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A REAL VARIABLE CHARACTERIZATION OF GROMOV HYPERBOLICITY OF FLUTE SURFACES

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Abstract

In this paper we give a characterization of the Gromov hyperbolicity of trains (a large class of Denjoy domains which contains the flute surfaces) in terms of the behavior of a real function. This function describes somehow the distances between some remarkable geodesics in the train. This theorem has several consequences; in particular, it allows to deduce a result about stability of hyperbolicity, even though the original surface and the modified one are not quasi-isometric. In order to obtain these results we also prove some trigonometric lemmas that are interesting by themselves, since they provide very simple estimates on some hyperbolic distances.

1. Introduction

The theory of Gromov hyperbolic spaces is a useful tool in order to understand the connections between graphs and potential theory (see e.g. [4], [10], [13], [25], [26], [27], [28], [35], [36], [40]). Besides, the concept of Gromov hyperbolicity grasps the essence of negatively curved spaces, and has been successfully used in the theory of groups (see e.g. [15], [17], [18] and the references therein).

A geodesic metric space is called hyperbolic (in the Gromov sense) if there exists an upper bound of the distance of every point in a side of any geodesic triangle to the union of the two other sides (see Definition 2.2). The latter condition is known as Rips condition.

But, it is not easy to determine whether a given space is Gromov hyperbolic or not. Recently, there has been some research aimed to show that metrics used in geometric function theory are Gromov hyperbolic. Some specific examples are showing that the Klein–Hilbert metric ([8], [29]) is Gromov hyperbolic (under particular conditions on the domain of definition), that the Gehring–Osgood metric ([20]) is Gromov hyperbolic, and that the Vuorinen metric ([20]) is not Gromov hyperbolic (except for

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a particular case). Recently, some interesting results by Balogh and Buckley [5] about the hyperbolicity of Euclidean bounded domains with their quasihyperbolic metric have made significant progress in this direction (see also [9], [41] and the references therein). Another interesting instance is that of a Riemann surface endowed with the Poincaré metric. With such metric structure a Riemann surface is always negatively curved, but not every Riemann surface is Gromov hyperbolic, since topological obstacles may impede it: for instance, the two-dimensional jungle-gym (a $\mathbb{Z}^2$-covering of a torus with genus two) is not hyperbolic.

We are interested in studying when Riemann surfaces equipped with their Poincaré metric are Gromov hyperbolic (see e.g. [3], [21], [22], [23], [24], [30], [31], [32], [33], [34], [37], [38], [39]). To be more precise, in the current paper our main aim is to study the hyperbolicity of Denjoy domains, that is to say, plane domains Ω with $\partial\Omega \subset \mathbb{R}$. This kind of surfaces are becoming more and more important in geometric theory of functions, since, on the one hand, they are a very general type of Riemann surfaces, and, on the other hand, they are more manageable due to its symmetry. For instance, Garnett and Jones have proved the Corona theorem for Denjoy domains ([14]), and in [2] the authors have got the characterization of Denjoy domains which satisfy a linear isoperimetric inequiality.

Denjoy domains are such a wide class of Riemann surfaces that characterization criteria are not straightforward to apply. That is the main reason that led us to focus on a particular type of Denjoy domain, which we have called *train*. A train can be defined as the complement of a sequence of ordered closed intervals (see Definition 2.3). Trains do include a especially important case of surfaces which are the flute surfaces (see, e.g. [6], [7]). These ones are the simplest examples of infinite ends, and besides, in a flute surface it is possible to give a fairly precise description of the ending geometry (see, e.g. [19]). In [3] there are some results on hyperbolicity of trains.

This paper is a natural continuation of [3]. Although some of the theorems in the current work might seem alike to some of the results in the preceding paper, the truth is that they are much more powerful and the proofs developed are completely new. Without a doubt, the main contribution of this paper is Theorem 3.2, that provides a characterization of the hyperbolicity of trains in terms of the behavior of a real function with two integer parameters. (In [3] we give either necessary or sufficient conditions, and there is a characterization, but much more difficult to apply than the one presented here). This function describes somehow the distances between some remarkable geodesics (called *fundamental geodesics*) in the train. At first sight, Theorem 3.2 might not seem very user-friendly. However, in practice, this tool let us deduce a result about stability of hyperbolicity, even for cases when the original surface and the modified one are not quasi-isometric (see Theorem 3.8).

Theorem 3.2 also allows to deduce both sufficient and necessary conditions that either guarantee or discard hyperbolicity (see Corollary 3.14, Theorems 3.16 and 3.17). Besides, these three theorems give a much simpler characterization than Theorem 3.2 for an interesting case of trains: those for which the lengths of their fundamental geodesics are a quasi-increasing sequence. We are talking about Theorem 3.18, another crucial result in this paper.

In order to obtain these results we also prove some trigonometric lemmas that are interesting by themselves, since they provide very simple estimates on some hyperbolic distances (see Propositions 4.8 and 4.9).

For the sake of clarity and readability, we have opted for moving all the technical lemmas to the last section of the paper. This makes the proof of Theorem 3.2, our main result, much more understandable.

NOTATIONS. We denote by X a geodesic metric space. By d_X and L_X we shall denote, respectively, the distance and the length in the metric of X. From now on, when there is no possible confusion, we will not write the subindex X.

We denote by Ω a train with its Poincaré metric.

Given a subset F of the complex plane, we define $F^+ = F \cap \{z \in \mathbb{C} : \Im z \geq 0\}$, where $\Im z$ is the imaginary part of z.

If E is either a function or a constant related to a domain Ω, we will denote by E' or E^j the same function or constant related to a domain Ω' or Ω^j, respectively.

As usual, we denote by x_+ the positive part of x: $x_+ := x$ if $x \geq 0$ and $x_+ := 0$ if $x < 0$.

If "a is *comparable* to b", i.e. if there exists a constant c such that $c^{-1}a \leq b \leq ca$, we will denote it by $a \asymp b$.

Finally, we denote by c and c_i, positive constants which can assume different values in different theorems.

2. Background in Gromov spaces and Riemann surfaces

In our study of hyperbolic Gromov spaces we use the notations of [15]. We give now the basic facts about these spaces. We refer to [15] for more background and further results.

DEFINITION 2.1. If $\gamma \colon [a,b] \to X$ is a continuous curve in a metric space (X,d), the *length* of γ is

$$L(\gamma) := \sup\left\{ \sum_{i=1}^{n} d(\gamma(t_{i-1}), \gamma(t_i)) \colon a = t_0 < t_1 < \cdots < t_n = b \right\}.$$

We say that γ is a *geodesic* if it is an isometry, i.e. $L(\gamma|_{[t,s]}) = d(\gamma(t), \gamma(s)) = |t-s|$ for every $s, t \in [a,b]$. We say that X is a *geodesic metric space* if for every $x, y \in X$ there exists a geodesic joining x and y; we denote by $[x,y]$ any of such geodesics (since we do not require uniqueness of geodesics, this notation is ambiguous, but convenient as well).

DEFINITION 2.2. Consider a geodesic metric space X. If $x_1, x_2, x_3 \in X$, a *geodesic triangle* $T = \{x_1, x_2, x_3\}$ is the union of three geodesics $[x_1, x_2]$, $[x_2, x_3]$ and $[x_3, x_1]$. We say that T is δ*-thin* if for every $x \in [x_i, x_j]$ we have that $d(x, [x_j, x_k] \cup [x_k, x_i]) \leq \delta$. The space X is δ*-hyperbolic* if every geodesic triangle in X is δ-thin.

We would like to point out that deciding whether or not a space is hyperbolic is usually extraordinarily difficult: Notice that, first of all, we have to consider an arbitrary geodesic triangle T, and calculate the minimum distance from an arbitrary point P of T to the union of the other two sides of the triangle to which P does not belong to. And then we have to take supremum over all the possible choices for P and then over all the possible choices for T. It means that if our space is, for instance, an n-dimensional manifold and we select two points P and Q on different sides of a triangle T, the function F that measures the distance between P and Q is a $(3n+2)$-variable function. In order to prove that our space is hyperbolic we would have to take the minimum of F over the variable that describes Q, and then the supremum over the remaining $3n+1$ variables, or at least prove that it is finite. Without disregarding the difficulty of solving a $(3n+2)$-variable minimax problem, notice that the main obstacle is that we do not even know in an approximate way the location of geodesics in the space.

EXAMPLES. (1) Every bounded metric space X is $(\operatorname{diam} X)$-hyperbolic (see e.g. [15, p. 29]).
(2) Every complete simply connected Riemannian manifold with sectional curvature which is bounded from above by $-k$, with $k > 0$, is hyperbolic (see e.g. [15, p. 52]).
(3) Every tree with edges of arbitrary length is 0-hyperbolic (see e.g. [15, p. 29]).

A *non-exceptional* Riemann surface S is a Riemann surface whose universal covering space is the unit disk $\mathbb{D} = \{z \in \mathbb{C}\colon |z| < 1\}$, endowed with its Poincaré metric, i.e. the metric obtained by projecting the Poincaré metric of the unit disk $ds = 2|dz|/(1-|z|^2)$. Therefore, any simply connected subset of S is isometric to a subset of $\mathbb{D}$. With this metric, S is a geodesically complete Riemannian manifold with constant curvature -1, and therefore S is a geodesic metric space. The only Riemann surfaces which are left out are the *exceptional* Riemann surfaces, that is to say, the sphere, the plane, the punctured plane and the tori. It is easy to study the hyperbolicity of these particular cases. The Poincaré metric is natural and useful in complex analysis: for instance, any holomorphic function between two domains is Lipschitz with constant 1, when we consider the respective Poincaré metrics.

A *Denjoy domain* is a domain Ω in the Riemann sphere with $\partial\Omega \subset \mathbb{R} \cup \{\infty\}$. As we mentioned in the introduction of this paper, Denjoy domains are becoming more and more interesting in geometric function theory (see e.g. [1], [2], [14], [16]).

