{(0,-1)})$$

and

$$\frac{\partial_{\xi_j}\omega(t, x, \xi)}{\omega(t, x, \xi)} = \frac{a(t)\langle x\rangle^{\sigma}\partial_{\xi_j}\langle\xi\rangle^2}{2\omega^2(t, x, \xi)} \in C([0, T]; SG^{(-1,0)})$$

we obtain that $[\tilde{\lambda}, \omega]\omega^{-1}$ has order $(0, 0)$. Moreover,

$$(D_t\omega)(t, x, D_x)(\omega(t, x, D_x))^{-1} = S \circ \tilde{s}(t, x, D_x) + r_3(t, x, D_x) \tag{3.8}$$

where

$$\tilde{s}(t, x, \xi) = \frac{-ia'(t)}{2(a(t) + \langle x\rangle_h^{-\sigma}\langle\xi\rangle^{-2})} \tag{3.9}$$

and $r_3 \in C([0, T], LG^0)$. On the other hand, since $s_1(t, x, D_x) \in C([0, T], LG^{(1,\sigma/2)})$, by (3.4) we can write

$$s_1(t, x, D_x)(\omega(t, x, D_x))^{-1} = S \circ \tilde{s}_1(t, x, D_x) + r_4(t, x, D_x)$$

where $r_4 \in C([0, T], LG^0)$ and

$$\tilde{s}_1(t, x, D_x) = \mathrm{op}\left(\frac{ia'(t)\sqrt{Q(t, x, \xi)}}{2\sqrt{(a(t) + \langle x\rangle_h^{-\sigma}\langle\xi\rangle^{-2})(1 + a(t)\langle x\rangle_h^{\sigma}\langle\xi\rangle^{2})}}\right)$$
(3.10)
$$= \mathrm{op}\left(\frac{ia'(t)\langle x\rangle_h^{-\sigma/2}\langle\xi\rangle^{-1}\sqrt{Q(t, x, \xi)}}{2(a(t) + \langle x\rangle_h^{-\sigma}\langle\xi\rangle^{-2})}\right).$$

Finally, $c(t,x)(\omega(t,x,D_x))^{-1}$, $s_2(t,x,D_x)\circ(\omega(t,x,D_x))^{-1}$ and $r_1(t,x,D_x)\circ(\omega(t,x,D_x))^{-1}$ belong to $C([0, T], LG^0)$. Hence, problem (1.8) is reduced to the equivalent problem

$$\begin{cases} L_1 U = 0, \\ U(0, x) = U_0(x), \end{cases} \tag{3.11}$$

where $U = (u_0, u_1)$ and

$$L_1 = \partial_t - i\begin{pmatrix} -\tilde{\lambda}(t, x, D_x) & \omega(t, x, D_x) \\ 0 & \tilde{\lambda}(t, x, D_x)\end{pmatrix} + A_1(t, x, D_x)b(t, x, D_x)(\omega(t, x, D_x))^{-1}$$
$$+ A_2(t, x, D_x)\tilde{s}(t, x, D_x) + A_3(t, x, D_x)\tilde{s}_1(t, x, D_x) + R(t, x, D_x)$$

where A_1, A_2, A_3, R are matrices of pseudodifferential operators in $C([0, T], LG^0)$. Notice now that the matrix

$$\begin{pmatrix} -\tilde{\lambda}(t, x, \xi) & \omega(t, x, \xi) \\ 0 & \tilde{\lambda}(t, x, \xi)\end{pmatrix}$$

can be diagonalized by

$$M = \begin{pmatrix} 1 & \dfrac{\langle x\rangle_h^{\sigma/2}\langle\xi\rangle}{2\sqrt{Q(t, x, \xi)}} \\ 0 & 1 \end{pmatrix}$$

which is of order $(0, 0)$ thanks to the condition (1.11). Then, problem (3.11) is equivalent to

