Konno, K. and Nagashima, D.
Osaka J. Math.
49 (2012), 225–245

MAXIMAL IDEAL CYCLES OVER NORMAL SURFACE SINGULARITIES OF BRIESKORN TYPE

KAZUHIRO KONNO and DAISUKE NAGASHIMA

(Received May 12, 2010, revised October 5, 2010)

Abstract

For normal two dimensional hypersurface singularities of Brieskorn type, concrete descriptions are given to both the fundamental cycle and the maximal ideal cycle on a star-shaped good resolution space. It is determined when these two cycles coincide.

Introduction

Let (V, o) be a germ of a normal surface singularity and $\phi\colon (X, \mathrm{E}) \to (V, o)$ a resolution, where $\mathrm{E} = \phi^{-1}(o)$ denotes the exceptional set. Let $\mathrm{E} = \bigcup_{i=1}^{r} E_i$ be the irreducible decomposition of E. A formal sum $Y = \sum_{i=1}^{r} \lambda_i E_i$ $(\lambda_i \in \mathbb{Z})$ is called a cycle on E. For a cycle Y, $-Y$ is said to be nef on E if $YE_i \leq 0$ for all i. Since the intersection form is negative definite on E, the set $\{Y \succ 0 \mid -Y \text{ is nef on } \mathrm{E}\}$ is non-empty and has the smallest element Z_{E}, the *fundamental cycle* on E. The arithmetic genus of Z_{E} is called the fundamental genus of (V, o) and we denote it by $p_f(V, o)$. Let $\mathfrak{m}$ be the maximal ideal of $\mathcal{O}_{V,o}$. For any non-zero $f \in \mathfrak{m}$, the zero divisor of $f \circ \phi$ can be written as $(f \circ \phi) = (f \circ \phi)_X + D$, where $(f \circ \phi)_X$ is a cycle on E and D is an effective divisor which does not involve any of E_i's. We call $(f \circ \phi)_X$ the cycle on E led by $f \in \mathfrak{m}$. The divisorial part M_{E} of the scheme theoretic fiber $\phi^* o$ is said to be the *maximal ideal cycle* on E. If $f_1, \dots, f_\sigma \in \mathfrak{m}$ generate $\mathfrak{m}$, then $M_{\mathrm{E}} = \inf_{1 \leq i \leq \sigma} (f_i \circ \phi)_X$ by [14, Proposition 2.12]. Since $-M_{\mathrm{E}}$ is nef, we always have $0 \prec Z_{\mathrm{E}} \preceq M_{\mathrm{E}}$.

It sometimes happens that $M_{\mathrm{E}} = Z_{\mathrm{E}}$, as one can observe for rational singular points, Kodaira singular points and singularities of type $\{z^n = f(x, y)\}$ $(n \geq 2,\ f \in \mathbb{C}\{x, y\})$ when $n \gg 0$. As for the last type, Tomaru proved in [10, Theorem 4.1] that two cycles coincide on any resolution when n divides $\mathrm{ord}(f)$, extending the well-known result for $n = 2$ due to Dixon [2, Theorem 1]. However, even for a particular class of singularities, a more systematic study will be required in order to clarify when such a coincidence of important cycles occurs.

2010 Mathematics Subject Classification. Primary 14J17; Secondary 32S25.

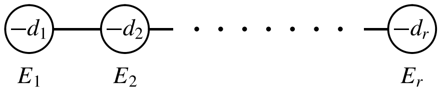

Fig. 1.

In this paper, we consider the normal 2-dimensional hypersurface singularities of Brieskorn type,

$$(V_{a_0,a_1,a_2}, o) = \{(x_0, x_1, x_2) \in U \mid x_0^{a_0} + x_1^{a_1} = x_2^{a_2}\},$$

where $U \subset \mathbb{C}^3$ is a small neighborhood of the origin $o = (0, 0, 0)$ and the a_i's are integers with $2 \leq a_0 \leq a_1 \leq a_2$, and give a necessary and sufficient condition for the coincidence of the maximal ideal cycle and the fundamental cycle.

Before stating the results, let us introduce some notation which will be used throughout the paper. We put $\lceil x \rceil := \min\{n \in \mathbb{Z} \mid n \geq x\}$ for $x \in \mathbb{R}$. For integers $d_1, d_2, \ldots, d_r$ ($d_i \geq 2$ for all i), we denote by $[[d_1, d_2, \ldots, d_r]]$ the continued fraction:

$$[[d_1, d_2, \ldots, d_r]] := d_1 - \cfrac{1}{d_2 - \cfrac{1}{d_3 - \cfrac{1}{\ddots - \cfrac{1}{d_r}}}}.$$

Let n and μ be positive integers that are relatively prime and $0 < \mu < n$. The singularity

$$C_{n,\mu} := \mathbb{C}^2 \Big/ \left\langle \begin{pmatrix} \epsilon_n & 0 \\ 0 & \epsilon_n^{\mu} \end{pmatrix} \right\rangle,$$

where $\epsilon_n := \exp(2\pi\sqrt{-1}/n)$ denotes the primitive n-th root of unity, is called a cyclic quotient singularity. It is well-known (e.g., [4]) that, if $\mathrm{E} = \bigcup_{i=1}^{r} E_i$ is the exceptional set for the minimal resolution of $C_{n,\mu}$, then $E_i \simeq \mathbb{P}^1$ and the weighted dual graph of E is chain-shaped as in Fig. 1, where $n/\mu = [[d_1, d_2, \ldots, d_r]]$.

To the singularity $(V_{a_0,a_1,a_2}, o) = \{x_0^{a_0} + x_1^{a_1} = x_2^{a_2}\}$, $2 \leq a_0 \leq a_1 \leq a_2$, we associate the integers

$$l := \gcd(a_0, a_1, a_2), \quad l_i := \frac{\gcd(a_j, a_k)}{l}, \quad \alpha_i := \frac{a_i}{l_j l_k l} \quad (\{i, j, k\} = \{0, 1, 2\}).$$

Furthermore, we let p_0, p_1, p_2 be the integers determined by

$$p_i \alpha_j \alpha_k l_i + 1 \equiv 0 \pmod{\alpha_i}, \quad 0 \leq p_i < \alpha_i,$$

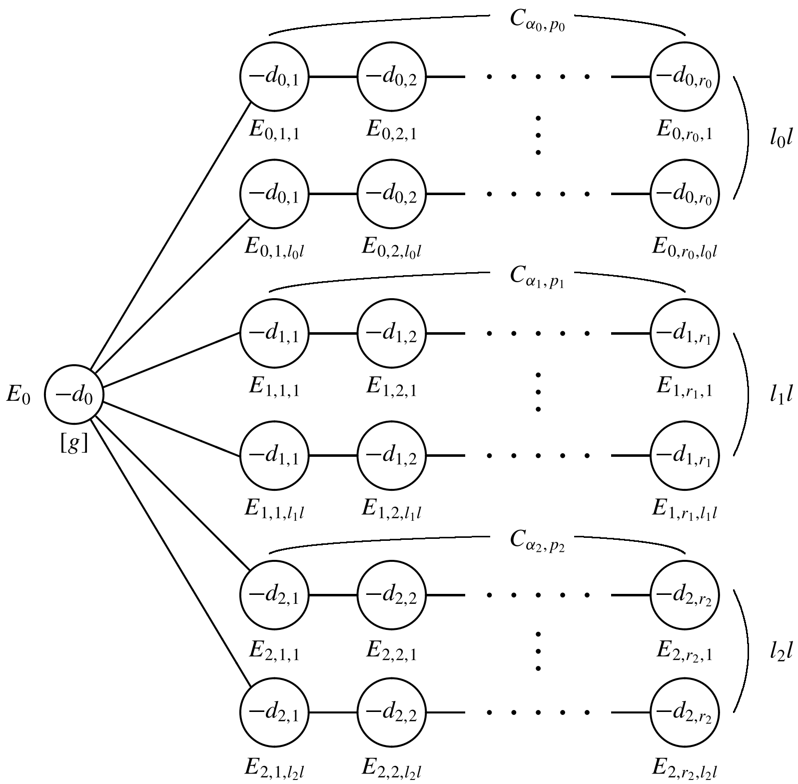

Fig. 2. Weighted dual graph of E_π.

where $\{i, j, k\} = \{0, 1, 2\}$. When $\alpha_w > 1$, we also put

$$\alpha_w/p_w = [[d_{w,1}, d_{w,2}, \ldots, d_{w,r_w}]]$$

and $e_{w,\nu} := [[d_{w,\nu}, d_{w,\nu+1}, \ldots, d_{w,r_w}]]$ $(1 \le \nu \le r_w;\ w = 0,1,2)$. With this notation, by [7], there exists a resolution $\pi : (X, \mathrm{E}_\pi) \to (V_{a_0,a_1,a_2}, o)$ such that the weighted dual graph of E_π is as in Fig. 2. It is star-shaped with $l_w l$ branches of type C_{α_w,p_w} $(w = 0, 1, 2)$ starting from a unique vertex. The non-singular curve E_0 corresponding to that vertex will be referred to as the central curve. We shall mainly work on the resolution π. Note that, when $\alpha_w = 1$, the corresponding branches do not appear and we understand $C_{1,0}$ as a non-singular point on E_0.

Now, we are going to state our results in this paper. First, we concretely describe the fundamental cycle over (V_{a_0,a_1,a_2}, o). We remark here that an algorithm computing the fundamental cycle from the exponents a_0, a_1, a_2 was established by Tomari (cf. [8, (3.3)]).