It is obvious that as we focus on more particular kind of surfaces, we can obtain more powerful results. For this reason we introduce now a new type of space.

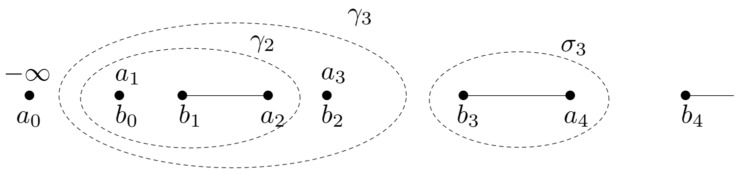

Fig. 1. Train seen as a subset of the complex plane.

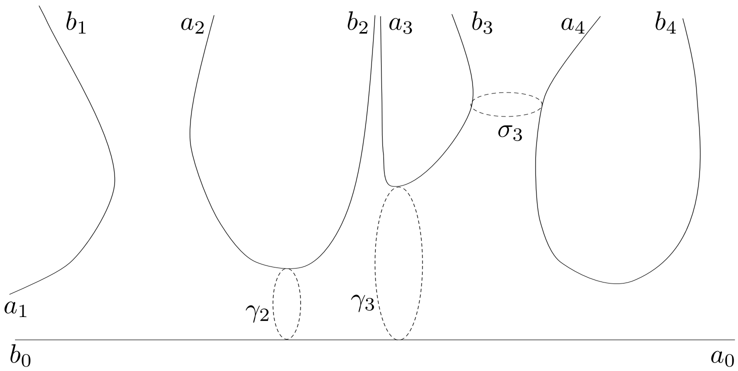

Fig. 2. The same train seen with “Euclidean eyes”.

We have used the word *geodesic* in the sense of Definition 2.1, that is to say, as a global geodesic or a minimizing geodesic; however, we need now to deal with a special type of local geodesics: simple closed geodesics, which obviously can not be minimizing geodesics. We will continue using the word geodesic with the meaning of Definition 2.1, unless we are dealing with closed geodesics.

DEFINITION 2.3. A *train* is a Denjoy domain $\Omega \subset \mathbb{C}$ with $\Omega \cap \mathbb{R} = \bigcup_{n=0}^{\infty}(a_n, b_n)$, such that $-\infty \leq a_0$ and $b_n \leq a_{n+1}$ for every n. A *flute surface* is a train with $b_n = a_{n+1}$ for every n.

We say that a curve in a train Ω is a *fundamental geodesic* if it is a simple closed geodesic which just intersects $\mathbb{R}$ in (a_0, b_0) and (a_n, b_n) for some $n > 0$; we denote by γ_n the fundamental geodesic corresponding to n and $2l_n := L_\Omega(\gamma_n)$. A curve in a train Ω is a *second fundamental geodesic* if it is a simple closed geodesic which just intersects $\mathbb{R}$ in (a_n, b_n) and (a_{n+1}, b_{n+1}) for some $n \geq 0$; we denote by σ_n the second fundamental geodesic corresponding to n and $2r_n := L_\Omega(\sigma_n)$ (see the figures above). If $b_n = a_{n+1}$, we define σ_n as the puncture at this point and $r_n = 0$. Given $z \in \Omega$, we define the *height* of z as $h(z) := d_\Omega(z, (a_0, b_0))$.

REMARK. Recall that in every free homotopy class there exists a single simple closed geodesic, assuming that punctures are simple closed geodesics with length equal to zero. That is why both the fundamental geodesic and the second fundamental geodesic are unique for every n.

A train is a flute surface if and only if every second fundamental geodesic is a puncture.

Flute surfaces are the simplest examples of infinite ends; furthermore, in a flute surface it is possible to give a fairly precise description of the ending geometry (see, e.g. [19]).

3. The main results

It is not difficult to see that the values of $\{l_n\}$ and $\{r_n\}$ determine a train, since for every n there exists a single fundamental geodesic and a single second fundamental geodesic (see Remark to Definition 2.3). Then, there must exist a characterization of hyperbolicity in terms of the lengths of the fundamental geodesics. It would be desirable to obtain such a characterization, since these lengths describe the Denjoy domain from a simple geometric viewpoint.

In order to obtain this characterization, we need to introduce the following functions.

(We refer to the next section for the details of the proofs of technical lemmas. We think that this structure makes the paper more readable, because it shortens considerably the proof of Theorem 3.2).

DEFINITION 3.1. Let us consider a sequence of positive numbers $\{l_n\}_{n=1}^{\infty}$ and a sequence of non-negative numbers $\{r_n\}_{n=1}^{\infty}$. Denote by x_+ the positive part of x: $x_+ := \max\{x, 0\}$. Consider $n \geq 1$ and $0 \leq h \leq l_n$. We define $A_n(h) := \max\{m < n : l_m \leq h\}$ if this set is non-empty and $A_n(h) := 1$ in other case, $B_n(h) := \min\{m > n : l_m \leq h\}$ if this set is non-empty and $B_n(h) := \infty$ in other case,

$$\Delta(k) := e^{-l_k} + e^{-l_{k+1}} + e^{-(1/2)(l_k + l_{k+1} - r_k)_+} + (r_k - l_k - l_{k+1})_+,$$

and

$$\Gamma_{nm}(h) := \begin{cases} (r_m + h - l_{m+1})_+ + e^h \displaystyle\sum_{k=m+1}^{n-1} \Delta(k), & \text{if} \quad m < n \quad \text{and} \quad l_m \leq h, \\ l_m - h + e^h \displaystyle\sum_{k=m}^{n-1} \Delta(k), & \text{if} \quad m < n \quad \text{and} \quad l_m > h, \\ \min\{h, l_n - h\}, & \text{if} \quad m = n, \\ l_m - h + e^h \displaystyle\sum_{k=n}^{m-1} \Delta(k), & \text{if} \quad m > n \quad \text{and} \quad l_m > h, \\ (r_{m-1} + h - l_{m-1})_+ + e^h \displaystyle\sum_{k=n}^{m-2} \Delta(k), & \text{if} \quad m > n \quad \text{and} \quad l_m \leq h. \end{cases}$$

The functions $\Gamma_{nm}(h)$ are naturally associated to trains by taking $\{l_n\}_{n=1}^{\infty}$ and $\{r_n\}_{n=1}^{\infty}$ as the half-lengths of their fundamental geodesics.

Theorem 3.2. *A train Ω is hyperbolic if and only if*

$$K := \sup_{n\ge1} \sup_{h\in[0,l_n]} \min_{m\in[A_n(h),B_n(h)]} \Gamma_{nm}(h) < \infty.$$

Furthermore, if Ω is δ-hyperbolic, then K is bounded by a constant which only depends on δ; if $K < \infty$, then Ω is δ-hyperbolic, with δ a constant which only depends on K.

REMARKS. (1) Notice that this is a real variable characterization of the hyperbolicity.
(2) Theorem 3.2 clearly improves [3, Theorem 5.3]: we need to know the lengths of the fundamental geodesics instead of the precise location of these geodesics and the distances to $\mathbb{R}$ from their points.
(3) The proof of Theorem 3.2 gives that its conclusion also holds if we replace K by

$$K(l_0) := \sup_{n\ge1} \sup_{h\in[l_0,l_n]} \min_{m\in[A_n(h),B_n(h)]} \Gamma_{nm}(h) < \infty,$$

for any fixed $l_0 > 0$. In this case, the constant δ depends on $K(l_0)$ and l_0.

Proof. By [3, Theorem 5.3], Ω is δ-hyperbolic if and only if

$$K_1 := \sup_{n\ge1} \sup_{z\in\gamma_n} \inf_{m\ge0} d_\Omega(z, (a_m, b_m)) < \infty,$$

with the appropriate dependence of the constants (if Ω is δ-hyperbolic, then K_1 is bounded by a constant which only depends on δ; if $K_1 < \infty$, then Ω is δ-hyperbolic, with δ a constant which only depends on K_1).

Fix any constant $l_0 > 0$. Notice that:
(1) $d_\Omega(z, (a_0, b_0)) = h(z)$ and $d_\Omega(z, (a_n, b_n)) = l_n - h(z)$. Since any z with $h(z) < l_0$ verifies

$$\inf_{m\ge0} d_\Omega(z, (a_m, b_m)) \le d_\Omega(z, (a_0, b_0)) = h(z) < l_0,$$

we only need to consider z with $l_0 \le h(z) \le l_n$.

From now on, let us fix $n \ge 1$ and $z \in \gamma_n$ with $l_0 \le h(z) \le l_n$.
(2) If $k < m < n$, with $l_m \le h(z)$, let us consider the geodesic σ which gives the minimum distance between z and (a_k, b_k). Define the point $w := \sigma \cap \gamma_m$; hence $d_\Omega(z, w) < d_\Omega(z, (a_k, b_k))$ and Lemma 4.3 give

$$d_\Omega(z, (a_m, b_m)) \le d_2(z, (a_m, b_m) \cap \gamma_m) \le d_2(z, w) \le 3d_\Omega(z, w) < 3d_\Omega(z, (a_k, b_k)),$$

where d_2 is the function in Definition 4.2. In a similar way, if $k > m > n$, with $l_m \le h(z)$, then $d_\Omega(z, (a_m, b_m)) < 3d_\Omega(z, (a_k, b_k))$. Hence we only need to consider $d_\Omega(z, (a_m, b_m))$ with $m \in \{0\} \cup [A_n(h(z)), B_n(h(z))]$, in order to study if K_1 is finite.
(3) If $m \in (A_n(h(z)), n)$, then $l_0 \le h(z) < l_m$. By Lemma 4.4, we can replace $d_\Omega(z, (a_m, b_m))$ by $d_1(z, \gamma_m \cap (a_m, b_m))$, where d_1 is the function in Definition 4.2. If z_m is the point in γ_m with $h(z_m) = h(z)$, then $d_1(z, \gamma_m \cap (a_m, b_m)) := d_\Omega(z, z_m) + l_m - h(z)$. Standard hyperbolic trigonometry in quadrilaterals (see e.g. [12, p. 88]) gives that

$$d_\Omega(z, z_m) = 2 \operatorname{Arcsinh}\Big(\sinh \frac{1}{2} d_\Omega(\gamma_m, \gamma_n) \cosh h(z) \Big).$$

