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STOCHASTIC EVOLUTION EQUATIONS DRIVEN BY LÉVY PROCESSES

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Abstract

Existence of weak (martingale) solutions and pathwise uniqueness are established for stochastic evolution equations driven by Lévy processes.

1. Introduction

In this paper, we are concerned with H-valued weak (martingale) solutions and pathwise uniqueness of stochastic evolution equations driven by Lévy processes:

$$\begin{cases} dY_t = -\mathcal{A}Y_t\,dt + \sum_{i=1}^{d} \sigma_i(Y_{t-})\,dZ_t^i, \\ Y_0 = h \in H, \end{cases} \tag{1.1}$$

in the framework of a Gelfand triple:

$$V \subset H \cong H^* \subset V^*, \tag{1.2}$$

where $(H, \|\cdot\|_H)$, $(V, \|\cdot\|_V)$ are two Hilbert spaces, $\mathcal{A}\colon V \to V^*$ is a bounded coercive linear operator, σ_i, $i = 1, \dots, d$, are bounded continuous mappings from H into H, and $Z = (Z^1, \dots, Z^d)$ is a $\mathbb{R}^d$-valued Lévy process. The equation (1.1) includes examples of stochastic partial differential equations, when the operator $\mathcal{A}$ is taken to be a differential operator. While there is a great amount of work on stochastic evolution equations and stochastic partial differential equations driven by Wiener processes, there has not been much study of stochastic partial differential equations driven by jump processes, especially very little on their weak (martingale) solutions. However, there have been growing interests on the topic recently. The motivation comes from mathematical physics and financial mathematics. In many applications, jump processes provide more realistic models than continuous processes.

In [2], existence and uniqueness for solutions of stochastic reaction equations driven by Poisson random measures are given. In [7], Malliavin calculus was applied to study

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the absolute continuity of the law of the solutions of stochastic reaction equations driven by Poisson random measures. In [11], a minimal solution was obtained for the stochastic heat equation driven by non-negative Lévy noise with coefficients of polynomial growth. In [12], a weak solution is established for stochastic heat equation driven by stable noise with coefficients of polynomial growth. The weak solutions of stochastic differential equations driven by stable processes in finite dimensions were recently obtained by Bass and Chen in [4]. We particularly mention the recent book [13] on SPDEs driven by Lévy noise.

The main purpose of this paper is to study weak (martingale) solutions of equation (1.1). Because Lévy processes such as α-stable processes may not have finite second moment for every $\alpha \in (0,2)$ (they do not even have finite first moment when $\alpha \in (0,1]$), we separate the small jumps from the big ones in our approach. The existence of weak solutions are established under merely continuity assumptions on the coefficients. The idea is to approximate coefficients by functions which are Lipschitz continuous. The hard part is to establish the tightness of the approximating solutions because of the infinite dimensional feature of the space and the jumps of the driving processes. Our method of proving the tightness seems to be new even for stochastic partial differential equations driven by white noise. To identify the limit as a weak solution of equation (1.1), we follow the approach in [4]. The uniqueness of weak solutions remains largely open. However, we give some results on pathwise uniqueness beyond the Lipschitz conditions, extending the corresponding results in [18] and [6] to the jump case. We remark that, in contrast to the existing literature, our method of proving uniqueness does not require the control function $sr(s)$ (see (4.2)) to be concave.

2. Framework

For our framework and notations, we follow closely the ones in [4] and [17]. Let $(V, \|\cdot\|_V)$, $(H, \|\cdot\|_H)$ be two separable Hilbert spaces such that V is continuously and densely imbedded in H. Identifying H with its dual we have

$$V \subset H \cong H^* \subset V^*, \tag{2.1}$$

where V^* stands for the topological dual of V. Let $\mathcal{A}$ be a bounded linear operator from V to V^* satisfying the following coercivity hypothesis: There exist constants $\delta_0 > 0$ and $\lambda_0 \geq 0$ such that

$$2\langle \mathcal{A}u, u\rangle + \lambda_0 \|u\|_H^2 \geq \delta_0 \|u\|_V^2 \quad \text{for all} \quad u \in V, \tag{2.2}$$

where $\langle \mathcal{A}u, u\rangle$ denotes the inner product between $\mathcal{A}u \in V$ and $u \in V$ (or, equivalently, denotes the action of $\mathcal{A}u \in V^*$ on $u \in V$).

We point out that in general $\mathcal{A}$ is not bounded as an operator from H into H. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space equipped with a filtration $\{\mathcal{F}_t\}$ satisfying the usual conditions. Recall that an $\mathbb{R}^d$-valued Lévy process $Z = \{Z_t,\ t \ge 0\}$ with Z_0 is characterized by its Lévy exponent Φ:

$$\mathbb{E}[e^{i\xi\cdot Z_t}] = e^{-t\Phi(\xi)} \quad \text{for every} \quad \xi \in \mathbb{R}^d.$$

By Lévy–Khintchine formula (cf. [5]), Φ can be uniquely expressed as

$$\begin{aligned}\Phi(\xi) = {} & -ib\cdot\xi + \sum_{i,j=1}^{d} a_{i,j}\xi_i\xi_j \\ & + \int_{\mathbb{R}^d} (1 - e^{\xi\cdot y} + i\xi\cdot y\mathbf{1}_{\{|y|\le 1\}})J(dy), \quad \xi = (\xi_1, \dots, \xi_d) \in \mathbb{R}^d,\end{aligned} \tag{2.3}$$

where $b \in \mathbb{R}^d$ is a constant vector, $(a_{ij})_{1\le i,j\le d}$ is a non-negative definite symmetric constant matrix and J is a non-negative measure on $\mathbb{R}^d$ satisfying $\int_{\mathbb{R}^d}(1\wedge|y|^2)J(dy) < \infty$. The measure J is called the Lévy measure of Z that describes the jumps of Z. Suppose $F(x, s, \omega)$ is jointly measurable with respect to $\mathcal{B}(\mathbb{R}^d) \times \mathcal{P}$ and such that $F(0, s, \omega) = 0$ and $\int_0^t \int_{\mathbb{R}^d} |F(x, s, \omega)|J(dx)\,ds < \infty$ for every $t > 0$ a.s., where $\mathcal{B}(\mathbb{R}^d)$ is the Borel σ-field on $\mathbb{R}$ and $\mathcal{P}$ is the predictable σ-field generated by Z. The Lévy system formula (see [5]) says that

