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# A POLYNOMIAL INVARIANT OF VIRTUAL MAGNETIC LINK DIAGRAMS

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**Abstract**

We introduce a polynomial invariant of virtual magnetic link diagrams. For virtual links, this polynomial equals the polynomial invariant defined in [3]. We show that it gives lower bounds of the classical crossing number and the virtual crossing number of virtual links. Also we give various properties of this polynomial and examples.

## 1. Introduction

Virtual knot theory is introduced by L.H. Kauffman as a generalization of classical knot theory in the sense that if two classical link diagrams are equivalent as virtual links, then they are equivalent as classical links [8]. A virtual link diagram is a link diagram in $\mathbb{R}^2$ possibly with some encircled crossings without over/under information, called virtual crossings. A virtual link is the equivalence class of such a link diagram by generalized Reidemeister moves, which consist of (classical) Reidemeister moves of type $R_1, R_2$ and $R_3$ and virtual Reidemeister moves of type $VR_1$, $VR_2$, $VR_3$ and the *semivirtual move* $VR_4$ as shown in Fig. 1. In [4], A. Ishii, N. Kamada and S. Kamada introduce a *virtual magnetic link diagram* as an oriented link diagram which may have virtual crossings and nodes shown in Fig. 2. Two virtual magnetic link diagrams are equivalent if they are related by a finite sequence of the generalized Reidemeister moves or nodal moves which consist of $N_1$-moves and $N_2$-moves as in Fig. 3. The nine moves, which are generalized Reidemeister moves and nodal moves, are called the *basic local* moves in this paper. A *virtual magnetic link* is the equivalence class of a virtual magnetic link diagram. Kauffman [7] used such a diagram to give an oriented state model for the Jones polynomial of classical links. Note that a virtual magnetic link diagram can be regarded as a generalized virtual link diagram because a virtual magnetic link diagram without nodes is a virtual link diagram.

Until now, many invariants of virtual links are defined without regard to virtual crossings, for example, the fundamental group, the Alexander polynomial, the Jones polynomial, the quantum link invariants and the Vassiliev invariant, see [1, 2, 8, 10]. So, in many cases a classical invariant extends to an invariant of virtual links that is

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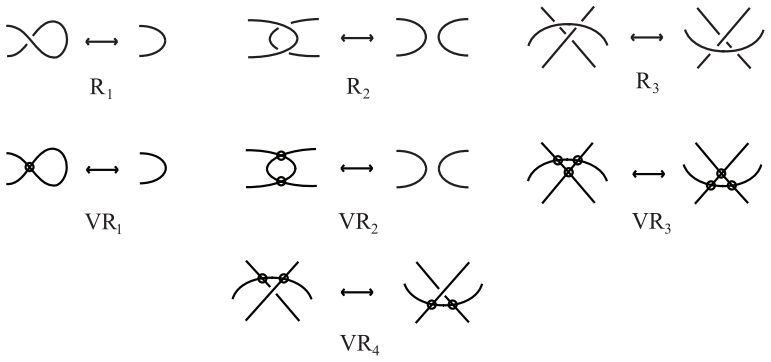



Fig. 1.

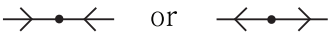



Fig. 2.

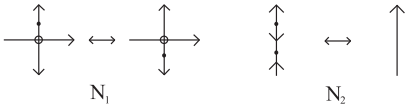



Fig. 3.

an extension of ideas from classical knot theory. However, in some cases one has an invariant of virtual links that vanishes for classical links. These are the polynomial invariants studied by Sawollek [11], Silver and Williams [12] and by Kauffman and Radford [9], which are eventually the same.

Also, A. Henrich [2] introduces a polynomial invariant of virtual knots which vanishes for classical knots. Later, Im, Lee and Lee [3] extend Henrich's polynomial to virtual links. The purpose of this paper is to introduce a polynomial invariant of virtual magnetic links which is a generalization of the polynomial defined by Im, Lee and Lee, and give various properties of this polynomial and examples. Also this polynomial can be used to find lower bounds on the classical crossing number and the virtual crossing number of virtual links.

This paper is organized as follows. Section 2 gives the basic definitions and results of virtual links and virtual magnetic links. In Section 3, we introduce a polynomial invariant for virtual magnetic links and give the proof of the invariance. In Section 4, we provide various properties of the polynomial invariant and some examples.

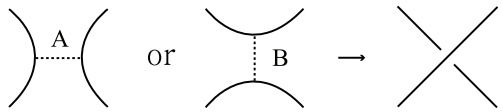



Fig. 4.

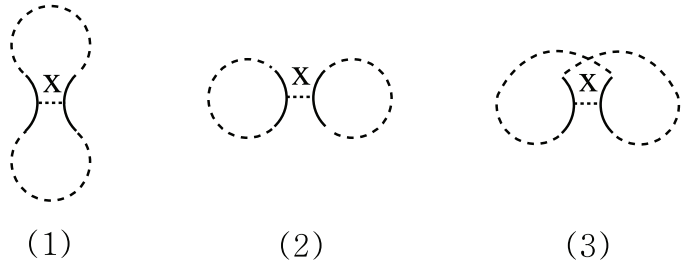



Fig. 5.

## 2. Preliminaries

In this section, we give basic definitions and results which are needed throughout this paper.

A *state* of a virtual magnetic link diagram $D$ is a union of immersed loops in $\mathbb{R}^2$ with only virtual crossings and nodes, which is obtained by splicing all classical crossings of $D$. At each spliced crossing we attach a chord labeled $A$ or $B$ to represent the splicing direction as shown in Fig. 4. A state $S$ of a virtual magnetic link diagram $D$ is *normal* if for any classical crossing $x$ of $D$, the loops of $S$ spliced at $x$ are of type (1) or (2) in Fig. 5. A virtual magnetic link diagram $D$ is *normal* if every state of $D$ is normal. A virtual magnetic link $L$ is *normal* if $L$ is the equivalence class of a normal diagram under the equivalence relation generated by generalized Reidemeister moves, $N_1$ and $N_2$ moves. An edge of a virtual magnetic link diagram $D$ means an arc of $D$ divided by nodes or a loop of $D$ without nodes. For a virtual magnetic link diagram $D$, we denote by $\bar{D}$ the union of immersed circles in $\mathbf{R}^2$ obtained by ignoring nodes and orientations of the edges of $D$, and by replacing all classical crossings of $D$ with 4-valent vertices and leaving the virtual information unchanged.

DEFINITION 2.1 ([6]). $\bar{D}$ admits an alternate orientation if all edges (when regarding $\bar{D}$ as a 4-valent planar graph) can be oriented as shown in Fig. 6.

**Proposition 2.2** (cf. [6, Proposition 6]). *A virtual magnetic link diagram $D$ is normal if and only if $\bar{D}$ admits an alternate orientation.*

Fig. 6.

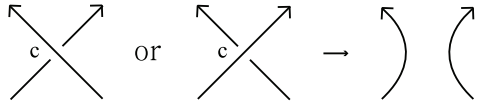



Fig. 7.