Recall that (a_0, b_0) contains the shortest geodesic joining γ_m and γ_n. By Corollary 4.7 we can replace $d_\Omega(z, z_m)$ by $d_\Omega(\gamma_m, \gamma_n) e^{h(z)}$, and therefore $d_1(z, \gamma_m \cap (a_m, b_m))$ by $d_\Omega(\gamma_m, \gamma_n) e^{h(z)} + l_m - h(z)$. Standard hyperbolic trigonometry in right-angled hexagons (see e.g. [12, p. 86]) gives that

$$d_\Omega(\gamma_k, \gamma_{k+1}) = \operatorname{Arccosh} \frac{\cosh r_k + \cosh l_k \cosh l_{k+1}}{\sinh l_k \sinh l_{k+1}}$$

for every $k \ge 1$. Proposition 4.8 gives

$$\begin{aligned} d_\Omega(\gamma_k, \gamma_{k+1}) = &\ f(l_k, l_{k+1}, r_k) \\ &\asymp e^{-l_k} + e^{-l_{k+1}} + e^{-(1/2)(l_k + l_{k+1} - r_k)_+} + (r_k - l_k - l_{k+1})_+ = \Delta(k), \end{aligned}$$

for every $k \in (A_n(h(z)), n)$, since then $l_k, l_{k+1} \ge h(z) \ge l_0$. Therefore we can replace $d_\Omega(z, (a_m, b_m))$ by

$$l_m - h(z) + e^{h(z)} \sum_{k=m}^{n-1} \Delta(k).$$

A symmetric argument gives that if $m \in (n, B_n(h(z)))$, then we can replace $d_\Omega(z, (a_m, b_m))$ by

$$l_m - h(z) + e^{h(z)} \sum_{k=n}^{m-1} \Delta(k).$$

(4) If $m = A_n(h(z))$, then $h(z) \ge l_m$. If z_{m+1} is the point in γ_{m+1} with $h(z_{m+1}) = h(z)$, by Lemma 4.5, we can replace $d_\Omega(z, (a_m, b_m))$ by $d_\Omega(z, z_{m+1}) + d_\Omega(z_{m+1}, (a_m, b_m))$. We have seen in (3) that we can replace $d_\Omega(z, z_{m+1})$ by

$$e^{h(z)} \sum_{k=m+1}^{n-1} \Delta(k).$$

Standard hyperbolic trigonometry in pentagons (see e.g. [12, p. 87]) gives that

$$\sinh d_\Omega(z_{m+1}, (a_m, b_m)) = -\cosh l_m \sinh h(z) + \sinh l_m \cosh h(z) \cosh d_\Omega(\gamma_m, \gamma_{m+1}).$$

Standard hyperbolic trigonometry in right-angled hexagons (see e.g. [12, p. 86]) gives that

$$\cosh d_\Omega(\gamma_m, \gamma_{m+1}) = \frac{\cosh r_m + \cosh l_m \cosh l_{m+1}}{\sinh l_m \sinh l_{m+1}},$$

and hence

$$\begin{aligned}
&\sinh d_\Omega(z_{m+1}, (a_m, b_m)) \\
&= -\cosh l_m \sinh h(z) + \cosh h(z) \frac{\cosh r_m + \cosh l_m \cosh l_{m+1}}{\sinh l_{m+1}} \\
&= \frac{\cosh l_m(\cosh l_{m+1} \cosh h(z) - \sinh l_{m+1} \sinh h(z)) + \cosh r_m \cosh h(z)}{\sinh l_{m+1}} \\
&= \frac{\cosh l_m \cosh(l_{m+1} - h(z)) + \cosh r_m \cosh h(z)}{\sinh l_{m+1}} = \sinh F(l_m, l_{m+1}, r_m, h(z)),
\end{aligned}$$

where F is the function in Proposition 4.9. Therefore, Corollary 4.10 gives that we can replace $d_\Omega(z_{m+1}, (a_m, b_m))$ by $(r_m + h(z) - l_{m+1})_+$. Consequently, we can substitute $d_\Omega(z, (a_m, b_m))$ by

$$(r_m + h(z) - l_{m+1})_+ + e^{h(z)} \sum_{k=m+1}^{n-1} \Delta(k).$$

A symmetric argument gives that if $m = B_n(h(z))$, then we can replace $d_\Omega(z, (a_m, b_m))$ by

$$(r_{m-1} + h(z) - l_{m-1})_+ + e^{h(z)} \sum_{k=n}^{m-2} \Delta(k).$$

Notice that each time that we replace a quantity by another in this proof, the constants are under control. Let us remark that (1), (2), (3) and (4) give the result, with $\inf_{m \in [A_n(h), B_n(h)]} \Gamma_{nm}(h)$ instead of $\min_{m \in [A_n(h), B_n(h)]} \Gamma_{nm}(h)$.

Let us see now that this infimum is attained. Seeking for a contradiction, suppose that the latest statement is not true. Therefore, $B_n(h) = \infty$ and $l_m > h$ for every $m > n$. Then, there exists an increasing sequence of integer numbers $\{m_j\}$ with $\lim_{j\to\infty} \Gamma_{nm_j}(h) = \inf_{m \in [A_n(h), \infty)} \Gamma_{nm}(h)$. By choosing a subsequence if it is necessary, we can assume that $\{\Gamma_{nm_j}(h)\}_j$ is a decreasing sequence. Hence,

$$\Gamma_{nm_{j+1}}(h) = l_{m_{j+1}} - h + e^h \sum_{k=n}^{m_{j+1}-1} \Delta(k) < \Gamma_{nm_j}(h) = l_{m_j} - h + e^h \sum_{k=n}^{m_j - 1} \Delta(k).$$

Consequently, we have that $l_{m_{j+1}} < l_{m_j} < l_{m_1}$ for every j, and

$$\Gamma_{nm_j}(h) = l_{m_j} - h + e^h \sum_{k=n}^{m_j-1} \Delta(k) \geq e^h \sum_{k=n}^{m_j} e^{-l_k} \geq e^h \sum_{k=1}^{j} e^{-l_{m_k}} \geq e^h j e^{-l_{m_1}}.$$

Hence, $\lim_{j\to\infty} \Gamma_{nm_j}(h) = \lim_{j\to\infty} e^h j e^{-l_{m_1}} = \infty$, which is a contradiction. This finishes the proof. □

Lemma 3.3. *For every $r_k \geq 0$ and $0 < l_k \leq h \leq l_{k+1}$, we have*

$$(r_k + h - l_{k+1})_+ < e^h \Delta(k).$$

Proof. Let us remark that it is sufficient to prove

$$r_k + h - l_{k+1} < e^h (e^{-(1/2)(l_k + l_{k+1} - r_k)_+} + (r_k - l_k - l_{k+1})_+),$$

for every $r_k \geq 0$ and $0 < l_k \leq h \leq l_{k+1}$.

Since the left hand side of the inequality does not depend on l_k and the right hand side is a decreasing function on l_k, it is sufficient to prove

$$r_k + h - l_{k+1} < e^h (e^{-(1/2)(h + l_{k+1} - r_k)_+} + (r_k - h - l_{k+1})_+),$$

for every $r_k \geq 0$ and $0 < h \leq l_{k+1}$.

If $r_k \leq h + l_{k+1}$, then the inequality is

$$r_k + h - l_{k+1} < e^h e^{-(1/2)(h + l_{k+1} - r_k)} = e^{(1/2)(r_k + h - l_{k+1})},$$

which trivially holds since $t < e^{t/2}$ for every real number t.

If $r_k \geq h + l_{k+1}$, then the inequality is

$$r_k + h - l_{k+1} < e^h (1 + r_k - h - l_{k+1}).$$

Since $e^h \geq 1$, it is clear that the function

$$U(r_k) := e^h (1 + r_k - h - l_{k+1}) - r_k - h + l_{k+1}$$

is increasing in $r_k \in [h + l_{k+1}, \infty)$. Then $U(r_k) \geq U(h + l_{k+1}) = e^h - 2h > 0$, and the inequality holds. □

Proposition 3.4. *In any train Ω we have*

$$\min_{m\in[A_n(h), B_n(h)]} \Gamma_{nm}(h) = \min_{m\geq 1} \Gamma_{nm}(h),$$

for every $n \geq 1$ and $0 \leq h \leq l_n$.

Proof. Fix $n \geq 1$ and $0 \leq h \leq l_n$. If $m < A_n(h)$, then Lemma 3.3 gives $\Gamma_{nm}(h) > \Gamma_{nA_n(h)}(h)$:

$$\Gamma_{nm}(h) \geq e^h \sum_{k=m+1}^{n-1} \Delta(k) \geq e^h \sum_{k=A_n(h)}^{n-1} \Delta(k) = e^h \Delta(A_n(h)) + e^h \sum_{k=A_n(h)+1}^{n-1} \Delta(k)$$

$$> (r_{A_n(h)} + h - l_{A_n(h)+1})_+ + e^h \sum_{k=A_n(h)+1}^{n-1} \Delta(k) = \Gamma_{nA_n(h)}(h).$$

The case $m > B_n(h)$ is similar. □

Proposition 3.5. *If for some n we have $l_m \geq l_n$ for every $m \geq n$, then the conclusion of* Theorem 3.2 *also holds if we replace $[A_n(h), B_n(h)]$ by $[A_n(h), n]$ for this n.*

Proof. It suffices to remark that for every $z \in \gamma_n$ and $m > n$, we have $d_\Omega(z, (a_n, b_n)) = l_n - h(z) \leq l_m - h(z) < d_\Omega(z, (a_m, b_m))$. □

Although to compute the minimum and the supremum in Theorem 3.2 can be difficult in the general case, Theorem 3.2 is the main tool in order to obtain the remaining results of this paper. We start with an elementary corollary.