$$\sum_{0<s\le t} F(\Delta Z_s, s) - \int_0^t \int_{\mathbb{R}^d} F(x, s)(\omega)J(dx)\,ds \tag{2.4}$$

is a local martingale. Here $\Delta Z_s := Z_s - Z_{s-}$. It will be a martingale if F is bounded and $\mathbb{E}\big[\int_0^t \int_{\mathbb{R}^d} |F(x, s, \omega)|J(dx)\,ds\big] < \infty$ for every $t > 0$. When $(a_{ij}) = 0$, $b = 0$ and $J(dx) = c|x|^{-d-\alpha}$ for some $\alpha \in (0, 2)$ and $c > 0$ in (2.3), $\Phi(\xi) = a|\xi|^\alpha$ and the corresponding Lévy process Z is called a (rotationally) symmetric α-stable process in $\mathbb{R}^d$. For each $1 \le k \le d$, let μ_k be the measure concentrated on the x_k-axis defined by

$$\mu_k(A) = \begin{cases} m(A \cap L_k) & \text{if} \quad A \cap L_k \ne \emptyset, \\ 0 & \text{otherwise,} \end{cases}$$

where L_k denotes the x_k-axis and m is the 1-dimensional Lebesgue measure. When $(a_{ij}) = 0$, $b = 0$ and $J(dx) = \sum_{k=1}^d c_k|x_k|^{-d-\alpha_k}\mu_k(dx)$ for some $\alpha_k \in (0, 2)$ and $c_k > 0$, $1 \le k \le d$, in (2.3), $\Phi(\xi) = \sum_{k=1}^d a_k|\xi_k|^{\alpha_k}$ and the i-th component of the corresponding Lévy process $Z = (Z^1, \dots, Z^d)$ is a one-dimensional symmetric α_i-stable process and they are independent to each other.

Although the results of this paper hold for any Lévy process Z on $\mathbb{R}^d$, for simplicity, we assume throughout this paper that $Z := (Z^1, \dots, Z^d)$ is a Lévy process Z defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ that has $(a_{ij}) = 0$ and $b = 0$ in (2.3); that is, Z has no diffusion component nor drift.

For a separable Hilbert space L, denote by $\mathbb{D}([0,T],L)$ the space of all cadlag paths from $[0,T]$ into L. Let $\sigma_i(\cdot)\colon H \to H$, $i=1,\dots,d$, be continuous mappings. Consider the following stochastic evolution equation:

$$\begin{cases} dY_t = -\mathcal{A}Y_t\,dt + \displaystyle\sum_{i=1}^{d} \sigma_i(Y_{t-})\,dZ_t^i, \\ Y_0 = h \in H. \end{cases} \tag{2.5}$$

DEFINITION 2.1. An H-valued stochastic process $Y=\{Y_t,\ t\geq 0\}$ is said to be a pathwise solution of equation (2.5) if

(i) Y_t is progressively measurable with respect to the minimal admissible filtration generated by the Lévy process $\{Z^i,\ i=1,\dots,d\}$,

(ii) Y_t is right continuous with left limits in space H,

(iii) for any $v\in V$,

$$\langle Y_t, v\rangle = \langle h, v\rangle - \int_0^t \langle Y_s, \mathcal{A}^* v\rangle\,ds + \sum_{i=1}^{d}\int_0^t \langle \sigma_i(Y_{s-}), v\rangle\,dZ_s^i \tag{2.6}$$

almost surely for $t\geq 0$, where $\mathcal{A}^*\colon V\to V$ is the adjoint operator of $\mathcal{A}\colon V\to V$.

We say that pathwise uniqueness holds for equation (2.5) if Y^1 and Y^2 are two pathwise solutions of (2.5) driven by the same Lévy process $Z=(Z^1,\dots,Z^d)$ and with the same initial value, then

$$\mathbb{P}(Y_t^1 = Y_t^2 \text{ for every } t\geq 0) = 1.$$

DEFINITION 2.2. We say that a weak (martingale) solution to equation (2.5) exists if one can find processes $(\tilde{Y},\tilde{Z})$ on some probability space such that $\tilde{Z}$ is a Lévy process on $\mathbb{R}^d$ that has the same distribution as Z given in (2.5) and that $(\tilde{Y},\tilde{Z})$ satisfies (i)–(iii) in Definition 2.1 with $(\tilde{Y},\tilde{Z})$ in place of (Y,Z).

We end this section by two examples.

EXAMPLE 2.3 (Stochastic partial differential equations (SPDEs) driven by Lévy processes). Let D be a bounded regular domain in $\mathbb{R}^d$. Put $H=L^2(D,dx)$, and let $V=W_0^{1,2}(D)$ be the closure of $C_c^\infty(D)$ under the Sobolev norm

$$\|v\|_V^2 = \int_D v(x)^2 dx + \int_D |\nabla u(x)|^2\,dx.$$

Here $C_c^\infty(D)$ is the space of smooth functions with compact support in D. Denote by $A(x) = (a_{ij}(x))$ a matrix-valued function on D satisfying the uniform ellipticity condition:

$$\frac{1}{c} I_{d\times d} \le A(x) \le c I_{d\times d} \quad \text{for some constant} \quad c \in [1, \infty).$$

Here $I_{d\times d}$ denotes the $d \times d$ identity matrix. Let $b(x)$ be a vector field on D with $|b| \in L^p(D, dx)$ for some $p > d$. Define

$$\mathcal{A}u(x) = -\operatorname{div}(A(x)\nabla u(x)) + b(x) \cdot \nabla u(x).$$

Observe that for $u \in V$,

$$v \mapsto \mathcal{A}u(v) := \langle \mathcal{A}u, v\rangle := \int_D \nabla u(x) \cdot A(x)\nabla v(x)\, dx + \int_D (b(x) \cdot \nabla u(x))v(x)\, dx$$

defines a continuous linear functional on V. So $\mathcal{A}$: $V \to V^*$ is a bounded linear operator; moreover, condition (2.2) is fulfilled for $(H, V, \mathcal{A})$. The following SPDE

$$\frac{\partial u(t, x)}{\partial t} = \operatorname{div}(a(x)\nabla u(t, x)) - b(x) \cdot \nabla u(t, x) + \sum_{i=1}^{d} \sigma_i(u(t-, \cdot))\, dZ_t^i$$

is a stochastic evolution equation of the type (1.1).