**Proposition 2.3** (cf. [6, Corollary 7]). *Let $D = D_1 \cup D_2 \cup \cdots \cup D_n$ be an $n$-component virtual magnetic link diagram. If $D$ is normal, then the number of all classical crossings between $D_i$ and $D \setminus D_i$ is even.*

## 3. The polynomial invariant $P_L(t)$ of virtual magnetic links

In this section, we define a polynomial of a virtual magnetic link and give the proof of the invariance.

A classical crossing of a virtual magnetic link diagram $D$ is said to be a *self* crossing if the crossing is composed of two arcs which belong to the same component of $D$. We denote the set of self crossings of $D$ by $SC(D)$. A classical crossing of $D$ is said to be *mixed* crossing if the crossing is composed of two edges which belong to different components of $D$. We denote the set of mixed crossings of $D$ by $MC(D)$. If we denote the set of classical crossings of $D$ by $CC(D)$, then it is obvious that $CC(D) = SC(D) \sqcup MC(D)$. The *writhe* of $D$ denoted by $wr(D)$ is the sum of signatures of classical crossings of $D$, that is $\sum_{c \in CC(D)} \text{sign}(c)$.

We introduce a virtual magnetic link diagram derived from $D$. Suppose that a classical crossing $c$ of $D$ is a self crossing. Then, there exists a unique component of $D$ which constitutes $c$. We denote it by $D_c$. Let $\hat{D}_c$ be a virtual magnetic link diagram obtained from $D_c$ by smoothing $c$ as shown in Fig. 7. If $c$ is a mixed crossing of $D$, then we define $\hat{D}_c$ as the subdiagram of $D$ which consists of two components constituting $c$. An example is illustrated in Fig. 8. A *weight map* of $\hat{D}_c$ is a map $\sigma : e(\hat{D}_c) \to \{\pm 1\}$ such that $\sigma(e) \neq \sigma(e')$ for adjacent edges $e$ and $e'$ of $\hat{D}_c$, where $e(\hat{D}_c)$ is the set of edges of $\hat{D}_c$. There exist $2^{\#(\hat{D}_c)}$ weight maps, where $\#(\hat{D}_c)$ is the number of components of $\hat{D}_c$. Note that $\#(\hat{D}_c)$ is either one or two. We denote the set of weight maps of $\hat{D}_c$ by $W(\hat{D}_c)$.

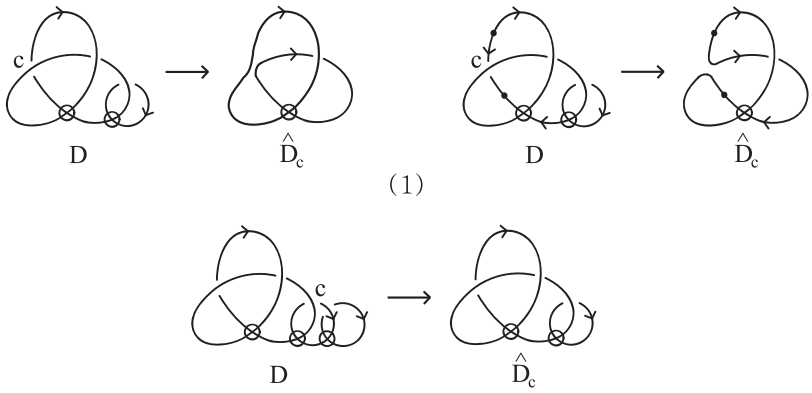



Fig. 8.

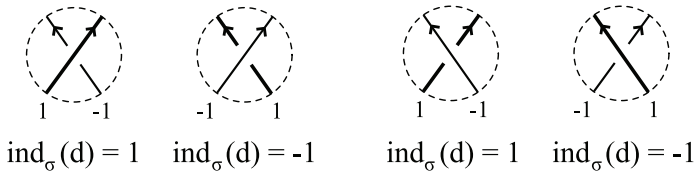



Fig. 9.

A classical crossing $d$ of $\hat{D}_c$ is composed of two arcs. The arc passing over at $d$ is called the over path at $d$ and the other arc is called the under path at $d$. Let $e_d^o$ and $e_d^u$ be edges including the over path and the under path at $d$, respectively. The *index number* of $d$ with respect to a weight map $\sigma$ is given by $(1/2)\,\mathrm{sign}(d)(\sigma(e_d^o) - \sigma(e_d^u))$ and is denoted by $\mathrm{ind}_\sigma(d)$. A diagrammatic description is given in Fig. 9 and note that $\mathrm{ind}_\sigma(d) \in \{0, \pm 1\}$ [3].

DEFINITION 3.1. Let $D$ be a virtual magnetic link diagram and $c$ a classical crossing of $D$. Let $\sigma$ be a weight map of $\hat{D}_c$. The intersection index of $c$ with respect to $\sigma$, denoted by $i_\sigma(c)$, is given by $i_\sigma(c) = \sum_{d \in CC(\hat{D}_c)} \mathrm{ind}_\sigma(d)$.

We will define polynomials $\bar{P}_D(t)$ and $P_D(t)$ for a virtual magnetic link diagram $D$ and show that $\bar{P}_D(t)$ is an invariant under the basic local moves except the Reidemeister move of type I and $P_D(t)$ is an invariant for virtual magnetic link diagrams.

DEFINITION 3.2. Let $D$ be a virtual magnetic link diagram. Then, the polynomials $\bar{P}_D(t)$ and $P_D(t)$ in $\mathbb{Q}[t^{\pm 1}]$ for $D$ are given by

$$\bar{P}_D(t) = \sum_{c \in CC(D)} \left( \frac{\text{sign}(c)}{2^{\#(\hat{D}_c)}} \sum_{\sigma \in W(\hat{D}_c)} t^{i_\sigma(c)} \right)$$

and

$$P_D(t) = \sum_{c \in CC(D)} \left( \frac{\text{sign}(c)}{2^{\#(\hat{D}_c)}} \sum_{\sigma \in W(\hat{D}_c)} (t^{i_\sigma(c)} - 1) \right).$$

REMARK 3.3. Note that $P_D(t) = \bar{P}_D(t) - \sum_{c \in CC(D)} \text{sign}(c) = \bar{P}_D(t) - wr(D)$.

**Theorem 3.4.** *The polynomial $\bar{P}_D(t)$ is an invariant for virtual magnetic link diagrams under the basic local moves except the Reidemeister move of type* I.

Let $D$ be a virtual magnetic link diagram and $D'$ a virtual magnetic link diagram obtained from $D$ by applying a local move in a local disk $B$. Then, generally speaking, we cannot regard a weight map of $D$ as that of $D'$ because there exists a local move, which changes the number of edges of a diagram, like the $N_2$-move. However, we note that $D \setminus B$ and $D' \setminus B$ are identical. If the numbers of weight maps of $D$ and $D'$ are equal, and, for a weight map $\sigma$ of $D$, there exists a unique weight map $\sigma'$ of $D'$ such that the weight of each edge of $D'$ outside $B$ is equal to that of the corresponding edge of $D$, then we can give a bijection from $W(D)$ to $W(D')$ by sending $\sigma$ to $\sigma'$. We call it the *canonical* map from $W(D)$ to $W(D')$ denoted by $cm\colon W(D) \to W(D')$. For example, if a local move is a basic one, then we see that there exists the canonical map.