Proposition 3.6. *Let us consider a train Ω with $l_n \leq c$ for every n. Then Ω is δ-hyperbolic, where δ is a constant which only depends on c.*

Proof. For each positive integer n, we have $\Gamma_{nn}(h) := \min\{h, l_n - h\} \leq l_n \leq c$ for every $h \in [0, l_n]$. Hence, $K \leq c$ and Theorem 3.2 finishes the proof. □

One of the important problems in the study of any property is to obtain its stability under appropriate deformations. Theorem 3.2 allows to prove a result which shows that hyperbolicity is stable under bounded perturbations of the lengths of the fundamental geodesics. Theorem 3.8 is particularly remarkable since there are very few results on hyperbolic stability which do not involve quasi-isometries. We need a previous lemma; it deals with some kind of reverse inequality to the one in Lemma 3.3.

Lemma 3.7. *For every $r_k, l_{k+1} \geq 0$ and $0 \leq h \leq l_k$, we have*

$$e^h(e^{-(1/2)(l_k+l_{k+1}-r_k)_+} + (r_k - l_k - l_{k+1})_+) \leq (1 + (r_k + h - l_{k+1})_+)e^{(1/2)(r_k+h-l_{k+1})_+}.$$

Proof. Since the right hand side of the inequality does not depend on l_k and the left hand side is a decreasing function on l_k, it is sufficient to prove

$$e^h(e^{-(1/2)(h+l_{k+1}-r_k)_+} + (r_k - h - l_{k+1})_+) \leq (1 + (r_k + h - l_{k+1})_+)e^{(1/2)(r_k+h-l_{k+1})_+},$$

for every $r_k, l_{k+1}, h \geq 0$.

If $h + l_{k+1} - r_k \ge 0$, the inequality is direct since

$$e^h(e^{-(1/2)(h+l_{k+1}-r_k)_+} + (r_k - h - l_{k+1})_+) = e^h e^{-(1/2)(h+l_{k+1}-r_k)} = e^{(1/2)(r_k+h-l_{k+1})}.$$

If $h + l_{k+1} - r_k < 0$, then $r_k - l_{k+1} > h$ and $(r_k + h - l_{k+1})_+ > 2h$; consequently,

$$\begin{aligned} &e^h(e^{-(1/2)(h+l_{k+1}-r_k)_+} + (r_k - h - l_{k+1})_+) \\ &= e^h(1 + r_k - h - l_{k+1}) \\ &< (1 + (r_k + h - l_{k+1})_+)e^{(1/2)(r_k+h-l_{k+1})_+}. \end{aligned}$$ □

Next, the result about stability that we have talked about before Lemma 3.7. Theorem 3.8 is both a qualitative and a quantitative result.

Theorem 3.8. *Let us consider two trains Ω, Ω' and a constant c such that $|r'_n - r_n| \le c$, and $|l'_n - l_n| \le c$ for every $n \ge 1$. Then Ω is hyperbolic if and only if Ω' is hyperbolic. Furthermore, if Ω is δ-hyperbolic, then Ω' is δ'-hyperbolic, with δ' a constant which only depends on δ and c.*

This result is a significant improvement with respect to [3, Theorem 5.33], since, in that paper, the lengths r_n and r'_n were required to be bounded, whereas Theorem 3.8 only requires $r_n - r'_n$ to be bounded. Notice that this is a much weaker condition. Furthermore, the argument in the proof is completely new.

REMARKS. (1) Notice that in many cases Ω and Ω' are not quasi-isometric (for example, if there exists a subsequence $\{n_k\}_k$ with $\lim_{k\to\infty} l_{n_k} = 0$ and $l'_{n_k} \ge c_0 > 0$).
(2) We have examples which show that Theorem 3.8 is sharp: if we change the constants in Theorem 3.8 by any function growing slowly to infinity, then the conclusion of Theorem 3.8 does not hold. For instance, if $\{r_n\}$ is bounded and $\{r'_n\}$ is not bounded, then there exists $\{l_n\} = \{l'_n\}$ with Ω hyperbolic and Ω' not hyperbolic.

Proof. By symmetry, it is sufficient to prove that if Ω is δ-hyperbolic, then Ω' is δ'-hyperbolic, with δ' a constant which only depends on δ and c. Therefore, let us assume that Ω is δ-hyperbolic.

Notice that $e^{-l_k} + e^{-l_{k+1}} \le e^c(e^{-l'_k} + e^{-l'_{k+1}})$.

If $l_k + l_{k+1} \le r_k$, then $e^{-(1/2)(l_k+l_{k+1}-r_k)_+} + (r_k - l_k - l_{k+1})_+ = 1 + r_k - l_k - l_{k+1}$ and

$$\begin{aligned} &e^{-(1/2)(l'_k+l'_{k+1}-r'_k)_+} + (r'_k - l'_k - l'_{k+1})_+ \\ &\le 1 + 3c + r_k - l_k - l_{k+1} \\ &\le (1 + 3c)(e^{-(1/2)(l_k+l_{k+1}-r_k)_+} + (r_k - l_k - l_{k+1})_+). \end{aligned}$$

If $l'_k + l'_{k+1} \geq r'_k$, then

$$e^{-(1/2)(l'_k+l'_{k+1}-r'_k)_+} + (r'_k - l'_k - l'_{k+1})_+ = e^{-(1/2)(l'_k+l'_{k+1}-r'_k)_+}$$
$$\leq e^{3c/2}(e^{-(1/2)(l_k+l_{k+1}-r_k)_+} + (r_k - l_k - l_{k+1})_+).$$

If $l_k + l_{k+1} > r_k$ and $l'_k + l'_{k+1} < r'_k$, then

$$l_k + l_{k+1} - r_k \leq l'_k + l'_{k+1} - r'_k + 3c < 3c,$$
$$r'_k - l'_k - l'_{k+1} \leq r_k - l_k - l_{k+1} + 3c < 3c,$$

and consequently

$$e^{-(1/2)(l'_k+l'_{k+1}-r'_k)_+} + (r'_k - l'_k - l'_{k+1})_+$$
$$= 1 + r'_k - l'_k - l'_{k+1} < (1+3c)e^{3c/2}e^{-3c/2}$$
$$< (1+3c)e^{3c/2}(e^{-(1/2)(l_k+l_{k+1}-r_k)_+} + (r_k - l_k - l_{k+1})_+).$$

Therefore

$$e^{-l'_k} + e^{-l'_{k+1}} + e^{-(1/2)(l'_k+l'_{k+1}-r'_k)_+} + (r'_k - l'_k - l'_{k+1})_+$$
$$\leq (1+3c)e^{3c/2}(e^{-l_k} + e^{-l_{k+1}} + e^{-(1/2)(l_k+l_{k+1}-r_k)_+} + (r_k - l_k - l_{k+1})_+),$$

i.e. $\Delta'(k) \leq (1+3c)e^{3c/2}\Delta(k)$. We also have

$$(r'_m + h - l'_{m+1})_+ \leq 2c + (r_m + h - l_{m+1})_+,$$
$$l'_m - h \leq c + l_m - h,$$
$$\min\{h, l'_n - h\} \leq c + \min\{h, l_n - h\}.$$

Hence, we conclude

$$(\Gamma_{nm})'(h) \leq (1+3c)e^{3c/2}\Gamma_{nm}(h) + 2c,$$

for every $n, m \geq 1$ and $h \geq 0$ with either $m = n$ or $l_m, l'_m \leq h$ or $l_m, l'_m > h$.

We deal now with the other cases. Let us assume that $m \in [A'_n(h), n)$. The case $m \in (n, B'_n(h)]$ is similar.

If $l'_m \leq h < l_m$, then $m = A'_n(h)$ and $l'_m \leq h < l'_{m+1}$. Applying Lemma 3.3 we obtain

$$(\Gamma_{nm})'(h) = (r'_m + h - l'_{m+1})_+ + e^h \sum_{k=m+1}^{n-1} \Delta'(k) < e^h \sum_{k=m}^{n-1} \Delta'(k)$$
$$\leq l_m - h + (1+3c)e^{3c/2}e^h \sum_{k=m}^{n-1} \Delta(k) \leq (1+3c)e^{3c/2}\Gamma_{nm}(h).$$

If $l_m \le h < l'_m$, then $m > A'_n(h)$ and $h < l'_{m+1}$. We also have $l'_m - h \le l'_m - l_m \le c$. Applying Lemma 3.7 we obtain

$$
\begin{aligned}
&(\Gamma_{nm})'(h) \\
&= l'_m - h + e^{h-l'_m} + e^{h-l'_{m+1}} + e^h\big(e^{-(1/2)(l'_m+l'_{m+1}-r'_m)_+} + (r'_m - l'_m - l'_{m+1})_+\big) + e^h \sum_{k=m+1}^{n-1} \Delta'(k) \\
&\le c + 2 + \big(1 + (r'_m + h - l'_{m+1})_+\big)e^{(1/2)(r'_m+h-l'_{m+1})_+} + (1+3c)e^{3c/2}e^h \sum_{k=m+1}^{n-1} \Delta(k) \\
&\le c + 2 + \big(1 + 2c + (r_m + h - l_{m+1})_+\big)e^c e^{(1/2)(r_m+h-l_{m+1})_+} + (1+3c)e^{3c/2}e^h \sum_{k=m+1}^{n-1} \Delta(k) \\
&\le c + 2 + (1 + 2c + \Gamma_{nm}(h))e^c e^{(1/2)\Gamma_{nm}(h)} + (1+3c)e^{3c/2}\Gamma_{nm}(h).
\end{aligned}
$$

We can conclude in any case

$$
\begin{aligned}
&\sup_{h\in[0,\min\{l_n,l'_n\}]} \ \min_{m\in[A'_n(h),\,B'_n(h)]} (\Gamma_{nm})'(h) \\
&= \sup_{h\in[0,\min\{l_n,l'_n\}]} \ \min_{m\ge 1} (\Gamma_{nm})'(h) \\
&\le \sup_{h\in[0,l_n]} \ \min_{m\ge 1} \big(c + 2 + (1 + 2c + \Gamma_{nm}(h))e^c e^{(1/2)\Gamma_{nm}(h)} + (1+3c)e^{3c/2}\Gamma_{nm}(h)\big) \\
&\le c + 2 + (1 + 2c + K)e^c e^{(1/2)K} + (1+3c)e^{3c/2}K,
\end{aligned}
$$

for every $n \ge 1$, where K only depends on δ, by Theorem 3.2 and Proposition 3.4.

