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TORSIONFREE DIMENSION OF MODULES AND SELF-INJECTIVE DIMENSION OF RINGS

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Abstract

Let R be a left and right Noetherian ring. We introduce the notion of the torsionfree dimension of finitely generated R-modules. For any $n \geq 0$, we prove that R is a Gorenstein ring with self-injective dimension at most n if and only if every finitely generated left R-module and every finitely generated right R-module have torsionfree dimension at most n, if and only if every finitely generated left (or right) R-module has Gorenstein dimension at most n. For any $n \geq 1$, we study the properties of the finitely generated R-modules M with $\mathrm{Ext}^i_R(M, R) = 0$ for any $1 \leq i \leq n$. Then we investigate the relation between these properties and the self-injective dimension of R.

1. Introduction

Throughout this paper, R is a left and right Noetherian ring (unless stated otherwise) and $\mathrm{mod}\, R$ is the category of finitely generated left R-modules. For a module $M \in \mathrm{mod}\, R$, we use $\mathrm{pd}_R M$, $\mathrm{fd}_R M$, $\mathrm{id}_R M$ to denote the projective, flat, injective dimension of M, respectively.

For any $n \geq 1$, we denote ${}^{\perp_n}{}_R R = \{M \in \mathrm{mod}\, R \mid \mathrm{Ext}^i_R({}_R M, {}_R R) = 0$ for any $1 \leq i \leq n\}$ (resp. ${}^{\perp_n} R_R = \{N \in \mathrm{mod}\, R^{op} \mid \mathrm{Ext}^i_{R^{op}}(N_R, R_R) = 0$ for any $1 \leq i \leq n\}$), and ${}^{\perp}{}_R R = \bigcap_{n \geq 1} {}^{\perp_n}{}_R R$ (resp. ${}^{\perp} R_R = \bigcap_{n \geq 1} {}^{\perp_n} R_R$).

For any $M \in \mathrm{mod}\, R$, there exists an exact sequence:

$$P_1 \xrightarrow{f} P_0 \to M \to 0$$

in $\mathrm{mod}\, R$ with P_0 and P_1 projective. Then we get an exact sequence:

$$0 \to M^* \to P_0^* \xrightarrow{f^*} P_1^* \to \mathrm{Tr}\, M \to 0$$

in $\mathrm{mod}\, R^{op}$, where $(-)^* = \mathrm{Hom}(-, R)$ and $\mathrm{Tr}\, M = \mathrm{Coker}\, f^*$ is the *transpose* of M ([1]).

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Auslander and Bridger generalized the notions of finitely generated projective modules and the projective dimension of finitely generated modules as follows.

DEFINITION 1.1 ([1]). Let $M \in \operatorname{mod} R$.
(1) M is said to have *Gorenstein dimension zero* if $M \in {}^{\perp}{}_{R}R$ and $\operatorname{Tr} M \in {}^{\perp}R_{R}$.
(2) For a non-negative integer n, the *Gorenstein dimension* of M, denoted by $\text{G-dim}_R M$, is defined as $\inf\{n \geq 0 \mid$ there exists an exact sequence $0 \to M_n \to \cdots \to M_1 \to M_0 \to M \to 0$ in $\operatorname{mod} R$ with all M_i having Gorenstein dimension zero$\}$. We set $\text{G-dim}_R M$ infinity if no such integer exists.

Huang introduced in [7] the notion of the left orthogonal dimension of modules as follows, which is "simpler" than that of the Gorenstein dimension of modules.

DEFINITION 1.2 ([7]). For a module $M \in \operatorname{mod} R$, the *left orthogonal dimension* of a module $M \in \operatorname{mod} R$, denoted by ${}^{\perp}{}_{R}R\text{-dim}_R M$, is defined as $\inf\{n \geq 0 \mid$ there exists an exact sequence $0 \to X_n \to \cdots \to X_1 \to X_0 \to M \to 0$ in $\operatorname{mod} R$ with all $X_i \in {}^{\perp}{}_{R}R\}$. We set ${}^{\perp}{}_{R}R\text{-dim}_R M$ infinity if no such integer exists.

Let $M \in \operatorname{mod} R$. It is trivial that ${}^{\perp}{}_{R}R\text{-dim}_R M \leq \text{G-dim}_R M$. On the other hand, by [14], we have that ${}^{\perp}{}_{R}R\text{-dim}_R M \neq \text{G-dim}_R M$ in general.

Recall that R is called a *Gorenstein ring* if $\operatorname{id}_R R = \operatorname{id}_{R^{op}} R < \infty$. The following result was proved by Auslander and Bridger in [1, Theorem 4.20] when R is a commutative Noetherian local ring. Hoshino developed in [4] Auslander and Bridger's arguments and applied obtained the obtained results to Artinian algebras. Then Huang generalized in [7, Corollary 3] Hoshino's result with the left orthogonal dimension replacing the Gorenstein dimension of modules.

Theorem 1.3 ([4, Theorem] and [7, Corollary 3]). *The following statements are equivalent for an Artinian algebra R.*
(1) *R is Gorenstein.*
(2) *Every module in* $\operatorname{mod} R$ *has finite Gorenstein dimension.*
(3) *Every module in* $\operatorname{mod} R$ *and every module in* $\operatorname{mod} R^{op}$ *have finite left orthogonal dimension.*

One aim of this paper is to generalize this result to left and right Noetherian rings. On the other hand, note that the left orthogonal dimension of modules is defined by the least length of the resolution composed of the modules in ${}^{\perp}{}_{R}R$, which are the modules satisfying one of the two conditions in the definition of modules having Gorenstein dimension zero. So, a natural question is: If a new dimension of modules is defined by the least length of the resolution composed of the modules satisfying the other condition in the definition of modules having Gorenstein dimension zero, then can one give an equivalent characterization of Gorenstein rings similar to the above result in terms of the new dimension of modules? The other aim of this paper is to give a positive answer to this question. This paper is organized as follows.

