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AN INTEGRAL INVARIANT FROM THE KNOT GROUP

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Abstract

For a knot K in S^3, J. Ma and R. Qiu defined an integral invariant $a(K)$ which is the minimal number of elements that generate normally the commutator subgroup of the knot group, and showed that it is a lower bound of the unknotting number. We prove that it is also a lower bound of the tunnel number. If the invariant were additive under connected sum, then we could deduce something about additivity of both the unknotting numbers and the tunnel numbers. However, we found a sequence of examples that the invariant is not additive under connected sum. Let $T(2, p)$ be the $(2, p)$-torus knot, and $K_{p,q} = T(2, p) \sharp T(2, q)$. Then we have $a(K_{p,q}) = 1$ if and only if $\gcd(p, q) = 1$.

1. Introduction

For a knot K in S^3, many integral invariants were defined and studied. Let $G(K)$ be the knot group of K, and $G'(K)$ its commutator subgroup. J. Ma and R. Qiu [10] defined an invariant $a(K)$ as the minimal number of elements that generate $G'(K)$ *normally* in $G(K)$. We call it the *MQ* (*Ma–Qiu*) *index* of K. In this paper, we mainly deal with it.

H. Schubert [20] showed that both the Seifert genus and the bridge index of knots are additive under connected sum. Y. Nakanishi [14] studied the unknotting number of knots. As a lower bound for it, he defined an invariant which is called now the *Nakanishi index* (see [6, 8, 14] or Section 2). M. Scharlemann [18] showed that unknotting number one knots are prime. As a corollary of the theorem, "$1 + 1$" should be two for the unknotting numbers of knots. However the additivity problem in the general case is still open. T. Kobayashi [9] and K. Morimoto [12] constructed examples such that the tunnel number is not additive under conncted sum. M. Scharlemann and J. Schultens [19] showed that the tunnel number of the sum of n non-trivial knots is at least n. J. Ma and R. Qiu [10] defined an integral invariant of a knot. Roughly speaking, it is an extended version of the Nakanishi index to the commutator subgroup of the knot group. They showed that it is a lower bound of the unknotting number by using a modified Wirtinger presentation (see Theorem 2.2).

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Let $r(K)$ be the rank of the knot group $G(K)$ of a knot K. Then the following is our first main theorem.

Theorem 1.1. $a(K) \leq r(K) - 1$.

Let K be a knot, $t(K)$ the tunnel number of K, $u(K)$ the unknotting number of K, and $m(K)$ the Nakanishi index of K (see Section 2). If K is a 2-bridge knot, then $t(K) = 1$. Since T. Kanenobu and H. Murakami [5] characterized 2-bridge knots with unknotting number one, there is an example of a knot K' with $u(K') \geq 2$ and $t(K') = 1$. On the other hand, for prime knots up to ten crossings, K. Morimoto, M. Sakuma and Y. Yokota [13] determined the tunnel numbers, and P. Ozsváth and Z. Szabó [16] determined the unknotting number one knots except 10_{153} (We also remark that C. Kearton and M.J. Wilson [8] determined the Nakanishi indices for the class). Hence we can find an example of a knot K'' with $u(K'') = 1$ and $t(K'') = 2$. The examples above would imply that there are no relations between the unknotting number and the tunnel number. However, by combining Theorem 1.1 and known results, we have the following as a corollary.

Corollary 1.2. $m(K) \leq a(K) \leq \min\{r(K) - 1, u(K)\} \leq \min\{t(K), u(K)\}$.

If the MQ index were additive under connected sum, then we could deduce something about the additivity problem of both the unknotting number and the tunnel number. For example, we could reprove both Scharlemann's result [18], and Scharlemann and Schultens' result [19] purely algebraically. However we found a sequence that the MQ index is not additive under connected sum. Let $T(2, p)$ be the $(2, p)$-torus knot for an odd number p ($|p| \geq 3$), and $K_{p,q} = T(2, p) \sharp T(2, q)$ ($3 \leq p \leq |q|$). Then the following is our second theorem.

Theorem 1.3. *The following three statements are equivalent*:
(1) $\gcd(p, q) = 1$.
(2) $m(K_{p,q}) = 1$.
(3) $a(K_{p,q}) = 1$.