If for some n we have $l_n < l'_n$ and $h \in [l_n, l'_n]$, then $(\Gamma_{nn})'(h) \le l'_n - h \le l'_n - l_n \le c$ and

$$
\sup_{h\in[l_n,l'_n]} \ \min_{m\in[A'_n(h),\,B'_n(h)]} (\Gamma_{nm})'(h) \le c.
$$

Therefore, $K' \le c + 2 + (1 + 2c + K)e^c e^{(1/2)K} + (1+3c)e^{3c/2}K$, and the conclusion holds by Theorem 3.2. □

Theorem 3.8 has the following direct consequence.

Corollary 3.9. *Let us consider two trains Ω, Ω' such that $r'_n = r_n$, and $l'_n = l_n$ for every $n \ge N$. Then Ω is hyperbolic if and only if Ω' is hyperbolic.*

Theorems 3.11 and 3.12 are simpler versions of Theorem 3.2, which can be applied in many occasions, and are obtained by replacing $\Gamma_{nm}(h)$ for $\Gamma^*_{nm}(h)$ and $\Gamma^0_{nm}(h)$, respectively. We define now these functions.

DEFINITION 3.10. Let us consider a sequence of positive numbers $\{l_n\}_{n=1}^{\infty}$ and a sequence of non-negative numbers $\{r_n\}_{n=1}^{\infty}$. Consider $n \ge 1$ and $0 \le h \le l_n$. We define

$$\Gamma_{nm}^*(h) := \begin{cases} (r_m + h - l_{m+1})_+ + e^h \displaystyle\sum_{k=m+1}^{n} e^{-l_k}, & \text{if} \quad m < n \quad \text{and} \quad l_m \le h, \\ l_m - h + e^h \displaystyle\sum_{k=m}^{n} e^{-l_k}, & \text{if} \quad m < n \quad \text{and} \quad l_m > h, \\ \min\{h, l_n - h\}, & \text{if} \quad m = n, \\ l_m - h + e^h \displaystyle\sum_{k=n}^{m} e^{-l_k}, & \text{if} \quad m > n \quad \text{and} \quad l_m > h, \\ (r_{m-1} + h - l_{m-1})_+ + e^h \displaystyle\sum_{k=n}^{m-1} e^{-l_k}, & \text{if} \quad m > n \quad \text{and} \quad l_m \le h, \end{cases}$$

and

$$\Gamma_{nm}^0(h) := \begin{cases} e^h \displaystyle\sum_{k=m+1}^{n} e^{-l_k}, & \text{if} \quad m < n \quad \text{and} \quad l_m \le h, \\ e^h \displaystyle\sum_{k=n}^{m-1} e^{-l_k}, & \text{if} \quad m > n \quad \text{and} \quad l_m \le h, \\ \Gamma_{nm}^*(h), & \text{in other case.} \end{cases}$$

The functions $\Gamma_{nm}^*(h)$ and $\Gamma_{nm}^0(h)$ are naturally associated to trains by taking $\{l_n\}_{n=1}^{\infty}$ and $\{r_n\}_{n=1}^{\infty}$ as the half-lengths of their fundamental geodesics.

Theorem 3.11. *Let us consider a train Ω such that there exists a constant $c > 0$ with $r_n \le 2c + |l_n - l_{n+1}|$ for every $n \ge 1$. Then Ω is hyperbolic if and only if*

$$K^* := \sup_{n\ge 1} \sup_{h\in[0,l_n]} \min_{m\in[A_n(h),\, B_n(h)]} \Gamma_{nm}^*(h) < \infty.$$

Furthermore, if Ω is δ-hyperbolic, then K^* is bounded by a constant which only depends on δ and c; if $K^* < \infty$, then Ω is δ-hyperbolic, with δ a constant which only depends on K^* and c.

Proof. First, let us consider the integer numbers k with $l_k + l_{k+1} \ge r_k$. The inequality $r_k - l_k - l_{k+1} \le 2c - 2\min\{l_k, l_{k+1}\}$ (which is equivalent to $r_k \le 2c + |l_k - l_{k+1}|$) gives

$$\begin{aligned} e^{-(1/2)(l_k+l_{k+1}-r_k)_+} + (r_k - l_k - l_{k+1})_+ &= e^{(1/2)(r_k-l_k-l_{k+1})} \\ &\le e^{c-\min\{l_k,l_{k+1}\}} \le e^c(e^{-l_k} + e^{-l_{k+1}}). \end{aligned}$$

And now, consider the integer numbers k with $l_k + l_{k+1} \le r_k$. The inequality $0 \le r_k - l_k - l_{k+1} \le 2c - 2\min\{l_k, l_{k+1}\}$ gives $\min\{l_k, l_{k+1}\} \le c$, and consequently

$$e^{-c} \le e^{-\min\{l_k, l_{k+1}\}}, \quad 1 \le e^c (e^{-l_k} + e^{-l_{k+1}}).$$

Hence

$$\begin{aligned} e^{-(1/2)(l_k+l_{k+1}-r_k)_+} + (r_k - l_k - l_{k+1})_+ &= 1 + r_k - l_k - l_{k+1} \\ &\le 1 + 2c \le (1+2c)e^c(e^{-l_k} + e^{-l_{k+1}}). \end{aligned}$$

Then

$$\begin{aligned} e^{-(1/2)(l_k+l_{k+1}-r_k)_+} + (r_k - l_k - l_{k+1})_+ &\le (1+2c)e^c(e^{-l_k} + e^{-l_{k+1}}), \\ e^{-l_k} + e^{-l_{k+1}} \le \Delta(k) &\le (1 + (1+2c)e^c)(e^{-l_k} + e^{-l_{k+1}}), \end{aligned}$$

for every $k \ge 1$. Hence, if we apply Theorem 3.2 we obtain the conclusion, with $\inf_{m\in[A_n(h),B_n(h)]} \Gamma^*_{nm}(h)$ instead of $\min_{m\in[A_n(h),B_n(h)]} \Gamma^*_{nm}(h)$. In order to see that the infimum is attained we can follow an argument similar to the one at the end of the proof of Theorem 3.2. □

Theorem 3.12. *Let us consider a train Ω such that there exists a constant $c > 0$ with $r_n \le c$ for every $n \ge 1$. Then Ω is hyperbolic if and only if*

$$K^0 := \sup_{n\ge1} \sup_{h\in[0,l_n]} \min_{m\in[A_n(h),B_n(h)]} \Gamma^0_{nm}(h) < \infty.$$

Furthermore, if Ω is δ-hyperbolic, then K^0 is bounded by a constant which only depends on δ and c; if $K^0 < \infty$, then Ω is δ-hyperbolic, with δ a constant which only depends on K^0 and c.

REMARK. Notice that Γ^0_{nm} is much simpler than Γ_{nm}:

Firstly, the four terms in the definition of $\Delta(k)$ are replaced by its first term.

Furthermore, in the first and fifth cases in the definition of Γ^0_{nm} we remove the first term in the corresponding definition of Γ_{nm}.

In order to obtain these simplifications, we must pay with the hypothesis $r_n \le c$, but this is a usual hypothesis: for instance, every flute surface satisfies it.

Proof. Notice that $(r_m + h - l_{m+1})_+ \le r_m \le c$ if $m = A_n(h)$ (since $l_{m+1} > h$) and $(r_{m-1} + h - l_{m-1})_+ \le r_{m-1} \le c$ if $m = B_n(h)$.

Hence, if we apply Theorem 3.11 we obtain the conclusion, with $\inf_{m\in[A_n(h),B_n(h)]} \Gamma^0_{nm}(h)$ instead of $\min_{m\in[A_n(h),B_n(h)]} \Gamma^0_{nm}(h)$.

In order to see that the infimum is attained we can follow an argument similar to the one at the end of the proof of Theorem 3.2. □

Proposition 3.13. *In any train Ω we have*

$$\min_{m\in[A_n(h),B_n(h)]} \Gamma^0_{nm}(h) = \min_{m\geq 1} \Gamma^0_{nm}(h),$$

for every $n \geq 1$ and $0 \leq h \leq l_n$.

Proof. Fix $n \geq 1$ and $0 \leq h \leq l_n$. If $m < A_n(h)$, then $\Gamma^0_{nm}(h) > \Gamma^0_{nA_n(h)}(h)$:

$$\Gamma^0_{nm}(h) \geq e^h \sum_{k=m+1}^{n} e^{-l_k} > e^h \sum_{k=A_n(h)+1}^{n} e^{-l_k} = \Gamma^0_{nA_n(h)}(h).$$

The case $m > B_n(h)$ is similar. □

Theorem 3.12 let us obtain an alternative proof of a result that appears in [3], but using now a completely new argument. It is a simple sufficient condition for the hyperbolicity.

Corollary 3.14. *Let us consider a train Ω with $l_1 \leq l^0$, $r_n \leq c_1$ for every n and*

$$\sum_{k=n}^{\infty} e^{-l_k} \leq c_2 e^{-l_n}, \quad \textit{for every} \quad n > 1. \tag{3.1}$$

Then Ω is δ-hyperbolic, where δ is a constant which only depends on c_1, c_2 and l^0.

EXAMPLES. Let us consider an increasing C^1 function f with $\lim_{x\to\infty} f(x) = \infty$, and define $l_n := f(n)$ for every n. A direct computation gives that $\{l_n\}$ satisfies (3.1) if and only if there exist constants c, M with $f'(x) \geq c > 0$ for every $x \geq M$.