In Section 2, we give the definition of n-torsionfree modules, and investigate the properties of such modules. We prove that a module in mod R is n-torsionfree if and only if it is an n-syzygy of a module in ${}^{\perp_n}{}_RR$.

In Section 3, we introduce the notion of the torsionfree dimension of modules. Then we give some equivalent characterizations of Gorenstein rings in terms of the properties of the torsionfree dimension of modules. The following is the main result in this paper.

Theorem 1.4. *For any $n \geq 0$, the following statements are equivalent.*
(1) *R is a Gorenstein ring with* $\mathrm{id}_R R = \mathrm{id}_{R^{op}} R \leq n$.
(2) *Every module in* mod R *has Gorenstein dimension at most n.*
(3) *Every module in* mod R^{op} *has Gorenstein dimension at most n.*
(4) *Every module in* mod R *and every module in* mod R^{op} *have torsionfree dimension at most n.*
(5) *Every module in* mod R *and every module in* mod R^{op} *have left orthogonal dimension at most n.*

In Section 4, for any $n \geq 1$, we first prove that every module in ${}^{\perp_n}{}_RR$ is torsionless (in this case, ${}^{\perp_n}{}_RR$ is said to have the *torsionless property*) if and only if every module in ${}^{\perp_n}{}_RR$ is ∞-torsionfree, if and only if every module in ${}^{\perp_n}{}_RR$ has torsionfree dimension at most n, if and only if every n-torsionfree module in mod R is ∞-torsionfree, if and only if every n-torsionfree module in mod R^{op} is in ${}^{\perp}R_R$, if and only if ${}^{\perp_n}R_R = {}^{\perp}R_R$. Note that if $\mathrm{id}_{R^{op}} R \leq n$, then ${}^{\perp_n}{}_RR$ has the torsionless property. As some applications of the obtained results, we investigate when the converse of this assertion holds true. Assume that n and k are positive integers and ${}^{\perp_n}{}_RR$ has the torsionless property. If R is $g_n(k)$ or $g_n(k)^{op}$ (see Section 4 for the definitions), then $\mathrm{id}_{R^{op}} R \leq n+k-1$. As a corollary, we have that if $\mathrm{id}_R R \leq n$, then $\mathrm{id}_R R = \mathrm{id}_{R^{op}} R \leq n$ if and only if ${}^{\perp_n}{}_RR$ has the torsionless property.

In view of the results obtained in this paper, we pose in Section 5 the following two questions: (1) Is the subcategory of mod R consisting of modules with torsionfree dimension at most n closed under extensions or under kernels of epimorphisms? (2) If $\mathrm{id}_{R^{op}} R \leq n$, does then every module $M \in$ mod R has torsionfree dimension at most n?

2. Preliminaries

Let $M \in$ mod R and $n \geq 1$. Recall from [1] that M is called *n-torsionfree* if $\mathrm{Tr}\, M \in {}^{\perp_n}R_R$; and M is called *∞-torsionfree* if M is n-torsionfree for all n. We use $\mathcal{T}_n(\mathrm{mod}\, R)$ (resp. $\mathcal{T}(\mathrm{mod}\, R)$) to denote the subcategory of mod R consisting of all n-torsionfree modules (resp. ∞-torsionfree modules). It is well-known that M is 1-torsionfree (resp. 2-torsionfree) if and only if M is torsionless (resp. reflexive) (see [1]). Also recall from [1] that M is called an n-syzygy module (of A), denoted by $\Omega^n(A)$, if there exists an exact sequence $0 \to M \to P_{n-1} \to \cdots \to P_1 \to P_0 \to A \to 0$ in $\mathrm{mod}\, R$ with all P_i projective. In particular, set $\Omega^0(M) = M$. We use $\Omega^n(\mathrm{mod}\, R)$ to denote the subcategory of $\mathrm{mod}\, R$ consisting of all n-syzygy modules. It is easy to see that $\mathcal{T}_n(\mathrm{mod}\, R) \subseteq \Omega^n(\mathrm{mod}\, R)$, and in general, this inclusion is strict when $n \geq 2$ (see [1]).

Jans proved in [13, Corollary 1.3] that a module in $\mathrm{mod}\, R$ is 1-torsionfree if and only if it is an 1-syzygy of a module in ${}^{\perp_1}{}_RR$. We generalize this result as follows.

Proposition 2.1. *For any $n \geq 1$, a module in* $\mathrm{mod}\, R$ *is n-torsionfree if and only if it is an n-syzygy of a module in ${}^{\perp_n}{}_RR$.*

Proof. Assume that $M \in \mathrm{mod}\, R$ is an n-syzygy of a module A in ${}^{\perp_n}{}_RR$. Then there exists an exact sequence:

$$0 \to M \to P_{n-1} \to \cdots \to P_1 \xrightarrow{f} P_0 \to A \to 0$$

in $\mathrm{mod}\, R$ with all P_i projective. Let

$$P_{n+1} \to P_n \to M \to 0$$

be a projective presentation of M in $\mathrm{mod}\, R$. Then the above two exact sequences yield the following exact sequence:

$$0 \to A^* \to P_0^* \xrightarrow{f^*} \cdots \to P_n^* \to P_{n+1}^* \to \mathrm{Tr}\, M \to 0.$$

By the exactness of $P_{n+1} \to P_n \to \cdots \to P_1 \xrightarrow{f} P_0$, we get that $\mathrm{Tr}\, M \in {}^{\perp_n}R_R$. Thus M is n-torsionfree.