In Section 3, we prove the equivalence of (1) and (2) in Theorem 1.3 by the commutative ring theory. In Section 4, we prove the equivalence of (1) and (3) in Theorem 1.3 by Reidemeister–Schreier's method from the combinatorial group theory [11]. It may be a fundamental method for studying the MQ index. In Section 5, we raise some questions.

2. Proof of Theorem 1.1

Proof of Theorem 1.1. Let $\pi\colon G(K) \to \mathbb{Z}$ be the abelianization map. We regard $\mathbb{Z}$ as an additive group (i.e. 1 is a generator of $\mathbb{Z}$).

Let $\{x_i\ (i = 1, \dots, r)\}$ be the set of generators of $G(K)$ realizing $r = r(K)$, and y_i an integer satisfying $\pi(x_i) = y_i$. Since $y_1, \dots, y_r$ generate $\mathbb{Z}$, we have $\gcd(y_1, \dots, y_r) = 1$.

Claim. *We can take $x_1, \dots, x_r$ satisfying $y_1 = 1$ and $y_2 = \cdots = y_r = 0$.*

Proof. We may assume $y_1 \geq y_2 \geq \cdots \geq y_r \geq 0$ without loss of generality.

Suppose $N = \sum_{i=1}^{r} y_i$ is minimal. We show $N = 1$ which is equivalent to our claim. Note that $N \geq 1$ because $G(K)/G'(K)$ is non-trivial.

Suppose $N > 1$. Then $y_2 > 0$ by $\gcd(y_1, \dots, y_r) = 1$. Set $x_1' = x_1 x_2^{-1}$ and $\pi(x_1') = y_1'$. Then $x_1', x_2, \dots, x_r$ also generate $G(K)$, $y_1' = y_1 - y_2 \geq 0$, and

$$0 \leq y_1' + \sum_{i=2}^{r} y_i < N.$$

This contradicts that N is minimal. Hence we have $N = 1$. □

Suppose $x_1, \dots, x_r$ satisfy the condition stated in the claim. Then we show that $x_2, \dots, x_r$ generate $G'(K)$ normally.

It is easy to see that $x_2, \dots, x_r \in G'(K)$. Take any element $z \in G'(K)$. Then z is expressed as a word of $x_1, \dots, x_r$. Since $\pi(z) = 0$, the sum of powers of x_1 in the word of z is zero. This implies that z is a product of conjugations of $x_2, \dots, x_r$ conjugated by some power of x_1.

Therefore we have $a(K) \leq r(K) - 1$. □

Since the knot group of a torus knot is generated by two elements, we have the following corollary.

Corollary 2.1. *Let K be a torus knot in S^3. Then $a(K) = 1$.*

Let $E(K)$ be the exterior of a knot K. Then we have $H := H_1(E(K); \mathbb{Z}) \cong \mathbb{Z}$ and H is generated by an element represented by a meridian of K. Let $\tilde{E}(K)$ be the universal abelian covering of $E(K)$, and $\Lambda = \mathbb{Z}[t, t^{-1}]$ the one-variable Laurent polynomial ring over $\mathbb{Z}$ with a variable t where we take t as a generator of H. Then $H' := H_1(\tilde{E}(K); \mathbb{Z})$ has a Λ-module structure, and it is called the *Alexander module*. The *Nakanishi index* of K, denoted by $m(K)$, is the minimal number of generators for H' as a Λ-module.

Ma and Qiu [10] showed the following.

Lemma 2.2. *Let K be a knot in S^3. Then we have $m(K) \leq a(K) \leq u(K)$.*

The following is well-known.

Lemma 2.3. *Let K be a knot in S^3. Then we have $r(K) - 1 \leq t(K)$.*

By combining Theorem 1.1, Lemma 2.2 and 2.3, we have Corollary 1.2.

3. Equivalence of (1) and (2) in Theorem 1.3

Let K be a knot, and $\Delta_K(t)$ the Alexander polynomial of K.