EXAMPLE 2.4. Stochastic evolution equations associated with fractional Laplacian:

$$\begin{cases} dY_t = \Delta^{\alpha/2} Y_t\, dt + \sum_{i=1}^{d} \sigma_i(Y_{t-})\, dZ_t^i, \\ Y_0 = h \in H, \end{cases} \tag{2.7}$$

where $\Delta^{\alpha/2} := -(-\Delta)^{\alpha/2}$ is the fractional Laplacian for some $0 < \alpha < 2$, which is the infinitesimal generator of a rotationally symmetric α-stable process in $\mathbb{R}^d$. It is well-known that the Dirichlet form associated with $\Delta^{\alpha/2}$ is given by

$$\mathcal{E}(u, v) = c_{d,\alpha} \int_{\mathbb{R}^d\times\mathbb{R}^d} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{d+\alpha}}\, dx\, dy,$$
$$\mathcal{D}(\mathcal{E}) = \left\{u \in L^2(\mathbb{R}^d)\colon \int_{\mathbb{R}^d\times\mathbb{R}^d} \frac{|u(x) - u(y)|^2}{|x - y|^{d+\alpha}}\, dx\, dy < \infty\right\}.$$

To see that equation (2.7) fits into our framework, take $H = L^2(\mathbb{R}^d, dx)$, and $V = \mathcal{D}(\mathcal{E})$ with the inner product $\langle u, v\rangle = \mathcal{E}(u, v) + (u, v)_{L^2(\mathbb{R}^d)}$. Define $\mathcal{A}u = \Delta^{\alpha/2}$. Since for $u \in V$,

$$v \mapsto \langle \mathcal{A}u, v\rangle := c_{d,\alpha} \int_{\mathbb{R}^d\times\mathbb{R}^d} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{d+\alpha}}\, dx\, dy$$

is a continuous linear functional on V, $\mathcal{A}$: $V \to V^*$ is a bounded linear operator. Moreover condition (2.2) is fulfilled for $(H, V, \mathcal{A})$. See [8] for more details about the fractional Laplacian operator.

3. Existence of weak solutions

In this section we assume that the adjoint operator $\mathcal{A}^*$: $V \to V^*$ admits a complete system of eigenvectors; that is, there exists a sequence $\{e_k,\ k \geq 1\} \subset V$ that forms an orthonormal basis of the Hilbert space H and satisfy

$$\mathcal{A}^* e_k = \lambda_k e_k \quad \text{for} \quad k \geq 1.$$

Let σ_i, $i = 1, \ldots, d$, be bounded continuous maps from H into H. Let $\sigma_i^n(\,\cdot\,)$ be a sequence of Lipschitz mappings: $H \to H$ such that

$$\lim_{n\to\infty} \sigma_i^n(x) = \sigma_i(x)$$

uniformly on compact sets and $\sup_n \|\sigma_i^n\|_\infty < \infty$ for every $1 \leq i \leq d$. Here $\|\sigma_i^n\|_\infty := \sup_{x\in H} \|\sigma_i^n(x)\|_H$. Such an approximating sequence always exists, for example, choose (see [15])

$$\sigma_i^n(x) = \int_{\mathbb{R}^n} \rho_n(\xi - Q_n x)\sigma_i\left(\sum_{k=1}^n \xi_k e_k\right) d\xi,$$

where $Q_n x = (\langle x, e_1\rangle, \ldots, \langle x, e_n\rangle)$, and ρ_n is a smooth non-negative function on $\mathbb{R}^n$ such that

$$\operatorname{supp}\{\rho_n\} \subset \left\{\xi \in \mathbb{R}^n \colon |\xi|_{\mathbb{R}^n} \leq \frac{1}{n}\right\} \quad \text{and} \quad \int_{\mathbb{R}^n} \rho_n(\xi)\, d\xi = 1.$$

$\sigma_i^n(x)$ converges to $\sigma_i(x)$ uniformly on compact sets because for any compact subset $K \subset H$,

$$G := K \cup \left(\bigcup_{n=1}^\infty \left\{\sum_{k=1}^n (\langle x, e_k\rangle + \xi_k)e_k \colon x \in K \text{ and } \sum_{k=1}^n \xi_k^2 \leq n^{-2}\right\}\right)$$

is still a compact set and $\sigma_i(x)$ is uniformly continuous on G.

First we state a theorem on existence and uniqueness of pathwise solution.

Theorem 3.1. *Assume that for each $i = 1, \ldots, d$, b_i is a Lipschitz map from H to itself. Then for every $h \in H$, there exists a unique H-valued progressively measurable process $Y = \{Y_t,\ t \geq 0\}$ such that*

(i) $Y \in \mathbb{D}([0, T], H) \cap L^2([0, T], V)$ *for any* $T > 0$,

(ii)

$$Y_t = h - \int_0^t \mathcal{A}Y_s\, ds + \sum_{i=1}^d \int_0^t b_i(Y_{s-})\, dZ_s^i, \quad t \geq 0. \tag{3.1}$$

Under the Lipschitz condition, the proof of this theorem is standard. We refer the readers to [13] or [17] for proofs of similar equations.

For $M > 0$, set

$$\bar{Z}_t^M = Z_t - \sum_{0<s\leq t} \Delta Z_s \mathbf{1}_{\{|\Delta Z_s|>M\}}.$$

It is known (cf. [5]) that $Z^M := (Z^{1,M}, \ldots, Z^{d,M})$ again is a Lévy process with Lévy measure $\mathbf{1}_{\{|y|\leq M\}} J(dy)$ and each $Z^{k,M}$ is a martingale having finite moment of any order. For $h \in H$, consider the equation:

$$\begin{cases} dY_t^{n,M} = -\mathcal{A}Y_t^{n,M}\, dt + \sum_{i=1}^d \sigma_i^n(Y_{t-}^{n,M})\, d\bar{Z}_t^{i,M}, \\ Y_0^{n,M} = h. \end{cases} \tag{3.2}$$

Theorem 3.1 remains true with $\bar{Z}^{i,M}$ in place of Z^i. So equation (3.2) admits a unique pathwise solution. Define $Y_t^{n,M,k} := \langle Y_t^{n,M}, e_k\rangle$ and $h_k := \langle h, e_k\rangle$. The following is a crucial tightness result of this section.