Conversely, assume that $M \in \mathrm{mod}\, R$ is n-torsionfree and

$$P_1 \xrightarrow{g} P_0 \xrightarrow{\pi} M \to 0$$

is a projective presentation of $M \in \mathrm{mod}\, R$. Then we get an exact sequence:

$$0 \to M^* \xrightarrow{\pi^*} P_0^* \xrightarrow{g^*} P_1^* \to \mathrm{Tr}\, M \to 0$$

in $\mathrm{mod}\, R^{op}$. Let

$$\cdots \xrightarrow{h_{n+1}} Q_n \xrightarrow{h_n} \cdots \xrightarrow{h_1} Q_0 \xrightarrow{h_0} M^* \to 0$$

be a projective resolution of M^* in $\mathrm{mod}\, R^{op}$. Then we get a projective resolution of $\mathrm{Tr}\, M$:

$$\cdots \xrightarrow{h_{n+1}} Q_n \xrightarrow{h_n} \cdots \xrightarrow{h_1} Q_0 \xrightarrow{\pi^* h_0} P_0^* \xrightarrow{g^*} P_1^* \to \mathrm{Tr}\, M \to 0.$$

Because M is n-torsionfree, $\mathrm{Tr}\, M \in {}^{\perp_n}R_R$ and we get the following exact sequence:

$$0 \to (\mathrm{Tr}\, M)^* \to P_1^{**} \xrightarrow{g^{**}} P_0^{**} \xrightarrow{h_0^*\pi^{**}} Q_0^* \xrightarrow{h_1^*} \cdots \xrightarrow{h_{n-1}^*} Q_{n-1}^* \to \mathrm{Coker}\, h_{n-1}^* \to 0.$$

It is easy to see that $M \cong \mathrm{Coker}\, g^{**}$. By the exactness of $Q_{n-1} \xrightarrow{h_{n-1}} \cdots \xrightarrow{h_1} Q_0 \xrightarrow{\pi^* h_0} P_0^* \xrightarrow{g^*} P_1^*$, we get that $\mathrm{Coker}\, h_{n-1}^* \in {}^{\perp_n}{}_RR$. The proof is finished. □

As an immediate consequence, we have the following

Corollary 2.2. *For any $n \geq 1$, an n-torsionfree module in* mod R *is a* 1*-syzygy of an* $(n-1)$*-torsionfree module A in* mod R *with $A \in {}^{\perp_1}{}_RR$. In particular, an ∞-torsionfree module in* mod R *is a* 1*-syzygy of an ∞-torsionfree module T in* mod R *with $T \in {}^{\perp_1}{}_RR$.*

We also need the following easy observation.

Lemma 2.3. *For any $n \geq 1$, both $\mathcal{T}_n$(mod R) and $\mathcal{T}$(mod R) are closed under direct summands and finite direct sums.*

3. Torsionfree dimension of modules

In this section, we will introduce the notion of the torsionfree dimension of modules in mod R. Then we will give some equivalent characterizations of Gorenstein rings in terms of the properties of this dimension of modules.

We begin with the following well-known observation.

Lemma 3.1 ([1, Lemma 3.9]). *Let $0 \to A \xrightarrow{f} B \to C \to 0$ be an exact sequence in* mod R. *Then we have exact sequences $0 \to C^* \to B^* \to A^* \to \mathrm{Coker}\, f^* \to 0$ and $0 \to \mathrm{Coker}\, f^* \to \mathrm{Tr}\, C \to \mathrm{Tr}\, B \to \mathrm{Tr}\, A \to 0$ in* mod R^{op}.

The following result is useful in this section.

Proposition 3.2. *Let*

$$0 \to M \to T_1 \xrightarrow{f} T_0 \to A \to 0$$

be an exact sequence in mod R *with both T_0 and T_1 in $\mathcal{T}$(mod R). Then there exists an exact sequence*:

$$0 \to M \to P \to T \to A \to 0$$

in mod R *with P projective and $T \in \mathcal{T}$(mod R).*

Proof. Let

$$0 \to M \to T_1 \xrightarrow{f} T_0 \to A \to 0$$

be an exact sequence in $\operatorname{mod} R$ with both T_0 and T_1 in $\mathcal{T}(\operatorname{mod} R)$. By Corollary 2.2, there exists an exact sequence $0 \to T_1 \to P \to W \to 0$ in $\operatorname{mod} R$ with P projective and $W \in {}^{\perp_1}{}_R R \cap \mathcal{T}(\operatorname{mod} R)$. Then we have the following push-out diagram:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M & \longrightarrow & T_1 & \longrightarrow & \operatorname{Im} f & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M & \longrightarrow & P & \longrightarrow & B & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & W & = & W & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Now, consider the following push-out diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \operatorname{Im} f & \longrightarrow & T_0 & \longrightarrow & A & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & B & \longrightarrow & T & \longrightarrow & A & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & W & = & W & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

Because $W \in {}^{\perp_1}{}_R R$, we get an exact sequence:

$$0 \to \operatorname{Tr} W \to \operatorname{Tr} T \to \operatorname{Tr} T_0 \to 0$$

by Lemma 3.1 and the exactness of the middle column in the above diagram. Because both W and T_0 are in $\mathcal{T}(\operatorname{mod} R)$, both $\operatorname{Tr} W$ and $\operatorname{Tr} T_0$ are in ${}^{\perp}R_R$. So $\operatorname{Tr} T$ is also in ${}^{\perp}R_R$ and hence $T \in \mathcal{T}(\operatorname{mod} R)$. Connecting the middle rows in the above two diagrams, then we get the desired exact sequence. □

Now we introduce the notion of the torsionfree dimension of modules as follows.