Lemma 3.1. *For a knot K, the Nakanishi index $m(K)$ is one if and only if $\Delta_K(t)$ is non-trivial and the Alexander module of K is isomorphic to $\Lambda/(\Delta_K(t))$ where $(\Delta_K(t))$ is an ideal of Λ generated by $\Delta_K(t)$.*

Let $\Delta_p(t)$ ($p \geq 3$ is odd) be the Alexander polynomial of the $(2, p)$-torus knot $T(2, p)$. Then we have

$$\Delta_p(t) \doteq \frac{(t^{2p}-1)(t-1)}{(t^p-1)(t^2-1)} = \frac{t^p+1}{t+1} = t^{p-1} - t^{p-2} + \cdots - t + 1. \tag{3.1}$$

Proof of equivalence of (1) and (2) in Theorem 1.3. Since the Alexander matrix of $K_{p,q}$ is

$$\begin{pmatrix} \Delta_p(t) & 0 \\ 0 & \Delta_q(t) \end{pmatrix}$$

which is a presentation matrix of the Alexander module,

$$\Lambda/(\Delta_p(t)) \oplus \Lambda/(\Delta_q(t)) \tag{3.2}$$

over Λ, it is easy to see that $m(K_{p,q}) = 1$ or 2. We may assume $q > 0$ because $m(K_{p,q}) = m(K_{p,-q})$.

Suppose $\gcd(p, q) = d \geq 2$. Then $\Delta_d(t)$ is a common divisor of $\Delta_p(t)$ and $\Delta_q(t)$. By the elementary divisor theory, the Alexander matrix does not reduce to a matrix $(\Delta_p(t)\Delta_q(t))$ by elementary moves over Λ, and we have $m(K_{p,q}) = 2$.

Suppose $\gcd(p, q) = 1$. Let a continued fraction expansion of q/p be

$$\frac{q}{p} = r_1 + \cfrac{1}{r_2 + \cfrac{1}{\ddots + \cfrac{1}{r_k}}}$$

where r_i $(i = 1, \ldots, k)$ is a positive integer. Then we have a set of integers $\{q_0, q_1, \ldots, q_k, q_{k+1}\}$ such that $q_{i-1} = q_i r_i + q_{i+1}$ for $i = 1, \ldots, k$ where $0 \leq q_{i+1} < q_i$, $q_0 = q$, $q_1 = p$, $q_k = 1$ and $q_{k+1} = 0$. We set

$$f_s(t) = t^{s-1} - t^{s-2} + \cdots + (-1)^s t + (-1)^{s+1}$$

for an integer $s \geq 0$ where $f_1(t) \equiv 1$ and $f_0(t) \equiv 0$. There are elements of Λ, $\{g_1(t), g_2(t), \dots, g_k(t)\}$, such that

$$f_{q_{i-1}}(t) = f_{q_i}(t)g_i(t) + (-1)^{r_i} f_{q_{i+1}}(t) \quad (i = 1, \dots, k).$$

We note that $f_{q_0}(t) = \Delta_q(t)$ and $f_{q_1}(t) = \Delta_p(t)$. Then we can find two elements $u(t), v(t) \in \Lambda$ such that $u(t)\Delta_p(t) + v(t)\Delta_q(t) = 1$. Thus $(\Delta_p(t), \Delta_q(t)) = \Lambda$. By the Chinese Remainder Theorem, the Alexander module in (3.2) is isomorphic to $\Lambda/(\Delta_p(t)\Delta_q(t))$. Hence $m(K_{p,q}) = 1$ by Lemma 3.1. □

4. Equivalence of (1) and (3) in Theorem 1.3

Let K_i $(i = 1, 2)$ be a knot, and $G_i := G(K_i)$ the knot group of K_i. We take an element of G_1 (resp. G_2) represented by a meridian of K_1 (resp. K_2), and we denote it by μ_1 (resp. μ_2). We set $K = K_1 \sharp K_2$ and $G = G(K)$. Then we have the following.

Lemma 4.1. (1) $G \cong G_1 * G_2/\langle\langle \mu_1\mu_2^{-1}\rangle\rangle$, *where* $\langle\langle \mu_1\mu_2^{-1}\rangle\rangle$ *is a normal subgroup of* $G_1 * G_2$ *generated by* $\mu_1\mu_2^{-1}$.
(2) $G' \cong G_1' * G_2'$.