Theorem 1.4. *Let*

$$Z = \theta_0 E_0 + \sum_{w=0}^{2} \sum_{\nu=1}^{r_w} \sum_{\xi=1}^{l_w l} \theta_{w,\nu,\xi} E_{w,\nu,\xi}$$

be the fundamental cycle for resolution π. *Then the sequence* $\{\theta_{w,\nu,\xi}\}_{\nu=0}^{r_w}$ $(w = 0, 1, 2)$ *is defined by the following recurrence formula*:

(1) $\theta_{w,0,\xi} := \theta_0 = \begin{cases} \alpha_0\alpha_1\alpha_2 & \text{if} \quad \alpha_2 \le l_2, \\ \alpha_0\alpha_1 l_2 & \text{if} \quad \alpha_2 \ge l_2, \end{cases}$

(2) $\theta_{w,\nu,\xi} = \lceil \theta_{w,\nu-1,\xi}/e_{w,\nu} \rceil$, $1 \le \nu \le r_w$.

To show it, we first study the chain-shaped configuration obtained by plugging one extra vertex to the configuration of type $C_{n,\mu}$. We consider the condition which should be satisfied by the smallest cycle $Y \succ 0$ such that $-Y$ is nef on the $C_{n,\mu}$ part, when the multiplicity at the extra vertex is given. Then we apply it to branches of the star-shaped configuration as in Fig. 2. To determine the multiplicity of the central curve is our final task. Our method is so simple that it may apply also to the other singularities with $\mathbb{C}^*$-action in determining their fundamental cycles. As a by-product, we can reprove in Theorem 1.7 the formula computing the fundamental genus which was originally obtained in [8] and [9] by an entirely different method.

Next, we turn our attention to the maximal ideal cycle. With the help of Tomaru's result in [12], we can determine the multiplicity of each component and give a formula similar to Theorem 1.4 also for $(x_w \circ \pi)_X$ $(w = 0,1,2)$. See, Theorem 2.1 for the precise statement. This enables us to show in Theorem 3.1 that $(x_2 \circ \pi)_X$ is the maximal ideal cycle for resolution π (here, the assumption $a_0 \le a_1 \le a_2$ is essential).

Using the concrete descriptions thus obtained, we compare the fundamental cycle and the maximal ideal cycle on E_π and get the following:

Theorem 3.2. *The maximal ideal cycle coincides with the fundamental cycle for resolution* π *if and only if* $\alpha_2 \ge l_2$.

In particular, this implies that both cycles coincide on the minimal resolution, when a_0 is a prime number. Similarly, we obtain the following:

Theorem 3.11. *The maximal ideal cycle coincides with the fundamental cycle for any resolution of* (V_{a_0,a_1,a_2}, o), $2 \le a_0 \le a_1 \le a_2$, *if and only if* $\alpha_2 \ge l_2$ *and* $1 < a_1/a_2 + \gcd(a_0, a_1)/a_0$. *Furthermore, if this is the case, then the fundamental cycle is led by the holomorphic function* x_2.

As an application, we give in Proposition 4.4 the necessary and sufficient condition for (V_{a_0,a_1,a_2}, o) to be a Kodaira singularity ([5], [6]), in order to supplement a result in [10]. We also describe the canonical cycle for π in Proposition 4.6.

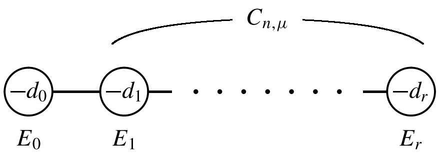

Fig. 3.

The paper grew out of the second named author's master thesis at Osaka University. The authors would like to express their deep gratitude to Professor Tadashi Tomaru for many helpful suggestions in the course of the study. Thanks are also due to Professors Masataka Tomari, Tomohiro Okuma and Tadashi Ashikaga for offering the second named author an opportunity to give a talk at their exciting seminar and for their precious comments.

1. The fundamental cycle

1.1. Minimal anti-nef cycle on a chain. Let n and μ be integers that are relatively prime and $0 < \mu < n$, and put $n/\mu = [[d_1, d_2, \dots, d_r]]$. We consider a connected bunch $\bigcup_{i=0}^{r} E_i$ of irreducible curves E_i on a smooth surface whose weighted dual graph is chain-shaped and $E_1 + \cdots + E_r$ forms the configuration of type $C_{n,\mu}$ as in Fig. 3. We put $e_i := [[d_i, d_{i+1}, \dots, d_r]]$ for $1 \le i \le r$. Then $d_i = e_i + 1/e_{i+1}$ for $1 \le i < r$, and $d_r = e_r$. For a positive integer λ_0, consider the set

$$D(\lambda_0) := \left\{ Y\colon \text{ cycle on } \bigcup_{i=0}^{r} E_i \;\middle|\; -Y \text{ is nef on } \bigcup_{i=1}^{r} E_i,\ \ \mathrm{mult}_{E_0}(Y) = \lambda_0 \right\}.$$

Lemma 1.1. *Take a positive integer λ_0 and define the sequence $\{\lambda_i\}_{i=0}^{r}$ by the recurrence formula $\lambda_i = \lceil \lambda_{i-1}/e_i \rceil$ for $1 \le i \le r$. Then the cycle $Y_0 := \sum_{i=0}^{r} \lambda_i E_i$ is the smallest element of $D(\lambda_0)$.*

Proof. Let ρ_i $(0 \le i \le r)$ be positive integers and put $Y = \sum_{i=0}^{r} \rho_i E_i$. We first claim that $\lceil \rho_{i-1}/e_i \rceil \le \rho_i$ holds for $1 \le i \le r$, if $-Y$ is nef on $\bigcup_{i=1}^{r} E_i$. This can be seen by induction as follows. For $i = r$, we have $0 \ge YE_r = \rho_{r-1} - \rho_r d_r$. Since $d_r = e_r$, we get $\rho_{r-1}/e_r \le \rho_r$ which implies $\lceil \rho_{r-1}/e_r \rceil \le \rho_r$. Take an index i with $1 \le i < r$ and assume that $\lceil \rho_i/e_{i+1} \rceil \le \rho_{i+1}$ holds. We have $0 \ge YE_i = \rho_{i-1} - \rho_i d_i + \rho_{i+1}$. Then $\rho_{i-1}/\rho_i \le d_i - \rho_{i+1}/\rho_i \le d_i - 1/e_{i+1} = e_i$, since $\rho_i/e_{i+1} \le \rho_{i+1}$ by the hypothesis. It follows $\rho_{i-1}/e_i \le \rho_i$ and, hence, $\lceil \rho_{i-1}/e_i \rceil \le \rho_i$.

We next consider the cycle Y_0. We have $Y_0 E_r = \lambda_{r-1} - d_r \lambda_r = \lambda_{r-1} - e_r \lambda_r \le 0$, because $\lambda_r = \lceil \lambda_{r-1}/e_r \rceil \ge \lambda_{r-1}/e_r$. Take any index i with $1 \le i < r$. Since $\lambda_i = \lceil \lambda_{i-1}/e_i \rceil \ge \lambda_{i-1}/e_i$, we get $\lambda_{i-1}/\lambda_i \le e_i = d_i - 1/e_{i+1}$, i.e., $\lambda_{i-1} \le d_i \lambda_i - \lambda_i/e_{i+1}$. It follows $\lambda_{i-1} \le d_i \lambda_i - \lceil \lambda_i/e_{i+1} \rceil = d_i \lambda_i - \lambda_{i+1}$, which shows $Y_0 E_i \le 0$. Hence $Y_0 \in$

$D(\lambda_0)$. Then, from the first half of the proof, it is clear that Y_0 is the smallest element of $D(\lambda_0)$. $\square$

Note that we did not specify the self-intersection number of E_0. As the proof shows, if Y_0 and Y_0' are the smallest elements in $D(\lambda_0)$ and $D(\lambda_0')$, respectively, then we have $Y_0 \preceq Y_0'$ if and only if $\lambda_0 \leq \lambda_0'$.

Since the proof of the following lemma is elementary, we leave it to the reader.

Lemma 1.2. *Let the sequence $\{\lambda_i\}_{i=0}^r$ be as in the previous lemma and, for $1 \leq i \leq r$, take relatively prime positive integers n_i and μ_i satisfying $n_i/\mu_i = e_i$. Put $\lambda_{r+1} := \lambda_r d_r - \lambda_{r-1}$.*
(1) *If $\lambda_{i-1} = \lambda_i d_i - \lambda_{i+1}$ holds for $1 \leq i \leq r$, then $\lambda_1 = (\mu\lambda_0 + \lambda_{r+1})/n$.*
(2) *If $\lambda_0 \equiv 0 \pmod n$, then $\lambda_i = \mu_i\lambda_{i-1}/n_i$ for $1 \leq i \leq r$. If $\mu\lambda_0 + 1 \equiv 0 \pmod n$, then $\lambda_i = (\mu_i\lambda_{i-1} + 1)/n_i$ for $1 \leq i \leq r$.*
(3) *If either $\lambda_0 \equiv 0 \pmod n$ or $\mu\lambda_0 + 1 \equiv 0 \pmod n$, then $\lambda_{i-1} = \lambda_i d_i - \lambda_{i+1}$ holds for $1 \leq i \leq r$. Furthermore, $\lambda_{r+1} = 0$ when $\lambda_0 \equiv 0 \pmod n$, and $\lambda_{r+1} = 1$ when $\mu\lambda_0 + 1 \equiv 0 \pmod n$.*
(4) *If $\lambda_0 \equiv 0 \pmod n$, then $\lambda_r = \lambda_0/n$. If $\mu\lambda_0 + 1 \equiv 0 \pmod n$, then $\lambda_r = \lceil \lambda_0/n \rceil$.*

1.2. The fundamental cycle.

We keep the notation in Introduction.