Proposition 3.2. *Assume that $\lambda_k \geq 0$ for every $k \geq 1$ and that $\sum_{k=1}^\infty e^{-\lambda_k \delta} < \infty$ for any $\delta > 0$. Then for every $M > 0$,*

$$\lim_{m\to\infty} \sup_n \mathbb{E}\left[\sum_{k=m}^\infty \sup_{0\leq t\leq T} (Y_t^{n,M,k})^2\right] = 0. \tag{3.3}$$

Proof. For simplicity, assume $T = 1$ and $d = 1$. Write $\bar{Z}_t^i$ for $\bar{Z}_t^{i,M}$, Y_t^n for $Y_t^{n,M}$, and $Y_t^{n,k}$ for $Y_t^{n,M,k}$. In the sequel, c denotes a generic constant whose value may differ from line to line. By Ito's formula and the Lévy system formula (2.4),

$$\begin{aligned} (Y_t^{n,k})^2 &= (h_k)^2 + 2\int_0^t Y_{s-}^{n,k} \langle \sigma^n(Y_{s-}^n), e_k\rangle\, d\bar{Z}_s \\ &\quad - 2\lambda_k \int_0^t (Y_s^{n,k})^2\, ds + \sum_{0<s\leq t} \langle \sigma^n(Y_{s-}^n), e_k\rangle^2 (\Delta \bar{Z}_s)^2 \\ &= (h_k)^2 + N_t^k + M_t^k + c\int_0^t \langle \sigma^n(Y_s^n), e_k\rangle^2\, ds - 2\lambda_k \int_0^t (Y_s^{n,k})^2\, ds, \end{aligned} \tag{3.4}$$

where

$$N_t^k = 2\int_0^t Y_{s-}^{n,k}\langle\sigma^n(Y_{s-}^n), e_k\rangle\, d\bar{Z}_s, \quad c = \int_{-M}^{M} x^2 J(dx),$$

and M_t^k is a purely discontinuous square integrable martingale with

$$[M^k]_t = \sum_{0<s\le t}\langle\sigma^n(Y_{s-}^n), e_k\rangle^4|\Delta\bar{Z}_s|^4.$$

Using the Davis' inequality (see [9]), we have

$$\begin{aligned}\mathbb{E}\Big[\sup_{0\le u\le t}|M_u^k|\Big] &\le 6\sqrt{2}\mathbb{E}[\sqrt{[M^k]_t}] \le 6\sqrt{2}\mathbb{E}\Big[\sum_{0<s\le t}\langle\sigma^n(Y_{s-}^n), e_k\rangle^2|\Delta\bar{Z}_s|^2\Big]\\ &= 6\sqrt{2}\mathbb{E}\Big[\int_0^t\int_{-M}^{M}\langle\sigma^n(Y_{s-}^n), e_k\rangle^2|y|^2 J(dy)\, ds\Big]\\ &\le c\int_0^t \mathbb{E}[\langle\sigma^n(Y_s^n), e_k\rangle^2]\, ds.\end{aligned}\tag{3.5}$$

Similarly,

$$\begin{aligned}\mathbb{E}\Big[\sup_{0\le u\le t}|N_u^k|\Big] &\le c\int_0^t \mathbb{E}[|Y_s^{n,k}\langle\sigma^n(Y_s^n), e_k\rangle|]\, ds\\ &\le c\int_0^t \mathbb{E}[\langle\sigma^n(Y_s^n), e_k\rangle^2]\, ds + c\int_0^t \mathbb{E}[(Y_s^{n,k})^2]\, ds.\end{aligned}\tag{3.6}$$

Noting that λ_k are non-negative, it follows from (3.4)–(3.6),

$$\mathbb{E}\Big[\sup_{0\le u\le t}(Y_u^{n,k})^2\Big] \le (h_k)^2 + c\int_0^t \mathbb{E}[(Y_s^{n,k})^2]\, ds + c\int_0^t \mathbb{E}[\langle\sigma^n(Y_s^n), e_k\rangle^2]\, ds.$$

Applying the Gronwall's inequality we obtain that

$$\mathbb{E}\Big[\sup_{0\le u\le t}(Y_u^{n,k})^2\Big] \le e^{ct}\Big(c\int_0^t \mathbb{E}[\langle\sigma^n(Y_s^n), e_k\rangle^2]\, ds + (h_k)^2\Big).\tag{3.7}$$

Hence,

$$\begin{aligned}\sum_{k=m}^{\infty}\mathbb{E}\Big[\sup_{0\le u\le t}(Y_u^{n,k})^2\Big] &\le e^{ct}\Big(c\int_0^t\sum_{k=m}^{\infty}\mathbb{E}[\langle\sigma^n(Y_s^n), e_k\rangle^2]\, ds + \sum_{k=m}^{\infty}(h_k)^2\Big)\\ &\le e^{ct}\Big(c\|\sigma^n\|_\infty^2 t + \sum_{k=m}^{\infty}(h_k)^2\Big).\end{aligned}\tag{3.8}$$

On the other hand, it also follows from (3.4) that

$$(Y_t^{n,k})^2 \leq (h_k)^2 + \sup_{0\leq u\leq 1}|M_u^k| + \sup_{0\leq u\leq 1}|N_u^k| + c\int_0^1 \langle \sigma^n(Y_s^n), e_k\rangle^2\, ds \\ - 2\lambda_k \int_0^t (Y_s^{n,k})^2\, ds.$$

By the Gronwall's inequality this implies that

$$(Y_t^{n,k})^2 \leq e^{-2\lambda_k t}\biggl[(h_k)^2 + \sup_{0\leq u\leq 1}|M_u^k| + \sup_{0\leq u\leq 1}|N_u^k| + c\int_0^1 \langle \sigma^n(Y_s^n), e_k\rangle^2\, ds\biggr]. \tag{3.9}$$

Hence, for any $\delta > 0$, we have

$$\sup_{\delta\leq t\leq 1}(Y_t^{n,k})^2 \leq e^{-2\lambda_k \delta}\biggl[(h_k)^2 + \sup_{0\leq u\leq 1}|M_u^k| + \sup_{0\leq u\leq 1}|N_u^k| + c\int_0^1 \langle \sigma^n(Y_s^n), e_k\rangle^2\, ds\biggr].$$