Lemma 4.1 can be proved by using van Kampen's theorem (see Z. Yang [21] for the proof of (2)).

Since there are natural isomorphisms for both cases in Lemma 4.1, we denote them by

$$G = G_1 * G_2/\langle\langle \mu_1\mu_2^{-1}\rangle\rangle$$

and

$$G' = G_1' * G_2'.$$

By Lemma 4.1, we have the following.

Lemma 4.2 ([10]). *Let* K_1 *and* K_2 *be knots in* S^3. *Then we have*

$$\max\{a(K_1), a(K_2)\} \leq a(K_1 \sharp K_2) \leq a(K_1) + a(K_2).$$

Let $T(\alpha, \beta)$ be the (α, β)-torus knot where we assume $2 \leq \alpha < |\beta|$ and $\gcd(\alpha, \beta) = 1$. Then its knot group is

$$G(T(\alpha, \beta)) = \langle x, y \mid x^\alpha y^\beta \rangle,$$

and $t = x^\gamma y^\delta$ is represented by a meridian where $\alpha\delta - \beta\gamma = 1$. We can find corresponding loops in the exterior of $T(\alpha, \beta)$ to x and y respectively.

Suppose $\alpha = 2$ and $\beta = 2k + 1$ ($k \geq 1$). Then $\gamma = 1$ and $\delta = k + 1$. We present $G'(T(2, 2k + 1))$ ($k \geq 1$) by Reidemeister–Schreier's method. We set $t = xy^{k+1}$, $a_i = t^{-i}(xt^{2k+1})t^i$ and $b_i = t^{-i}(yt^{-2})t^i$ ($i \in \mathbb{Z}$). Then we rewrite x^2y^{2k+1} and $xy^{k+1}t^{-1}$ by a_i and b_i.

$$\begin{aligned} x^2y^{2k+1} &= (xt^{2k+1})\{t^{-(2k+1)}(xt^{2k+1})t^{2k+1}\}\{t^{-(4k+2)}(yt^{-2})t^{4k+2}\} \\ &\quad \times \{t^{-4k}(yt^{-2})t^{4k}\}\cdots\{t^{-4}(yt^{-2})t^4\}\{t^{-2}(yt^{-2})t^2\} \\ &= a_0a_{2k+1}b_{4k+2}b_{4k}\cdots b_4b_2, \end{aligned} \tag{4.1}$$

$$\begin{aligned} xy^{k+1}t^{-1} &= (xt^{2k+1})\{t^{-(2k+1)}(yt^{-2})t^{2k+1}\} \\ &\quad \times \{t^{-(2k-1)}(yt^{-2})t^{2k-1}\}\cdots\{t^{-1}(yt^{-2})t\} \\ &= a_0b_{2k+1}b_{2k-1}\cdots b_1. \end{aligned} \tag{4.2}$$

Since $y = y^{-(2k+1)+2(k+1)} = x^2(x^{-1}t)^2$, we have

$$b_0 = xtx^{-1}t^{-1} = (xt^{2k+1})(t^{-2k}x^{-1}t^{-1}) = a_0a_{-1}^{-1}. \tag{4.3}$$

By (4.3), we have $b_i = a_ia_{i-1}^{-1}$, and substitute it to (4.1) and (4.2). Then we have

$$x^2y^{2k+1} = a_0a_{2k+1}a_{4k+2}a_{4k+1}^{-1}a_{4k}a_{4k-1}^{-1}\cdots a_2a_1^{-1}, \tag{4.4}$$

$$xy^{k+1}t^{-1} = a_0a_{2k+1}a_{2k}^{-1}a_{2k-1}a_{2k-2}^{-1}\cdots a_1a_0^{-1}. \tag{4.5}$$