Let $c$ be a crossing of $D$ outside $B$. Since $D \setminus B$ and $D' \setminus B$ are identical, there exists a unique crossing $c'$ of $D'$ corresponding to $c$ by the identity map from $D \setminus B$ to $D' \setminus B$. We call the crossing $c'$ the *double* of $c$ and denote it by $db(c)$.

**Lemma 3.5.** *Let $D$ be a virtual magnetic link diagram and $D'$ a virtual magnetic link diagram obtained from $D$ by applying a basic local move in a local disk $B$. Let $c$ be a classical crossing of $D$ outside $B$ and $\sigma$ a weight map of $D$. Then,* $\text{ind}_\sigma(c) = \text{ind}_{cm(\sigma)}(db(c))$.

Proof. By definitions of the canonical map $cm$ and the crossing $db(c)$, we have $\sigma(d_c^o) = cm(\sigma)(e_{db(c)}^o)$ and $\sigma(d_c^u) = cm(\sigma)(e_{db(c)}^u)$. Since $\text{sign}(c) = \text{sign}(db(c))$, we obtain the claim. □

**Lemma 3.6.** *Let $D$ be a virtual magnetic link diagram and $D'$ a virtual magnetic link diagram obtained from $D$ by applying a basic local move in a local disk $B$. Let*

*$\sigma$ be a weight map of $D$. Then,*

$$\sum_{c\in CC(D\cap B)} \mathrm{ind}_\sigma(c) = \sum_{c\in CC(D'\cap B)} \mathrm{ind}_{cm(\sigma)}(c),$$

*where $CC(E \cap B)$ denotes the set of classical crossings of a diagram $E$ in $B$.*

Proof. The proof is divided into five cases. To begin with, we suppose that the basic local move is any of virtual Reidemeister moves. Then, $CC(D \cap B) = CC(D' \cap B) = \emptyset$. Thus, the claim is true. Next, we suppose that the basic local move is a semivirtual move. Then, each of the sets $CC(D \cap B)$ and $CC(D' \cap B)$ consists of one element. Let $c$ and $c'$ be the elements of $CC(D \cap B)$ and $CC(D' \cap B)$, respectively. Since $\sigma(e^o_c) = cm(\sigma)(e^o_{c'})$, $\sigma(e^u_c) = cm(\sigma)(e^u_{c'})$ and $\mathrm{sign}(c) = \mathrm{sign}(c')$, we have $\mathrm{ind}_\sigma(c) = \mathrm{ind}_{cm(\sigma)}(c')$. Thus, the claim is true. Next, we suppose that the basic local move is a Reidemeister move of type I. We may assume that the number of crossings of $D$ is less than that of $D'$. Then, $CC(D\cap B) = \emptyset$ and $CC(D' \cap B)$ consists of one crossing $a$. Since $cm(\sigma)(e^o_a) = cm(\sigma)(e^u_a)$, we have $\mathrm{ind}_{cm(\sigma)}(a) = 0$. Thus, the claim is true. Furthermore, we suppose that the basic local move is a Reidemeister move of type II. We may assume that the number of crossings of $D$ is less than that of $D'$. Then, $CC(D \cap B) = \emptyset$ and $CC(D' \cap B)$ consists of two crossings $a$ and $b$. Since $cm(\sigma)(e^o_a) = cm(\sigma)(e^o_b)$, $cm(\sigma)(e^u_a) = cm(\sigma)(e^u_b)$ and $\mathrm{sign}(a) = -\mathrm{sign}(b)$, we have $\mathrm{ind}_{cm(\sigma)}(a) + \mathrm{ind}_{cm(\sigma)}(b) = 0$. Thus, the claim is true. Finally, we suppose that the basic local move is a Reidemeister move of type III. Then, each of the sets $CC(D \cap B)$ and $CC(D' \cap B)$ consists of three crossings. Let $c_1$, $c_2$ and $c_3$ (resp. $c'_1$, $c'_2$ and $c'_3$) be the crossings of $D \cap B$ (resp. $D' \cap B$) between the top and the middle arcs, the top and the bottom arcs, and the middle and the bottom arcs in $B$, respectively. Since $\sigma(e^o_{c_i}) = cm(\sigma)(e^o_{c'_i})$, $\sigma(e^u_{c_i}) = cm(\sigma)(e^u_{c'_i})$ and $\mathrm{sign}(c_i) = \mathrm{sign}(c'_i)$, $1 \le i \le 3$, we obtain $\mathrm{ind}_\sigma(c_i) = \mathrm{ind}_{cm(\sigma)}(c'_i)$ for each $i$, and $\sum_{c\in CC(D\cap B)} \mathrm{ind}_\sigma(c) = \sum_{c\in CC(D'\cap B)} \mathrm{ind}_{cm(\sigma)}(c)$. This completes the proof. □

**Lemma 3.7.** *Let $D$ be a virtual magnetic link diagram and $D'$ a virtual magnetic link diagram obtained from $D$ by applying a basic local move in a local disk $B$. Let $c$ be a classical crossing of $D$ outside $B$ and $\sigma$ a weight map of $D$. Then, $i_\sigma(c) = i_{cm(\sigma)}(db(c))$.*

Proof. Considering two diagrams $\hat{D}_c$ and $\hat{D}'_{db(c)}$, we have the following two cases: One is the case that $\hat{D}_c$ and $\hat{D}'_{db(c)}$ are identical. The other is the case that $\hat{D}'_{db(c)}$ is obtained from $\hat{D}_c$ by applying the local move. First, we suppose that $\hat{D}_c = \hat{D}'_{db(c)}$. Then, for a classical crossing $d$ of $\hat{D}_c$, we easily obtain $\mathrm{ind}_\sigma(d) = \mathrm{ind}_{cm(d)}(db(d))$, which implies $i_\sigma(c) = i_{cm(\sigma)}(db(c))$. Next, we suppose that $\hat{D}'_{db(c)}$ is obtained from $\hat{D}_c$ by applying the local move in $B$. Note that $CC(\hat{D}_c) = CC(\hat{D}_c \setminus B) \sqcup CC(\hat{D}_c \cap B)$ and $CC(\hat{D}'_{db(c)}) = CC(\hat{D}'_{db(c)} \setminus B) \sqcup CC(\hat{D}'_{db(c)} \cap B)$, where $CC(E \setminus B)$ denotes the set

of classical crossings of a diagram $E$ outside $B$. If a crossing $d$ of $\hat{D}_c$ is an element of $CC(\hat{D}_c \setminus B)$, then Lemma 3.5 shows that $\mathrm{ind}_\sigma(d) = \mathrm{ind}_{cm(\sigma)}(db(d))$. Thus, we obtain $\sum_{d\in CC(\hat{D}_c\setminus B)} \mathrm{ind}_\sigma(d) = \sum_{d\in CC(\hat{D}'_{db(c)}\setminus B)} \mathrm{ind}_{cm(\sigma)}(d)$. By Lemma 3.6, we also obtain $\sum_{d\in CC(\hat{D}_c\cap B)} \mathrm{ind}_\sigma(d) = \sum_{d\in CC(\hat{D}'_{db(c)}\cap B)} \mathrm{ind}_{cm(\sigma)}(d)$. The above two equalities give $i_\sigma(c) = i_{cm(\sigma)}(db(c))$, completing the proof. □