Consequently, for $a, b > 0$ and $c \in \mathbb{R}$, the sequence $l_n := an^b + c$ satisfies (3.1) if and only if $b \geq 1$.

Proof. Let us consider $n \geq 1$ and $h \in [l^0, l_n]$. Since $l_1 \leq l^0 \leq h$, we have that $m = A_n(h)$ satisfies $l_m \leq h < l_{m+1}$ and

$$\Gamma^0_{nm}(h) = e^h \sum_{k=m+1}^{n} e^{-l_k} \leq e^h c_2 e^{-l_{m+1}} < c_2.$$

If $h \in [0, l^0]$, then $\Gamma^0_{nn}(h) \leq h \leq l^0$. Hence, $K^0 \leq \max\{c_2, l^0\}$, and Theorem 3.12 gives the result. □

Lemma 3.15. (1) *Let us consider a sequence $\{l_n\}$ such that $l_m \leq l_n + c$ for every positive integer number $m \leq n$. Then there exists a non-decreasing sequence $\{l'_n\}$, such that $|l_n - l'_n| \leq c$ for every n.*

(2) *Let us consider a non-decreasing sequence* $\{l'_n\}$. *If* $\{l_n\}$ *is a sequence with* $|l_n - l'_n| \le c$ *for every* n, *then* $l_m \le l_n + 2c$ *for every positive integer number* $m \le n$.

Proof. We prove now the first part of the lemma. We define a sequence $\{l'_n\}$ in the following way: $l'_n := \max\{l_1, l_2, \dots, l_n\}$. It is clear that $\{l'_n\}$ is a non-decreasing sequence. Since $l_m \le l_n + c$ for every $m = 1, 2, \dots, n$, we have $l_n \le l'_n \le l_n + c$. Consequently, $|l_n - l'_n| \le c$ for every n.

In order to prove the second part, notice that if $m \le n$, then $l_m \le l'_m + c \le l'_n + c \le l_n + 2c$. □

The two following theorems provide necessary conditions for hyperbolicity.

Theorem 3.16. *Let us consider a hyperbolic train* Ω *with* $l_m \le l_n + c_1$ *for every positive integer number* $m \le n$. *If* K *is the constant defined in* Theorem 3.2, *then*

$$r_n \le 2\max\{K, 1\} + 2\log\max\{K, 1\} + 3c_1, \quad \textit{for every} \quad n \quad \textit{with} \quad l_{n+1} > 4(K + c_1).$$

Proof. Let us define $M := \max\{K, 1\}$ and fix n with $l_{n+1} > 4(K + c_1)$.

Let us assume that $r_n \le l_{n+1}$. Consider $\varepsilon \in (0, 1/2)$ and $h_{n+1} := l_{n+1} - \varepsilon r_n$. Then

$$\begin{aligned}
&\Gamma_{n+1,n+1}(h_{n+1}) = \min\{l_{n+1} - \varepsilon r_n, \varepsilon r_n\} = \varepsilon r_n,\\
&\Gamma_{n+1,m}(h_{n+1}) \ge l_m - h_{n+1} \ge l_{n+1} - c_1 - h_{n+1} = \varepsilon r_n - c_1, \quad \text{if} \quad m > n+1,\\
&\Gamma_{n+1,n}(h_{n+1}) \ge (r_n + h_{n+1} - l_{n+1})_+ = (1-\varepsilon) r_n, \quad \text{if} \quad l_n \le h_{n+1},\\
&\Gamma_{n+1,m}(h_{n+1}) \ge e^{h_{n+1}}\Delta(n) \ge e^{l_{n+1}-\varepsilon r_n} e^{-(1/2)(l_n + l_{n+1} - r_n)} \ge e^{l_{n+1}-\varepsilon r_n} e^{-(1/2)(l_{n+1} + l_{n+1} + c_1 - r_n)}\\
&\qquad\qquad = e^{-(1/2)c_1 + ((1/2)-\varepsilon) r_n}, \quad \text{if either} \quad m < n \quad \text{or} \quad m = n \quad \text{and} \quad l_n > h_{n+1}.
\end{aligned}$$

Since $\varepsilon \in (0, 1/2)$

$$\begin{aligned}
M &\ge \min\{\varepsilon r_n, \varepsilon r_n - c_1, (1-\varepsilon) r_n, e^{-(1/2)c_1 + ((1/2)-\varepsilon) r_n}\}\\
&= \min\{\varepsilon r_n - c_1, e^{-(1/2)c_1 + ((1/2)-\varepsilon) r_n}\},
\end{aligned}$$

and we deduce

$$r_n \le \max\left\{\frac{M + c_1}{\varepsilon}, \frac{\log M + c_1/2}{1/2 - \varepsilon}\right\}.$$

Taking $\varepsilon = (M + c_1)/(2M + 2\log M + 3c_1)$ (notice that $\varepsilon \in (0, 1/2)$, since $\log M \ge 0$), we obtain the equality of the two terms inside the maximum, and therefore $r_n \le 2M + 2\log M + 3c_1$.

We prove now that $r_n \le l_{n+1}$. Seeking for a contradiction, assume that $r_n > l_{n+1}$, and consider $h^{n+1} := (3/4) l_{n+1}$. A similar argument, with h^{n+1} instead of h_{n+1}, gives:

If $l_n + l_{n+1} < r_n$, since $l_{n+1} > 4(K + c_1)$,

$$K \geq \min\left\{\frac{1}{4}l_{n+1},\ \frac{1}{4}l_{n+1} - c_1,\ \frac{3}{4}l_{n+1},\ e^{(3/4)l_{n+1}}\right\} = \frac{1}{4}l_{n+1} - c_1 > K,$$

since $l_{n+1} > 4(K + c_1)$, and this is a contradiction. If $l_n + l_{n+1} \geq r_n$, we obtain with a similar argument

$$\begin{aligned} K &\geq \min\left\{\frac{1}{4}l_{n+1},\ \frac{1}{4}l_{n+1} - c_1,\ \frac{3}{4}l_{n+1},\ e^{(1/4)l_{n+1}-(1/2)c_1}\right\} \\ &= \min\left\{\frac{1}{4}l_{n+1} - c_1,\ e^{(1/4)l_{n+1}-(1/2)c_1}\right\} > K, \end{aligned}$$

and this is the contradiction we are looking for. □

Condition $l_m \leq l_n + c_1$ for every positive integer number $m \leq n$ in Theorem 3.16 can seem superfluous, but we have examples which prove that, in fact, if it is removed, then the conclusion of the theorem is not true.

The following theorem obtains a similar inequality to (3.1) but with an explicit control of the constants involved.

Theorem 3.17. *Let us consider a hyperbolic train Ω with $l_m \leq l_n + c_1$ for every positive integer number $m \leq n$. If K is the constant defined in* Theorem 3.2, *then*

$$\sum_{k=n}^{\infty} e^{-l_k} \leq K e^{K+c_1} e^{-l_n}, \quad \textit{for every} \quad n \quad \textit{with} \quad l_n > 2K + c_1.$$

Proof. Theorem 3.2 and Proposition 3.4 give that

$$\min_{m\geq 1} \Gamma_{nm}(h) \leq K, \quad \text{for every} \quad n \geq 1 \quad \text{and} \quad h \in [0, l_n].$$

Let us fix n with $l_n > 2K + c_1$ and $n_0 \geq n$. Consider $\varepsilon > 0$ with $l_n \geq 2K + c_1 + \varepsilon$. If we define $h := l_n - K - c_1 - \varepsilon/2 \geq K + \varepsilon/2 > K$, then for any $m \geq n$ we have $l_m - h \geq l_n - h - c_1 = K + \varepsilon/2 > K$ and

$$\Gamma_{n_0 m}(h) \geq \Gamma^0_{n_0 m}(h) \geq K + \frac{\varepsilon}{2} > K.$$

If $m < n$, we obtain

$$\Gamma_{n_0 m}(h) \geq \Gamma^0_{n_0 m}(h) \geq e^h \sum_{k=n}^{n_0} e^{-l_k}.$$

Consequently,

$$K \geq \min_{m \geq 1} \Gamma_{n_0 m}(h) = \min_{1 \leq m < n} \Gamma_{n_0 m}(h) \geq e^{l_n - K - c_1 - \varepsilon/2} \sum_{k=n}^{n_0} e^{-l_k},$$

for every $n_0 \geq n$ and ε small enough. Therefore

$$K \geq e^{l_n - K - c_1} \sum_{k=n}^{\infty} e^{-l_k},$$

which finishes the proof. □

Corollary 3.14, Theorems 3.16, 3.17, 3.2 and Proposition 3.6 give the following powerful and simple characterization. In particular, this result characterizes hyperbolicity of trains for which l_n is a non-decreasing sequence.

Theorem 3.18. *Let us consider a train Ω with $l_m \leq l_n + c_1$ for every positive integer number $m \leq n$.*
(1) *If $\{l_n\}$ is a bounded sequence, then Ω is hyperbolic.*
(2) *If $\lim_{n\to\infty} l_n = \infty$, then Ω is hyperbolic if and only if $\{r_n\}$ is a bounded sequence and* (3.1) *holds for some constant c_2.*

REMARK. Note that Theorem 3.18 deals with every case under the hypothesis "$l_m \leq l_n + c_1$ for $m \leq n$": $\{l_n\}$ is either a bounded sequence or a sequence with limit ∞.

4. Trigonometric lemmas

In this section some technical lemmas are collected. All of them have been used in Section 3 in order to simplify the proof of Theorem 3.2.

DEFINITION 4.1. Given a surface M, a geodesic γ in M, and a continuous unit vector field ξ along γ, orthogonal to γ, we define the *Fermi coordinates* based on γ as the map $E(u, v) := \exp_{\gamma(u)} v\xi(u)$.

