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A REMARK ON THE EMBEDDING THEOREM ASSOCIATED TO COMPLEX CONNECTIONS OF MIXED TYPE

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Abstract

Let M be a compact complex manifold and let (L, H) be a holomorphic Hermitian line bundle over M such that the curvature form of h is nondegenerate and splits into the difference $\Theta_+ - \Theta_-$ of two semipositive forms Θ_+ and Θ_- whose null spaces define mutually transverse holomorphic foliations $\mathcal{F}_-$ and $\mathcal{F}_+$, respectively. Then L^m admits, for sufficiently large $m \in \mathbb{N}$, C^∞ sections whose ratio embeds M into $\mathbb{CP}^N$ holomorphically (resp. antiholomorphically) along $\mathcal{F}_+$ (resp. along $\mathcal{F}_-$).

Introduction

In the theory of complex manifolds, geometric structures defined by the subbundles of tangent bundles are basic in analyzing submanifolds and holomorphic maps. Foliations is one of such structures.

In [6], and embedding theorem was established for those manifolds equipped with two mutually transverse holomorphic foliations. Namely, it was proved that, given a compact complex manifold M with holomorphic foliations $\mathcal{F}_+$ and $\mathcal{F}_-$ such that the tangent bundle of M is the direct sum of those of $\mathcal{F}_+$ and $\mathcal{F}_-$, M is embeddable into $\mathbb{CP}^N$ by a C^∞ map in such a way that the map is holomorphic along $\mathcal{F}_+$ and anti-holomorphic along $\mathcal{F}_-$, if M admits a holomorphic line bundle L whose curvature form is everywhere nondegenerate and splits into the difference of two semipositive forms say Θ_+ and Θ_-, in such a way that Θ_+ (resp. Θ_-) is definite along $\mathcal{F}_+$ (resp. $\mathcal{F}_-$) and zero along $\mathcal{F}_-$ (resp. $\mathcal{F}_+$).

Example of such quadruples $(M, \mathcal{F}_+, \mathcal{F}_-, L)$ arose naturally in the study of bundle-valued $\bar\partial$-cohomology groups on complex tori (cf. [5]). For other examples, see Section 3 below.

In such a situation, it was proved that there exist $m \in \mathbb{N}$ and C^∞ sections $s_0, \dots, s_N$ of $K_+ \otimes \overline{K}_- \otimes L^m$ such that the ratio $(s_0 : \cdots : s_N)$ gives such and embedding, where $K_\pm$ denote the canonical bundles of $\mathcal{F}_\pm$ and $\overline{K}_-$ the complex conjugate of K_-.

In view of this result, a natural question is whether or not, under the same situation as above, one can embed M similarly by L^m. A significance of this question lies in the fact that the method of [6] consisting in applying an L^2 estimate for the twisted $\bar\partial$-operator does not work any more. The purpose of the present note is to overcome

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this difficulty by establishing an analogue of Serre's vanishing theorem for algebraic cohomology groups on projective algebraic varieties. For the statement of our vanishing theorem and the resulting embedding theorem, see Section 1. The proofs of these results will be given in Section 2.

1. Preliminaries and results

Let M be a compact complex manifold of dimension n and let TM denote the tangent bundle of M. We shall say that a foliation $\mathcal{F}$ on M is holomorphic if the tangent bundle $T\mathcal{F}$ of $\mathcal{F}$ is a holomorphic subbundle of TM. We shall say that M is holomorphically woven if M is endowed with two holomorphic foliations say $\mathcal{F}_+$ and $\mathcal{F}_-$ whose tangent bundles $T\mathcal{F}_\pm \subset TM$ satisfy $T\mathcal{F}_+ + T\mathcal{F}_- = TM$ and $T\mathcal{F}_+ \cap T\mathcal{F}_- = \{0\} \times M$.

Typical examples of holomorphically woven manifolds are naturally induced from orthogonal decompositions of the tangent bundles of complex tori, with respect to constant nondegenerate indefinite (1, 1)-forms (cf. [5]).

Let $L \to M$ be a holomorphic line bundle equipped with a C^∞ fiber metric h. Let $T_+^{0,1}$ (resp. $T_-^{1,0}$) denote the antiholomorphic (resp. holomorphic) tangent bundle of $\mathcal{F}_+$ (resp. $\mathcal{F}_-$), and let $F^{p,q}(L)$ $(= F^{p,q}(M, L))$ be the set of C^∞ sections over M of the bundle $L \otimes \bigwedge^p (T_+^{0,1})^* \otimes \bigwedge^q (T_-^{1,0})^*$, where $(\,\cdot\,)^*$ denote the duals.