**Lemma 3.8.** *Let $D$ be a virtual magnetic link diagram and $D'$ a virtual magnetic link diagram obtained from $D$ by applying a Reidemeister move of type* II. *Then, $\bar{P}_D(t) = \bar{P}_{D'}(t)$.*

Proof. We may assume that the number of classical crossings of $D$ is less than that of $D'$. Let $a$ and $b$ be the two new crossings of $D'$ in a local disk $B$ where the local move is applied. Let $c$ be a classical crossing of $D$ and $\sigma$ a weight map of $D$. Then, Lemma 3.7 shows $i_\sigma(c) = i_{cm(\sigma)}(db(c))$. Since $\mathrm{sign}(c) = \mathrm{sign}(db(c))$ and $\#(\hat{D}_c) = \#(\hat{D}'_{db(c)})$, we have

$$\begin{aligned}
\bar{P}_D(t) &= \sum_{c\in CC(D)} \left( \frac{\mathrm{sign}(c)}{2^{\#(\hat{D}_c)}} \sum_{\sigma\in W(\hat{D}_c)} t^{i_\sigma(c)} \right) \\
&= \sum_{c\in CC(D')} \left( \frac{\mathrm{sign}(c)}{2^{\#(\hat{D}'_c)}} \sum_{\sigma\in W(\hat{D}'_c)} t^{i_\sigma(c)} \right) - \sum_{c\in\{a,b\}} \left( \frac{\mathrm{sign}(c)}{2^{\#(\hat{D}'_c)}} \sum_{\sigma\in W(\hat{D}'_c)} t^{i_\sigma(c)} \right) \\
&= \bar{P}_{D'}(t) - \sum_{c\in\{a,b\}} \left( \frac{\mathrm{sign}(c)}{2^{\#(\hat{D}'_c)}} \sum_{\sigma\in W(\hat{D}'_c)} t^{i_\sigma(c)} \right).
\end{aligned}$$

To prove the lemma, we only have to show that the second term of the last expression in the above equality is equal to zero. If $a \in MC(D')$, then $b \in MC(D')$. In this case, we obtain $\hat{D}'_a = \hat{D}'_b$, and thus $i_\sigma(a) = i_\sigma(b)$ for any weight map $\sigma$ of $D'_a$. Since $\#(\hat{D}'_a) = \#(\hat{D}'_b)$ and $\mathrm{sign}(a) = -\mathrm{sign}(b)$, we have

$$\sum_{c\in\{a,b\}} \left( \frac{\mathrm{sign}(c)}{2^{\#(\hat{D}'_c)}} \sum_{\sigma\in W(\hat{D}'_c)} t^{i_\sigma(c)} \right) = 0.$$

Suppose that $a \in SC(D')$. Then, $b \in SC(D')$. We have two cases. First, we consider the case that $\hat{D}'_b$ is obtained from $\hat{D}'_a$ by changing the crossing $b$. For the local move, we have the canonical map from $W(\hat{D}'_a)$ to $W(\hat{D}'_b)$. Since $\mathrm{ind}_\sigma(b) = \mathrm{ind}_{cm(c)}(a)$ for $b \in CC(\hat{D}'_a)$, $a \in CC(\hat{D}'_b)$ and a weight map $\sigma$ of $\hat{D}'_a$, we obtain $i_\sigma(a) = i_{cm(\sigma)}(b)$. Since $\#(\hat{D}'_a) = \#(\hat{D}'_b)$ and $\mathrm{sign}(a) = -\mathrm{sign}(b)$, we have the desired result. Next, we consider the case that $\hat{D}'_a$ and $\hat{D}'_b$ have self crossings which come from local curls in $B$. In this case, we also have the canonical map from $W(\hat{D}'_a)$ to $W(\hat{D}'_b)$. Since a

self crossing is composed of two edges with the same weight, we see that $\mathrm{ind}_\sigma(b) = \mathrm{ind}_{cm(\sigma)}(a) = 0$ for $b \in CC(\hat{D}'_a)$, $a \in CC(\hat{D}'_b)$ and a weight map $\sigma$ of $\hat{D}'_a$. Thus, we obtain $i_\sigma(a) = i_{cm(\sigma)}(b)$. Two facts $\#(\hat{D}'_a) = \#(\hat{D}'_b)$ and $\mathrm{sign}(a) = -\mathrm{sign}(b)$ complete the proof. □

Let $D$ be a virtual magnetic link diagram and $D'$ a virtual magnetic link diagram obtained from $D$ by applying a Reidemeister move of type III in a local disk $B$. Let $d_1$, $d_2$ and $d_3$ be three crossings of $D$ in $B$ between the top and the middle arcs, the top and the bottom arcs, and the middle and the bottom arcs in $B$, respectively. We denote three crossings of $D'$ in $B$ by $d'_1$, $d'_2$ and $d'_3$, similarly. Suppose that all the signs of the three crossings $d_1$, $d_2$ and $d_3$ are $+1$. Such a Reidemeister move of type III is called *standard*. Let $E$ and $E'$ be virtual magnetic link diagrams obtained from $D$ and $D'$ by smoothing $d_2$ and $d'_2$, respectively. Note that $E \setminus B = E' \setminus B$ though $E$ and $E'$ are not related by a sequence of basic local moves in $B$. Thus, we may suppose that $E'$ is obtained from $E$ by applying a local move in $B$. Then, we see that there exists the canonical map from $W(E)$ to $W(E')$ and have the following two lemmas.

**Lemma 3.9.**

$$\sum_{d \in CC(E)} \mathrm{ind}_\sigma(d) = \sum_{d \in CC(E')} \mathrm{ind}_{cm(\sigma)}(d).$$

Proof. Since $E \setminus B = E' \setminus B$ and $\sigma_{|E \setminus B} = cm(\sigma)_{|E' \setminus B}$, it is easy to see that $\sum_{d \in CC(E \setminus B)} \mathrm{ind}_\sigma(d) = \sum_{d \in CC(E' \setminus B)} \mathrm{ind}_{cm(\sigma)}(d)$. Thus, we only have to show that $\sum_{d \in CC(E \cap B)} \mathrm{ind}_\sigma(d) = \sum_{d \in CC(E' \cap B)} \mathrm{ind}_{cm(\sigma)}(d)$. The set $CC(E \cap B)$ consists of two crossings $a$ and $b$. Since $\sigma(e^o_a) = \sigma(e^u_b)$, $\sigma(e^u_a) = \sigma(e^o_b)$ and $\mathrm{sign}(a) = \mathrm{sign}(b)$, we have $\sum_{d \in CC(E \cap B)} \mathrm{ind}_\sigma(d) = \mathrm{ind}_\sigma(a) + \mathrm{ind}_\sigma(b) = 0$.