Consequently, by (3.5) and (3.6)

$$\begin{aligned}\mathbb{E}\Bigl[\sup_{\delta\leq t\leq 1}(Y_t^{n,k})^2\Bigr] &\leq e^{-2\lambda_k\delta}\biggl((h_k)^2 + 3c\mathbb{E}\Bigl[\int_0^1 \langle \sigma^n(Y_s^n), e_k\rangle^2\, ds\Bigr] + c\mathbb{E}\Bigl[\int_0^1 (Y_s^{n,k})^2\, ds\Bigr]\biggr)\\ &\leq ce^{-2\lambda_k\delta}\biggl((h_k)^2 + \mathbb{E}\Bigl[\int_0^1 \langle \sigma^n(Y_s^n), e_k\rangle^2\, ds\Bigr]\biggr),\end{aligned}$$

where in the last inequality we used (3.7) with $t = 1$. Hence,

$$\sum_{k=m}^{\infty}\mathbb{E}\Bigl[\sup_{\delta_0\leq t\leq 1}(Y_t^{n,k})^2\Bigr] \leq c\sum_{k=m}^{\infty} e^{-2\lambda_k\delta_0}. \tag{3.10}$$

Given any $\varepsilon > 0$. By (3.8), there is $\delta_0 \in (0, 1)$ so that

$$\sum_{k=m}^{\infty}\mathbb{E}\Bigl[\sup_{0\leq u\leq \delta_0}(Y_u^{n,k})^2\Bigr] \leq \frac{\varepsilon}{4} + 2\sum_{k=m}^{\infty}(h_k)^2 \quad \text{for every} \quad m \geq 1. \tag{3.11}$$

For the fixed δ_0, it follows from (3.10) and the assumption that $\sum_{k=1}^{\infty} e^{-\lambda_k\delta_0} < \infty$, that there exists $m_1 \geq 1$ so that

$$\sum_{k=m_1}^{\infty}\mathbb{E}\Bigl[\sup_{\delta_0\leq t\leq 1}(Y_t^{n,k})^2\Bigr] \leq \frac{\varepsilon}{4}. \tag{3.12}$$

Combining (3.11) and (3.12), we can find $m_0 \geq m_1$ so that $\sum_{k=m_0}^{\infty}\mathbb{E}[\sup_{0\leq t\leq 1}(Y_t^{n,k})^2] \leq \varepsilon$. Since ε is arbitrary, (3.3) follows. $\square$

For every $h \in H$, consider the equation:

$$\begin{cases} dY_t^n = -\mathcal{A}Y_t^n\,dt + \sum_{i=1}^d \sigma_i^n(Y_{t-}^n)\,dZ_t^i, \\ Y_0^n = h. \end{cases} \tag{3.13}$$

Since σ^n is Lipschitz, the existence of the solution of the above equation is guaranteed by Theorem 3.1. Let $\mathbb{P}_n$ denote the law of Y^n on $\mathbb{D}([0,\infty), H)$.

Proposition 3.3. *Assume $\lambda_k \geq 0$ for every $k \geq 1$ and that $\sum_{k=1}^\infty e^{-\lambda_k \delta} < \infty$ for any $\delta > 0$. Then the family $\{\mathbb{P}_n, n \geq 1\}$ is tight on $\mathbb{D}([0,\infty), H)$.*

Proof. Let $V_t^{n,k} := \langle Y_t^n, e_k\rangle$ and $h_k = \langle h, e_k\rangle$. It follows from (3.13) that

$$V_t^{n,k} = h_k - \lambda_k \int_0^t V_s^{n,k}\,ds + \sum_{i=1}^d \int_0^t \langle \sigma_i^n(Y_{s-}^n), e_k\rangle\,dZ_s^i.$$

Suppose $f \in C_c^2(\mathbb{R}^m)$. Applying Ito's formula and using the Lévy system formula, we obtain that

$$\begin{aligned} &f(V_t^{n,1}, \dots, V_t^{n,m}) - f(V_0^{n,1}, \dots, V_0^{n,m}) \\ &\quad = M_t^f - \sum_{k=1}^m \lambda_k \int_0^t \frac{\partial f}{\partial x_k}(V_s^{n,1}, \dots, V_s^{n,m}) V_s^{n,k}\,ds \\ &\qquad + \sum_{i=1}^d \int_0^t \biggl(\int_{\mathbb{R}^d \setminus \{0\}} \biggl(f(V_{s-}^{n,1} + \langle \sigma_i^n(Y_{s-}^n), e_1\rangle w, \dots, V_{s-}^{n,m} + \langle \sigma_i^n(Y_{s-}^n), e_m\rangle w) \\ &\qquad\qquad - f(V_{s-}^{n,1}, \dots, V_{s-}^{n,m}) \\ &\qquad\qquad - \sum_{k=1}^m \frac{\partial f}{\partial x_k}(V_{s-}^{n,1}, \dots, V_{s-}^{n,m}) \langle \sigma_i^n(Y_{s-}^n), e_k\rangle w \mathbf{1}_{\{|w|\leq 1\}} \biggr) J(dx) \biggr) ds, \end{aligned} \tag{3.14}$$

where M_t^f is a martingale. Since the integrand inside the integral of the bounded variation term in (3.14) is uniformly bounded, there exists a constant $c_{f,m}$ such that

$$f(V_t^{n,1}, \dots, V_t^{n,m}) - f(V_0^{n,1}, \dots, V_0^{n,m}) - c_{f,m} t$$

is a supermartingale. We can apply the argument in [3] to conclude that for any bounded stopping time τ_n and $\eta > 0$,

$$\lim_{\delta \to 0} \sup_{n \geq 1} \mathbb{P}\biggl(\sup_{\tau_n \leq s \leq \tau_n + \delta} \|\hat{Y}_s^{n,m} - \hat{Y}_{\tau_n}^{n,m}\|_H \geq \eta \biggr) = 0, \tag{3.15}$$

where $\hat{Y}_t^{n,m} := \sum_{k=1}^m V_t^{n,k} e_k$. Clearly for each fixed t, $\hat{Y}_t^{n,m}$ converges in H to Y_t^n. Next we show that the tails of the processes Y^n is uniformly small in the sense that for any $\varepsilon > 0$,

$$\lim_{m\to\infty} \sup_{n\geq 1} \mathbb{P}\left(\sum_{k=m}^{\infty} \sup_{0\leq t\leq T} (V_t^{n,k})^2 > \varepsilon \right) = 0. \tag{3.16}$$

