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MULTIPLE SOLUTIONS FOR SUPERLINEAR p-LAPLACIAN NEUMANN PROBLEMS

Dedicated to the memory of Stefan Mirica

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Abstract

Our main goal is to prove the existence of multiple solutions with precise sign information for a Neumann problem driven by the p-Laplacian differential operator with a $(p-1)$-superlinear term which does not satisfy the Ambrosetti–Rabinowitz condition. Using minimax methods we show that the problem has five nontrivial smooth solutions, two positive, two negative and the fifth nodal. In the semilinear case ($p=2$), using Morse theory, we produce a second nodal solution (for a total of six nontrivial smooth solutions).

1. Introduction

In a recent paper [2], we studied the following nonlinear Neumann problem

$$\begin{cases} -\triangle_p u(z) + \beta |u(z)|^{p-2}u(z) = f(z, u(z)) & \text{in} \quad \Omega, \\ \dfrac{\partial u}{\partial n} = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subseteq \mathbb{R}^N$ is a bounded domain with a C^2 boundary $\partial\Omega$, n is the outward unit normal on $\partial\Omega$, $\beta > 0$, $2 \le p < \infty$ and $\triangle_p$ stands for the p-Laplacian differential operator defined by

$$\triangle_p u(z) = \operatorname{div}\bigl(\|Du(z)\|_{\mathbb{R}^N}^{p-2} Du(z)\bigr).$$

Also $f(z, x)$ is a Caratheodory function which exhibits a $(p-1)$-superlinear growth near $\pm\infty$. More precisely, it satisfies the so-called Ambrosetti–Rabinowitz condition (AR-condition, for short), which says that there exist $\mu > p$ and $M > 0$ such that

$$0 < \mu F(z, x) \le f(z, x)x \quad \text{for a.a.} \quad z \in \Omega, \quad \text{all} \quad |x| \ge M, \tag{1.2}$$

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where $F(z,x) = \int_0^x f(z,s)\,ds$. Integrating (1.2) we obtain the weaker condition

$$c_0|x|^\mu \le F(z,x) \quad \text{for a.a.} \quad z \in \Omega, \quad \text{all} \quad |x| \ge M, \quad \text{and some} \quad c_0 > 0. \tag{1.3}$$

From (1.3) we infer the much weaker condition

$$\lim_{|x|\to\infty} \frac{F(z,x)}{|x|^p} = +\infty \quad \text{uniformly for a.a.} \quad z \in \Omega. \tag{1.4}$$

This condition dictates a p-superlinear growth for $F(z,\,\cdot\,)$ for a.a. $z \in \Omega$. It is easy to see that it is satisfied if $f(z,\cdot)$ is $(p-1)$-superlinear near $\pm\infty$, uniformly for a.a. $z \in \Omega$.

In (1.1), the presence of the term $\beta|x|^{p-2}x$ on the left-hand side facilitates considerably the study of the equation, since the corresponding nonlinear operator of the problem is maximal monotone and coercive. In [2], we did not address the question of what happens if $\beta = 0$, in which case the nonlinear operator is no longer coercive. In this paper we consider this limit case. So, here the problem under consideration is

$$\begin{cases} -\triangle_p u(z) = f(z,u(z)) & \text{in} \quad \Omega, \\ \dfrac{\partial u}{\partial n} = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1.5}$$

with $1 < p < \infty$. Note that in contrast to [2] we do not need the restriction $2 \le p$. Again, we consider a $(p-1)$-superlinear perturbation $f(z,\,\cdot\,)$ but we no longer use the AR-condition (see (1.2). Instead, we use a weaker condition allowing a larger class of nonlinearities. We prove a multiplicity result for problem (1.5) by producing five nontrivial smooth solutions with precise sign information for all of them. In the semilinear case ($p = 2$), we obtain six nontrivial smooth solutions with precise sign information. Our approach combines variational methods based on the critical point theory, with truncation techniques, the method of upper and lower solutions, and Morse theory. Our strategy for proving the existence of the second nodal solution in the case $p = 2$ is comparable to that used by Dancer–Du [13] for semilinear Dirichlet problems.