Similarly, we obtain $\sum_{d \in CC(E' \cap B)} \mathrm{ind}_{cm(\sigma)}(d) = 0$. This completes the proof. □

**Lemma 3.10.** $i_\sigma(d_j) = i_{cm(\sigma)}(d'_j)$, $1 \leq i \leq 3$.

Proof. First, we consider the case of $j = 1$. Then, we have two cases: One is the case of $\hat{D}_{d_1} = \hat{D}'_{d'_1}$. The other is the case that $\hat{D}'_{d'_1}$ is obtained from $\hat{D}_{d_1}$ by applying the standard Reidemeister move of type III. If $\hat{D}_{d_1} = \hat{D}'_{d'_1}$, then it is clear that $i_\sigma(d_1) = i_{cm(\sigma)}(d'_1)$. Suppose that $\hat{D}_{d_1} = \hat{D}'_{d'_1}$ are related by the local move. Then, Lemma 3.5 and 3.6 give $i_\sigma(d_1) = i_{cm(\sigma)}(d'_1)$. Next, we consider the case of $j = 3$. Then, we can show the claim in the same way as the above case. Finally, we consider the case of $j = 2$. Then, we have three cases. Two of them are similar to the above cases. So, we omit these cases. We deal with the other case that $\hat{D}_{d_2}$ and $\hat{D}'_{d'_2}$ are identical with $E$ and $E'$ respectively. Then, by Lemma 3.5 and 3.9, we obtain $i_\sigma(d_2) = i_{cm(\sigma)}(d'_2)$, completing the proof. □

**Lemma 3.11.** *Let $D$ be a virtual magnetic link diagram and $D'$ a virtual magnetic link diagram obtained from $D$ by applying a single Reidemeister move of type* III. *Then, $\bar{P}_D(t) = \bar{P}_{D'}(t)$.*

Proof. Since any Reidemeister move of type III can be realized by a sequence of a standard Reidemeister move of type III and some Reidemeister moves of type II, by Lemma 3.8, we only have to show that $\bar{P}_D(t)$ is invariant under the standard Reidemeister move of type III. Let $c$ be a classical crossing of $D$ and $c'$ the corresponding crossing of $D'$ to $c$, which means the double of $c$ if $c$ is a crossing of $D$ outside the local disk where the local move is applied. By Lemmas 3.7 and 3.10, for a weight map $\sigma$ of $D$, we have $i_\sigma(c) = i_{cm(\sigma)}(c')$. Since $\mathrm{sign}(c) = \mathrm{sign}(c')$ and $\#(\hat{D}_c) = \#(\hat{D}'_{c'})$, we obtain

$$\begin{aligned}\bar{P}_D(t) &= \sum_{c \in CC(D)} \left( \frac{\mathrm{sign}(c)}{2^{\#(\hat{D}_c)}} \sum_{\sigma \in W(\hat{D}_c)} t^{i_\sigma(c)} \right) \\ &= \sum_{c' \in CC(D')} \left( \frac{\mathrm{sign}(c')}{2^{\#(\hat{D}'_{c'})}} \sum_{cm(\sigma) \in W(\hat{D}'_{c'})} t^{i_{cm(\sigma)}(c')} \right) = \bar{P}_{D'}(t). \qquad \square\end{aligned}$$

**Lemma 3.12.** *Let $D$ be a virtual magnetic link diagram and $D'$ a virtual magnetic link diagram obtained from $D$ by applying any of virtual Reidemeister moves or of nodal moves. Then, $\bar{P}_D(t) = \bar{P}_{D'}(t)$.*

Proof. Let $c$ be a classical crossing of $D$. Since there are no classical crossings in a local disk $B$ where the local move is applied, $c$ is a crossing outside $B$. We have two cases. If $\hat{D}_c = \hat{D}'_{db(c)}$, then it is clear that $i_\sigma(c) = i_{cm(\sigma)}(db(c))$ for any weight map $\sigma$ of $D$. Since $\mathrm{sign}(c) = \mathrm{sign}(db(c))$ and $\#(\hat{D}_c) = \#(\hat{D}'_{db(c)})$, we obtain the claim. If $\hat{D}'_{db(c)}$ is obtained from $\hat{D}_c$ by applying the local move, then Lemma 3.7 shows $i_\sigma(c) = i_{cm(\sigma)}(db(c))$ for any weight map $\sigma$ of $D$. Since $\mathrm{sign}(c) = \mathrm{sign}(db(c))$ and $\#(\hat{D}_c) = \#(\hat{D}'_{db(c)})$, we have the claim. $\square$

**Lemma 3.13.** *Let $D$ be a virtual magnetic link diagram and $D'$ a virtual magnetic link diagram obtained from $D$ by applying a semivirtual move in a local disk $B$. Let $c$ and $c'$ the classical crossings of $D$ and $D'$ in $B$, respectively. Then, for a weight map $\sigma$ of $D$, $i_\sigma(c) = i_{cm(\sigma)}(c')$.*

Proof. Considering two diagrams $\hat{D}_c$ and $\hat{D}'_{c'}$, we have three cases. First, we suppose that $\hat{D}_c$ and $\hat{D}'_{c'}$ are related by two virtual Reidemeister moves of type II. Since each of the Reidemeister moves can be applied in $B$ and there are no classical crossing in $B$, by Lemma 3.5, we obtain $i_\sigma(c) = i_{cm(\sigma)}(db(c))$. Next, we consider the remaining two cases: One is the case of $\hat{D}_c = \hat{D}'_{c'}$. The other is the case that $\hat{D}'_{c'}$ is obtained

from $\hat{D}_c$ by the semivirtual move, which changes $D$ to $D'$. The proof of these cases are similar to that of Lemma 3.7. Hence, we have the claim. □

**Lemma 3.14.** *Let $D$ be a virtual magnetic link diagram and $D'$ a virtual magnetic link diagram obtained from $D$ by applying a semivirtual move. Then, $\bar{P}_D(t) = \bar{P}_{D'}(t)$.*

Proof. Let $c$ be a classical crossing of $D$ and $\sigma$ a weight map of $D$. If $c$ is outside the local disk $B$ where the local move is applied, Lemma 3.7 give $i_\sigma(c) = i_{cm(\sigma)}(db(c))$. Then, we see that $\mathrm{sign}(c) = \mathrm{sign}(db(c))$ and $\#(\hat{D}_c) = \#(\hat{D}'_{db(c)})$. Suppose that $c$ is in $B$. Then, there is the corresponding crossing $c'$ of $D'$ in $B$. By Lemma 3.13, we obtain $i_\sigma(c) = i_{cm(\sigma)}(c')$. We also see that $\mathrm{sign}(c) = \mathrm{sign}(c')$ and $\#(\hat{D}_c) = \#(\hat{D}'_{c'})$. From the definition of $\bar{P}_D(t)$ and the above facts, we have the result. □

Let $L$ be a virtual magnetic link and $D$ a virtual magnetic link diagram representing $L$. We define a polynomial for $L$ by $P_D(t)$ and denote it by $P_L(t)$.

**Theorem 3.15.** *The polynomial $P_L(t)$ is an invariant for virtual magnetic links.*

Proof. Since the writhe of a virtual magnetic link diagram is an invariant under basic local moves except Reidemeister moves of type I, by Theorem 3.4 and Remark 3.3, it suffices to show that $P_L(t)$ is an invariant under a Reidemeister move of type I.