Hence $G'(T(2, 2k + 1))$ is generated by $\{a_i\ (i \in \mathbb{Z})\}$. We fix it as a set of generators. Then its relations are generated by conjugations of the righthand sides of both (4.4) and (4.5). We set

$$c_0 = a_{2k}a_{2k-1}^{-1}a_{2k-2}a_{2k-3}^{-1}\cdots a_1^{-1}a_0$$

and $c_j = t^{-j}c_0t^j$ ($j \in \mathbb{Z}$). Then the righthand side of (4.4) is

$$a_0\{(a_{2k+1}c_{2k+2}a_{2k+1}^{-1})c_0\}a_0^{-1},$$

and the righthand side of (4.5) is $a_0c_1a_0^{-1}$. Since (4.5) is conjugate to c_1 as a word of $\{a_i\ (i \in \mathbb{Z})\}$, every c_j ($j \in \mathbb{Z}$) is trivial in $G'(T(2, 2k+1))$. Since (4.4) can be generated by conjugations of c_j ($j \in \mathbb{Z}$), $G'(T(2, 2k + 1))$ is presented as

$$G'(T(2, 2k + 1)) = \langle a_i\ (i \in \mathbb{Z}) \mid c_j\ (j \in \mathbb{Z})\rangle. \tag{4.6}$$

We also denote (4.6) by

$$G'(T(2, 2k + 1)) = \langle\langle a_0 \mid (c_0 =)\ a_{2k}a_{2k-1}^{-1}\cdots a_1^{-1}a_0\rangle\rangle. \tag{4.7}$$

By the similar way, we have

$$G'(T(2, -(2k + 1))) = \langle\langle a_0 \mid a_{2k}a_{2k-1}^{-1}\cdots a_1^{-1}a_0\rangle\rangle \tag{4.8}$$

where $t = x^{-1}y^{k+1}$ and $a_i = t^{-i}(x^{-1}t^{2k+1})t^i$.

Proof of equivalence of (1) and (3) in Theorem 1.3. If $\gcd(p, q) \geq 2$, then we have $a(K_{p,q}) = 2$ by Corollary 1.2 and Section 3. Hence we suppose $\gcd(p, q) = 1$.

By Lemma 4.1 (2), (4.7) and (4.8), we have

$$G'(K_{p,q}) = \langle\langle a_0, z_0 \mid a_{p-1}a_{p-2}^{-1}\cdots a_1^{-1}a_0,\ z_{q-1}z_{q-2}^{-1}\cdots z_1^{-1}z_0\rangle\rangle \tag{4.9}$$

and we may assume $q > 0$.

We set $w_i = a_i z_i^{-1}$ $(i \in \mathbb{Z})$, and denote a subgroup of G generated by $\{w_i\ (i \in \mathbb{Z})\}$ normally by $\langle\langle w\rangle\rangle$ where $w = w_0$. Since $a_i \equiv z_i$ (mod $\langle\langle w\rangle\rangle$), we have

$$a_{p-1}a_{p-2}^{-1}\cdots a_1^{-1}a_0 \equiv z_{p-1}z_{p-2}^{-1}\cdots z_1^{-1}z_0 \equiv 1 \pmod{\langle\langle w\rangle\rangle}. \tag{4.10}$$

Let a continued fraction expansion of q/p be

$$\frac{q}{p} = r_1 + \cfrac{1}{r_2 + \cfrac{1}{\ddots + \cfrac{1}{r_k}}}$$

where r_i $(i = 1, \dots, k-1)$ is an even integer. Then we have a set of integers $\{q_0, q_1, \dots, q_k, q_{k+1}\}$ such that $q_{i-1} = q_i r_i + q_{i+1}$ for $i = 1, \dots, k$ where $0 \leq |q_{i+1}| < |q_i|$, $q_0 = q$, $q_1 = p$, $q_k = \pm 1$ and $q_{k+1} = 0$. Note that q_i $(i = 0, \dots, k)$ is an odd number. We set

$$\begin{aligned}
\sigma_i &= z_{|q_i|-1}z_{|q_i|-2}^{-1}\cdots z_1^{-1}z_0, \quad \overline{\sigma}_i = z_{|q_i|-1}^{-1}z_{|q_i|-2}\cdots z_1 z_0^{-1},\\
\rho_i &= (z_{2|q_i|-1}z_{2|q_i|-2}^{-1}\cdots z_{|q_i|+1}^{-1}z_{|q_i|})(z_{|q_i|-1}^{-1}z_{|q_i|-2}\cdots z_1 z_0^{-1}),\\
\overline{\rho}_i &= (z_{2|q_i|-1}^{-1}z_{2|q_i|-2}\cdots z_{|q_i|+1}z_{|q_i|}^{-1})(z_{|q_i|-1}z_{|q_i|-2}^{-1}\cdots z_1^{-1}z_0)
\end{aligned}$$