DEFINITION 3.3. For a module $M \in \operatorname{mod} R$, the *torsionfree dimension* of M, denoted by $\mathcal{T}\text{-dim}_R M$, is defined as $\inf\{n \geq 0 \mid$ there exists an exact sequence $0 \to X_n \to \cdots \to X_1 \to X_0 \to M \to 0$ in $\operatorname{mod} R$ with all $X_i \in \mathcal{T}(\operatorname{mod} R)\}$. We set $\mathcal{T}\text{-dim}_R M$ infinity if no such integer exists.

Let $M \in \operatorname{mod} R$. It is trivial that $\mathcal{T}\text{-dim}_R M \leq \text{G-dim}_R M$. On the other hand, by [14], we have that $\mathcal{T}\text{-dim}_R M \neq \text{G-dim}_R M$ in general.

Proposition 3.4. *Let $M \in \operatorname{mod} R$ and $n \geq 0$. If $\mathcal{T}\text{-dim}_R M \leq n$, then there exists an exact sequence $0 \to H \to T \to M \to 0$ in $\operatorname{mod} R$ with $\operatorname{pd}_R H \leq n-1$ and $T \in \mathcal{T}(\operatorname{mod} R)$.*

Proof. We proceed by induction on n. If $n = 0$, then $H = 0$ and $T = M$ give the desired exact sequence. If $n = 1$, then there exists an exact sequence:

$$0 \to T_1 \to T_0 \to M \to 0$$

in $\operatorname{mod} R$ with both T_0 and T_1 in $T \in \mathcal{T}(\operatorname{mod} R)$. Applying Proposition 3.2, with $A = 0$, we get an exact sequence:

$$0 \to P \to T_0' \to M \to 0$$

in $\operatorname{mod} R$ with P projective and $T_0' \in \mathcal{T}(\operatorname{mod} R)$.

Now suppose $n \geq 2$. Then there exists an exact sequence:

$$0 \to T_n \to T_{n-1} \to \cdots \to T_0 \to M \to 0$$

in $\operatorname{mod} R$ with all $T_i \in \mathcal{T}(\operatorname{mod} R)$. Set $K = \operatorname{Im}(T_1 \to T_0)$. By the induction hypothesis, we get the following exact sequence:

$$0 \to P_n \to P_{n-1} \to P_{n-2} \to \cdots \to P_2 \to T_1' \to K \to 0$$

in $\operatorname{mod} R$ with all P_i projective and $T_1' \in \mathcal{T}(\operatorname{mod} R)$. Set $N = \operatorname{Im}(P_2 \to T_1')$. By Proposition 3.2, we get an exact sequence:

$$0 \to N \to P_1 \to T \to M \to 0$$

in $\operatorname{mod} R$ with P_1 projective and $T \in \mathcal{T}(\operatorname{mod} R)$. Thus we get the desired exact sequence:

$$0 \to P_n \to P_{n-1} \to P_{n-2} \to \cdots \to P_1 \to T \to M \to 0$$

and the assertion follows. □

Christensen, Frankild and Holm proved in [2, Lemma 2.17] that a module with Gorenstein dimension at most n can be embedded into a module with projective dimension at most n, such that the cokernel is a module with Gorenstein dimension zero. The following result extends this result.

Corollary 3.5. *Let* $M \in \operatorname{mod} R$ *and* $n \geq 0$. *If* $\mathcal{T}\text{-}\dim_R M \leq n$, *then there exists an exact sequences* $0 \to M \to N \to T \to 0$ *in* $\operatorname{mod} R$ *with* $\operatorname{pd}_R N \leq n$ *and* $T \in {}^{\perp_1}{}_R R \cap \mathcal{T}(\operatorname{mod} R)$.

Proof. Let $M \in \operatorname{mod} R$ with $\mathcal{T}\text{-}\dim_R M \leq n$. By Proposition 3.4, there exists an exact sequence $0 \to H \to T' \to M \to 0$ in $\operatorname{mod} R$ with $\operatorname{pd}_R H \leq n-1$ and $T' \in \mathcal{T}(\operatorname{mod} R)$. By Corollary 2.2, there exists an exact sequence $0 \to T' \to P \to T \to 0$ in $\operatorname{mod} R$ with P projective and $T \in {}^{\perp_1}{}_R R \cap \mathcal{T}(\operatorname{mod} R)$. Consider the following push-out diagram:

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H & \longrightarrow & T' & \longrightarrow & M & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H & \longrightarrow & P & \longrightarrow & N & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & T & = & T & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

Then the third column in the above diagram is as desired. □

The following result plays a crucial role in proving the main result in this paper.