Let $D$ and $D'$ be virtual magnetic link diagrams. We consider the case that $D'$ is obtained from $D$ by applying a Reidemeister move of type I. We may assume that the number of crossings of $D$ is less than that of crossings of $D'$. Then, we add a new crossing $c$ to $D'$, and therefore add a new term to the sum of the polynomial of $D$. If we smooth at the new crossing $c$ of $D'$, we obtain a two component diagram $\hat{D}'_c$ which is a disjoint union of a virtual magnetic link diagram and a trivial loop. Then $i_\sigma(c) = 0$ for the new crossing and any weight map $\sigma$. Thus the term corresponding to the new crossing vanishes. The remaining terms corresponding to the other classical crossings of $D$ and $D'$ are the same since the Reidemeister move of type I does not affect the intersection indices of virtual magnetic link diagrams arising from other crossings. Thus, $P_D(t) = P_{D'}(t)$. □

**Corollary 3.16.** *The polynomial $P_L(t)$ is an invariant for virtual links.*

Proof. Virtual link diagrams are virtual magnetic link diagrams without nodes. □

## 4. The properties of $P_L(t)$ and examples

In this section, we give several various important properties of the polynomial invariant $P_L(t)$ for virtual magnetic links. First, we mention that $P_L(t)$ is the same as the

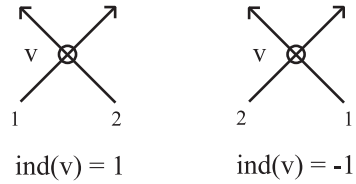



Fig. 10.

polynomial invariant $Q_L(t)$ defined in [3] for virtual links. Recall the polynomial $Q_L(t)$.

Let $D$ be a virtual link diagram, and $SC(D)$ (resp., $MC(D)$) be the set of self crossings of $D$ (resp., the set of mixed crossings of $D$). If we denote the set of classical crossings of $D$ by $CC(D)$, then it is obvious that $CC(D) = SC(D) \sqcup MC(D)$.

Suppose that a classical crossing $c$ of $D$ is a self crossing. Then, there exists a unique component of $D$ which constitutes $c$. We denote it by $D_c$. Let $\hat{D}_c$ be a virtual magnetic link diagram obtained from $D_c$ by smoothing $c$ as shown in Fig. 7.

This smoothing gives us a two-component virtual link diagram $\hat{D}_c$, which is a part of the smoothed virtual link. We choose an ordering $(1, 2)$ for the components of the virtual link diagram $\hat{D}_c$ and let $1 \cap 2$ denote the set of virtual crossings between the two components. For each virtual crossing $v \in 1 \cap 2$, we assign 1 or $-1$, called the *index number* of $v$ and denoted by $\mathrm{ind}(v)$ as shown in Fig. 10. And for each virtual crossing $v$ in one component of $\hat{D}_c$, we assign 0.

If $c$ is a mixed crossing of $D$, then we define $\hat{D}_c$ as the subdiagram of $D$ which consists of two components constituting $c$.

DEFINITION 4.1 ([3]). Let $D$ be a virtual link diagram and $c$ be a classical crossing of $D$. The *virtual intersection index*, denoted by $i(c)$, is given by

$$i(c) = \sum_{d \in 1 \cap 2} \mathrm{ind}(d).$$

DEFINITION 4.2 ([3]). Let $D$ be a virtual link diagram. Then the polynomial $Q_D(t) \in \mathbf{Z}[t]$ for a virtual link diagram $D$ is defined by

$$Q_D(t) = \sum_{c \in CC(D)} \mathrm{sign}(c)(t^{|i(c)|} - 1).$$

By using the virtual intersection index which is the sum of indices of virtual crossings instead of the intersection index used in the definition of the polynomial invariant $P_L(t)$ of a virtual magnetic link $L$, we can find a new polynomial $\tilde{Q}_L(t)$. Then it is easy to check that $P_L(t) = \tilde{Q}_L(t) = Q_L(t)$ by replacing $t^{|i(c)|} - 1$ by $(1/4)(t^{i(c)} + t^{-i(c)} - 2)$ for any classical crossing $c$ of a virtual link $L$ via the Jordan curve theorem.

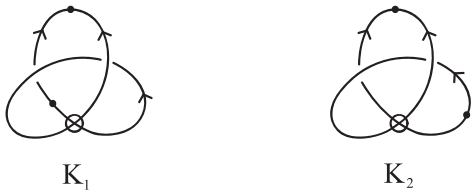



Fig. 11.

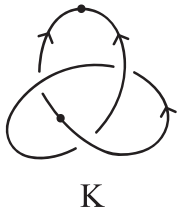



Fig. 12.

However $\tilde{Q}_L(t)$ is not preserved under an $N_1$ move so that it is not an invariant of virtual magnetic links as follows.

EXAMPLE 4.3. Consider the following virtual magnetic links $K_1$ and $K_2$ as in Fig. 11. By the quick computation, $\tilde{Q}_{K_1}(t) = -(1/4)(t+t^{-1}-2)$ and $\tilde{Q}_{K_2}(t) = (1/4)(t+t^{-1}-2)$. However, $K_1$ and $K_2$ are equivalent.

Like some polynomials defined by Sawollek [11], Silver and Williams [12] and by Kauffman and Radford [9], the polynomial $P_L(t)$ is trivial for any classical links.

**Proposition 4.4** (cf. [3, Proposition 4.1]). *The polynomial $P_L(t)$ of any classical link $L$ is zero. Therefore if $P_L(t) \neq 0$, then the link $L$ is not a classical link.*

**Proposition 4.5** (cf. [3, Proposition 4.2]). *If $K$ is a virtual knot which is normal, then $P_K(t) \in \mathbf{Q}[t^{\pm 2}]$.*

However, the above propositions are not true any more in the case of magnetic links.

EXAMPLE 4.6. Consider a classical magnetic knot $K$ as shown in Fig. 12. Then $K$ is normal. But, it is easy to compute $P_L(t) = (1/2)(t+t^{-1}-2)$ that is non-zero and not in $\mathbf{Q}[t^{\pm 2}]$.

As we can see in the following examples, the polynomial $P_L(t)$ for virtual magnetic links provides us the advantage to distinguish whether given virtual magnetic links are distinct or not via a quick computation.

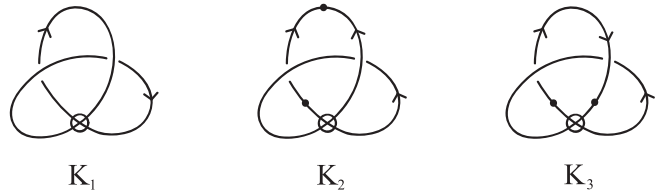



Fig. 13.

Fig. 14.

EXAMPLE 4.7. Consider the following virtual knots in Fig. 13.

(1) For the virtual trefoil knot $K_1$, $P_{K_1}(t) = (1/2)(t + t^{-1} - 2)$. Thus, $K_1$ is a virtual knot which is not normal.
(2) For $K_2$, $P_{K_2}(t) = -(1/4)(t + t^{-1} - 2)$.
(3) For $K_3$, $P_{K_3}(t) = 0$.

Thus, we find that $K_1$, $K_2$ and $K_3$ are distinct.

For a virtual magnetic link diagram $D$, we denote by $D^*$ the virtual magnetic link diagram obtained by interchanging the over- and under-information at all classical crossings of $D$ while keeping the orientation of $D$. If $D$ and $D^*$ represent the same virtual magnetic link diagrams, then the virtual magnetic link is called *amphicheiral*.

We have the following result.