Proposition 1.3. *There exists a resolution $\pi\colon (X, \mathrm{E}_\pi) \to (V_{a_0,a_1,a_2}, o)$ such that the weighted dual graph of E_π is as in* Fig. 2. *Furthermore, the genus g and the self-intersection number $-d_0$ of the central curve E_0 are given respectively by*

$$2g - 2 = l(l_0l_1l_2l - l_0 - l_1 - l_2),$$
$$d_0 = l\left(\sum_{w=0}^{2} \frac{p_w l_w}{\alpha_w} + \frac{1}{\alpha_0\alpha_1\alpha_2}\right).$$

Proof. See [7, Proposition (3.5.1) and Theorem (3.6.1)]. We shall prove it in the course of the proof of Theorem 2.1 below. $\square$

In the sequel, we will work on the resolution space X in Proposition 1.3, unless otherwise stated explicitly.

Theorem 1.4. *Let*

$$Z = \theta_0 E_0 + \sum_{w=0}^{2}\sum_{\nu=1}^{r_w}\sum_{\xi=1}^{l_w l} \theta_{w,\nu,\xi} E_{w,\nu,\xi}$$

be the fundamental cycle for resolution π. Then the sequence $\{\theta_{w,\nu,\xi}\}_{\nu=0}^{r_w}$ $(w = 0, 1, 2)$ is defined by the following recurrence formula:

(1) $\theta_{w,0,\xi} := \theta_0 = \begin{cases} \alpha_0\alpha_1\alpha_2 & \textit{if} \quad \alpha_2 \le l_2, \\ \alpha_0\alpha_1 l_2 & \textit{if} \quad \alpha_2 \ge l_2, \end{cases}$

(2) $\theta_{w,\nu,\xi} = \lceil \theta_{w,\nu-1,\xi}/e_{w,\nu} \rceil$ $(1 \le \nu \le r_w)$.

Proof. By Lemma 1.1 applied to each branch of type C_{α_w,p_w} plugged to E_0, we obtain (2) once θ_0 is given. So, it suffices to show (1). Let u_w $(w = 0,1,2)$ be the integer determined by $p_w\theta_0 + u_w \equiv 0 \pmod{\alpha_w}$,] $0 \le u_w < \alpha_w$. Then $\theta_{w,1,\xi} = \lceil \theta_0/e_{w,1} \rceil = (p_w\theta_0 + u_w)/\alpha_w$ by (2). By substituting the formula for d_0 in Proposition 1.3, the inequality $\theta_0 d_0 \ge \sum_{w=0}^{2} \sum_{\xi=1}^{l_w l} \theta_{w,1,\xi}$ coming from $-ZE_0 \ge 0$ becomes

$$\theta_0 l \left(\sum_{w=0}^{2} \frac{p_w l_w}{\alpha_w} + \frac{1}{\alpha_0\alpha_1\alpha_2} \right) \ge \sum_{w=0}^{2} l_w l \frac{p_w\theta_0 + u_w}{\alpha_w}.$$

It follows $\theta_0 \ge \alpha_1\alpha_2 l_0 u_0 + \alpha_0\alpha_2 l_1 u_1 + \alpha_0\alpha_1 l_2 u_2$. Put

$$\Lambda := \left\{ \lambda \ge 1 \;\middle|\; \begin{array}{l} \lambda \ge \alpha_1\alpha_2 l_0 u_0 + \alpha_0\alpha_2 l_1 u_1 + \alpha_0\alpha_1 l_2 u_2, \\ p_w\lambda + u_w \equiv 0 \pmod{\alpha_w},\ 0 \le u_w < \alpha_w\ (w = 0,\,1,\,2) \end{array} \right\}.$$

Then, since Z is the fundamental cycle, $\theta_0 = \min \Lambda$. Since α_0, α_1 and α_2 are mutually coprime, we have $\min\{\lambda \in \Lambda \mid u_0 = u_1 = u_2 = 0\} = \alpha_0\alpha_1\alpha_2$. Also, since $\alpha_0\alpha_1 l_2 \le \alpha_0\alpha_2 l_1 \le \alpha_1\alpha_2 l_0$ by $a_0 \le a_1 \le a_2$, we get $\min\{\lambda \in \Lambda \mid (u_0, u_1, u_2) \ne (0,0,0)\} = \alpha_0\alpha_1 l_2$. Therefore,

$$\theta_0 = \begin{cases} \alpha_0\alpha_1\alpha_2 & \text{if} \quad \alpha_2 \le l_2, \\ \alpha_0\alpha_1 l_2 & \text{if} \quad \alpha_2 \ge l_2. \end{cases}$$

This shows (1). □

We put $\theta_{w,\nu} := \theta_{w,\nu,\xi}$, because it does not depend on ξ, and $\theta_{w,r_w+1} := \theta_{w,r_w} d_{w,r_w} - \theta_{w,r_w-1}$. Furthermore, we sometimes write $E_{w,\nu}$ for $E_{w,\nu,\xi}$, when the index ξ is not important. By Lemma 1.2, we get the following:

Lemma 1.5. *Let the situation be as above. Then*

$$\theta_{w,\nu-1} = \theta_{w,\nu} d_{w,\nu} - \theta_{w,\nu+1} \quad (w = 0,\,1,\,2;\ 1 \le \nu \le r_w).$$

Furthermore, the following hold:

(1) *If* $\alpha_2 \le l_2$, *then* $\theta_{i,r_i} = \alpha_j\alpha_k$ $(\{i, j, k\} = \{0, 1, 2\})$ *and* $\theta_{w,r_w+1} = 0$ $(w = 0, 1, 2)$.

(2) *If* $\alpha_2 \ge l_2$, *then* $\theta_{0,r_0} = \alpha_1 l_2$, $\theta_{1,r_1} = \alpha_0 l_2$, $\theta_{2,r_2} = \lceil \alpha_0\alpha_1 l_2/\alpha_2 \rceil$, $\theta_{0,r_0+1} = \theta_{1,r_1+1} = 0$ *and* $\theta_{2,r_2+1} = 1$.

Proposition 1.6. *The self-intersection number of the fundamental cycle is given by*

$$-Z^2 = \begin{cases} l\alpha_0\alpha_1\alpha_2 & \text{if} \quad \alpha_2 \le l_2, \\ l_2 l\lceil \alpha_0\alpha_1 l_2/\alpha_2\rceil & \text{if} \quad \alpha_2 \ge l_2. \end{cases}$$

Proof. By Lemma 1.5, we have $ZE_{w,\nu} = 0$ for $w = 0, 1, 2$, $1 \le \nu \le r_w - 1$.

We first consider the case where $\alpha_2 \le l_2$. We already know that $ZE_{w,r_w} = \theta_{w,r_w+1} = 0$ $(w = 0, 1, 2)$. Since

$$\begin{aligned} -ZE_0 &= \theta_0 d_0 - \sum_{w=0}^{2} l_w l\theta_{w,1} \\ &= l(p_0\alpha_1\alpha_2 l_0 + p_1\alpha_0\alpha_2 l_1 + p_2\alpha_0\alpha_1 l_2 + 1) - l_0 l p_0\alpha_1\alpha_2 - l_1 l p_1\alpha_0\alpha_2 - l_2 l p_2\alpha_0\alpha_1 \\ &= l, \end{aligned}$$

we obtain $-Z^2 = l\theta_0 = l\alpha_0\alpha_1\alpha_2$.

Next, we consider the case where $\alpha_2 \ge l_2$. We have $ZE_{0,r_0} = \theta_{0.r_0+1} = 0$, $ZE_{1,r_1} = \theta_{1.r_1+1} = 0$ and $ZE_{2,r_2} = \theta_{2.r_2+1} = 1$. Furthermore,

$$\begin{aligned} -ZE_0 &= \theta_0 d_0 - \sum_{w=0}^{2} l_w l\theta_{w,1} \\ &= \alpha_0\alpha_1 l_2 l\left(\sum_{w=0}^{2} \frac{p_w l_w}{\alpha_w} + \frac{1}{\alpha_0\alpha_1\alpha_2}\right) - l_0 l p_0\alpha_1 l_2 - l_1 l p_1\alpha_0 l_2 - l_2 l \frac{p_2\alpha_0\alpha_1 l_2 + 1}{\alpha_2} \\ &= 0. \end{aligned}$$

Therefore, $-Z^2 = l_2 l\theta_{2,r_2} = l_2 l\lceil \alpha_0\alpha_1 l_2/\alpha_2\rceil$. □

Theorem 1.7 ([9, Theorem 2]). *The fundamental genus p_f of (V_{a_0,a_1,a_2}, o), $2 \le a_0 \le a_1 \le a_2$, is given as follows.*

(1) *If $\alpha_2 \le l_2$, then*

$$p_f = \frac{1}{2}l\{\operatorname{lcm}(a_0, a_1, a_2) - \alpha_1\alpha_2 l_0 - \alpha_0\alpha_2 l_1 - \alpha_0\alpha_1 l_2 - \alpha_0\alpha_1\alpha_2 + 1\} + 1.$$

(2) *If $\alpha_2 \ge l_2$, then*

$$p_f = \frac{1}{2}\left\{(a_0 - 1)(a_1 - 1) - \left(2\left\lceil \frac{\alpha_0\alpha_1 l_2}{\alpha_2}\right\rceil - 1\right)\gcd(a_0, a_1) + 1\right\}.$$