and

$$\varepsilon_i = \operatorname{sign}(q_i)\cdot\operatorname{sign}(r_i)\cdot\operatorname{sign}(q_{i+1}) = \operatorname{sign}(q_{i-1})\cdot\operatorname{sign}(q_{i+1})$$

for $i = 0, \dots, k$, and $\sigma_{-1} = \overline{\sigma}_{-1} = \rho_{-1} = \overline{\rho}_{-1} = 1$.

By the definition, we have the following:

$$\sigma_{i-1} = (t^{-s_1}\rho_i t^{s_1})(t^{-s_2}\rho_i t^{s_2})\cdots(t^{-s_{r_i'}}\rho_i t^{s_{r_i'}})(\sigma'_{i+1})^{\varepsilon_i}, \tag{4.11}$$

$$\overline{\sigma}_{i-1} = (t^{-s_1}\overline{\rho}_i t^{s_1})(t^{-s_2}\overline{\rho}_i t^{s_2})\cdots(t^{-s_{r_i'}}\overline{\rho}_i t^{s_{r_i'}})(\overline{\sigma}'_{i+1})^{\varepsilon_i}, \tag{4.12}$$

$$\rho_i = (t^{-|q_i|}\sigma_i t^{|q_i|})\overline{\sigma}_i \tag{4.13}$$

and

$$\overline{\rho}_i = (t^{-|q_i|}\overline{\sigma}_i t^{|q_i|})\sigma_i \tag{4.14}$$

where $s_j = |q_{i-1}| - 2j|q_i|$ $(j = 1, \dots, |r_i|/2)$, $r_i' = |r_i|/2$,

$$\sigma_{i+1}' = \begin{cases} \sigma_{i+1} & (\varepsilon_i = 1), \\ t^{|q_{i+1}|-1}\sigma_{i+1}t^{-|q_{i+1}|+1} & (\varepsilon_i = -1) \end{cases}$$

and

$$\overline{\sigma}_{i+1}' = \begin{cases} \overline{\sigma}_{i+1} & (\varepsilon_i = 1), \\ t^{|q_{i+1}|-1}\overline{\sigma}_{i+1}t^{-|q_{i+1}|+1} & (\varepsilon_i = -1). \end{cases}$$

We need a key lemma for the proof.

Lemma 4.3. *The following equations hold for* $i = 0, \dots, k-1$.

(1) $t^{-j}\sigma_i t^j \equiv \sigma_i$, $t^{-j}\overline{\sigma}_i t^j \equiv \overline{\sigma}_i$, $t^{-j}\rho_i t^j \equiv \rho_i$, $t^{-j}\overline{\rho}_i t^j \equiv \overline{\rho}_i$ (mod $\langle\langle w\rangle\rangle$) $(j \in \mathbb{Z})$.

(2) $\sigma_{i-1} \equiv \rho_i^{|r_i|/2}\sigma_{i+1}^{\varepsilon_i}$, $\overline{\sigma}_{i-1} \equiv \overline{\rho}_i^{|r_i|/2}\,\overline{\sigma}_{i+1}^{\varepsilon_i}$ (mod $\langle\langle w\rangle\rangle$) *where* $r_0 = 0$.

(3) $\rho_i \equiv \sigma_i\overline{\sigma}_i$, $\overline{\rho}_i \equiv \overline{\sigma}_i\sigma_i$ (mod $\langle\langle w\rangle\rangle$).

Proof. We prove them by induction on i. We note that it is sufficient to show only the case $j = 1$ for (1).

(i) The case $i = 0$ and 1.