**Proposition 4.8** (cf. [3, Proposition 4.6]). *For any virtual magnetic link diagram $D$, $P_{D^*}(t) = -P_D(t)$ and $P_{-D}(t) = P_D(t)$, where $-D$ is the diagram with the reversed orientation.*

**Corollary 4.9** (cf. [3, Corollary 4.7]). *For any amphicheiral virtual magnetic link $L$, $P_L(t) = 0$.*

The operation on virtual link diagram depicted in Fig. 14 is called a *Kauffman's flype*. Jones polynomial is preserved under a *Kauffman's flype*, but our invariant is not preserved.

EXAMPLE 4.10. For two virtual knots $K_1$, $K_2$ in Fig. 15, we obtain $P_{K_1}(t) = -(1/2)(t^2 + t^{-2} - 2)$ and $P_{K_2}(t) = 0$, so that the polynomial invariant $P_L(t)$ is not preserved under Kauffman's flypes.

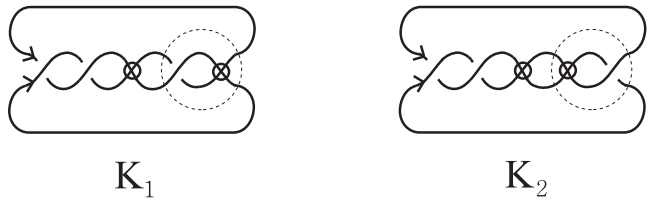



Fig. 15.

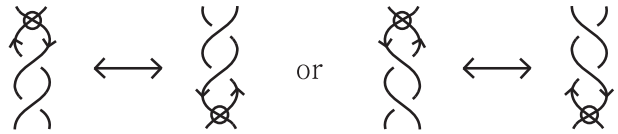



Fig. 16.

Now we introduce an operation which preserves our invariant $P_L(t)$. Let us call the operations shown in Fig. 16 *double flypes* [5].

**Theorem 4.11.** *The polynomial $P_D(t)$ of virtual magnetic link diagrams is invariant under the double flypes.*

Proof. Let $D$ and $D'$ be virtual magnetic link diagrams which differ by a single double flype and $d_1$ and $d_2$ (resp. $d_1{}'$ and $d_2{}'$) be classical crossings of $D$ (resp. $D'$) in a local disk $B$ (resp. $B'$) where the double flype is applied. Then, we have the canonical map from $W(D)$ to $W(D')$

For any classical crossing $c$ in $D \setminus B$ and any weight map $\sigma$ of $D$, we have $i_\sigma(c) = i_{cm(\sigma)}(db(c))$ according to whether $c$ belongs to the same component of $D$ (resp., $D'$) or $c$ belongs to the different components of $D$ (resp., $D'$) since $\mathrm{ind}_\sigma(d_1) + \mathrm{ind}_\sigma(d_2) = 0 = \mathrm{ind}_{cm(\sigma)}(d_1') + \mathrm{ind}_{cm(\sigma)}(d_2')$.

For a classical crossing $d_1$ (resp., $d_1{}'$) in $B$ (resp., $B'$), which belongs to the same component, we can calculate

$$i_\sigma(d_1) = \sum_{\substack{j \neq 1,2 \\ d_j \in SC(\hat{D}_{d_1})}} \mathrm{ind}_\sigma(d_j) + \mathrm{ind}_\sigma(d_2),$$

$$i_{cm(\sigma)}(d_1{}') = \sum_{\substack{j \neq 1,2 \\ d_j{}' \in SC(\hat{D}'_{d_1{}'})}} \mathrm{ind}_{cm(\sigma)}(d_j{}') + \mathrm{ind}_{cm(\sigma)}(d_2{}')$$

for any weight map $\sigma$. Since $\mathrm{ind}_\sigma(d_2) = \mathrm{ind}_{cm(\sigma)}(d_2{}') = 0$ and $\mathrm{ind}_\sigma(d_j) = \mathrm{ind}_{cm(\sigma)}(d_j{}')$ for $d_j \in SC(\hat{D}_{d_1})$ and $d_j{}' \in SC(\hat{D}'_{d_1'})$, then $i_\sigma(d_1) = i_{cm(\sigma)}(d_1{}')$. By the same reason, for a

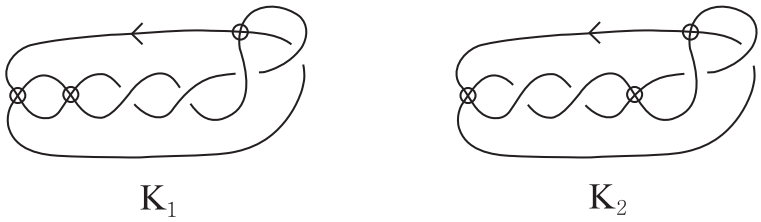



Fig. 17.

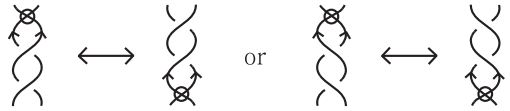



Fig. 18.

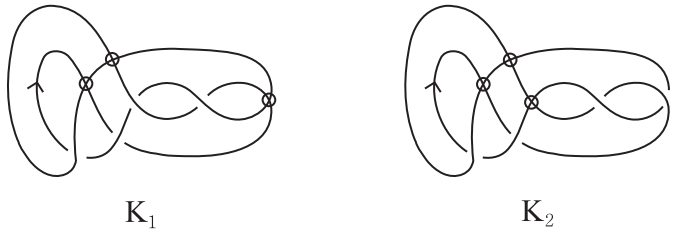



Fig. 19.

classical crossing $d_2$ (resp., $d_2{}'$) in $B$ (resp., $B'$), which belongs to the same component, we have $i_\sigma(d_2) = i_{cm(\sigma)}(d_2{}')$ for any weight map $\sigma$.

Finally, for classical crossings $d_1$, $d_2$ (resp., $d_1{}'$, $d_2{}'$) in $B$ (resp., $B'$), which belong to different components, we get $i_\sigma(d_j) = i_{cm(\sigma)}(d'_j)$ $(j = 1, 2)$ because $\mathrm{ind}_\sigma(d_1) + \mathrm{ind}_\sigma(d_2) = 0 = \mathrm{ind}_{cm(\sigma)}(d_1{}') + \mathrm{ind}_{cm(\sigma)}(d_2{}')$.

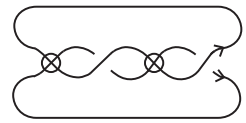
Therefore, we have $P_D(t) = P_{D'}(t)$. □

We give an example which explains Theorem 4.11.

EXAMPLE 4.12. For virtual knots $K_1$ and $K_2$ in Fig. 17 [5], we find that $K_2$ is obtained from $K_1$ by a *double flype* and $P_{K_1}(t) = P_{K_2}(t) = (t + t^{-1} - 2)$ by a computation.

REMARK 4.13. The operations in Fig. 18 are not called double flypes so that they may not preserve the polynomial $P_L(t)$.

For virtual knots $K_1$ and $K_2$ in Fig. 19 [5], $P_{K_1}(t) = (1/2)(t + t^{-1} - 2) + (1/2)(t^2 + t^{-2} - 2)$ and $P_{K_2}(t) = (1/2)(t + t^{-1} - 2)$.



L

Fig. 20.