Proof. We consider the case where $\alpha_0, \alpha_1, \alpha_2 \ge 2$. The other cases can be treated similarly. Let K be the canonical line bundle on X. Since $E_{w,\nu} \simeq \mathbb{P}^1$ for $\nu \ne 0$, we have $KE_{w,\nu} = -(E_{w,\nu})^2 + 2 \times 0 - 2 = d_{w,\nu} - 2$. Similarly, since E_0 is of genus g, $KE_0 = -E_0^2 + 2g - 2 = d_0 + 2g - 2$. It follows

$$\begin{aligned}
KZ &= \theta_0(d_0 + 2g - 2) + \sum_{w=0}^{2} l_w l \sum_{\nu=1}^{r_w} \theta_{w,\nu}(d_{w,\nu} - 2) \\
&= \theta_0\{d_0 + l(l_0 l_1 l_2 l - l_0 - l_1 - l_2)\} + \sum_{w=0}^{2} l_w l \left(\sum_{\nu=1}^{r_w} (\theta_{w,\nu-1} + \theta_{w,\nu+1}) - \sum_{\nu=1}^{r_w} 2\theta_{w,\nu} \right) \\
&= \theta_0\{d_0 + l(l_0 l_1 l_2 l - l_0 - l_1 - l_2)\} + \sum_{w=0}^{2} l_w l (\theta_0 - \theta_{w,1} - \theta_{w,r_w} + \theta_{w,r_w+1}) \\
&= \theta_0 l_0 l_1 l_2 l^2 + \left(\theta_0 d_0 - \sum_{w=0}^{2} l_w l \theta_{w,1} \right) - \sum_{w=0}^{2} l_w l \theta_{w,r_w} + l_2 l \theta_{2,r_2} \\
&= \theta_0 l_0 l_1 l_2 l^2 - ZE_0 - \sum_{w=0}^{2} l_w l \theta_{w,r_w} + l_2 l \theta_{2,r_2}.
\end{aligned}$$

(1) If $\alpha_2 \leq l_2$, then $\theta_0 = \alpha_0\alpha_1\alpha_2$, $-ZE_0 = l$, $\theta_{0,r_0} = \alpha_1\alpha_2$, $\theta_{1,r_1} = \alpha_0\alpha_2$, $\theta_{2,r_2} = \alpha_0\alpha_1$, $\theta_{2,r_2+1} = 0$ and $Z^2 = -l\alpha_0\alpha_1\alpha_2$ by Lemma 1.5 and Proposition 1.6. The assertion follows from the formula $2p_f - 2 = KZ + Z^2$.

(2) If $\alpha_2 \geq l_2$, then $\theta_0 = \alpha_0\alpha_1 l_2$, $ZE_0 = 0$, $\theta_{0,r_0} = \alpha_1 l_2$, $\theta_{1,r_1} = \alpha_0 l_2$, $\theta_{2,r_2} = \lceil \alpha_0\alpha_1 l_2/\alpha_2 \rceil$, $\theta_{2,r_2+1} = 1$ and $Z^2 = -l_2 l \lceil \alpha_0\alpha_1 l_2/\alpha_2 \rceil$ by Lemma 1.5 and Proposition 1.6. Therefore, we obtain the assertion. □

REMARK 1.8. An algorithm computing Z from the exponents a_0, a_1, a_2 was first obtained by Masataka Tomari (cf. [8, (3.3)]). Based on it, the formula for p_f was shown by Tomaru in [8, Theorem 4.3] in the special case: $\mathrm{lcm}(a_0, a_1) \leq a_2$, and later completed in [9, Theorem 2]. However, the hypothesis of [9, Theorem 2 (1)] needs a small correction from "$m = l_2$" to "$m = l_2 \geq 2$". Indeed, when $\alpha_2 = l_2 = 1$, [9, Theorem 2 (1)] yields $2p_f - 2 = (a_0 - 1)(a_1 - 1) - (2\alpha_0\alpha_1 + 1)l - 1$, while it should be $2p_f - 2 = (a_0 - 1)(a_1 - 1) - (2\alpha_0\alpha_1 - 1)l - 1$ according to Theorem 1.7.

2. Cycles led by coordinate functions

The purpose of the section is to show the following:

Theorem 2.1. *Let $Z^{(k)} := (x_k \circ \pi)_X$ be the cycle on E_π led by x_k ($k = 0, 1, 2$), and put*

$$Z^{(k)} = \lambda_0^{(k)} E_0 + \sum_{w=0}^{2} \sum_{\nu=1}^{r_w} \sum_{\xi=1}^{l_w l} \lambda_{w,\nu,\xi}^{(k)} E_{w,\nu,\xi}.$$

Then the sequence $\{\lambda^{(k)}_{w,\nu,\xi}\}$ $(k = 0, 1, 2)$ *is determined by the following recurrence formula.*

$$\lambda^{(k)}_{w,\nu-1,\xi} = \lambda^{(k)}_{w,\nu,\xi} d_{w,\nu} - \lambda^{(k)}_{w,\nu+1,\xi},$$
$$\lambda^{(k)}_{w,0,\xi} := \lambda^{(k)}_0 = \alpha_i \alpha_j l_k \quad (\{i,\ j,\ k\} = \{0, 1, 2\}),$$
$$\lambda^{(k)}_{w,r_w+1,\xi} = \begin{cases} 1 & \textit{if} \quad w = k, \\ 0 & \textit{if} \quad w \neq k. \end{cases}$$

In particular, for $\{i,\ j,\ k\} = \{0, 1, 2\}$,

$$\begin{cases} \lambda^{(k)}_{i,1,\xi} = p_i \alpha_j l_k, & \lambda^{(k)}_{k,1,\xi} = \dfrac{p_k \alpha_i \alpha_j l_k + 1}{\alpha_k}, \\ \lambda^{(k)}_{i,r_i,\xi} = \alpha_j l_k, & \lambda^{(k)}_{k,r_k,\xi} = \left\lceil \dfrac{\alpha_i \alpha_j l_k}{\alpha_k} \right\rceil. \end{cases}$$

We divide the proof into three steps. During the proof, we will construct the resolution π and show Proposition 1.3. Put $\{i,\ j,\ k\} = \{0, 1, 2\}$ and denote the primitive n-th root of unity by ϵ_n.

Step 1. The resolution of the branch locus. We put $C := \{x_i^{a_i} + x_j^{a_j} = 0\} \subset \mathbb{C}^2$. First, we compute the minimal embedded resolution of C. Though there are several methods for computing such resolutions (see [1]), we use a result in [11] here. Put $d := \mathrm{lcm}(a_i, a_j)$, $n_1 := a_i/\gcd(a_i, a_j)$, $n_2 := a_j/\gcd(a_i, a_j)$. Furthermore we put $\bar{C} := \{\bar{x}_i^d + \bar{x}_j^d = 0\} \subset \mathbb{C}^2$, and let $\Psi\colon \mathbb{C}^2_{(\bar{x}_i, \bar{x}_j)} \to \mathbb{C}^2_{(x_i, x_j)}$ be the holomorphic map defined by $x_i = \bar{x}_i^{n_2}$, $x_j = \bar{x}_j^{n_1}$. Since $d = a_i a_j/\gcd(a_i, a_j) = a_i n_2 = a_j n_1$, we have $\Psi(\bar{C}) = C$. The map Ψ can be regarded as the quotient map by the natural action to $\mathbb{C}^2$ of the group

$$G = \left\langle \begin{pmatrix} \epsilon_{n_2} & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & \epsilon_{n_1} \end{pmatrix} \right\rangle.$$

Let $\bar{\Phi}\colon \bar{N} \to \mathbb{C}^2$ be the blowing-up at the origin of the $(\bar{x}_i, \bar{x}_j)$-plane. We denote by $\bar{E}$ the exceptional (-1)-curve for $\bar{\Phi}$. Then $\bar{N}$ is covered by two open sets U_0 and U_1 each of which is isomorphic to $\mathbb{C}^2$. The action of G is lifted onto $\bar{N}$ through $\bar{\Phi}$. Let μ_1, μ_2 be non-negative integers defined by

$$n_2\mu_1 + 1 \equiv 0 \pmod{n_1}, \quad 0 \le \mu_1 < n_1,$$
$$n_1\mu_2 + 1 \equiv 0 \pmod{n_2}, \quad 0 \le \mu_2 < n_2.$$

Then, from [11, Theorem 2.3], we can easily see that the quotient space $\bar{N}/G$ is covered by two cyclic quotient singularity spaces U_0/G and U_1/G whose respective types are C_{n_1,μ_1} and C_{n_2,μ_2}; also those singular points are located on $\psi(\bar{E}) \simeq \mathbb{P}^1$, where $\psi\colon \bar{N} \to \bar{N}/G$ is the quotient map. Let $\eta\colon N \to \bar{N}/G$ be the minimal resolution

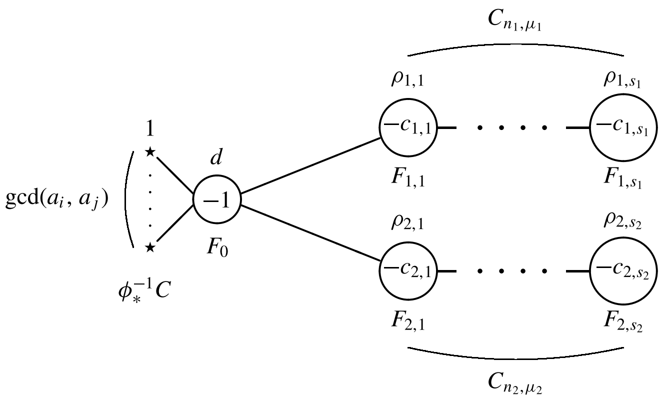

Fig. 4. Weighted dual graph of ϕ^*C.

of those two cyclic quotient singularities and $\Phi\colon \bar{N}/G \to \mathbb{C}^2$ the natural map to the (x_i, x_j)-plane. Then the composite $\phi = \Phi \circ \eta\colon N \to \mathbb{C}^2$ gives us the minimal embedded resolution of C.