By (4.9) and (4.10), we have $\sigma_0 = 1$ and $\sigma_1 \equiv 1$ (mod $\langle\langle w\rangle\rangle$). Then the first equation of (1) holds for the cases $i = 0$ and 1. Since $\sigma_{-1} = 1$ and $\sigma_1 \equiv 1$ (mod $\langle\langle w\rangle\rangle$), the first equation of (2) holds for the case $i = 0$. Since $\sigma_0 = 1$, we have

$$\begin{aligned}\overline{\sigma}_0 &= z_{q-1}^{-1}z_{q-2}\cdots z_1 z_0^{-1} \\ &= (z_{q-2}^{-1}z_{q-3}\cdots z_1^{-1}z_0)(z_{q-2}z_{q-3}^{-1}\cdots z_1 z_0^{-1})\end{aligned}$$

and

$$\begin{aligned}t^{-1}\overline{\sigma}_0 t &= (z_{q-1}^{-1}z_{q-2}\cdots z_2^{-1}z_1)(z_{q-1}z_{q-2}^{-1}\cdots z_2 z_1^{-1}) \\ &= \{(z_{q-2}^{-1}z_{q-3}\cdots z_1^{-1}z_0)z_{q-2}\cdots z_2^{-1}z_1\} \\ &\quad \times \{(z_{q-2}^{-1}z_{q-3}\cdots z_1^{-1}z_0)^{-1}z_{q-2}^{-1}\cdots z_2 z_1^{-1}\} \\ &= (z_{q-2}^{-1}z_{q-3}\cdots z_1^{-1}z_0)(z_{q-2}z_{q-3}^{-1}\cdots z_1 z_0^{-1}) = \overline{\sigma}_0.\end{aligned}$$

Then the second equation of (1) holds for the case $i = 0$, and (3) holds for the case $i = 0$ by (4.13) and (4.14). Hence the third and the fourth equations of (1) hold for the case $i = 0$.

By the similar way as the case $i = 0$, we have the second equation of (1) for the case $i = 1$. Then (3) holds for the case $i = 1$ by (4.13) and (4.14). Hence (2) and the third and the fourth equations of (1) hold for the case $i = 1$.

(ii) The case $2 \le i \le k-1$.

Suppose (1), (2) and (3) hold if i is replaced with $i' < i$.

Since (1) holds for the case $i-2$, and (2) and (3) holds for the case $i-1$, the first and the second equations of (1) hold for the case i. Hence (3) holds for the case i by (4.13) and (4.14), and the third and the fourth equations of (1) hold for the case i. By (4.11), (4.12) and (1) for the cases $i-1$ and i, (2) holds for the case i. Therefore the proof is completed. □

By Lemma 4.3 (1) and (2) for the cases $i = k-1$ and $k-2$, we have $t^{-1}\sigma_k t \equiv \sigma_k$ (mod $\langle\langle w\rangle\rangle$). Since $\sigma_k = z_0$, we have $z_0 \equiv z_j$ (mod $\langle\langle w\rangle\rangle$) for every integer j and $z_0 \equiv 1$ (mod $\langle\langle w\rangle\rangle$) by (4.9). Therefore we have $G'(K_{p,q}) = \langle\langle w\rangle\rangle$ and $a(K_{p,q}) = 1$. □

REMARK 4.4. By (4.8), we have $a(K_{p,q}) = a(K_{p,-q})$. Since $T(2, p)$ is ambient isotopic to $-T(2, p)$, we have $a(K_{p,q}) = a(T(2, p) \sharp (-T(2, q)))$.

Let K^* be the mirror image of a knot K. For general cases, we raise the following question.

QUESTION 4.5. Let K_1 and K_2 be knots in S^3. Then $a(K_1 \sharp K_2) = a(K_1 \sharp K_2^*) = a(K_1 \sharp (-K_2^*))$?

REMARK 4.6. F.H. Norwood [15] showed that if $r(K) = 2$ for a knot K, then K is a prime knot. Hence we have $r(K_{p,q}) = 3$, and $a(K_{p,q}) < r(K_{p,q}) - 1$ if $\gcd(p, q) = 1$. Norwood's theorem deduces the statement that if $t(K) = 1$, then K is a prime knot. Let $K' = \sharp^n K_{p,q}$ be a knot which is a connected sum of n copies of $K_{p,q}$. Then it has $2n$ prime factors and satisfies $m(K') = a(K') = n$. By Scharlemann and Schultens' result, we have $t(K') = 2n$. Therefore $t(K') - a(K')$ can be arbitrary large.