The *classical crossing number* of a virtual magnetic link $L$, denoted by $c(L)$, is the minimum number of classical crossings of all virtual magnetic link diagrams representing $L$. Our invariant $P_L(t)$ can be used to find a lower bound on the classical crossing number of a virtual magnetic link as follows.

**Theorem 4.14.** *Let $L$ be a virtual magnetic link. Then, the maximal degree of $P_L(t)$ is less than or equal to $c(L)$.*

Proof. Let $D$ be a virtual magnetic link diagram representing $L$. Suppose that the number of classical crossings of $D$ is equal to the classical crossing number of $L$. By the definition of $P_L(t)$, the possible maximal degree of $P_L(t)$ is the classical crossing number of $L$. This completes the proof. □

**Proposition 4.15.** *Let $L$ be a virtual link. Then the maximal degree of $P_L(t)$ is less than or equal to $v(L)$, where $v(L)$ is the virtual crossing number of $L$.*

Proof. For a virtual link $L$, $P_L(t)$ can be regarded as $Q_L(t)$ which is a polynomial invariant defined in [3]. By [3, Corollary 4.14], the maximal degree of $Q_L(t)$ is less than or equal to $v(L)$. □

We give an example which explains Proposition 4.15.

EXAMPLE 4.16. For a virtual link $L$ in Fig. 20, we have $P_L(t) = -(1/2)(t^2 + t^{-2} - 2)$. Thus, the *virtual crossing number* and *classical crossing number* of $L$ is exactly 2, because the maximal degree of $P_L(t)$ is 2 and $v(L), c(L) \leq 2$.

Finally, we give families of virtual knots and their polynomial $P_K(t)$.

EXAMPLE 4.17. Consider the following families of virtual knots.
(1) Let $K_n$ be a classical knot with $n + 2$, $n \geq 0$, crossings as in Fig. 21 (1). Then $P_{K_n}(t) = 0$ by Proposition 4.4.
(2) Let $K'_n$ be a virtual knot obtained from $K_n$ by replacing a classical crossing with a virtual one as in Fig. 21 (2). Then, $P_{K'_{2n}}(t) = (n/2)(t + t^{-1} - 2)$ and $P_{K'_{2n+1}}(t) = ((n+1)/2)(t + t^{-1} - 2)$ for any integer $n$. Thus, $K'_{2n}$ and $K'_{2m}$ (resp. $K'_{2n+1}$ and $K'_{2m+1}$)

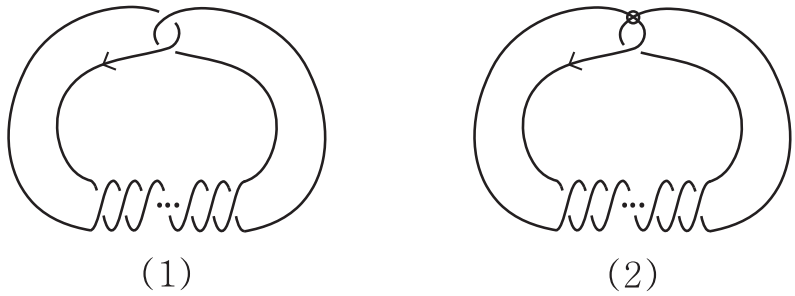



Fig. 21.

are distinct if $n \neq m$. Furthermore, $K'_{2n}$ and $K'_{2m+1}$ are distinct except the case of $n-m=1$.

(3) Let $J_n$ be a virtual knot with $n$ virtual crossings as in Fig. 22 (1), where the box $P_j$, $1 \leq j \leq n$, means $p_j$ horizontal half twists. For any classical crossing $c$ of the lower part of the diagram and any weight map $\sigma$, it is immediate that we have $i_\sigma(c) = 0$ by the smoothing operation.

Let $c$ be a classical crossing of the upper part of the diagram and $\sigma$ be a weight map. We consider two cases according to the number $m = n + |p_1| + |p_2| + \cdots + |p_n|$ is odd or even.

If $m$ is odd, then the crossing $c$ has positive sign. By the smoothing operation at $c$, we have a virtual link diagram of $L$ with two components. Note that if the number of classical crossings between two consecutive virtual crossings $v$ and $v'$ is even (resp., odd), then in the computation of $i_\sigma(c)$, $\mathrm{ind}_\sigma(v) = -\mathrm{ind}_{cm(\sigma)}(v')$ (resp., $\mathrm{ind}_\sigma(v) = \mathrm{ind}_{cm(\sigma)}(v')$). Thus, we have $i_\sigma(c) = \sum_{j=1}^{n}(-1)^{N_j}$, where $N_1 = 0$ and $N_j$, $1 \leq j \leq n$, is the number of boxes with even numbers of classical crossings in the left of the $j$-th virtual crossing from the left in the lower part of the diagram. Therefore, we have

$$P_{J_n}(t) = \frac{1}{2}\left(t^{\sum_{j=1}^{n}(-1)^{N_j}} + t^{-\sum_{j=1}^{n}(-1)^{N_j}} - 2\right).$$

If $m$ is even, then the crossing $c$ has negative sign. By the smoothing operation at $c$, we have a virtual link diagram of $L$ with two components. By the same way, we obtain

$$P_{J_n}(t) = -\frac{1}{2}\left(t^{\sum_{j=1}^{n}(-1)^{N_j}} + t^{-\sum_{j=1}^{n}(-1)^{N_j}} - 2\right).$$

(4) Let $J'_n$ be a virtual knot obtained from $J_n$ by replacing a classical crossing with a virtual one in Fig. 22 (2). For any classical crossing $c$ of the lower part of the diagram and any weight map $\sigma$, it is immediate that we have $|i_\sigma(c)| = 1$ by the smoothing operation.

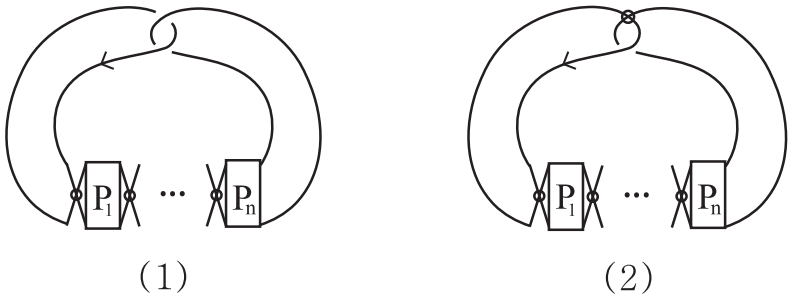



Fig. 22.

By the same way as (3), we have

$$P_{J_n'}(t) = \frac{1}{4}(p_1 + p_2 + \cdots + p_n)(t + t^{-1} - 2) + \frac{1}{4}\left(t^{-1+\sum_{j=1}^{n}(-1)^{N_j}} + t^{1-\sum_{j=1}^{n}(-1)^{N_j}} - 2\right)$$

if $m$ is odd.

And we have

$$P_{J_n'}(t) = \frac{1}{4}(p_1 + p_2 + \cdots + p_n)(t + t^{-1} - 2) - \frac{1}{4}\left(t^{\sum_{j=1}^{n}(-1)^{N_j}} + t^{-\sum_{j=1}^{n}(-1)^{N_j}} - 2\right)$$

if $m$ is even.

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