Second, we describe ϕ^*C. The strict transform $\bar{\Phi}_*^{-1}\bar{C}$ of $\bar{C}$ by $\bar{\Phi}$ consists of disjoint d branches each of which intersects $\bar{E}$ transversally at a point. Then $\psi(\bar{\Phi}_*^{-1}\bar{C})$ consists of $\gcd(a_i, a_j)$ irreducible components each of which intersects $\psi(\bar{E})$ transversally at a point. For $\bar{f} := \bar{x}_i^d + \bar{x}_j^d$, the multiplicity of $\bar{f} \circ \bar{\Phi}$ along $\bar{E}$ is d. If f denotes the holomorphic function on $\bar{N}/G$ induced by $\bar{f}$, then the multiplicity of f along $\psi(\bar{E})$ is also d. Furthermore, the multiplicity of f along each component of $\psi(\bar{\Phi}_*^{-1}\bar{C})$ is one. Since $f = (x_i^{a_i} + x_j^{a_j}) \circ \Phi$, the dual graph of the divisor ϕ^*C becomes as in Fig. 4. In that figure, F_0 is the strict transform of $\psi(\bar{E})$ by η and F_{m,ν_m} $(m = 1, 2; 1 \le \nu_m \le s_m)$ is the exceptional curve arising from C_{n_m,μ_m} with self-intersection number $-c_{m,\nu_m}$, where $n_m/\mu_m = [[c_{m,1}, c_{m,2}, \dots, c_{m,s_m}]]$. For $m = 1, 2$, we denote by ρ_{m,ν_m} the multiplicity of ϕ^*C along F_{m,ν_m}. Since $F_{m,\nu_m}\phi^*C = 0$, we have

$$\rho_{m,\nu_m-1} = \rho_{m,\nu_m} c_{m,\nu_m} - \rho_{m,\nu_m+1}, \quad 1 \le \nu_m \le s_m, \tag{2.1}$$

with $\rho_{m,0} = d$ and $\rho_{m,s_m+1} = 0$. Then, by Lemma 1.2 (1), we get $\rho_{m,1} = \mu_m d/n_m$, that is,

$$\rho_{1,1} = \mu_1 a_j, \quad \rho_{2,1} = \mu_2 a_i. \tag{2.2}$$

We also have $\rho_{1,1} + \rho_{2,1} + \gcd(a_i, a_j) = d$ by $F_0\phi^*C = 0$, since $-F_0^2 = F_0F_{1,1} = F_0F_{2,1} = 1$ and $F_0\phi_*^{-1}C = \gcd(a_i, a_j)$.

Step 2. The resolution of the cyclic covering. We consider the resolution of $\{x_i^{a_i} + x_j^{a_j} = x_k^{a_k}\}$ regarding V_{a_0,a_1,a_2} as an a_k-fold cyclic covering of $\mathbb{C}^2$. Let $\phi\colon N \to \mathbb{C}^2$ be the holomorphic map constructed in Step 1. We consider the normalization W of

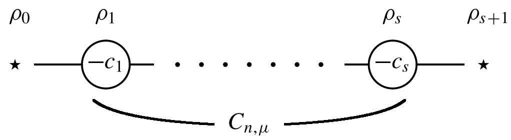

Fig. 5.

the fiber product $V_{a_0,a_1,a_2} \times_{\mathbb{C}^2} N$. For this purpose, we use the following result due to Tomaru [12].

Theorem 2.2 ([12]). *Let (U, o) be the cyclic quotient singularity of type $C_{n,\mu}$, and $\mathfrak{m}$ the maximal ideal of $\mathcal{O}_{U,o}$. Assume that the zero divisor of the pull-back of $h \in \mathfrak{m}$ on the minimal resolution of (U, o) has the weighted dual graph as in* Fig. 5, *where $n/\mu = [[c_1, \dots, c_s]]$ and the ρ_i's denote multiplicities. For a positive integer a, put*

$$\bar{a} = \frac{a}{\gcd(a, \operatorname{lcm}(\rho_0, \rho_{s+1}))}, \quad \bar{n} = \frac{\gcd(a, \rho_0, \rho_1, \dots, \rho_{s+1})n}{\gcd(a, \rho_0, \rho_{s+1})}$$

and $\alpha = \bar{a}\bar{n}$. Furthermore, let p be the integer defined by

$$p \equiv \frac{a}{\gcd(a, \rho_{s+1})}\mu\beta + \frac{\rho_{s+1}}{\gcd(a, \rho_{s+1})}\gamma \pmod{\alpha}, \quad 0 \le p < \alpha,$$

where β and γ are integers determined by

$$\frac{a}{\gcd(a, \rho_0)}\beta \equiv 1 \pmod{\rho_0/\gcd(a, \rho_0)}, \quad 0 \le \beta < \frac{\rho_0}{\gcd(a, \rho_0)},$$
$$\frac{\rho_0}{\gcd(a, \rho_0)}\gamma = \frac{a}{\gcd(a, \rho_0)}\beta - 1.$$

Then the normalization of the a-fold covering of U defined by $z^a = h$ has exactly $\gcd(a, \rho_0, \dots, \rho_{s+1})$ *cyclic quotient singularities of type $C_{\alpha,p}$.*

In our application, we always have $a = a_k$ and $\rho_0 = d = \operatorname{lcm}(a_i, a_j)$. So, β and γ are determined by

$$\alpha_k\beta \equiv 1 \pmod{\alpha_i\alpha_j l_k},\ 0 \le \beta < \alpha_i\alpha_j l_k; \quad \alpha_i\alpha_j l_k\gamma = \alpha_k\beta - 1. \tag{2.3}$$

CASE 1. We study W over a neighborhood of $F_0 \cap \phi_*^{-1}C$ on N. Let $u = 0$ and $v = 0$ be local analytic equations of F_0, $\phi_*^{-1}C$ in a small neighborhood of each intersection point of F_0 and $\phi_*^{-1}C$. Recall that there are $\gcd(a_i, a_j) = l_k l$ such points in total. The a_k-fold cyclic covering is locally isomorphic to the singularity $\{u^d v = x_k^{a_k}\}$. Then, by Theorem 2.2 applied to $(n, \mu) = (1, 0)$, $s = 0$ and $h = u^d v$ ($\rho_0 = d$, $\rho_1 = 1$), we see that W has one cyclic quotient singularity of type C_{α_k, p_k} for each point of $F_0 \cap \phi_*^{-1}C$, where p_k is the integer defined by $p_k \equiv \gamma \pmod{\alpha_k}$ (see also [10, Lemma 2.5]). Hence it is determined by the property $p_k\alpha_i\alpha_j l_k + 1 \equiv 0 \pmod{\alpha_k}$, $0 \le p_k < \alpha_k$ by (2.3). By resolving these $l_k l$ singular points according to [3], we easily see that the central curve E_0, which is nothing more than the proper inverse image of F_0, has a simple intersection with the curve $E_{k,1,\xi}$ $(1 \le \xi \le l_k l)$ of self-intersection number $-d_{k,1}$ as in Fig. 2, where we put $\alpha_k/p_k = [[d_{k,1}, \dots, d_{k,r_k}]]$ as before. Similarly, the proper inverse image of $\phi_*^{-1}C$ has a simple intersection with each $E_{k,r_k,\xi}$.

CASE 2. We study W over a neighborhood of C_{n_1,μ_1} and C_{n_2,μ_2} on N. We consider W over a neighborhood of C_{n_1,μ_1} by applying Theorem 2.2 to $F_0 + F_{1,1} + \cdots + F_{1,s_1}$, that is, the curve $\star$ on the left side in Fig. 5 is F_0 and $\rho_{s_1+1} = 0$. We take the pull-back to N of the equation $x_i^{a_i} + x_j^{a_j}$ of C as the function h. Then $\rho_0 = d$, $\rho_i = \rho_{1,i}$ $(1 \le i \le s_1)$. By (2.1) and (2.2), we have $\gcd(a_k, d, \rho_{1,1}, \dots, \rho_{1,s_1}, \rho_{1,s_1+1}) = \gcd(a_k, d, \rho_{1,1}) = l_i l$. Then, since $\bar{n}_1 = \alpha_i$ and $\bar{a}_k = 1$, Theorem 2.2 implies that W has $l_i l$ cyclic quotient singularities of type C_{α_i,p_i}, where p_i is the integer defined by $p_i \equiv \mu_1\beta \pmod{\alpha_i}$, $0 \le p_i < \alpha_i$. Note that p_i satisfies $p_i\alpha_j\alpha_k l_i + 1 \equiv 0 \pmod{\alpha_i}$ by the choice of μ_1 and (2.3). Similarly, by considering C_{n_2,μ_2}, we see that W also has $l_j l$ cyclic quotient singularities of type C_{α_j,p_j}, where p_j is the integer defined by $p_j \equiv \mu_2\beta \pmod{\alpha_j}$, $0 \le p_j < \alpha_j$. Then, p_j satisfies $p_j\alpha_i\alpha_k l_j + 1 \equiv 0 \pmod{\alpha_j}$.

From Cases 1 and 2, we know that W has $l(l_0+l_1+l_2)$ cyclic quotient singularities in total. Then we obtain the desired resolution $\pi\colon X \to V_{a_0,a_1,a_2}$ by performing the minimal resolutions of all such cyclic quotient singularities of W. Now, it is clear that the resolution dual graph is just as in Fig. 2.