For $M > 0$, define

$$\tau_M = \inf\{t > 0\colon |\Delta Z_t| \geq M\}.$$

Let $Y^{n,M}$ be the strong solution of (3.2) and $Y_t^{n,M,k} := \langle Y_t^{n,M}, e_k\rangle$. Then $Y_t^n = Y_t^{n,M}$ and consequently $V_t^{n,k} = Y_t^{n,M,k}$ for $t < \tau_M$ and $k \geq 1$. For any given $T > 0$ and $\delta > 0$, since $\lim_{M\to\infty} \tau_M = \infty$ a.s., there is $M_0 > 0$ so that $\mathbb{P}(\tau_{M_0} \leq T) \leq \delta/2$. Observe that

$$\begin{aligned}
&\mathbb{P}\left(\sum_{k=m}^{\infty} \sup_{0\leq t\leq T} (V_t^{n,k})^2 > \varepsilon \right) \\
&\leq \mathbb{P}\left(\sum_{k=m}^{\infty} \sup_{0\leq t\leq T} (V_t^{n,k})^2 > \varepsilon,\ T < \tau_{M_0} \right) + \mathbb{P}(T \geq \tau_{M_0}) \\
&= \mathbb{P}\left(\sum_{k=m}^{\infty} \sup_{0\leq t\leq T} (Y_t^{n,M_0,k})^2 > \varepsilon,\ T < \tau_{M_0} \right) + \mathbb{P}(T \geq \tau_{M_0}) \\
&\leq \frac{1}{\varepsilon}\mathbb{E}\left[\sum_{k=m}^{\infty} \sup_{0\leq t\leq T} (Y_t^{n,M_0,k})^2 \right] + \frac{\delta}{2}.
\end{aligned}$$

So by Proposition 3.2,

$$\lim_{m\to\infty} \sup_{n\geq 1} \mathbb{P}\left(\sum_{k=m}^{\infty} \sup_{0\leq t\leq T} (Y_t^{n,k})^2 > \varepsilon \right) \leq \frac{\delta}{2}.$$

Since δ is arbitrary, (3.16) follows. Now (3.15) together with (3.16) implies that

$$\lim_{\delta\to 0} \sup_{n\geq 1} \mathbb{P}\left(\sup_{\tau_n \leq s\leq \tau_n+\delta} \|Y_s^n - Y_{\tau_n}^n\|_H \geq \eta \right) = 0.$$

This and (3.16) implies that the law of the processes $\{Y_t^n;\ t\in [0, T]\}$, $n \geq 1$, is tight in $\mathbb{D}([0,\infty), H)$. This gives the tightness of $\{\mathbb{P}_n, n \geq 1\}$. □

Theorem 3.4. *Assume that*

(i) $\lambda_k \geq 0$ *for every* $k \geq 1$ *and* $\sum_{k=1}^{\infty} e^{-\lambda_k \delta} < \infty$ *for any* $\delta > 0$,

(ii) σ_i *are bounded and continuous.*

Then for every $h \in H$, there exists a weak solution to stochastic evolution equation:

$$\begin{cases} dY_t = -\mathcal{A}Y_t\, dt + \sum_{i=1}^{d} \sigma_i(Y_{t-})\, dZ_t^i, \\ Y_0 = h. \end{cases} \tag{3.17}$$

Proof. Let $U_t = \sum_{0<s\le t} \Delta Z_s \mathbf{1}_{\{|\Delta Z_s|>1\}}$ and $\bar{Z}_t = Z_t - U_t$. Write $U_t = (U_t^1, \dots, U_t^d)$ and $\bar{Z}_t = (\bar{Z}_t^1, \dots, \bar{Z}_t^d)$. Let Y^n be the strong solution of equation (3.13). Let μ_n be the law of the process $(Y^n, \bar{Z}, U)$. It follows from Proposition 3.3 that $\{\mu_n,\ n \ge 1\}$ is tight on $\mathbb{D}([0,\infty), H \times \mathbb{R}^d \times \mathbb{R}^d)$. Let $\tilde{\mathbb{P}}$ be the limit of a convergent subsequence μ_{n_k}. We will show that the canonical coordinate process $(\tilde{Y}, \hat{Z}, \tilde{U})$ under $\tilde{\mathbb{P}}$ is a solution of the following equation:

$$\begin{cases} d\tilde{Y}_t = -\mathcal{A}\tilde{Y}_t\, dt + \sum_{i=1}^{d} \sigma_i(\tilde{Y}_{t-})\, d\tilde{Z}_t^i, \\ \tilde{Y}_0 = h, \end{cases} \tag{3.18}$$

where $\tilde{Z} := \hat{Z} + \tilde{U}$ has the same distribution as the Lévy process Z. By the Skorokhod theorem, we can find a probability space $(\Omega', \mathcal{F}', \mathbb{P}')$ and processes $\{(\tilde{Y}^n, \hat{Z}^n, \tilde{U}^n),\ n \ge 1\}$ and $(\tilde{Y}, \hat{Z}, \tilde{U})$ such that the law of $\{(\tilde{Y}^n, \hat{Z}^n, \tilde{U}^n),\ n \ge 1\}$ under $\mathbb{P}'$ is the same as the law of $(Y^n, \bar{Z}, U)$ under $\mathbb{P}$, and the law of $(\tilde{Y}, \hat{Z}, \tilde{U})$ under $\mathbb{P}'$ is the same as the law of the canonical process under $\tilde{\mathbb{P}}$. Moreover $\{(\tilde{Y}^n, \hat{Z}^n, \tilde{U}^n),\ n \ge 1\}$ converges to $(\tilde{Y}, \hat{Z}, \tilde{U})$ in the space $\mathbb{D}([0,\infty), H \times \mathbb{R}^d \times \mathbb{R}^d)$. Clearly under $\mathbb{P}'$, each component of $\tilde{Z}_t := \hat{Z}_t + \tilde{U}_t$ has the same law as the Lévy process Z. It suffices to show that $\tilde{Y}$ solves equation (3.18). To this end, we need to prove that for every $k \ge 1$,

$$\begin{aligned} \langle \tilde{Y}_t, e_k\rangle &= h_k - \lambda_k \int_0^t \langle \tilde{Y}_s, e_k\rangle\, ds \\ &\quad + \sum_{i=1}^{d} \int_0^t \langle \sigma_i(\tilde{Y}_{s-}), e_k\rangle\, d\hat{Z}_s^i + \sum_{i=1}^{d} \int_0^t \langle \sigma_i(\tilde{Y}_{s-}), e_k\rangle\, d\tilde{U}_s^i. \end{aligned} \tag{3.19}$$