$$\begin{cases} LV = 0, \\ V(0, x) = V_0(x), \end{cases} \tag{3.12}$$

where

$$V = MU \tag{3.13}$$

and

$$L = \partial_t - i\begin{pmatrix} -\tilde{\lambda}(t, x, D_x) & 0 \\ 0 & \tilde{\lambda}(t, x, D_x)\end{pmatrix} + \tilde{A}_1(t, x, D_x)b(t, x, D_x)(\omega(t, x, D_x))^{-1}$$
$$+ \tilde{A}_2(t, x, D_x)\tilde{s}(t, x, D_x) + \tilde{A}_3(t, x, D)\tilde{s}_1(t, x, D_x) + \tilde{R}(t, x, D_x)$$

with $\tilde{A}_1, \tilde{A}_2, \tilde{A}_3, \tilde{R} \in C([0,T], LG^0)$. Now by (3.9) and (3.10) we get

$$|\tilde{s}(t,x,\xi)| + |\tilde{s}_1(t,x,\xi)| \le \frac{|a'(t)|}{a(t) + \langle x\rangle_h^{-\sigma}\langle\xi\rangle^{-2}}.$$

Furthermore,

$$b(t,x,D_x)(\omega(t,x,D_x))^{-1} = S\circ \operatorname{op}\left(\frac{b(t,x,\xi)}{\sqrt{1+a(t)\langle x\rangle_h^{\sigma}\langle\xi\rangle^2}}\right) + r_5(t,x,D_x), \tag{3.14}$$

with $r_5 \in C([0,T], LG^0)$. Finally, by the Levi condition (1.12)

$$\begin{aligned}\frac{|b(t,x,\xi)|}{\sqrt{1+a(t)\langle x\rangle^{\sigma}\langle\xi\rangle^2}} &= \frac{|b(t,x,\xi)|}{\langle x\rangle^{\sigma/2}\langle\xi\rangle(a(t)+\langle x\rangle_h^{-\sigma}\langle\xi\rangle^{-2})^{\gamma}}\cdot\frac{1}{(a(t)+\langle x\rangle_h^{-\sigma}\langle\xi\rangle^{-2})^{1/k}}\\ &\le C\frac{1}{(a(t)+\langle x\rangle_h^{-\sigma}\langle\xi\rangle^{-2})^{1/k}}.\end{aligned}\tag{3.15}$$

Hence, L in (3.12) can be written in the form

$$L = \partial_t - i\begin{pmatrix} -\tilde{\lambda}(t,x,D_x) & 0\\ 0 & \tilde{\lambda}(t,x,D_x)\end{pmatrix} + A(t,x,D_x), \tag{3.16}$$

where $A(t,x,D_x)$ is a 2×2 matrix of pseudodifferential operators with symbol $A(t,x,\xi) \in C([0,T], SG^{(1,1)})$ thanks to the condition $\sigma \in [0,2]$. Moreover

$$|A(t,x,\xi)| \le \varphi(t,x,\xi), \tag{3.17}$$

for

$$\varphi(t,x,\xi) = C\left(\frac{|a'(t)|}{a(t)+\langle x\rangle_h^{-\sigma}\langle\xi\rangle^{-2}} + \frac{1}{(a(t)+\langle x\rangle_h^{-\sigma}\langle\xi\rangle^{-2})^{1/k}}\right), \tag{3.18}$$

C a positive constant.

In proving an energy estimate for the system (3.16), (3.17), the symbol $\int_0^T \varphi(t,x,\xi)\,dt$ will play an important role. We conclude this section by giving the following lemma.

Lemma 3.1. *Let us consider the function $\varphi(t,x,\xi)$ in* (3.18)*. Then, for every $\alpha, \beta \in \mathbb{Z}_+^n$ there exists a positive constant $\delta_{\alpha\beta}$ independent of h such that*

$$\int_0^T |\partial_\xi^\alpha \partial_x^\beta \varphi(t,x,\xi)|\,dt \le \delta_{\alpha\beta}\langle x\rangle_h^{-|\beta|}\langle\xi\rangle^{-|\alpha|}\log(\langle x\rangle_h^{\sigma}\langle\xi\rangle^2). \tag{3.19}$$