Then we define the differential operators

$$\overline{\partial}_+ \colon F^{p,q}(L) \to F^{p+1,q}(L)$$

and

$$\partial_{-,h} \colon F^{p,q}(L) \to F^{p,q+1}(L)$$

as the leafwise complex exterior differentiations for the L-valued forms, of type (0, 1) (resp. (1, 0)) with respect to $\mathcal{F}_+$ (resp. $\mathcal{F}_-$ and h).

We put $\check{\partial} = \overline{\partial}_+ + \partial_{-,h}$. Then $\check{\partial}$ maps the space $\bigoplus_{p,q} F^{p,q}(L)$ to itself. We shall say that (L, h) is $\mathcal{F}_\pm$-integrable if $\check{\partial}^2 = 0$.

Clearly $\check{\partial}^2 = 0$ holds if and only if the curvature form of h splits into the sum of a section of $\overline{(T_+^{0,1})^*} \otimes (T_+^{0,1})^*$ say Θ_+ and that of $(T_-^{1,0})^* \otimes \overline{(T_-^{1,0})^*}$ say Θ_-. Here $\overline{(T_+^{0,1})^*} \otimes (T_+^{0,1})^*$ and $(T_-^{1,0})^* \otimes \overline{(T_-^{1,0})^*}$ are naturally identified with subbundles of $\bigwedge^2 (TM \otimes \mathbb{C})^*$ and $\overline{(\,\cdot\,)}$ denote the complex conjugates.

Theorem 1.1. *Let $(M, \mathcal{F}_\pm)$ be a compact holomorphically woven manifold and let (L, h) be a Hermitian holomorphic line bundle over M which is $\mathcal{F}_\pm$-integrable. Suppose that the curvature form $\Theta_+ + \Theta_-$ of h, where $\Theta_\pm$ are as above, satisfies $\Theta_+|\mathcal{F}_+ > 0$ and $\Theta_-|\mathcal{F}_- < 0$. Then there exists $m_0 \in \mathbb{N}$ such that one can find, for any integer $m \geq m_0$, C^∞ sections $s_0, \ldots, s_N$ of L^m satisfying $\check{\partial} s_k = 0$ $(0 \leq k \leq N)$*

such that the ratio $(s_0 : \cdots : s_N)$ *embeds* M *into* $\mathbb{CP}^N$. *Here* L^m *is equipped with the fiber metric* h^m.

The proof of this assertion is based on the following vanishing theorem with L^2 norm estimates.

Theorem 1.2. *Let* $(M, \mathcal{F}_\pm)$ *and* (L, h) *be as in* Theorem 1.1, *and let* g *be any* C^∞ *Hermitian metric on* M. *Then there exist* $m_1 \in \mathbb{N}$ *and* $C > 0$ *such thar, for any integer* $m \geq m_1$ *and for any* $v \in \operatorname{Ker} \check{\partial} \cap \bigoplus_{p+q=r} F^{p,q}(L^m)$ *with* $r \geq 1$, *one can find* $u \in \bigoplus_{p+q=r-1} F^{p,q}(L^m)$ *satisfying* $\check{\partial} u = v$ *and* $m\|u\|^2 \leq C\|v\|^2$. *Here* $\|\cdot\|$ *denotes the* L^2 *norm with respect to* h^m *and* g.

2. Solving the $\check{\partial}$-equation

Admitting the validity of Theorem 1.2, the proof of Theorem 1.1 is carried over similarly as in the case of $K_+ \otimes \overline{K}_- \otimes L^m$ in [6, §3. Proof of Theorem 0.1], so that we shall dispense with this part.

Proof of Theorem 1.2. Let $v \in \bigoplus_{p+q=r} F^{p,q}(L^m) \cap \operatorname{Ker} \check{\partial}$ $(r \geq 1)$, where m will be specified later.

Let $U \subset M$ be a local coordinate neighbourhood with a holomorphic coordinate $(z_+, z_-)\colon U \to \mathbb{D}^n$, where $\mathbb{D} = \{\zeta \in \mathbb{C} \mid |\zeta| < 1\}$, such that $T_+^{0,1}|U = \operatorname{Ker} d\overline{z}_-$ and $T_-^{1,0}|U = \operatorname{Ker} dz_+$ hold. We choose $z_\pm$ in such a way that they are holomorphic on a neighbourhood of the closure of U in M. We put $z_+ = (z_1, \ldots, z_s)$, $z_- = (z_{s+1}, \ldots, z_n)$ and $v|U = \sum_{I,J} v_{IJ}\, d\overline{z}_I \otimes dz_J$. We note that $|I| + |J| = r$ and the components of multi-indices I (resp. J) are contained in $\{1, \ldots, s\}$ (resp. $\{s+1, \ldots, n\}$).