This can be done along the same line as the proof of Theorem 4.3 in [4]. One only needs to notice that in view of (3.15) and (3.16), for any $\varepsilon > 0$ there exists a compact subset $K \subset H$ such that

$$\mathbb{P}(\tilde{Y}_s^n \in K \text{ for all } t \le T) \ge 1 - \varepsilon \tag{3.20}$$

for all $n \ge 1$. We refer to [4] for details. □

4. Pathwise uniqueness

Theorem 3.1 gives existence and uniqueness of pathwise solution to equation (1.1) under the Lipschitz condition of σ_i. We will show in this section that pathwise uniqueness to equation (1.1) still hold under a condition that is weaker than Lipschitz continuity. Consider the following condition:

(H.1) There exist constants $\delta \in (0,1)$ and $C < \infty$ such that

$$\|\sigma_i(y_1) - \sigma_i(y_2)\|_H^2 \le C\|y_1 - y_2\|_H^2 r(\|y_1 - y_2\|_H^2) \tag{4.1}$$

for every $i = 1, \dots, d$ and $y_1, y_2 \in H$ with $\|y_1 - y_2\|_H \le \delta$, where $r(\cdot)\colon (0,\delta] \to [0,\infty)$ is a C^1 function satisfying

$$sr(s) \quad \text{decreases to} \quad 0 \quad \text{as} \quad s \downarrow 0 \quad \text{and} \quad \int_0^a \frac{1}{sr(s)}\, ds = +\infty, \tag{4.2}$$

for every $a \in (0, \delta]$.

Theorem 4.1. *Assume* (H.1) *and that one of the following conditions is fulfilled.*
(i) $s \mapsto sr(s)$ *is concave on* $(0,\delta]$;
(ii) $g(s) := r(s) + sr'(s) - 1$ *is non-negative on* $(0,\delta]$.
Then the pathwise uniqueness holds for the stochastic evolution equation (1.1).

Proof. As before, define

$$\bar{Z}_t = Z_t - \sum_{0<s\le t} \Delta Z_s \mathbf{1}_{\{|\Delta Z_s|>1\}}. \tag{4.3}$$

Then any pathwise solution $Y = \{Y_t,\ t \ge 0\}$ of the equation (1.1) can be constructed uniquely from the solution of the following equation by adding back the jumps of Z of size larger than 1:

$$\begin{cases} dX_t = -\mathcal{A}X_t\, dt + \displaystyle\sum_{i=1}^d \sigma_i(X_{t-})\, d\bar{Z}_t^i, \\ X_0 = h. \end{cases} \tag{4.4}$$

Therefore, it is sufficient to prove the pathwise uniqueness for equation (4.4). Let $X^1 = \{X_t^1,\ t \ge 0\}$ and $X^2 = \{X_t^2,\ t \ge 0\}$ be any two solutions of equation (4.4). Let $\xi_t :=$

$\|X_t^1 - X_t^2\|_H^2$. By Ito's formula, we have

$$\xi_t = -2\int_0^t \langle X_s^1 - X_s^2, \mathcal{A}(X_s^1 - X_s^2)\rangle\,ds + 2\sum_{i=1}^d \int_0^t \langle X_{s-}^1 - X_{s-}^2, \sigma_i(X_{s-}^1) - \sigma_i(X_{s-}^2)\rangle\,d\bar{Z}_s^i + \sum_{i=1}^d \sum_{0<s\le t} \|(\sigma_i(X_{s-}^1) - \sigma_i(X_{s-}^2))\Delta\bar{Z}_s^i\|_H^2. \tag{4.5}$$

Assume first that condition (i) holds. Define

$$\tau = \inf\{t > 0 \colon \|X_t^1 - X_t^2\|_H \ge \delta\}.$$

By virtue of (2.2), (H.1) and the Lévy system formula, it follows from (4.5) that

$$\begin{aligned}
\mathbb{E}[\xi_{t\wedge\tau}] &\le \mathbb{E}\biggl[\int_0^{t\wedge\tau} (\lambda_0\|X_s^1 - X_s^2\|_H^2 - \delta_0\|X_s^1 - X_s^2\|_V^2)\,ds\biggr] \\
&\quad + \sum_{i=1}^d \mathbb{E}\biggl[\sum_{0<s\le t\wedge\tau} \|(\sigma_i(X_{s-}^1) - \sigma_i(X_{s-}^2))\Delta\bar{Z}_s^i\|_H^2\biggr] \\
&\le \lambda_0\mathbb{E}\biggl[\int_0^{t\wedge\tau} \xi_s\,ds\biggr] + c\sum_{i=1}^d \mathbb{E}\biggl[\int_0^{t\wedge\tau} \|(\sigma_i(X_s^1) - \sigma_i(X_s^2))\|_H^2\,ds\biggr] \\
&\le \lambda_0\mathbb{E}\biggl[\int_0^t \xi_{s\wedge\tau}\,ds\biggr] + c\mathbb{E}\biggl[\int_0^t \xi_{s\wedge\tau} r(\xi_{s\wedge\tau})\,ds\biggr] \\
&\le \lambda_0 \int_0^t \mathbb{E}[\xi_{s\wedge\tau}]\,ds + c\int_0^t \mathbb{E}[\xi_{s\wedge\tau}] r(\mathbb{E}[\xi_{s\wedge\tau}])\,ds
\end{aligned} \tag{4.6}$$

where in the last inequality condition (i) and the Jensen's inequality are used. Now applying a generalized version of the Grownwall's inequality (cf. [10]), we conclude that $\mathbb{E}[\xi_{t\wedge\tau}] = 0$ and so $\xi_{t\wedge\tau} = 0$. This implies $\tau = \infty$ and $X^1 = X^2$.