Step 3. The cycle $Z^{(k)}$ and the central curve E_0. In this final step, we determine $Z^{(k)}$. We also calculate the genus g and the self-intersection number $-d_0$ of the central curve E_0, and complete Proposition 1.3.

As in Step 2, we regard V_{a_0,a_1,a_2} as an a_k-fold cyclic covering of the (x_i, x_j)-plane. We saw in Step 1 that $\phi_*^{-1}C$ meets the (-1)-curve F_0 at $\gcd(a_i,a_j) = l_k l$ distinct points, and the multiplicity of ϕ^*C along F_0 is $d = \operatorname{lcm}(a_i,a_j)$. Then we obtain E_0 as an $l_i l_j l$-fold cyclic covering of F_0, because $\gcd(a_k, \operatorname{lcm}(a_i,a_j)) = \gcd(\alpha_k l_i l_j l, \alpha_i\alpha_j l_0 l_1 l_2 l) = l_i l_j l$. Moreover, the vanishing order of $x_k \circ \pi$ along E_0, i.e., the multiplicity of $Z^{(k)}$ along E_0, is given by $d/l_i l_j l = \alpha_i\alpha_j l_k =: \lambda_0^{(k)}$. With this, we can determine the sequence $\{\lambda_{w,\nu,\xi}^{(k)}\}$ $(w = 0, 1, 2;\ \nu = 1, \dots, r_w;\ \xi = 1, \dots, l_w l)$. In fact, since the intersection number of $E_{w,\nu,\xi}$ with $(x_k \circ \pi)$ is zero, we obtain

$$\lambda_{w,\nu-1,\xi}^{(k)} = \lambda_{w,\nu,\xi}^{(k)} d_{w,\nu} - \lambda_{w,\nu+1,\xi}^{(k)},$$

with $\lambda_{w,0,\xi}^{(k)} := \lambda_0^{(k)} = \alpha_i\alpha_j l_k$ and $\lambda_{i,r_i+1,\xi}^{(k)} = \lambda_{j,r_j+1,\xi}^{(k)} = 0, \lambda_{k,r_k+1,\xi}^{(k)} = 1$ (recall that $E_{k,r_k,\xi}$ meets the proper inverse image of $\phi_*^{-1}C$ at a point). Note that one can compute all $\lambda_{w,\nu,\xi}^{(k)}$ from these data. In particular, $\lambda_{w,1,\xi}^{(k)}$ and $\lambda_{w,r_w,\xi}^{(k)}$ are determined by Lemma 1.2 (1) and (4).

Let us calculate g and d_0. In order to compute g, we observe how ramifies the $l_i l_j l$ sheeted covering map $E_0 \to F_0 \simeq \mathbb{P}^1$. Since there are $l_i l$ branches of type C_{α_i, p_i}, there exist $l_i l$ points on E_0 of ramification index l_j. Similarly, considering C_{a_j, p_j}, we have $l_j l$ points of ramification index l_i. Also, considering C_{a_k, p_k}, there exist $l_k l$ points of ramification index $l_i l_j l$ over $F_0 \cap \phi_*^{-1} C$. Hence, by the Riemann–Hurwitz formula, we get

$$\begin{aligned} 2g - 2 &= l_i l_j l(2 \times 0 - 2) + l_i l(l_j - 1) + l_j l(l_i - 1) + l_k l(l_i l_j l - 1) \\ &= l(l_0 l_1 l_2 l - l_0 - l_1 - l_2). \end{aligned}$$

Since the intersection number of E_0 with $(x_k \circ \pi)$ is zero, we get

$$\lambda_0 d_0 = \sum_{w=0}^{2} \sum_{\xi=1}^{l_w l} \lambda_{w,1,\xi}.$$

Hence,

$$\begin{aligned} d_0 &= \frac{1}{\alpha_i \alpha_j l_k} \left(p_i \alpha_j l_k l_i l + p_j \alpha_i l_k l_j l + \frac{p_k \alpha_i \alpha_j l_k + 1}{\alpha_k} l_k l \right) \\ &= l \left(\sum_{w=0}^{2} \frac{p_w l_w}{\alpha_w} + \frac{1}{\alpha_0 \alpha_1 \alpha_2} \right). \end{aligned}$$

In sum, we have shown Theorem 2.1 and Proposition 1.3.

3. The maximal ideal cycle

We keep the notation in the previous section, but put $\lambda_{w,\nu}^{(k)} := \lambda_{w,\nu,\xi}^{(k)}$ for simplicity, because it does not depend on ξ.

Theorem 3.1. $Z^{(2)} \preceq Z^{(1)} \preceq Z^{(0)}$. *In particular, $Z^{(2)}$ is the maximal ideal cycle for resolution π.*

Proof. $\{\lambda_{w,\nu}^{(k)}\}_{\nu=0}^{r_w}$ satisfies $\lambda_{w,\nu}^{(k)} = \lceil \lambda_{w,\nu-1}^{(k)} / e_{w,\nu} \rceil$. Since $\lambda_0^{(2)} \le \lambda_0^{(1)} \le \lambda_0^{(0)}$ by $a_0 \le a_1 \le a_2$, we obtain inductively $Z^{(2)} \preceq Z^{(1)} \preceq Z^{(0)}$. Needless to say, the maximal ideal $\mathfrak{m}$ of $\mathcal{O}_{V_{a_0,a_1,a_2},o}$ is generated by x_0, x_1, x_2. It follows from [14, Proposition 2.12] that $Z^{(2)}$ is the maximal ideal cycle. □

Theorem 3.2. *The maximal ideal cycle coincides with the fundamental cycle for resolution π if and only if $\alpha_2 \ge l_2$.*

Proof. By Theorems 1.4, 2.1 and Lemma 1.5, we see that $Z^{(2)}$ is the fundamental cycle if and only if $\alpha_2 \ge l_2$. Since $Z^{(2)}$ is the maximal ideal cycle by Theorem 3.1, we obtain the assertion. □

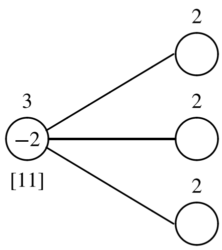

Fig. 6. Weighted dual graph of $Z^{(2)}$; $(a_0, a_1, a_2) = (6, 15, 20)$.

EXAMPLE 3.3 ($\alpha_2 \geq l_2$). If $(a_0, a_1, a_2) = (2, 3, 4)$, then $l = l_0 = l_2 = 1$, $l_1 = 2$, $\alpha_0 = 1$, $\alpha_1 = 3$, $\alpha_2 = 2$, $p_0 = 0$, $p_1 = 2$, $p_2 = 1$. The maximal ideal cycle $Z^{(2)}$ is nothing more than the fundamental cycle of a rational double point of type E_6.

EXAMPLE 3.4 ($\alpha_2 < l_2$). If $(a_0, a_1, a_2) = (6, 15, 20)$, then $l = 1$, $l_0 = 5$, $l_1 = 2$, $l_2 = 3$, $\alpha_0 = \alpha_1 = 1$, $\alpha_2 = 2$, $p_0 = p_1 = 0$, $p_2 = 1$. Hence the weighted dual graph of the maximal ideal cycle $Z^{(2)}$ is as in Fig. 6. It is clear that $Z^{(2)}$ is not the fundamental cycle.

Lemma 3.5. *If π is not the minimal resolution, then $l = 1$, $\{l_0, l_1, l_2\} = \{1, 1, n\}$ for some $n \geq 1$.*

Proof. Assume that π is not the minimal resolution. Since, in E_π, the self-intersection number of any component except E_0 is less than or equal to -2, we see that E_0 must be a (-1)-curve: $g = 0$ and $d_0 = 1$. By the formula in Proposition 1.3, we have $g = 0$ if and only if $(l_0, l_1, l_2, l) = (1, 1, 1, 2)$ or $l = 1$, $\{l_0, l_1, l_2\} = \{1, 1, n\}$.

Assume that $(l_0, l_1, l_2, l) = (1, 1, 1, 2)$. We have $2(p_0\alpha_1\alpha_2 + p_1\alpha_0\alpha_2 + p_2\alpha_0\alpha_1 + 1) = \alpha_0\alpha_1\alpha_2$ by $d_0 = 1$. Note that $\alpha_0\alpha_1\alpha_2$ divides $p_0\alpha_1\alpha_2 + p_1\alpha_0\alpha_2 + p_2\alpha_0\alpha_1 + 1$ by the choice of the p_w's. Hence one has $p_0\alpha_1\alpha_2 + p_1\alpha_0\alpha_2 + p_2\alpha_0\alpha_1 + 1 \geq \alpha_0\alpha_1\alpha_2 = 2(p_0\alpha_1\alpha_2 + p_1\alpha_0\alpha_2 + p_2\alpha_0\alpha_1 + 1)$, which is absurd. Therefore, $(l_0, l_1, l_2, l) \neq (1, 1, 1, 2)$ and we are left the case: $l = 1$, $\{l_0, l_1, l_2\} = \{1, 1, n\}$. □

Theorem 3.6. *The maximal ideal cycle coincides with the fundamental cycle for the minimal resolution of (V_{a_0,a_1,a_2}, o), $2 \leq a_0 \leq a_1 \leq a_2$, if and only if $\alpha_2 \geq l_2$.*

Proof. Assume that $\alpha_2 \geq l_2$. It is obvious that the fundamental cycle coincides with the maximal ideal cycle also on the minimal resolution by the assumption and Theorem 3.2.