It is well known that the Riemannian metric can be expressed in Fermi coordinates as $ds^2 = dv^2 + \eta^2(u, v)\, du^2$, where $\eta(u, v)$ is the solution of the scalar equation $\partial^2\eta/\partial v^2 + K\eta = 0$, $\eta(u, 0) = 1$, $\partial\eta/\partial v(u, 0) = 0$, and K is the curvature of M (see e.g. [11, p. 247]). Consequently, if M is a non-exceptional Riemann surface, the Poincaré metric in Fermi coordinates (based on any geodesic γ) is $ds^2 = dv^2 + \cosh^2 v\, du^2$, since $K = -1$ in the Poincaré metric. We always consider in a train the Fermi coordinates based on (a_0, b_0).

DEFINITION 4.2. Let us consider Fermi coordinates (u, v) in $\mathbb{D}$. We define the distances $d_1((u_1, v_1), (u_2, v_2))$, $d_2((u_1, v_1), (u_2, v_2))$ as follows: without loss of generality we can assume that $v_1 \geq v_2$; then

$$\begin{aligned} d_1((u_1, v_1), (u_2, v_2)) &:= d((u_1, v_1), (u_1, v_2)) + d((u_1, v_2), (u_2, v_2)) \\ &= v_1 - v_2 + d((u_1, v_2), (u_2, v_2)), \\ d_2((u_1, v_1), (u_2, v_2)) &:= d((u_1, v_1), (u_2, v_1)) + d((u_2, v_1), (u_2, v_2)) \\ &= d((u_1, v_1), (u_2, v_1)) + v_1 - v_2. \end{aligned}$$

The following lemma shows that the "cartesian distances" d_1 and d_2 are comparable to d.

Lemma 4.3. *Let us consider Fermi coordinates (u, v) in $\mathbb{D}$ and the distances d_1 and d_2. Then*

$$\frac{1}{2} d_1 \leq d \leq d_1, \quad \frac{1}{3} d_2 \leq d \leq d_2.$$

Proof. Triangle inequality gives directly $d \leq d_1$ and $d \leq d_2$. Let us consider $v_1 \geq v_2$. It is easy to check that

$$d((u_1, v_1), (u_1, v_2)) \leq d((u_1, v_1), (u_2, v_2)), \quad d((u_1, v_2), (u_2, v_2)) \leq d((u_1, v_1), (u_2, v_2))$$

and this implies $d_1 \leq 2d$.

We also have $d((u_2, v_1), (u_2, v_2)) \leq d((u_1, v_1), (u_2, v_2))$, and then

$$\begin{aligned} d((u_1, v_1), (u_2, v_1)) &\leq d((u_1, v_1), (u_2, v_2)) + d((u_2, v_1), (u_2, v_2)) \\ &\leq 2d((u_1, v_1), (u_2, v_2)), \\ d_2((u_1, v_1), (u_2, v_2)) &= d((u_1, v_1), (u_2, v_1)) + d((u_2, v_1), (u_2, v_2)) \\ &\leq 3d((u_1, v_1), (u_2, v_2)). \end{aligned}$$ □

Lemma 4.4. *Let Ω be a train and l_0 any positive constant. We have*

$$d_1(z, \gamma_n \cap (a_n, b_n)) \leq 2 d_\Omega(z, (a_n, b_n)) + 2 \operatorname{Arcsinh} \frac{1}{\sqrt{2 \tanh l_0}},$$

for every $n > 0$ and $z \in \Omega$ with $l_0 \leq h(z) \leq l_n$.

Proof. Let w be the nearest point in (a_n, b_n) to z, and define $v := \gamma_n \cap (a_n, b_n)$, let v_0 be the nearest point in (a_0, b_0) to v and w_0 the nearest point in (a_0, b_0) to w.

Consider the geodesic quadrilateral in Ω^+ with vertices v, w, w_0 and v_0. Standard hyperbolic trigonometry gives that

$$\tanh d_\Omega(w, w_0) = \tanh d_\Omega(v, v_0) \cosh d_\Omega(v_0, w_0) = \tanh l_n \cosh d_\Omega(v_0, w_0).$$

Denote by v' (respectively w') the point in $\gamma_n^+ = [v, v_0] \subset \Omega^+$ (respectively in $[w, w_0] \subset \Omega^+$) with $h(v') = h(z)$ (respectively $h(w') = h(z)$). Consider the geodesic quadrilateral in Ω with vertices v', w', w_0 and v_0. Standard hyperbolic trigonometry (see e.g. [12, p. 88]) gives that

$$\begin{aligned}\sinh \frac{d_\Omega(v', w')}{2} &= \sinh \frac{d_\Omega(v_0, w_0)}{2} \cosh h(z) = \cosh h(z) \sqrt{\frac{\cosh d_\Omega(v_0, w_0) - 1}{2}} \\ &= \frac{1}{\sqrt{2}} \cosh h(z) \sqrt{\frac{\tanh d_\Omega(w, w_0)}{\tanh l_n} - 1} \le \frac{1}{\sqrt{2}} \cosh h(z) \sqrt{\frac{1}{\tanh h(z)} - 1} \\ &= \frac{1}{\sqrt{2}} \cosh h(z) \sqrt{\frac{1 - \tanh^2 h(z)}{\tanh h(z)}} = \frac{1}{\sqrt{2 \tanh h(z)}} \le \frac{1}{\sqrt{2 \tanh l_0}}.\end{aligned}$$

This fact and Lemma 4.3 imply

$$\begin{aligned}d_1(z, v) &= d_\Omega(z, v') + d_\Omega(v', v) \le d_\Omega(v', w') + d_\Omega(z, w') + d_\Omega(w', w) \\ &\le 2 \operatorname{Arcsinh} \frac{1}{\sqrt{2 \tanh l_0}} + d_1(z, w) \le 2 d_\Omega(z, w) + 2 \operatorname{Arcsinh} \frac{1}{\sqrt{2 \tanh l_0}}. \quad \square\end{aligned}$$

Lemma 4.5. *Let us consider Fermi coordinates* (u, v) *in* $\mathbb{D}$. *Fix* $u_1 < u_4$, $g_1 := \{(u, v)\colon u = u_1,\ 0 \le v \le x\}$, $g_4 := \{(u, v)\colon u = u_4,\ v \ge 0\}$, *and* g_2 *the (infinite) geodesic orthogonal to* g_1 *in* (u_1, x). *We assume that* g_2 *does not intersect* g_4. *Consider* $(u_4, h) \in g_4$, *with* $h \ge x$, *and* $(u_2, v_2) \in g_2$, *with* $d((u_2, v_2), (u_4, h)) = d(g_2, (u_4, h))$. *Then*

$$d(g_2, (u_4, h)) \le d(g_2, (u_3, h)) + d((u_3, h), (u_4, h)) \le 6 d(g_2, (u_4, h)),$$

for every $u_2 \le u_3 \le u_4$.

Proof. We only need to prove the second inequality. Fix $u_3 \in [u_2, u_4]$. Let us assume that $v_2 \le h$. Then Lemma 4.3 implies

$$\begin{aligned}&d(g_2, (u_3, h)) + d((u_3, h), (u_4, h)) \\ &\le d((u_2, v_2), (u_2, h)) + d((u_2, h), (u_3, h)) + d((u_3, h), (u_4, h)) \\ &\le d((u_2, v_2), (u_2, h)) + 2d((u_2, h), (u_4, h)) \\ &\le 2d_2((u_2, v_2), (u_4, h)) \le 6d((u_2, v_2), (u_4, h)) = 6d(g_2, (u_4, h)).\end{aligned}$$

Let us assume now that $v_2 \geq h$. Lemma 4.3 also implies

$$\begin{aligned} &d(g_2, (u_3, h)) + d((u_3, h), (u_4, h)) \\ &\leq d((u_2, v_2), (u_2, h)) + d((u_2, h), (u_3, h)) + d((u_3, h), (u_4, h)) \\ &\leq d((u_2, v_2), (u_2, h)) + 2d((u_2, h), (u_4, h)) \\ &\leq 2d_1((u_2, v_2), (u_4, h)) \leq 4d((u_2, v_2), (u_4, h)) = 4d(g_2, (u_4, h)). \end{aligned}$$

□

Lemma 4.6. *Let us define F as*

$$F(a, x) := \begin{cases} \dfrac{1}{\sinh 1} \sinh a \cosh x, & \text{if} \quad 0 \leq a \leq 1, \\ \log(\sinh a \cosh x), & \text{if} \quad a \geq 1. \end{cases}$$

Then

$$F(a, x) \leq ae^x \leq 2 \sinh a \cosh x,$$

for every $a, x \geq 0$.

Proof. The last inequality is a direct consequence of $a \leq \sinh a$ and $e^x \leq 2 \cosh x$.

If $a \geq 1$, the function $h(x) := ae^x - a - x$ satisfies $h'(x) = ae^x - 1 \geq a - 1 \geq 0$ for every $x \geq 0$. Hence, $h(x) \geq h(0) = 0$ for every $x \geq 0$, and we conclude

$$ae^x \geq a + x = \log(e^a e^x) \geq \log(\sinh a \cosh x),$$

for $a \geq 1$ and $x \geq 0$.

Since the function $H(a) := \sinh a - a \sinh 1$ is convex in $[0, 1]$, it satisfies $H(a) \leq \max\{H(0), H(1)\} = 0$ for every $0 \leq a \leq 1$. Hence,

$$ae^x \geq \frac{1}{\sinh 1} \sinh ae^x \geq \frac{1}{\sinh 1} \sinh a \cosh x,$$

for $0 \leq a \leq 1$ and $x \geq 0$. □

This result has the following direct corollary.

Corollary 4.7. *For a set $E \subset \{(a, x)\colon a, x \geq 0\}$, we have* $\operatorname{Arcsinh}(\sinh a \cosh x) \leq c_1$, *for every $(a, x) \in E$ and some constant c_1, if and only if $ae^x \leq c_2$, for every $(a, x) \in E$ and some constant c_2.*

Furthermore, if one of the inequalities holds, the constant in the other inequality only depends on the first constant.