Since $\check{\partial} v = 0$, we have

$$(\overline{\partial}_+ + \partial_{-,h}) \sum_{I,J} v_{IJ}(dz_1 \wedge \cdots \wedge dz_s \wedge d\overline{z}_I) \otimes (d\overline{z}_{s+1} \wedge \cdots \wedge d\overline{z}_n \wedge d\overline{z}_J) = 0.$$

Here $\overline{\partial}_+$ and $\partial_{-,h}$ are defined leafwise with respect to $\mathcal{F}_\pm$ as before.

On the other hand, with respect to this extension of the mixed complex exterior derivative $\check{\partial} = \overline{\partial}_+ + \partial_{-,h}$, the leafwise application on Nakano's identity yields similarly as in [6, §2] an estimate

$$m\|\phi\|_*^2 \leq \|\check{\partial}\phi\|_*^2 + \|\check{\partial}^*\phi\|_*^2 \tag{2.1}$$

for compactly supported C^∞ L^m-valued differential forms ϕ on U of the form

$$\phi = \sum_{|I|+|J|=r} \phi_{IJ}(dz_1 \wedge \cdots \wedge dz_s \otimes d\overline{z}_I) \otimes (d\overline{z}_{s+1} \wedge \cdots \wedge d\overline{z}_n \wedge dz_J) \quad (r \geq 1).$$

Here $\check{\partial}^*$ denotes the adjoint of $\check{\partial}$ and the L^2 norms $\|\cdot\|_*$ are defined with respect to h^m and $\Theta_+ - \Theta_-$. We have used the assumptions that $\Theta_+|\mathcal{F}_+ > 0$ and $\Theta_-|\mathcal{F}_-$ to identify $\Theta_+ - \Theta_-$ with a Hermitian metric on U and exploited the fact that the canonical bundles of $\mathcal{F}_\pm$ are trivial on U. (For the computation, see (2.2)–(2.10) in [6].)

Since U is equivalent to the product of the polydisc $\mathbb{D}^s$ in the z_+-coordinate and the polydisc $\mathbb{D}^{n-s}$ in the z_--coordinate, by a standard argument as in [4] and [1] we obtain from (2.1), that there exists a smooth, L^m-valued form u on U satisfying $\check{\partial} u = v|U$ and $m\|u\|_*^2 \le C_0\|v|U\|_*^2$, where

$$C_0 = \frac{\sup_U |dz_1 \wedge \cdots \wedge dz_s|^2 |dz_{s+1} \wedge \cdots \wedge dz_n|^2}{\inf_U |dz_1 \wedge \cdots \wedge dz_s|^2 |dz_{s+1} \wedge \cdots \wedge dz_n|^2}.$$

Here the length of the forms is measured with respect to $\Theta_+ - \Theta_-$.

Since M is compact, there exist finitely many such coordinate neighbourhoods, say U_i $(i = 1, 2, \ldots, k)$ such that $M = \bigcup_{i=1}^k u_i$. Let $\{\chi_i\}_{i=1}^k$ be a C^∞ partition of unity subordinate to $\{U_i\}$. Since the given metric g is equivalent to $\Theta_+ - \Theta_-$, there exists a constant $C_1 > 0$ such that

$$C_1^{-1}\|\phi\|_*^2 \le \|\phi\|^2 \le C_1\|\phi\|_*^2$$

hold for all $\phi \in F^{p,q}(L^m)$, where the norm $\|\cdot\|$ is with respect to h^m and g.

Hence there exists a constant $C_2 > 0$ such that, for any i and $m \in \mathbb{N}$, one can find $u_i \in \bigoplus_{p+q=r-1} F^{p,q}(U_i, L^m|U_i)$ satisfying

$$\check{\partial} u_i = v|U_i \tag{2.2}$$

and

$$m\|u_i\|^2 \le C_2\|v\|^2. \tag{2.3}$$

We put

$$u_1 = \sum_{i=1}^k \chi_i u_i,$$

where $\chi_i u_i$ are defined to be zero outside U_i. Then, since $\sum \chi_i \check{\partial} u_i = \sum \chi_i v|U_i = v$, we have

$$\check{\partial} u_1 = v + \sum_{i=1}^k \check{\partial}\chi_i \wedge u_i,$$

where the wedge product of $\check{\partial}\chi_i$ and u_i is defined in the obvious manner.