Proof. For $\alpha = \beta = 0$, by (1.9) we immediately obtain the inequalities

$$\begin{cases} \displaystyle\int_0^T \frac{|a'(t)|}{a(t) + \langle x\rangle_h^{-\sigma}\langle\xi\rangle^{-2}}\,dt \le c\log(\langle x\rangle_h^{\sigma}\langle\xi\rangle^2), \\ \displaystyle\int_0^T \frac{1}{(a(t) + \langle x\rangle_h^{-\sigma}\langle\xi\rangle^{-2})^{1/k}}\,dt \le c\log(\langle x\rangle_h^{\sigma}\langle\xi\rangle^2), \quad c > 0. \end{cases} \tag{3.20}$$

Indeed, $a(t)$ has finitely many isolated zeroes of order k, so we can reduce ourselves to prove (3.20) not on the whole interval $[0, T]$ but only in a neighborhood $[t_0-\rho, t_0+\rho]$ of a zero t_0 such that $a(t) \ge c_0|t - t_0|^k$ for $t \in [t_0 - \rho, t_0 + \rho]$. Then, estimate

$$\int_{t_0-\rho}^{t_0+\rho} \frac{|a'(t)|}{\alpha(t) + \langle x\rangle_h^{-\sigma}\langle\xi\rangle^{-2}}\,dt \le c\log(\langle x\rangle_h^{\sigma}\langle\xi\rangle^2)$$

immediately follows; on the other hand,

$$\begin{aligned} \int_{t_0-\rho}^{t_0+\rho} \frac{1}{(\alpha(t) + \langle x\rangle_h^{-\sigma}\langle\xi\rangle^{-2})^{1/k}}\,dt &\le \int_{|t-t_0|<\langle x\rangle_h^{-\sigma/k}\langle\xi\rangle^{-2/k}} \frac{1}{\langle x\rangle_h^{-\sigma/k}\langle\xi\rangle^{-2/k}}\,dt \\ &\quad + \int_{\langle x\rangle_h^{-\sigma/k}\langle\xi\rangle^{-2/k}\le|t-t_0|\le\rho} \frac{1}{c_0^{1/k}|t-t_0|}\,dt \\ &\le c\log(\langle x\rangle_h^{\sigma}\langle\xi\rangle^2). \end{aligned}$$

For $(\alpha, \beta) \ne (0, 0)$, we only need to notice that

$$\begin{aligned} \partial_x^\beta\partial_\xi^\alpha\varphi(t, x, \xi) &= q_1^{(\alpha,\beta)}(t, x, \xi)\frac{|a'(t)|}{a(t) + \langle x\rangle_h^{-\sigma}\langle\xi\rangle^{-2}} \\ &\quad + q_2^{(\alpha,\beta)}(t, x, \xi)\frac{1}{(a(t) + \langle x\rangle_h^{-\sigma}\langle\xi\rangle^{-2})^{1/k}}, \end{aligned} \tag{3.21}$$

$q_i^{(\alpha,\beta)} \in C([0, T]; SG^{(-|\alpha|,-|\beta|)})$, $i = 1, 2$, and we immediately obtain (3.19) by (3.20) and (3.21). □

4. Energy estimate

In this section we give an energy estimate for the system $LV = 0$, with L in (3.16) under the conditions (3.17), (3.18). We are going to prove the following:

Theorem 4.1. *Consider L in* (3.16) *under assumptions* (3.17), (3.18). *There exists a constant $\delta > 0$ such that for all $V \in C^1([0, T]; H_{s_1,s_2}) \cap C([0, T]; H_{s_1+1,s_2+1})$ it holds the energy estimate*:

$$\|V(t)\|^2_{s_1-2\delta,s_2-\sigma\delta} \le C\left(\|V(0)\|^2_{s_1,s_2} + \int_0^t \|LV(\tau)\|^2_{s_1,s_2}\,d\tau\right). \tag{4.1}$$

In the proof of Theorem 4.1 we are going to use the following:

Lemma 4.1. *Let* $\Lambda(x, \xi) \geq 0$ *be a symbol such that*

$$|\partial_\xi^\alpha \partial_x^\beta \Lambda(x, \xi)| \leq \delta_{\alpha\beta} \langle x \rangle_h^{-|\beta|} \langle \xi \rangle^{-|\alpha|} \log(\langle x \rangle_h^\sigma \langle \xi \rangle^2), \tag{4.2}$$

for $\sigma \in [0, 2]$ *and for all* $\alpha, \beta \in \mathbb{Z}_+^n$, *with positive constants* $\delta_{\alpha\beta}$ *independent of* h. *Then, there exists* $\delta_0 > 0$ *such that*

$$|\partial_\xi^\alpha \partial_x^\beta e^{\Lambda(x,\xi)}| \leq \delta_{\alpha\beta} \langle x \rangle_h^{\sigma\delta_0 - |\beta|} \langle \xi \rangle^{2\delta_0 - |\alpha|}. \tag{4.3}$$

Thus, the operator $e^{\Lambda(x, D_x)} = \mathrm{op}(e^{\Lambda(x,\xi)}) \in LG^{(2\delta_0, \sigma\delta_0)}$.

Proof. By a direct computation, applying Faà di Bruno formula we get for $\alpha, \beta \in \mathbb{Z}_+^n$:

$$|\partial_x^\beta \partial_\xi^\alpha e^{\Lambda(x,\xi)}| \leq \delta_{\alpha\beta} e^{\Lambda(x,\xi)} \langle x \rangle_h^{-|\beta|} \langle \xi \rangle^{-|\alpha|} \log^{|\alpha|+|\beta|}(\langle x \rangle_h^\sigma \langle \xi \rangle^2).$$

Furthermore, by (4.2) with $\alpha = \beta = 0$ we have

$$e^{\Lambda(x,\xi)} \leq \langle x \rangle_h^{\sigma\delta_{00}} \langle \xi \rangle^{2\delta_{00}},$$

while for every $\epsilon > 0$

$$\log^{|\alpha|+|\beta|}(\langle x \rangle_h^\sigma \langle \xi \rangle^2) \leq c_{\alpha\beta} \langle x \rangle_h^{\sigma\epsilon} \langle \xi \rangle^{2\epsilon}.$$

Thus, (4.3) holds with $\delta_0 = \delta_{00} + \epsilon$. □

Proof of Theorem 4.1. We define $w_0(t, x, D_x)$ the operator with symbol

$$w_0(t, x, \xi) = e^{\int_0^t \varphi(\tau,x,\xi)\, d\tau}. \tag{4.4}$$

By (3.19) and Lemma 4.1, $w_0(t, x, D_x) \in LG^{(2\delta_0, \sigma\delta_0)}$ for some $\delta_0 > 0$. Moreover, arguing as in Proposition 3.1 and using (4.3), it is easy to prove that for h large enough $\omega_0(t, x, D_x)$ is invertible with inverse given by

$$(w_0(t, x, D_x))^{-1} = \mathrm{op}\Big(e^{-\int_0^t \varphi(\tau,x,\xi)\, d\tau}\Big) \circ (I + S_{-\eta}(t, x, D_x)),$$

$S_{-\eta} \in C([0, T]; SG^{(-\eta,-\eta)})$, $\eta \in (0, 1)$ arbitrary. We also observe that w_0^{-1} is of order $(2\epsilon, \sigma\epsilon)$ for every $\epsilon > 0$, since

$$\mathrm{op}\Big(e^{-\int_0^t \varphi(\tau,x,\xi)\, d\tau}\Big) \in LG^{(2\epsilon,\sigma\epsilon)}, \quad \epsilon > 0.$$

We now define

$$L_{w_0} = w_0^{-1} L w_0. \tag{4.5}$$

The transformation (4.5) carries a loss in the energy estimate of $2\delta_1$ in the first Sobolev index and of $\sigma\delta_1$ in the second one, with $\delta_1 = \delta_0 + \epsilon$. Computation gives

$$L_{w_0} = \partial_t - i\begin{pmatrix} -\tilde{\lambda} & 0 \\ 0 & \tilde{\lambda} \end{pmatrix} + A + \varphi(t, x, D_x)I + B_1, \tag{4.6}$$

where $B_1(t, x, \xi)$ satisfies an estimate of the form

$$|\partial_\xi^\alpha \partial_x^\beta B_1(t, x, \xi)| \le b_{\alpha,\beta}(t)\langle x\rangle^{-|\beta|}\langle \xi\rangle^{-|\alpha|} \log(\langle x\rangle^\sigma \langle \xi\rangle^2), \tag{4.7}$$

for some $b_{\alpha,\beta} \in C([0, T], \mathbb{R}_+)$.