Proposition 4.8. (1) *There exists a universal constant* c_1 *such that*

$$f(x, y, t) := \operatorname{Arccosh} \frac{\cosh t + \cosh x \cosh y}{\sinh x \sinh y}$$
$$\geq c_1(e^{-x} + e^{-y} + e^{-(1/2)(x+y-t)_+} + (t - x - y)_+),$$

for every $x, y, t \geq 0$.
(2) *For each* $l_0 > 0$, *there exists a constant* c_2, *which only depends on* l_0, *such that*

$$\operatorname{Arccosh} \frac{\cosh t + \cosh x \cosh y}{\sinh x \sinh y} \leq c_2(e^{-x} + e^{-y} + e^{-(1/2)(x+y-t)_+} + (t - x - y)_+),$$

for every $t \geq 0$ *and* $x, y \geq l_0$.

REMARK. This result is interesting by itself: if H is a right-angled hexagon in the unit disk for which three pairwise non-adjacent sides X, Y, T are given (with respective lengths x, y, t), then the opposite side of T in H has length $f(x, y, t)$ (see e.g. [12, p. 86], or the proof of Theorem 3.2).

Proof. First, we remark that if $x \geq l_0$, then $e^{-2l_0}e^{2x} \geq 1$ and $e^{2x} - 1 \geq (1 - e^{-2l_0})e^{2x}$. Therefore, if we define $c_3^{-1} := (1 - e^{-2l_0})/2$, we have for every $x \geq l_0$,

$$e^{2x} - 1 \geq 2c_3^{-1}e^{2x}, \quad \sinh x \geq c_3^{-1}e^x, \quad \coth x = 1 + \frac{2}{e^{2x} - 1} \leq 1 + c_3 e^{-2x}.$$

We also have

$$\coth x = 1 + \frac{2}{e^{2x} - 1} \geq 1 + 2e^{-2x}, \quad \text{for every} \quad x > 0.$$

Let us start with the proof of item (1).

If $f \geq 3$, then $f \geq e^{-x} + e^{-y} + e^{-(1/2)(x+y-t)_+}$. If $f \leq 3$, then $1 + (2/3)c_4^{-2} f^2 \geq \cosh f$, for some universal constant $c_4 \leq 1$, and

$$1 + \frac{2}{3}c_4^{-2} f^2 \geq \cosh f \geq 2e^{t-x-y} + \coth x \coth y$$
$$\geq 2e^{-(x+y-t)} + (1 + 2e^{-2x})(1 + 2e^{-2y}),$$
$$1 + \frac{2}{3}c_4^{-2} f^2 \geq 1 + 2(e^{-2x} + e^{-2y} + e^{-(x+y-t)_+}),$$
$$c_4^{-1} f \geq \sqrt{3}\sqrt{e^{-2x} + e^{-2y} + e^{-(x+y-t)_+}} \geq e^{-x} + e^{-y} + e^{-(1/2)(x+y-t)_+},$$
$$f \geq c_4(e^{-x} + e^{-y} + e^{-(1/2)(x+y-t)_+}),$$

where we have used the inequality $\sqrt{3}\sqrt{a + b + c} \geq \sqrt{a} + \sqrt{b} + \sqrt{c}$, for every $a, b, c \geq 0$.

This inequality is (1) if $t \le x + y$. If $t \ge x + y$, then

$$\cosh f > \frac{\cosh t}{\sinh x \sinh y} + 1 \ge 2e^{t-x-y} + 1$$
$$> \frac{4}{2}e^{t-x-y} + \frac{1}{4 \cdot 2}e^{-(t-x-y)} = \cosh(t - x - y + \log 4)$$

and $f > t - x - y + \log 4 > (t - x - y)_+ + e^{-(1/2)(x+y-t)_+}$.

Consequently we have

$$f \ge c_1(e^{-x} + e^{-y} + e^{-(1/2)(x+y-t)_+} + (t - x - y)_+),$$

for every $x, y, t \ge 0$, with $c_1 := c_4/2$, since $c_4 \le 1$.

Next, let us prove item (2). Fix $l_0 > 0$. We have seen that $\sinh x \ge c_3^{-1}e^x$ and $\coth x \le 1 + c_3e^{-2x}$, for every $x \ge l_0$.

Let us assume $t \ge x + y$. If $x, y \ge l_0$, then

$$\frac{1}{2}e^f \le \cosh f = \frac{\cosh t + \cosh x \cosh y}{\sinh x \sinh y} \le c_3^2e^{t-x-y} + \coth^2 l_0.$$

Consequently,

$$e^f \le 2c_3^2e^{t-x-y} + 2\coth^2 l_0 \le e^{t-x-y+c_5},$$

with $c_5 := \log\left(2c_3^2 + 2\coth^2 l_0\right)$, since $t - x - y \ge 0$. Hence, $f \le t - x - y + c_5 = (t - x - y)_+ + c_5e^{-(1/2)(x+y-t)_+}$, for every $t \ge 0$ and $x, y \ge l_0$ with $t \ge x + y$.

Let us assume $t \le x + y$. If $x, y \ge l_0$, then

$$1 + \frac{1}{2}f^2 \le \cosh f \le c_3^2e^{t-x-y} + \coth x \coth y \le c_3^2e^{t-x-y} + (1 + c_3e^{-2x})(1 + c_3e^{-2y}),$$
$$\frac{1}{2}f^2 \le c_3^2e^{t-x-y} + c_3e^{-2x} + c_3e^{-2y} + c_3^2e^{-2x-2y},$$
$$\frac{1}{2}f^2 \le c_3^2e^{t-x-y} + c_3e^{-2x} + c_3e^{-2y} + \frac{1}{2}c_3^2(e^{-2x} + e^{-2y}),$$
$$f^2 \le 2c_3^2e^{-(x+y-t)} + (2c_3 + c_3^2)e^{-2x} + (2c_3 + c_3^2)e^{-2y},$$
$$f^2 \le c_6^2(e^{-2x} + e^{-2y} + e^{-(x+y-t)_+}),$$
$$f \le c_6(e^{-x} + e^{-y} + e^{-(x+y-t)_+} + (t - x - y)_+),$$

where $c_6^2 := \max\{2c_3^2, 2c_3 + c_3^2\}$, for every $t \ge 0$ and $x, y \ge l_0$ with $t \le x + y$. Then we have (2) with $c_2 := \max\{1, c_5, c_6\}$. □

Proposition 4.9. *For each* $l_0 > 0$, *we have*

$$F(x, y, t, h) := \operatorname{Arcsinh} \frac{\cosh x \cosh(y-h) + \cosh t \cosh h}{\sinh y}$$
$$\asymp e^{-h+x} + e^{-(y-h-t)_+} + (t+h-y)_+,$$

for every $x, y, t, h \geq 0$, *verifying* $y \geq h \geq x$ *and* $y \geq l_0$. *Furthermore, the constants in the inequalities only depend on* l_0.

REMARK. This result is interesting by itself: if H is a right-angled hexagon in the unit disk for which three pairwise non-adjacent sides X, Y, T are given (with respective lengths x, y, t), P is the nearest point to X in Y, and P_h is the point in Y with $d(P_h, P) = h$, then $F(x, y, t, h)$ is the distance between P_h and the opposite side of Y in H (see the proof of Theorem 3.2).

Proof. We have seen that if $y \geq l_0$, and $c_3^{-1} := (1 - e^{-2l_0})/2$, we have $c_3^{-1} e^y \leq \sinh y \leq e^y/2$. We also have $e^z/2 \leq \cosh z \leq e^z$, for every $z \geq 0$.

Then $\sinh F \asymp e^{-h+x} + e^{-y+h+t}$, since $y \geq l_0$ and $y \geq h$, and the constants in the inequalities only depend on l_0.

If $h + t \leq y$, then $e^{-h+x} + e^{-y+h+t} \leq 2$, and

$$F \asymp \sinh F \asymp e^{-h+x} + e^{-(y-h-t)} = e^{-h+x} + e^{-(y-h-t)_+} + (t+h-y)_+.$$

If $h + t \geq y$, then $e^{-h+x} + e^{-y+h+t} \geq 1$, and

$$e^F \asymp \sinh F \asymp e^{-h+x} + e^{-y+h+t} \asymp e^{t+h-y} = e^{-1} e^{1+(t+h-y)_+}.$$

Since

$$F \geq \operatorname{Arcsinh} \frac{(e^x e^{y-h} + e^t e^h)/4}{e^y/2} \geq \operatorname{Arcsinh} \frac{1}{2}(e^{-h+x} + e^{-y+h+t}) \geq \operatorname{Arcsinh} \frac{1}{2} > 0,$$

and $1 + (t+h-y)_+ \geq 1 > 0$ for every $x, y, t, h \geq 0$, and $e^F \asymp e^{1+(t+h-y)_+}$ for every $x, y, t, h \geq 0$, verifying $h + t \geq y \geq h \geq x$ and $y \geq l_0$, we obtain that $F \asymp 1 + (t + h - y)_+$. Since $1 \leq e^{-h+x} + 1 = e^{-h+x} + e^{-(y-h-t)_+} \leq 2$, we also conclude that $F \asymp e^{-h+x} + e^{-(y-h-t)_+} + (t+h-y)_+$, if $h + t \geq y$. □

The following corollary can be directly deduced from this result.

Corollary 4.10. *For each* $l_0 > 0$, *let us consider a set* $E \subset \{(x, y, t, h)\colon x, y, t, h \geq 0,\ y \geq h \geq x,\ y \geq l_0\}$. *We have* $F(x, y, t, h) \leq c_1$, *for every* $(x, y, t, h) \in E$ *and some constant* c_1, *if and only if* $(t + h - y)_+ \leq c_2$, *for every* $(x, y, t, h) \in E$ *and some constant* c_2.

Furthermore, if one of the inequalities holds, the constant in the other inequality only depends on the first constant and l_0.

Obviously, we can replace condition $(t+h-y)_+ \leq c_2$ by $t+h-y \leq c_2$. We prefer the first one since F will be a distance and $(t+h-y)_+ \geq 0$.

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