Assume that $\alpha_2 < l_2$. If π is not the minimal resolution, then, by Lemma 3.5, $l_0 = l_1 = l = 1$, $l_2 \geq 2$, because $l_2 > \alpha_2 \geq 1$. Then we would have $a_2 = \alpha_2 < l_2 \leq a_0$, which is absurd. Hence π is minimal. By Theorems 1.4, 2.1 and 3.2, the maximal ideal cycle $Z^{(2)}$ cannot be the fundamental cycle. □

Corollary 3.7. *If a_0 is a prime number, then the maximal ideal cycle coincides with the fundamental cycle for the minimal resolution of* (V_{a_0,a_1,a_2}, o), $2 \le a_0 \le a_1 \le a_2$.

Proof. It can be checked directly that $\alpha_2 \ge l_2$ holds, when a_0 is prime. □

Lemma 3.8. $-(Z^{(k)})^2 = l_k l \lceil \alpha_i \alpha_j l_k/\alpha_k \rceil$, *where* $\{i, j, k\} = \{0, 1, 2\}$. *In particular,* $-(Z^{(2)})^2 = a_0 = \mathrm{mult}(\mathcal{O}_{V_{a_0,a_1,a_2},o})$ *holds if and only if* $\lceil \alpha_0\alpha_1 l_2/\alpha_2 \rceil = \alpha_0 l_1$, *i.e.,* $1 < a_1/a_2 + \gcd(a_0, a_1)/a_0$.

Proof. By Lemma 1.2 (4) and Theorem 2.1, we have $\lambda^{(k)}_{k,r_k,\xi} = \lceil \alpha_i \alpha_j l_k/\alpha_k \rceil$. Then the self-intersection number of $Z^{(k)}$ can be computed similarly as in the proof of Proposition 1.6. Hence we omit the detail. Note that

$$
\begin{aligned}
-(Z^{(2)})^2 = l_2 l \left\lceil \frac{\alpha_0\alpha_1 l_2}{\alpha_2} \right\rceil = a_0 \Leftrightarrow \left\lceil \frac{\alpha_0\alpha_1 l_2}{\alpha_2} \right\rceil = \alpha_0 l_1 \\
\Leftrightarrow \frac{\alpha_0\alpha_1 l_2}{\alpha_2} \le \alpha_0 l_1 < \frac{\alpha_0\alpha_1 l_2}{\alpha_2} + 1.
\end{aligned}
$$

In the last inequalities, we need not care the left hand side one, because it always holds true by $a_1 \le a_2$. As to the right hand side inequality, we have

$$
\alpha_0 l_1 < \frac{\alpha_0\alpha_1 l_2}{\alpha_2} + 1 \Leftrightarrow 1 < \frac{\alpha_1 l_2 l_0 l}{\alpha_2 l_1 l_0 l} + \frac{l_2 l}{\alpha_0 l_1 l_2 l} = \frac{a_1}{a_2} + \frac{\gcd(a_0, a_1)}{a_0}
$$

from which the assertion follows. □

Proposition 3.9. *Put* $\delta := \alpha_0 l_1 - \lceil \alpha_0\alpha_1 l_2/\alpha_2 \rceil \ge 0$. *The base points on* E_π *of the linear system* $|\mathcal{O}_X(-Z^{(2)})|$ *can be resolved by a succession of* $\delta l_2 l$ *simple blowing-ups. In particular, the linear system* $|\mathcal{O}_X(-Z^{(2)})|$ *has no base points on* E_π *if and only if* $\delta = 0$.

Proof. The second assertion is clear from Lemma 3.8, because $Z^{(2)}$ is the maximal ideal cycle and one has $\mathfrak{m}\mathcal{O}_X \simeq \mathcal{O}_X(-Z^{(2)})$. See, e.g., [13, Theorem 2.7].

Now, we prove the first assertion. Note that we have $\lambda^{(1)}_{2,r_2} = \alpha_0 l_1$ and $\lambda^{(2)}_{2,r_2} = \lceil \alpha_0\alpha_1 l_2/\alpha_2 \rceil$ by Theorem 2.1. Hence, δ is nothing but the difference of the multiplicities of $Z^{(1)}$ and $Z^{(2)}$ along $E_{2,r_2,\xi}$. Assume that $\delta > 0$ and put $D = (x_2 \circ \pi) - (x_2 \circ \pi)_X$. It is clear that the base points of $|\mathcal{O}_X(-Z^{(2)})|$ on E_π are $l_2 l$ intersection points $P_\xi = E_{2,r_2,\xi} \cap D$, $1 \le \xi \le l_2 l$. Let $\phi\colon \tilde{X} \to X$ be the composite of blowing-ups (performed δ times for each ξ) at P_ξ and the $\delta - 1$ points infinitely near to it on the proper transform of D (see, Fig. 7). Thus, ϕ blows up $\delta l_2 l$ points in total. Put $A := K_{\tilde{X}} - \phi^* K_X$ and let A_ξ be the (-1)-curve over P_ξ lastly appeared in ϕ. Then the multiplicity of A along A_ξ is δ, and the cycle on $\tilde{X}$ led by x_2 is $\tilde{Z}^{(2)} := \phi^* Z^{(2)} + A$. We have $\mathrm{mult}_{A_\xi}(\tilde{Z}^{(2)}) = \lambda^{(2)}_{2,r_2} + \delta = \lceil \alpha_0\alpha_1 l_2/\alpha_2 \rceil + \delta = \alpha_0 l_1$ for each ξ. Then, $x_1 \circ \pi \circ \phi$ gives us

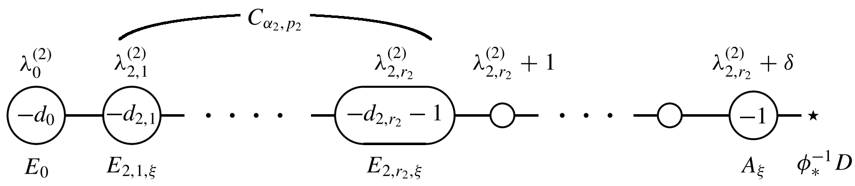

Fig. 7. A branch of the cycle led by x_2 on $\tilde{X}$.

a section of $\mathcal{O}_{\tilde{X}}(-\tilde{Z}^{(2)})$ that is a non-zero constant on each A_ξ, because $\tilde{Z}^{(2)} \preceq \phi^* Z^{(1)}$, $\mathrm{mult}_{A_\xi}(\phi^* Z^{(1)}) = \alpha_0 l_1$ and $(x_1 \circ \pi \circ \phi) = \phi^* Z^{(1)}$ in a neighborhood of the $l_2 l$ branches containing the A_ξ's. Therefore, $|\mathcal{O}_{\tilde{X}}(-\tilde{Z}^{(2)})|$ has no base points on $\mathrm{E}_{\pi\circ\phi}$. Needless to say, $\tilde{Z}^{(2)}$ is the maximal ideal cycle on $\tilde{X}$ and $-(\tilde{Z}^{(2)})^2 = a_0$. $\square$

EXAMPLE 3.10. If $(a_0,a_1,a_2) = (6,10,15)$, then $l = 1$, $l_0 = 5$, $l_1 = 3$, $l_2 = 2$, $\alpha_0 = \alpha_1 = \alpha_2 = 1$, $p_0 = p_1 = p_2 = 0$. The exceptional set is a non-singular curve E_0 of genus 11, and $d_0 = 1$, $\lambda_0 = 2$. Then $Z = E_0$ and $Z^{(2)} = 2E_0$. We have $-(Z^{(2)})^2 = 4$, while $\mathrm{mult}(\mathcal{O}_{V_{6,10,15},o}) = 6$. Two intersection points $E_0 \cap D$, where $D = (x_2 \circ \pi) - (x_2 \circ \pi)_X$, are base points of $|\mathcal{O}_X(-2E_0)|$. Indeed, since the vanishing order of $x_2 \circ \pi$ along E_0 is exactly 2, it induces a non-zero element of $H^0(X, -2E_0)/H^0(X, -3E_0) \subset H^0(E_0, -2E_0)$. On the other hand, $\dim H^0(E_0, -2E_0) \le 1$, because E_0 is a non-hyperelliptic curve. Therefore, $H^0(E_0, -2E_0)$ is generated by the image of $x_2 \circ \pi$ which vanishes at two intersection points P_1, P_2 mentioned above. Let ϕ be the blowing-up at P_1, P_2, and put $A = A_1 + A_2$, where $A_i = \phi^{-1}(P_i)$ for $i = 1, 2$. Then $2\phi^* E_0 + A$ is the cycle led by x_2 and we obtain $(2\phi^* E_0 + A)^2 = -6$.

Theorem 3.11. *The maximal ideal cycle coincides with the fundamental cycle for any resolution of* (V_{a_0,a_1,a_2}, o), $2 \le a_0 \le a_1 \le a_2$, *if and only if* $\alpha_2 \ge l_2$ *and* $1 < a_1/a_2 + \gcd(a_0, a_1)/a_0$. *If this is the case, then the fundamental cycle is led by the holomorphic function* x_2.

Proof. The fundamental cycle on a resolution is obtained as the pull-back of that on the minimal resolution. The same holds for the maximal ideal cycle, if the minus of it defines a free linear system on the minimal resolution. Therefore, the first assertion follows from Theorem 3.6, Lemma 3.8 and Proposition 3.9. The second assertion is clear, because $Z^{(2)}$ is led by x_2. $\square$

EXAMPLE 3.12. For $(V_{2,3,2n+1}, o)$, the above implies that the maximal ideal cycle coincides with the fundamental cycle for any resolution when $n = 1, 2$, while it holds not for all but for the minimal resolution when $n \ge 3$.