Theorem 3.6. *For any* $n \geq 0$, *if every module in* $\operatorname{mod} R$ *has torsionfree dimension at most* n, *then* $\operatorname{id}_{R^{op}} R \leq n$.

To prove this theorem, we need some lemmas. We use $\operatorname{Mod} R$ to denote the category of left R-modules.

Lemma 3.7 ([11, Proposition 1]). $\operatorname{id}_{R^{op}} R = \sup\{\operatorname{fd}_R E \mid E$ *is an injective module in* $\operatorname{Mod} R\} = \operatorname{fd}_R Q$ *for any injective cogenerator* Q *for* $\operatorname{Mod} R$.

Lemma 3.8. *For any* $n \geq 0$, $\operatorname{id}_{R^{op}} R \leq n$ *if and only if every module in* $\operatorname{mod} R$ *can be embedded into a module in* $\operatorname{Mod} R$ *with flat dimension at most* n.

Proof. Assume that $\operatorname{id}_{R^{op}} R \leq n$. Then the injective envelope of any module in $\operatorname{mod} R$ has flat dimension at most n by Lemma 3.7, and the necessity follows.

Conversely, let E be any injective module in $\operatorname{Mod} R$. Then by [15, Exercise 2.32], $E = \varinjlim_{i \in I} M_i$, where $\{M_i \mid i \in I\}$ is the set of all finitely generated submodules of E and I is a directed index set (in which the quasi-order is defined by $i \leq j$ if and only if $M_i \leq M_j$, the homomorphism $\lambda_j^i : M_i \to M_j$ is the canonical embedding). By assumption, for any $i \in I$ and $M_i \in \text{mod}\, R$, we have an exact sequence $0 \to M_i \stackrel{\alpha_i}{\longrightarrow} N_i$ with $N_i \in \text{Mod}\, R$ and $\text{fd}_R\, N_i \leq n$.

Put $K = \prod_{i \in I} N_i$ and $I_i = \{ j \in I \mid M_i \leq M_j \}$ for any $i \in I$. Since R is a left and right Noetherian ring, any direct product of flat modules is still flat. So $\text{fd}_R\, K \leq n$. Define $\beta_i = \prod_{k \in I} f_k^i$ with

$$f_k^i = \begin{cases} \alpha_k \lambda_k^i, & \text{if} \quad k \in I_i, \\ 0, & \text{if} \quad k \notin I_i \end{cases}$$

for any $i, k \in I$. Then $0 \to M_i \stackrel{\beta_i}{\to} K$ is exact for any $i \in I$. For any $i \leq j$ (determined by $M_i \leq M_j$), we have the following commutative and exact diagram:

$$\begin{array}{ccccc} & & 0 & & \\ & & \downarrow & & \\ 0 & \longrightarrow & M_i & \stackrel{\beta_i}{\longrightarrow} & K \\ & & \downarrow{\scriptstyle \lambda_j^i} & & \downarrow{\scriptstyle \varphi_j^i} \\ 0 & \longrightarrow & M_j & \stackrel{\beta_j}{\longrightarrow} & K \end{array}$$

where $\varphi_j^i = \prod_{k \in I} h_k$ with

$$h_k = \begin{cases} 1_{N_k}, & \text{if} \quad k \in I_j, \\ 0, & \text{if} \quad k \notin I_j \end{cases}$$

for any $k \in I$. It is clear that $\{K, \varphi_j^i\}$ is a direct system of the constant module K. It follows from [15, Theorem 2.18] that we get a monomorphism $0 \to E(= \varinjlim_{i \in I} M_i) \to \varinjlim_{i \in I} K$. Because the functor Tor commutes with $\varinjlim_{i \in I}$ by [15, Theorem 8.11], $\text{fd}_R \varinjlim_{i \in I} K \leq n$. So $\text{fd}_R\, E \leq n$ and hence $\text{id}_{R^{op}}\, R \leq n$ by Lemma 3.7. □

Proof of Theorem 3.6. By assumption and Corollary 3.5, we have that every module in $\text{mod}\, R$ can be embedded into a module in $\text{mod}\, R$ with projective dimension at most n. Then by Lemma 3.8, we get the assertion. □

Lemma 3.9. *For any* $M \in \text{mod}\, R$ *and* $n \geq 0$, ${}^{\perp}{}_R R\text{-dim}_R\, M \leq n$ *if and only if* $\text{Ext}_R^{n+i}(M, R) = 0$ *for any* $i \geq 1$.

Proof. For any $M \in \text{mod}\, R$, consider the following exact sequence:

$$\cdots \to W_n \to W_{n-1} \to \cdots \to W_0 \to M \to 0$$

in $\bmod R$ with all W_i in ${}^{\perp}{}_R R$. Then we have that $\mathrm{Ext}^i_R(\mathrm{Im}(W_n \to W_{n-1}), R) \cong \mathrm{Ext}^{n+i}_R(M,R)$ for any $i \geq 1$. So $\mathrm{Im}(W_n \to W_{n-1}) \in {}^{\perp}{}_R R$ if and only if $\mathrm{Ext}^{n+i}_R(M,R) = 0$ for any $i \geq 1$, and hence the assertion follows. □

Proposition 3.10. *For any $n \geq 0$, every module in* $\bmod R$ *has left orthogonal dimension at most n if and only if* $\mathrm{id}_R\, R \leq n$.