In view of (2.3) we have

$$
\begin{aligned}
&\|v - \check{\partial} u_1\| \\
&\leq \left\| \sum_{i=1}^{k} \check{\partial}\chi_i \wedge u_i \right\| \\
&\leq k \sup_i \left(\sup_{U_i} |\check{\partial}\chi_i| \cdot \|u_i\| \right) \\
&\leq k \sqrt{\frac{C_2}{m}} \left(\sup_i \sup_{U_i} |\check{\partial}\chi_i| \right) \|v\|.
\end{aligned}
\tag{2.4}
$$

Hence one can find $m_1 \in \mathbb{N}$ and $C < 4m_1$ such that if $m \geq m_1$,

$$4m\|u_1\|^2 \leq C\|v\|^2 \tag{2.5}$$

and

$$\|v - \check{\partial} u_1\| \leq \frac{1}{2}\|v\| \tag{2.6}$$

hold for v.

We put $v_0 = v$, $v_1 = v - \check{\partial} u_1$, and define $u_\mu \in \bigoplus_{p+q=r-1} F^{p,q}(L^m)$ and $v_\mu \in \bigoplus_{p+q=r} F^{p,q}(L^m)$ inductively for a fixed $m \geq m_1$ and for $\mu = 2, 3, \ldots$ by requiring

$$4m\|u_\mu\|^2 \leq C\|v_{\mu-1}\|^2, \tag{2.7}$$

$$\|v_{\mu-1} - \check{\partial} u_\mu\| \leq \frac{1}{2}\|v_{\mu-1}\| \tag{2.8}$$

and

$$v_\mu = v_{\mu-1} - \check{\partial} u_\mu. \tag{2.9}$$

Then, by letting $u_\infty = \sum_{\mu=1}^{\infty} u_\mu$ we have $\check{\partial} u_\infty = v$ and $m\|u_\infty\|^2 \leq C\|v\|^2$. Replacing u_∞ by the solution of $\check{\partial} u = v$ satisfying $u \perp \operatorname{Ker} \check{\partial}$, we obtain a required smooth solution for $\check{\partial} u = v$. □

REMARK. Similarly as above, given any holomorphic vector bundle $E \to M$, one can solve the $\check{\partial}$-equations for the $E \otimes L^m$-valued forms for sufficiently large m. A more refined successive approximation has already been used in [7] to prove a similar embedding theorem for "singly" complex-foliated manifolds.

3. Application

Theorem 1.1 implies the existence of nonzero sections of sufficiently high tensor powers of the product of a semipositive line bundle and a seminegative line bundle which are holomorphic (resp. antiholomorphic) along the positively curved (resp. negatively curved) directions, on certain fiber spaces. Such examples are described below.

Let X_j $(j = 1, 2)$ be two compact complex manifolds equipped with positive line bundles $L_j \to X_j$. Let $\tilde{X}_1 \xrightarrow{\pi} X_1$ be the universal covering of X_1, let ρ be a homomorphism from the group of covering transformations of π, say Γ, to the group of biholomorphic automorphisms of X_2, and let M be the quotient space of the product $\tilde{X}_1 \times X_2$ by the equivalence relation

$$(x, y) \sim (\gamma_x, \rho(\gamma)y) \quad (x \in \tilde{X}_1,\ y \in X_2 \text{ and } \gamma \in \Gamma).$$

Then M is a flat fiber bundle over X_1 with fiber X_2, and carries two mutually transverse holomorphic foliations $\mathcal{F}_1$ and $\mathcal{F}_2$ such that the leaves of $\mathcal{F}_1$ are the images of $\tilde{X}_1 \times \{y\}$ $(y \in X_2)$ in M and those of $\mathcal{F}_2$ are the fibers of the bundle $f \colon M \to X_1$.

If L_2 is invatiant under the action of $\rho(\Gamma)$, which is always the case is X_2 is simply connected, then L_2 naturally induces a semipositive line bundle say $\hat{L}_2$ over M.

It is obvious that the foliations $\mathcal{F}_1$ and $\mathcal{F}_2$ ans the line bundles $f^*L_1 \otimes \hat{L}_2^{-1}$ and $f^*L_1^{-1} \otimes \hat{L}_2$ both satisfy the assumptions of Theorem 1.1, up to the roles of $\mathcal{F}_1$ and $\mathcal{F}_2$.

It might be an intringuing question to describe a relation between the two embeddings with respect to L^m and L^{-m} in the situation of Theorem 1.1.

REMARK. Classification results are known for holomorphic foliations on some complex manifolds. For the case of ruled surfaces, see [3]. Holomorphic foliations of codimention one on complex tori are classified by Ghys [2].

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