We can now apply the sharp Gårding inequality for SG symbols, (see [16], Theorem 18.6.14 for the metric $g = |dx|^2/\langle x\rangle^2 + |d\xi|^2/\langle \xi\rangle^2$ and with $h(x, \xi) = \langle x\rangle^{-1}\langle \xi\rangle^{-1}$) to the matrix of operators $A+\varphi(t, x, D_x)I$, since it has non negative eigenvalues by (3.17). Hence there exist

$$Q \in C([0, T]; LG^{(1,1)})$$

a positive operator, that is

$$\langle QU, U\rangle \ge 0, \quad U \in C([0, T]; H_{1,1}), \tag{4.8}$$

and B_2 satisfying an estimate of the form (4.7) such that

$$A + \varphi(t, x, D_x)I = Q + B_2.$$

Thus,

$$L_{w_0} = \partial_t - i\begin{pmatrix} -\tilde{\lambda} & 0 \\ 0 & \tilde{\lambda} \end{pmatrix} + Q + B, \tag{4.9}$$

where $B = B_1 + B_2$, and again from (4.7) there exists $b \in C([0, T], \mathbb{R}_+)$ such that

$$|\langle BU(t), U(t)\rangle| \le b(t)\langle \mathrm{op}(\log(\langle x\rangle^\sigma \langle \xi\rangle^2))U(t), U(t)\rangle. \tag{4.10}$$

Next, we introduce a second change of variable defining $w_1(t, x, D_x)$ the operator with symbol

$$w_1(t, x, \xi) = \langle x\rangle^{\int_0^t b(\tau)\,d\tau}\langle \xi\rangle^{\int_0^t b(\tau)\,d\tau}, \tag{4.11}$$

which is obviously invertible with inverse

$$(w_1(t, x, D_x))^{-1} = \langle D_x\rangle^{-\int_0^t b(\tau)\,d\tau}\langle x\rangle^{-\int_0^t b(\tau)\,d\tau}.$$

We define

$$L_{w_1} = w_1^{-1} L_{w_0} w_1; \tag{4.12}$$

this second transformation brings a loss of $(2\delta_2, \sigma\delta_2)$ in the Sobolev indices, where $\delta_2 = \int_0^T b(t)\,dt$ and b is the function in (4.10). We have

$$L_{w_1} = \partial_t - i\begin{pmatrix} -\tilde{\lambda} & 0 \\ 0 & \tilde{\lambda} \end{pmatrix} + Q + B + b(t)\,\mathrm{op}(\log(\langle x\rangle^\sigma \langle \xi\rangle^2))I + R, \tag{4.13}$$

with $R \in C([0,T]; LG^0)$. Notice that, by (4.10), $B + b(t)\mathrm{op}(\log(\langle x\rangle^\sigma \langle\xi\rangle^2))I$ is a positive operator in the sense of (4.8).

Now, for $s_1 = s_2 = 0$ we consider, for any $W \in C^1([0, T]; L^2) \cap C([0, T]; H_{1,1})$

$$\frac{d}{dt}\|W(t)\|^2_{L^2} = 2\Re(W'(t), W(t))_{L^2} \le \beta(t)\|W(t)\|_0^2 + C\|L_{w_1}W(t)\|_0^2,$$

thanks to (4.13), (4.8), with β a function in $C([0, T], \mathbb{R}_+)$. An application of Gronwall's inequality gives

$$\|W(t)\|_0^2 \le C\left(\|W(0)\|_0^2 + \int_0^t \|L_{w_1}W(\tau)\|_0^2\,d\tau\right).$$