Proof. By Lemma 3.9, we have that $\mathrm{id}_R\, R \leq n$ if and only if $\mathrm{Ext}^{n+i}_R(M, R) = 0$ for any $M \in \bmod R$ and $i \geq 1$, if and only if ${}^{\perp}{}_R R\text{-}\mathrm{dim}_R\, M \leq n$ for any $M \in \bmod R$. □

Proof of Theorem 1.4. (1) $\Rightarrow$ (2) + (3) follows from [10, Theorem 3.5].

(2) $\Rightarrow$ (1) Let M be any module in $\bmod R$. Then by assumption, we have that $\mathrm{G}\text{-}\mathrm{dim}_R\, M \leq n$ and $\mathcal{T}\text{-}\mathrm{dim}_R\, M \leq n$. So $\mathrm{id}_{R^{op}}\, R \leq n$ by Theorem 3.6. On the other hand, because ${}^{\perp}{}_R R\text{-}\mathrm{dim}_R\, M \leq \mathrm{G}\text{-}\mathrm{dim}_R\, M$, $\mathrm{id}_R\, R \leq n$ by Proposition 3.10.

Symmetrically, we get (3) $\Rightarrow$ (1).

(4) $\Rightarrow$ (1) By Theorem 3.6 and its symmetric version.

(2) + (3) $\Rightarrow$ (4) Because $\mathcal{T}\text{-}\mathrm{dim}_R\, M \leq \mathrm{G}\text{-}\mathrm{dim}_R\, M$ and $\mathcal{T}\text{-}\mathrm{dim}_{R^{op}}\, N \leq \mathrm{G}\text{-}\mathrm{dim}_{R^{op}}\, N$ for any $M \in \bmod R$ and $N \in \bmod R^{op}$, the assertion follows.

(1) $\Leftrightarrow$ (5) By Proposition 3.10 and its symmetric version. □

4. The torsionless property and self-injective dimension

The following result plays a crucial role in this section, which generalizes [4, Lemma 4], [8, Lemma 2.1] and [13, Theorem 5.1].

Proposition 4.1. *For any $n \geq 1$, the following statements are equivalent.*

(1) ${}^{\perp_n}{}_R R \subseteq \mathcal{T}_1(\bmod R)$. *In this case,* ${}^{\perp_n}{}_R R$ *is said to have the torsionless property.*
(2) ${}^{\perp_n}{}_R R \subseteq \mathcal{T}(\bmod R)$.
(3) *Every module in* ${}^{\perp_n}{}_R R$ *has torsionfree dimension at most n.*
(4) $\mathcal{T}_n(\bmod R) = \mathcal{T}(\bmod R)$.
(5) $\mathcal{T}_n(\bmod R^{op}) \subseteq {}^{\perp}R_R$.
(6) ${}^{\perp_n}R_R = {}^{\perp}R_R$.

Proof. (2) $\Rightarrow$ (1) and (2) $\Rightarrow$ (3) are trivial, and (1) $\Leftrightarrow$ (6) follows from [8, Lemma 2.1]. Note that M and $\mathrm{Tr}\,\mathrm{Tr}\,M$ are projectively equivalent for any $M \in \bmod R$ or $\bmod R^{op}$. Then it is not difficult to verify (2) $\Leftrightarrow$ (5) and (4) $\Leftrightarrow$ (6). So it suffices to prove (1) $\Rightarrow$ (2) and (3) $\Rightarrow$ (2).

(1) $\Rightarrow$ (2) Assume that $M \in {}^{\perp_n}{}_R R$. Then M is torsionless by (1). So, by Proposition 2.1, we have an exact sequence $0 \to M \to P_0 \to M_1 \to 0$ in $\bmod R$ with P_0 projective and $M_1 \in {}^{\perp_1}{}_R R$, which yields that $M_1 \in {}^{\perp_{n+1}}{}_R R$. Then M_1 is torsionless by (1), and again by Proposition 2.1, we have an exact sequence $0 \to M_1 \to P_1 \to M_2 \to 0$

in mod R with P_1 projective and $M_2 \in {}^{\perp_1}{}_R R$, which yields that $M_1 \in {}^{\perp_{n+2}}{}_R R$. Repeating this procedure, we get an exact sequence:

$$0 \to M \to P_0 \to P_1 \to \cdots \to P_i \to \cdots$$

in mod R with all P_i projective and $\mathrm{Im}(P_i \to P_{i+1}) \in {}^{\perp_{n+i+1}}{}_R R \subseteq {}^{\perp_{i+1}}{}_R R$, which implies that M is ∞-torsionfree by Proposition 2.1.

(3) $\Rightarrow$ (2) Assume that $M \in {}^{\perp_n}{}_R R$. Then $\mathcal{T}\text{-dim}_R M \leq n$ by assumption. By Proposition 3.4, there exists an exact sequence:

$$0 \to H \to T \to M \to 0 \tag{1}$$

in mod R with $\mathrm{pd}_R H \leq n-1$ and $T \in \mathcal{T}(\mathrm{mod}\, R)$. Because $M \in {}^{\perp_n}{}_R R$, the sequence (1) splits, which implies that $M \in \mathcal{T}(\mathrm{mod}\, R)$ by Lemma 2.3. □

Similarly, we have the following result.

Proposition 4.2. *The following statements are equivalent.*
(1) ${}^{\perp}{}_R R \subseteq \mathcal{T}_1(\mathrm{mod}\, R)$. *In this case,* ${}^{\perp}{}_R R$ *is said to have the torsionless property.*
(2) ${}^{\perp}{}_R R \subseteq \mathcal{T}(\mathrm{mod}\, R)$.
(3) *Every module in* ${}^{\perp}{}_R R$ *has finite torsionfree dimension.*
(4) $\mathcal{T}(\mathrm{mod}\, R^{op}) \subseteq {}^{\perp}R_R$.