Now assume condition (ii) holds. Let C be the constant appeared in the hypothesis (H.1). Choose $a \in (0, 1)$ small enough so that

$$a^2\delta^2 + 2C\,d(a\delta)^2\sqrt{r(a^2\delta^2)} + d(a\delta)^2 r((a\delta)^2) < \delta, \tag{4.7}$$

and define

$$\tau_1 = \inf\{t > 0 \colon \|X_t^1 - X_t^2\|_H \ge a\delta\}.$$

For any $\rho > 0$, set

$$\Phi_\rho(y) = \exp\biggl(\int_0^y \frac{1}{sr(s) + \rho}\,ds\biggr).$$

Then by (H.1), for every $y \in (0, \delta]$, $\Phi_\rho(y) \to \Phi_0(y) = +\infty$ as $\rho \to 0$. Moreover for $0 < y < \delta$,

$$\Phi'_\rho(y) = \Phi_\rho(y)\frac{1}{yr(y) + \rho} \tag{4.8}$$

and

$$\Phi''_\rho(y) = \Phi_\rho(y)\frac{1}{(yr(y) + \rho)^2}(1 - r(y) - yr'(y)) \le 0. \tag{4.9}$$

By Ito's formula,

$$\begin{aligned}
\Phi_\rho(\xi_{t\wedge\tau_1}) = 1 + 2\sum_{i=1}^d \int_0^{t\wedge\tau_1} & \Phi'_\rho(\xi_{s-})\langle X^1_{s-} - X^2_{s-}, \sigma_i(X^1_{s-}) - \sigma_i(X^2_{s-})\rangle\, d\bar{Z}^i_s \\
& - 2\int_0^{t\wedge\tau_1} \Phi'_\rho(\xi_{s-})\langle X^1_{s-} - X^2_{s-}, \mathcal{A}(X^1_{s-} - X^2_{s-})\rangle\, ds \\
& + \sum_{i=1}^d \sum_{0<s\le t\wedge\tau_1} \Phi'_\rho(\xi_{s-})\|(\sigma_i(X^1_{s-}) - \sigma_i(X^2_{s-}))\Delta\bar{Z}^i_s\|^2_H \\
& + \sum_{0<s\le t\wedge\tau_1} (\Phi_\rho(\xi_s) - \Phi_\rho(\xi_{s-}) - \Phi'_\rho(\xi_{s-})\Delta\xi_s).
\end{aligned} \tag{4.10}$$

By (2.2), we have

$$-2\Phi'_\rho(\xi_s)\langle X^1_s - X^2_s, \mathcal{A}(X^1_s - X^2_s)\rangle \le \lambda_0\Phi_\rho(\xi_s)\frac{\xi_s}{\xi_s r(\xi_s) + \rho} - \delta_0\Phi'_\rho(\xi_s)\|\xi_s\|^2_V. \tag{4.11}$$

By virtue of (H.1), (4.8) and the Lévy system formula, it follows that

$$\begin{aligned}
& \sum_{i=1}^d \mathbb{E}\left[\sum_{0<s\le t\wedge\tau_1} \Phi'_\rho(\xi_{s-})\|(\sigma_i(X^1_{s-}) - \sigma_i(X^2_{s-}))\Delta\bar{Z}^i_s\|^2_H\right] \\
& = c\sum_{i=1}^d \mathbb{E}\left[\int_0^{t\wedge\tau_1} \Phi'_\rho(\xi_s)\|\sigma_i(X^1_s) - \sigma_i(X^2_s)\|^2_H\, ds\right] \\
& \le c\mathbb{E}\left[\int_0^{t\wedge\tau_1} \Phi_\rho(\xi_s)\frac{\xi_s r(\xi_s)}{\xi_s r(\xi_s) + \rho}\, ds\right] \\
& \le c\mathbb{E}\left[\int_0^{t\wedge\tau_1} \Phi_\rho(\xi_s)\, ds\right].
\end{aligned} \tag{4.12}$$

Note that for $s \le \tau_1$,

(4.13)

$$
\begin{aligned}
|\Delta\xi_s| &\le 2\sum_{i=1}^{d} |\langle X^1_{s-} - X^2_{s-}, \sigma_i(X^1_{s-}) - \sigma_i(X^2_{s-})\rangle \Delta \bar{Z}^i_s| + \sum_{i=1}^{d} \|(\sigma_i(X^1_{s-}) - \sigma_i(X^2_{s-}))\Delta \bar{Z}^i_s\|^2_H \\
&\le 2d(a\delta)^2\sqrt{r((a\delta)^2)} + d(a\delta)^2 r((a\delta)^2),
\end{aligned}
$$

where we have used assumption (H.1). Consequently, for $\theta \in [0, 1]$ and $s \le \tau_1$,

$$
\xi_{s-} + \theta|\Delta\xi_s| \le (a\delta)^2 + 2d(a\delta)^2\sqrt{r((a\delta)^2)} + d(a\delta)^2 r((a\delta)^2) < \delta, \tag{4.14}
$$

according to the choice of a. It follows from the mean value theorem and (4.9) that for $s \le \tau_1$,

$$
\Phi_\rho(\xi_s) - \Phi_\rho(\xi_{s-}) - \Phi'_\rho(\xi_{s-})\Delta\xi_s = \Phi''_\rho(\xi_{s-} + \theta\Delta\xi_s)(\Delta\xi_s)^2 \le 0. \tag{4.15}
$$

Taking expectation on both sides of (4.10) and using (4.11), (4.12) and (4.15), we obtain that

$$
\mathbb{E}[\Phi_\rho(\xi_{t\wedge\tau_1})] \le 1 + C\int_0^t \mathbb{E}[\Phi_\rho(\xi_{s\wedge\tau_1})]\, ds.
$$

By Gronwall's inequality, $\mathbb{E}[\Phi_\rho(\xi_{t\wedge\tau_1})] \le e^{Ct}$. Letting $\rho \to 0$, we have $\mathbb{E}[\Phi_0(\xi_{t\wedge\tau_1})] \le e^{Ct}$. This implies that $\xi_{t\wedge\tau_1} = 0$ for all $t \ge 0$ and consequently $\tau_1 = \infty$. The pathwise uniqueness of (1.1) holds. □

REMARK 4.2. Condition (i) in Theorem 4.1 is the same as the condition appeared in [18] for the pathwise uniqueness result of SDE on $\mathbb{R}^d$ driven by Brownian motion. Condition (ii) in Theorem 4.1 seems to be new. To the best knowledge of the authors, the pathwise results given by Theorem 4.1 for stochastic evolution equations driven by discontinuous Lévy processes are new even in the finite dimensional case. Examples of function r satisfying the conditions of Theorem 4.1 include $r(s) = \log(1/s)$, $r(s) = \log(1/s)\log\log(1/s)$, etc.

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