Superlinear equations were investigated primarily in the context of Dirichlet problems. We mention the works of Bartsch–Liu [8], García Azorero–Peral Alonso–Manfredi [17], Guo–Zhang [20], Motreanu–Motreanu–Papageorgiou [26], Papageorgiou–Papageorgiou [28] and Papageorgiou–Rocha–Staicu [29]. For the Neumann problem, to the best of our knowledge, the only such works are [2], [3], with the mention that in [3] we obtain only three nontrivial solutions of (1.5), with sign information for two of them. There have been some other multiplicity results for Neumann problems; see Anello [5], Binding–Drábek–Huang [9], Bonanno–Candito [10], Cammaroto–Chinnì–Di Bella [11], Filippakis–Gasiński–Papageorgiou [16], Motreanu–Papageorgiou [27], Ricceri [30] and Wu–Tan [32]. However, in the aforementioned works the authors impose restrictive symmetry or dimensionality (i.e., $N < p$) conditions, and in the nonsymmetric case, produce at most three nontrivial smooth solutions and do not provide sign information for all of them.

In the next section, for the convenience of the reader, we briefly review the main mathematical tools that we will use in the sequel.

2. Preliminaries

We start with elements of critical point theory. Let X be a Banach space and X^* its topological dual. By $\langle \cdot, \cdot \rangle$ we denote the duality brackets for the pair (X^*, X). Let $\varphi \in C^1(X)$ and $c \in \mathbb{R}$.

We say that φ satisfies the *Palais–Smale condition at level c* (the PS$_c$*-condition*, for short), if: every sequence $\{x_n\}_{n\geq 1} \subseteq X$ such that

$$\varphi(x_n) \to c \quad \text{and} \quad \varphi'(x_n) \to 0 \quad \text{in} \quad X^* \quad \text{as} \quad n \to \infty,$$

has a strongly convergent subsequence. We say that φ satisfies the *Palais–Smale condition* (the PS-*condition*, for short) if it satisfies the PS$_c$-condition at every level $c \in \mathbb{R}$.

Sometimes, it is more convenient to use a weaker compactness-type condition on φ. So, we say that φ satisfies the *Cerami condition at the level $c \in \mathbb{R}$* (the C$_c$*-condition*, for short), if every sequence $\{x_n\}_{n\geq 1} \subseteq X$ such that

$$\varphi(x_n) \to c \quad \text{and} \quad (1 + \|x_n\|)\varphi'(x_n) \to 0 \quad \text{in} \quad X^* \quad \text{as} \quad n \to \infty$$

has a strongly convergent subsequence. We say that φ satisfies the *Cerami-condition* (the C-*condition*, for short) if it verifies the C$_c$-condition at every level $c \in \mathbb{R}$.

It was shown by Bartolo–Benci–Fortunato [7] that the deformation lemma, and consequently the minimax theory of the critical values of a function $\varphi \in C^1(X)$, remain valid if instead of the PS-condition, one employs the weaker C-condition.

The next two theorems are two well known such minimax results. The first is known in the literature as *the mountain pass theorem*:

Theorem 1. *If X is a Banach space, $\varphi \in C^1(X)$, $x_0, x_1 \in X$ and $r > 0$ satisfy*

$$\max\{\varphi(x_0), \varphi(x_1)\} \leq \inf\{\varphi(x) \colon \|x - x_0\| = r\} =: \eta_r, \quad \|x_1 - x_0\| > r$$

and φ satisfies the C$_c$*-condition, where*

$$c = \inf_{\gamma \in \Gamma} \max_{t \in [-1,1]} \varphi(\gamma(t))$$

with

$$\Gamma = \{\gamma \in C([-1, 1], X) \colon \gamma(-1) = x_0, \gamma(1) = x_1\},$$

then $c \geq \eta_r$ and c is a critical value of φ. Moreover, if $c = \eta_r$, then there exists a critical point $x \in X$ of φ such that $\varphi(x) = c$ and $\|x - x_0\| = r$.

The second minimax theorem is known in the literature as the *saddle point theorem*. (Below, "*Id*" stands for identity.)

Theorem 2. *If* $\varphi \in C^1(X)$, $X = Y \oplus V$ *with* $\dim Y < \infty$, $r > 0$,

$$D = \{x \in Y : \|x\| \le r\}, \quad D_0 = \{x \in Y : \|x\| = r\} \quad \textit{and} \quad \max_{D_0} \varphi \le \inf_V \varphi =: \eta_0$$

and φ *satisfies the* C_c*-condition, where*

$$c = \inf_{\gamma \in \Gamma} \max_{x \in D} \varphi(\gamma(x)) \quad \textit{with} \quad \Gamma = \{\gamma \in C(D, X) : \gamma|_{D_0} = Id|_{D_0}\},$$

then $c \ge \eta_0$ *and* c *is a critical value of* φ. *Moreover, if* $c = \eta_0$, *then there exists a critical point* $x \in X$ *of* φ *such that* $\varphi(x) = c$ *and* $x \in V$.