4. Further remarks

4.1. Kodaira singularities. Let S be a non-singular complex surface and $D \subset \mathbb{C}$ a small open disc around the origin. A surjective holomorphic map $\Phi\colon S \to D$ is said to be a pencil of curves of genus g, if it is proper and connected, and fibers $S_t := \Phi^{-1}(t)$ $(t \neq 0)$ are smooth curves of genus g.

DEFINITION 4.1 ([5]). A normal surface singularity (V, o) is said to be a Kodaira singularity, if there exists a pencil of curves $\Phi\colon S \to D$ such that, after a finite number of blowing-ups at non-singular points in non-multiple components of the central fiber S_0, $\Psi\colon S' \to S$, there is a holomorphic map $\varphi\colon M \to V$ from an open neighborhood M of the proper transform of $\mathrm{Supp}(S_0)$ in S' which defines a resolution of (V, o).

Proposition 4.2 ([5, p. 46], [6]). *Let $\phi\colon (X, \mathrm{E}) \to (V, o)$ be the minimal good resolution of a normal surface singularity and $\mathfrak{m}$ the maximal ideal of $\mathcal{O}_{V,o}$. Then (V, o) is a Kodaira singularity if and only if $\mathrm{mult}_{E_j}(Z_{\mathrm{E}}) = 1$ holds for every component E_j satisfying $Z_{\mathrm{E}}E_j < 0$ and there exists an element $f \in \mathfrak{m}$ such that the divisor $(f \circ \phi)$ is normal crossing and $(f \circ \phi)_X = Z_{\mathrm{E}}$.*

Now, we return to the situation we are interested in. Consider the singularity of Brieskorn type and let $\pi\colon (X, \mathrm{E}_\pi) \to (V_{a_0,a_1,a_2}, o)$ be the resolution as before.

Lemma 4.3. *π is not the minimal good resolution if and only if $a_0 = a_1 = 2$, $a_2 = 2m + 1$ for a positive integer m (the rational double point of type A_{2m}).*

Proof. Clearly, π is not the minimal good resolution if and only if E_0 is a (-1)-curve and the number of branches plugged to it is at most two.

Assume that π is not the minimal good resolution. Then, since E_0 is a (-1)-curve, we have $l = 1$, $(l_i, l_j, l_k) = (1, 1, n)$ $(n \geq 1$, $\{i, j, k\} = \{0, 1, 2\})$ by Lemma 3.5. First, assume that $n = 1$. Then $\alpha_0, \alpha_1, \alpha_2 \geq 2$, because $2 \leq a_0 \leq a_1 \leq a_2$. But, this implies that there are three branches, a contradiction. Second, assume that $n \geq 3$. Then $\alpha_k \geq 2$ and, we obtain a contradiction, because the number of branches is at least $l_k l = n \geq 3$. Finally assume that $n = 2$. Then $\alpha_k \geq 2$. Furthermore, α_k is odd, because it is coprime to $l_k = 2$. We have $\alpha_i = \alpha_j = 1$, because $l_k l = 2$ and the number of branches must be at most two. Therefore, by $2 \leq a_0 \leq a_1 \leq a_2$, we see $a_0 = a_1 = 2$ and a_2 is an odd integer not less than three. Then it is a rational double point of type A_{2m} $(m \geq 1)$.

Conversely, assume that $a_0 = a_1 = 2$ and $a_2 = 2m + 1$ for some positive integer m. Then $l = l_0 = l_1 = 1$, $l_2 = 2$, $\alpha_0 = \alpha_1 = 1$, $\alpha_2 = 2m + 1$, $p_0 = p_1 = 0$, $p_2 = m$. Hence $g = 0$, $d_0 = 1$ by Proposition 1.3, and exactly two branches are plugged to E_0. So, we obtain the minimal good resolution by contracting E_0. □

The following shows that the sufficient condition given in [10, Corollary 4.6] is also necessary.

Proposition 4.4. *(V_{a_0,a_1,a_2}, o), $2 \leq a_0 \leq a_1 \leq a_2$, is a Kodaira singularity if and only if $\alpha_0\alpha_1 l_2 \leq \alpha_2$, i.e.,* $\operatorname{lcm}(a_0, a_1) \leq a_2$. *If this is the case, it is associated to a pencil of curves of genus* $(1/2)\{(a_0 - 1)(a_1 - 1) - \gcd(a_0, a_1) + 1\}$.

Proof. (i) We first assume that (V_{a_0,a_1,a_2}, o) is not a rational double point of type A_{2m} $(m \geq 1)$. Then π is the minimal good resolution by Lemma 4.3. Let Z denote the fundamental cycle for π. Assume that $\alpha_2 < l_2$. Let $\mathfrak{m}$ be the maximal ideal of $\mathcal{O}_{V_{a_0,a_1,a_2},o}$. We have no $f \in \mathfrak{m}$ such that $(f \circ \pi)_X = Z$, because Z is not the maximal ideal cycle by Theorem 3.2. Hence (V_{a_0,a_1,a_2}, o) is not a Kodaira singularity by Proposition 4.2. Assume that $\alpha_2 \geq l_2$. We already know that $(x_2 \circ \pi)$ is a normal crossing divisor, and $Z^{(2)} = (x_2 \circ \pi)_X = Z$ by Theorem 3.2. Furthermore, we have

$$ZE_{w,\nu,\xi} = \begin{cases} -1 & \text{if} \quad w = 2,\ \nu = r_2, \\ 0 & \text{otherwise.} \end{cases}$$

Hence, by Proposition 4.2, (V_{a_0,a_1,a_2}, o) is a Kodaira singularity if and only if $\operatorname{mult}_{E_{2,r_2,\xi}}(Z) = \lceil \alpha_0\alpha_1 l_2/\alpha_2 \rceil = 1$, i.e., $\alpha_0\alpha_1 l_2 \leq \alpha_2$.

(ii) Next, we consider $(V_{2,2,2m+1}, o)$. Then $\alpha_0\alpha_1 l_2 = 2 < \alpha_2 = 2m + 1$. Let Z' be the fundamental cycle on the minimal good resolution $\pi'\colon X' \to V_{2,2,2m+1}$. By Theorem 3.11, Z' is led by x_2 and it is clear that $(x_2 \circ \pi')$ is normal crossing. Hence $(V_{2,2,2m+1}, o)$ is a Kodaira singularity, by Proposition 4.2.

The last assertion for the genus follows from [8, Theorem 4.3] or Theorem 1.7 (2). □

4.2. Canonical cycle.

DEFINITION 4.5. Let $\phi\colon (X, \mathrm{E}) \to (V, o)$ be a resolution of a normal surface singularity. A $\mathbb{Q}$ divisor Z_K with support in $\mathrm{E} = \bigcup_{i=1}^{r} E_i$ is said to be the canonical cycle, if $-Z_K E_i = K E_i$ holds for any irreducible component E_i.

For the singularity of Brieskorn type, we can express Z_K in terms of some previously known cycles.

Proposition 4.6. *Let Z_K be the canonical cycle for the resolution $\pi\colon (X, \mathrm{E}_\pi) \to (V_{a_0,a_1,a_2}, o)$ in* Proposition 1.3. *Then*

$$Z_K = E + l_0 l_1 l_2 l Z_0 - Z^{(0)} - Z^{(1)} - Z^{(2)},$$

where E is the reduced exceptional divisor,

$$E = E_0 + \sum_{w=0}^{2} \sum_{\nu=1}^{r_w} \sum_{\xi=1}^{l_w l} E_{w,\nu,\xi},$$

and Z_0 *is the cycle with* $\mathrm{mult}_{E_0}(Z_0) = \alpha_0\alpha_1\alpha_2$ *appeared in* Theorem 1.4 *as the fundamental cycle for the case* $\alpha_2 \leq l_2$.

Proof. We only consider the case where $\alpha_0, \alpha_1, \alpha_2 \geq 2$, because the other cases can be carried out similarly. For short, we put $E_{w,\nu} := E_{w,\nu,\xi}$, and $E_{w,0} := E_0$. By Proposition 1.3 and the fact that $E_{w,\nu} \simeq \mathbb{P}^1$ when $\nu \neq 0$, we have

$$KE_{w,\nu} = \begin{cases} d_0 + l(l_0l_1l_2l - l_0 - l_1 - l_2) & \text{if} \quad \nu = 0, \\ d_{w,\nu} - 2 & \text{otherwise.} \end{cases}$$

On the other hand, as in the proof of Theorem 1.6, we have

$$-EE_{w,\nu} = \begin{cases} d_0 - l_0l - l_1l - l_2l & \text{if} \quad \nu = 0, \\ d_{w,\nu} - 1 & \text{if} \quad \nu = r_w, \\ d_{w,\nu} - 2 & \text{otherwise,} \end{cases}$$

$$-Z_0E_{w,\nu} = \begin{cases} l & \text{if} \quad \nu = 0, \\ 0 & \text{otherwise,} \end{cases}$$

$$-Z^{(k)}E_{w,\nu} = \begin{cases} 1 & \text{if} \quad w = k,\ \nu = r_w, \\ 0 & \text{otherwise} \end{cases}$$

in view of Theorems 1.4 and 2.1. Hence, for our Z_K, it can be checked directly that $-Z_KE_{w,\nu} = KE_{w,\nu}$ holds for all w and ν. □

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Kazuhiro Konno
Department of Mathematics
Graduate School of Science
Osaka University
Machikaneyama, Toyonaka
Osaka 560-0043
Japan
e-mail: konno@math.sci.osaka-u.ac.jp

Daisuke Nagashima
Meiji Yasuda Life Insurance Company
2-1-1 Marunouchi, Chiyoda-ku
Tokyo 100-0005
Japan
e-mail: sm4032nd@yahoo.co.jp