Let $N \in \mathrm{mod}\, R^{op}$ and

$$0 \to N \xrightarrow{\delta_0} E_0 \xrightarrow{\delta_1} E_1 \xrightarrow{\delta_2} \cdots \xrightarrow{\delta_i} E_i \xrightarrow{\delta_{i+1}} \cdots$$

be an injective resolution of N. For a positive integer n, recall from [3] that an injective resolution as above is called *ultimately closed* at n if $\mathrm{Im}\,\delta_n = \bigoplus_{j=0}^{m} W_j$, where each W_j is a direct summand of $\mathrm{Im}\,\delta_{i_j}$ with $i_j < n$. By [8, Corollary 2.3], if R_R has a ultimately closed injective resolution at n or $\mathrm{id}_{R^{op}}\, R \leq n$, then ${}^{\perp_n}{}_R R$ (and hence ${}^{\perp}{}_R R$) has the torsionless property.

The following result generalizes [16, Lemma A], which states that $\mathrm{id}_{R^{op}}\, R = \mathrm{id}_R\, R$ if both of them are finite.

Corollary 4.3. *If* $n = \min\{t \mid {}^{\perp_t}{}_R R$ *has the torsionless property*$\}$ *and* $m = \min\{s \mid {}^{\perp_s}R_R$ *has the torsionless property*$\}$, *then* $n = m$.

Proof. We may assume that $n \leq m$. Let $N \in {}^{\perp_n}R_R$. Then $N \in {}^{\perp}R_R$ $(\subseteq {}^{\perp_m}R_R)$ by Proposition 4.1. So $N \in \mathcal{T}(\mathrm{mod}\, R^{op})$ and ${}^{\perp_n}R_R$ has the torsionless property by the symmetric version of Proposition 4.1. Thus $n \geq m$ by the minimality of m. The proof is finished. □

In the following, we will investigate the relation between the torsionless property and the self-injective dimension of R. We have seen that if $\mathrm{id}_{R^{op}} R \leq n$, then ${}^{\perp_n}{}_RR$ has the torsionless property. In the rest of this section, we will investigate when the converse of this assertion holds true.

Proposition 4.4. *Assume that m and n be positive integers and $\Omega^m(\mathrm{mod}\, R^{op}) \subseteq \mathcal{T}_n(\mathrm{mod}\, R^{op})$. If ${}^{\perp_n}{}_RR$ has the torsionless property, then $\mathrm{id}_{R^{op}} R \leq m$.*

Proof. Let $M \in \Omega^m(\mathrm{mod}\, R^{op})$. Then $M \in \mathcal{T}_n(\mathrm{mod}\, R^{op})$ by assumption. Because ${}^{\perp_n}{}_RR$ has the torsionless property by assumption, $M \in {}^{\perp}R_R$ by Proposition 4.1. Then it is easy to verify that $\mathrm{id}_{R^{op}} R \leq m$. □

Assume that

$$0 \to {}_RR \to I^0(R) \to I^1(R) \to \cdots \to I^i(R) \to \cdots$$

is a minimal injective resolution of ${}_RR$.

Lemma 4.5. *If ${}^{\perp_n}{}_RR$ has the torsionless property, $\bigoplus_{i=0}^n I^i(R)$ is an injective cogenerator for* Mod R.

Proof. For any $S \in \mathrm{mod}\, R$, we claim that $\mathrm{Hom}_R\big(S, \bigoplus_{i=0}^n I^i(R)\big) \neq 0$. Otherwise, we have that $\mathrm{Ext}_R^i(S, R) \cong \mathrm{Hom}_R(S, I^i(R)) = 0$ for any $0 \leq i \leq n$. So $S \in {}^{\perp_n}{}_RR$ and hence S is reflexive by assumption and Proposition 4.1, which yields that $S \cong S^{**} = 0$. This is a contradiction. Thus we conclude that $\bigoplus_{i=0}^n I^i(R)$ is an injective cogenerator for Mod R. □

Proposition 4.6. *$\mathrm{id}_{R^{op}} R < \infty$ if and only if ${}^{\perp_n}{}_RR$ has the torsionless property for some $n \geq 1$ and $\mathrm{fd}_R \bigoplus_{i\geq 0} I^i(R) < \infty$.*

Proof. The sufficiency follows from Lemmas 4.5 and 3.7, and the necessity follows from Proposition 4.1 and Lemma 3.7. □

For any $n, k \geq 1$, recall from [9] that R is said to be $g_n(k)$ if $\mathrm{Ext}^j_{R^{op}}(\mathrm{Ext}_R^{i+k}(M, R), R) = 0$ for any $M \in \mathrm{mod}\, R$ and $1 \leq i \leq n$ and $0 \leq j \leq i - 1$; and R is said to be $g_n(k)^{op}$ if R^{op} is $g_n(k)$. It follows from [12, 6.1] that R is $g_n(k)$ (resp. $g_n(k)^{op}$) if $\mathrm{fd}_{R^{op}} I^i(R^{op})$ (resp. $\mathrm{fd}_R I^i(R)) \leq i + k$ for any $0 \leq i \leq n - 1$.