For the corresponding question of Remark 4.6, see Question 5.5, 5.6 and 5.7.

5. Final remarks

- We notice that tables in A. Kawauchi's book [6] includes many mistakes. For example, on the tables of the tunnel numbers, the book misread Morimoto, Sakuma and Yokota's result [13] in which the tunnel numbers of prime knots up to ten crossings are determined completely. For example, the set of prime knots with the crossing number eight and the tunnel number two in Theorem 2.6 of [13] is expressed as

$$8_n \quad \text{with} \quad n \in [16, 18]$$

which should be read as $\{8_{16}, 8_{17}, 8_{18}\}$ (i.e., In this case, [16, 18] implies the set of integers included by the closed interval from 16 to 18). However the book [6] read the set as $\{8_{16}, 8_{18}\}$ (i.e., Only the end points are read). Corrections of the tables can be found in [7], or a web page "KnotInfo" [1] maintained by J.C. Cha and C. Livingston.

• Let K be a knot in S^3, and $A(t)$ the Alexander matrix of K. For a prime number p, we denote the $\mathbb{Z}/p\mathbb{Z}$-dimension of the null space of $A(-1)$ (mod p) by $d_p(K)$. A. Ishii [3] defined $d_p(K)$ as (the dimension of the set of p-colorings)-1, and pointed out that $d_p(K) \leq t(K)$ holds. Since $m(K)$ is the minimal size of the Alexander matrix of K, it is easy to see that $d_p(K) \leq m(K)$ holds and that $d_p(K) \leq t(K)$ can be deduced from Corollary 1.2. It may be interesting to study $a(K)$ from a viewpoint of quandle.

• T. Kanenobu [4] discusses additivity problem of the *unknotting number* of 2-knots (cf. [2]). He defines the *weak unknotting number* which is determined from the 2-knot group, and is a lower bound of the unknotting number. Hence the unknotting number can be studied by combinatorial group theoretical methods. Our methods are parallel to his methods.

• To determine the MQ index is difficult in general. If K is a prime knot with the crossing number up to nine other than 8_{16}, 9_{29} and 9_{32}, then $m(K) = a(K)$ can be determined by Corollary 1.2 and [1, 6, 8]. Since it is pointed out in [10] that $a(K) = 0$ implies K is trivial, and $m(K) = 0$ if and only if the Alexander polynomial of K is trivial (cf. Lemma 3.1), there are infinitely many examples of K satisfying $m(K) = 0$ and $a(K) \geq 1$. For example, Kinoshita–Terasaka's knot satisfies the condition. We raise some questions about the MQ index.

QUESTION 5.1. Find examples of K satisfying $1 \leq m(K) < a(K)$.

QUESTION 5.2. Characterize the knots with $a(K) = 1$.

This is a combinatorial group theoretical problem like "Characterize normal subgroups of a given (knot) group generated normally by one element".

For the additivity problem on the MQ index, we raise the following questions.

QUESTION 5.3. Find a class of knots in which the MQ index is additive under connected sum.

We restrict Question 5.3.

QUESTION 5.4. For two prime non-fibered knots K_1 and K_2, $a(K_1 \sharp K_2) = a(K_1) + a(K_2)$ holds?

For relatively coprime n odd numbers $p_1, \ldots, p_n$, let K be a connected sum of $T(2, p_1), \ldots, T(2, p_n)$. By the argument in Section 3, we have $m(K) = 1$. In contrast with this and related with Remark 4.6, we raise the following questions.

QUESTION 5.5. Let K be a knot which is a connected sum of n non-trivial knots. Then $a(K) \geq n/2$ holds?

QUESTION 5.6. Let K be a knot which is a connected sum of n non-fibered knots. Then $a(K) \geq n$ holds?

Related with fiberedness, we raise the following question.

QUESTION 5.7. Let K be a fibered knot. Then $a(K) = m(K)$ holds?

Theorem 1.3 is a partial affirmative answer for the question. We note that both Question 5.5 and 5.7 cannot be affirmative.

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