The case $(s_1, s_2) \ne (0, 0)$ immediately follows, since

$$\langle x\rangle^{s_2}\langle D_x\rangle^{s_1} L_{w_1}\langle D_x\rangle^{-s_1}\langle x\rangle^{-s_2} = L_{w_1} + R, \quad R \in C([0, T]; LG^0).$$

We have so:

$$\|W(t)\|^2_{s_1,s_2} \le C\left(\|W(0)\|^2_{s_1,s_2} + \int_0^t \|L_{w_1}W(\tau)\|^2_{s_1,s_2}\,d\tau\right),$$

for any $W \in C^1([0, T]; H_{s_1,s_2}) \cap C([0, T]; H_{s_1+1,s_2+1})$. Finally, to recover (4.1) we only need to remind that

$$L_{w_1}W = 0 \Leftrightarrow LV = 0, \quad V = w_0 w_1 W,$$

and there exists a positive constant C such that for any $V \in C([0, T]; H_{s_1,s_2})$ it holds

$$\|V(t)\|_{s_1-2\delta, s_2-\sigma\delta} \le C\|W(t)\|_{s_1,s_2},$$

$\delta = \delta_1 + \delta_2$. Theorem 4.1 is completely proved. □

REMARK 1. The positive constant δ only depends on the $\delta_{\alpha\beta}$'s in (3.19); precisely, the Calderón–Vaillancourt's theorem gives for δ the estimate $\delta > \delta_{00} + \sup_{|\alpha|+|\beta|\le c_n}\delta_{\alpha\beta}$, with c_n a positive constant only depending on the space dimension n. Hence, by (1.2), Theorem 4.1 gives well posedness of the Cauchy problem for the system (3.16) in $\mathcal{S}(\mathbb{R}^n)$, $\mathcal{S}'(\mathbb{R}^n)$ with the loss of 2δ derivatives.

REMARK 2. We remark that the energy estimate (4.1) holds more generally for a system of the form (3.16) with $\tilde{\lambda}(t, x, \xi)$ real valued, $\tilde{\lambda}(t, x, \xi) \in L^1([0, T]; SG^{(1,1)})$ and $|A(t, x, \xi)| \leq \varphi(t, x, \xi)$ for a function φ satisfying (3.19). Hence, the regularity of $\tilde{\lambda}$ and A actually is not necessary to prove Theorem 4.1, while it is crucial to reduce the problem (1.8) to the form $LV = 0$.

REMARK 3. To derive the energy estimate (1.13) from (4.1), we need to go back to (3.7) and (3.13). In this way we obtain that (1.13) holds with $\delta_1 = 2\delta + 1$, $\delta_2 = \sigma\delta + 1$. Then in the case $\sigma = 0$, (1.13) has the form

$$\|u(t)\|^2_{s_1-2\delta-1,s_2-1} + \|\partial_t u(t)\|^2_{s_1-2\delta-2,s_2-2} \leq C(\|f\|^2_{s_1,s_2} + \|g\|^2_{s_1-1,s_2-1}).$$

On the other hand, for $\sigma = 0$ our approach to the proof of Theorem 1.1 reduces to the one adopted in [3], and condition (1.11) can be replaced by standard ellipticity as in (1.6). Moreover, all the operators involved in the changes of variables (3.7) and (3.13) are of order zero with respect to x. Then, taking any $f \in H_{s_1,s_2}(\mathbb{R}^n)$, $g \in H_{s_1-1,s_2}(\mathbb{R}^n)$ and repeating readily the argument of the proof, we obtain that $u(t, x)$ satisfies the energy estimate:

$$\|u(t)\|^2_{s_1-\delta_1,s_2} + \|\partial_t u(t)\|^2_{s_1-\delta_1-1,s_2} \leq C(\|f\|^2_{s_1,s_2} + \|g\|^2_{s_1-1,s_2}) \tag{4.14}$$

for every $t \in [0, T]$ and for any $s_1, s_2 \in \mathbb{R}$. In particular, for $s_2 = 0$ we recapture the result of [3] in standard Sobolev spaces. In general, besides $H^{\pm\infty}$ well posedness, we get also well posedness in $\mathcal{S}(\mathbb{R}^n)$, $\mathcal{S}'(\mathbb{R}^n)$.

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