Theorem 4.7. *Assume that n and k are positive integers and ${}^{\perp_n}{}_RR$ has the torsionless property. If R is $g_n(k)$ or $g_n(k)^{op}$, then $\mathrm{id}_{R^{op}} R \leq n + k - 1$.*

Proof. Assume that ${}^{\perp_n}{}_RR$ has the torsionless property.

If R is $g_n(k)$, then $\Omega^{n+k-1}(\mathrm{mod}\, R) \subseteq \mathcal{T}_n(\mathrm{mod}\, R) = \mathcal{T}(\mathrm{mod}\, R)$ by [9, Theorem 3.4] and Proposition 4.1, which implies that the torsionfree dimension of every module in mod R is at most $n+k-1$. So $\mathrm{id}_{R^{op}}\, R \leq n+k-1$ by Theorem 3.6.

If R is $g_n(k)^{op}$, then $\Omega^{n+k-1}(\mathrm{mod}\, R^{op}) \subseteq \mathcal{T}_n(\mathrm{mod}\, R^{op})$ by the symmetric version of [9, Theorem 3.4], which implies $\mathrm{id}_{R^{op}}\, R \leq n+k-1$ by Proposition 4.4. □

By Proposition 4.1 and Proposition 4.6 or Theorem 4.7, we immediately get the following

Corollary 4.8. *If* $\mathrm{fd}_R \bigoplus_{i=0}^{n} I^i(R) \leq n$, *then* $\mathrm{id}_{R^{op}}\, R \leq n$ *if and only if* ${}^{\perp_n}{}_R R$ *has the torsionless property.*

Recall that the Gorenstein symmetric conjecture states that $\mathrm{id}_R\, R = \mathrm{id}_{R^{op}}\, R$ for any Artinian algebra R, which remains still open. Hoshino proved in [5, Proposition 2.2] that if $\mathrm{id}_R\, R \leq 2$, then $\mathrm{id}_R\, R = \mathrm{id}_{R^{op}}\, R \leq 2$ if and only if ${}^{\perp_2}{}_R R$ has the torsionless property. As an immediate consequence of Theorem 4.7, the following corollary generalizes this result.

Corollary 4.9. *For any* $n \geq 1$, *if* $\mathrm{id}_R\, R \leq n$, *then* $\mathrm{id}_R\, R = \mathrm{id}_{R^{op}}\, R \leq n$ *if and only if* ${}^{\perp_n}{}_R R$ *has the torsionless property.*

Proof. The necessity follows from Proposition 4.1. We next prove the sufficiency. If $\mathrm{id}_R\, R \leq n$, then $\mathrm{fd}_{R^{op}} \bigoplus_{i=0}^{n} I^i(R^{op}) \leq n$ by Lemma 3.7, which implies that R is $g_n(n)$ by [12, 6.1]. Thus $\mathrm{id}_{R^{op}} \leq 2n-1$ by Theorem 4.7. It follows from [16, Lemma A] that $\mathrm{id}_{R^{op}}\, R \leq n$. □

5. Questions

In view of the results obtained above, the following two questions are worth being studied.

Note that both the subcategory of mod R consisting of modules with Gorenstein dimension at most n and that consisting of modules with left orthogonal dimension at most n are closed under extensions and under kernels of epimorphisms. So, it is natural to ask the following

QUESTION 5.1. Is the subcategory of mod R consisting of modules with torsion-free dimension at most n closed under extensions or under kernels of epimorphisms? In particular, Is $\mathcal{T}(\mathrm{mod}\, R)$ closed under extensions or under kernels of epimorphisms?

For any $n \geq 1$, $\mathcal{T}_n(\mathrm{mod}\, R)$ is not closed under extensions by [6, Theorem 3.3]. On the other hand, we have the following

Claim. *If* $^{\perp}R_R$ *has the torsionless property, then the answer to* Question 5.1 *is positive.*

In fact, if $^{\perp}R_R$ has the torsionless property, then, by the symmetric version of Proposition 4.2, we have that $\mathcal{T}(\mathrm{mod}\, R) \subseteq {}^{\perp}{}_RR$ and every module in $\mathcal{T}(\mathrm{mod}\, R)$ has Gorenstein dimension zero. So the torsionfree dimension and the Gorenstein dimension of any module in mod R coincide, and the claim follows.

By the symmetric version of [8, Corollary 2.3], if ${}_RR$ has a ultimately closed injective resolution at n or $\mathrm{id}_R\, R \leq n$, then the condition in the above claim is satisfied. This fact also means that the above claim extends [6, Corollary 2.5].

It is also interesting to know whether the converse of Theorem 3.6 holds true. That is, we have the following

QUESTION 5.2. Does $\mathrm{id}_{R^{op}}\, R \leq n$ imply that every module $M \in \mathrm{mod}\, R$ has torsionfree dimension at most n?

Claim. *When* $n = 1$, *the answer to* Question 5.2 *is positive.*

Assume that $\mathrm{id}_{R^{op}}\, R \leq 1$ and $0 \to K \to P \to M \to 0$ is an exact sequence in mod R with P projective. Then $\mathrm{Ext}^i_{R^{op}}(\mathrm{Tr}\, K, R) = 0$ for any $i \geq 2$. Notice that K is torsionless, so $\mathrm{Ext}^1_{R^{op}}(\mathrm{Tr}\, K, R) = 0$ and $K \in \mathcal{T}(\mathrm{mod}\, R)$, which implies $\mathcal{T}\text{-}\mathrm{dim}_R\, M \leq 1$. Consequently the claim is proved.

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