Another variational result that we will use in the study of problem (1.5) is the so called *second deformation theorem* (see, for example, Gasinski–Papageorgiou [18, p. 628]). Let us introduce the following sets:

$$K_\varphi = \{x \in X : \varphi'(x) = 0\} \quad \text{(the critical set of } \varphi\text{)},$$

$$K_\varphi^c = \{x \in K_\varphi : \varphi(x) = c\} \quad \text{(the critical set of } \varphi \text{ at the level } c \in \mathbb{R}\text{)},$$

$$\varphi^c = \{x \in X : \varphi(x) \le c\} \quad \text{(the sublevel set of } \varphi \text{ at } c \in \mathbb{R}\text{)}.$$

Theorem 3. *If* $\varphi \in C^1(X)$, $a \in \mathbb{R}$, $a < b \le \infty$, φ *satisfies the* PS_c*-condition at every level* $c \in (a, b)$, φ *has no critical values in* (a, b) *and* $\varphi^{-1}(a)$ *contains at most a finite number of critical points of* φ, *then there exists a homotopy* $h: [0,1] \times (\varphi^b \setminus K_\varphi^b) \to \varphi^b$ *such that*

(a) $h(t, \cdot)|_{\varphi^a} = Id|_{\varphi^a}$ *for all* $t \in [0, 1]$;

(b) $h(1, \varphi^b \setminus K_\varphi^b) \subseteq \varphi^a$;

(c) $\varphi(h(t, x)) \le \varphi(h(s, x))$ *for all* $t, s \in [0, 1]$, $0 \le s \le t \le 1$, *all* $x \in \varphi^b \setminus K_\varphi^b$.

REMARK. In particular, this theorem implies that φ^a is a strong deformation retract of $\varphi^b \setminus K_\varphi^b$.

The following notion from the theory of nonlinear operators of monotone type will help us verify the PS and C conditions. (Here and in the sequel, $\xrightarrow{w}$ designates weak convergence.)

DEFINITION 4. A map $A: X \to X^*$ is said to be of type $(S)_+$, if for every sequence $\{x_n\}_{n \ge 1} \subseteq X$ such that $x_n \xrightarrow{w} x$ in X and

$$\limsup_{n \to \infty} \langle A(x_n), x_n - x \rangle \le 0,$$

one has

$$x_n \to x \quad \text{in} \quad X \quad \text{as} \quad n \to \infty.$$

In the analysis of problem (1.5) we will use the following spaces:

$$C_n^1(\overline{\Omega}) = \left\{ u \in C^1(\overline{\Omega}) \colon \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega \right\}$$

and

$$W_n^{1,p}(\Omega) = \overline{C_n^1(\overline{\Omega})}^{\|\cdot\|},$$

$\|\cdot\|$ being the norm of the Sobolev space $W^{1,p}(\Omega)$. As usual, if $p = 2$, then we write $H_n^1(\Omega) = W_n^{1,2}(\Omega)$.

The Banach space $C_n^1(\overline{\Omega})$ is an ordered Banach space with the positive cone

$$C_+ = \{u \in C_n^1(\overline{\Omega}) \colon u(z) \geq 0 \text{ for all } z \in \overline{\Omega}\}.$$

This cone has a nonempty interior, given by

$$\operatorname{int} C_+ = \{u \in C_+ \colon u(z) > 0 \text{ for all } z \in \overline{\Omega}\}.$$

Let $X = W_n^{1,p}(\Omega)$ and $X^* = W_n^{1,p}(\Omega)^*$. Consider the nonlinear operator $A\colon W_n^{1,p}(\Omega) \to W_n^{1,p}(\Omega)^*$ defined by

$$\langle A(u), y \rangle = \int_\Omega \|Du\|_{\mathbb{R}^N}^{p-2} (Du, Dy)_{\mathbb{R}^N}\, dz \quad \text{for all} \quad u, y \in W_n^{1,p}(\Omega). \tag{2.1}$$

The following result is well-known (see, e.g., [2]):

Proposition 5. *The map $A\colon W_n^{1,p}(\Omega) \to W_n^{1,p}(\Omega)^*$ defined by* (2.1) *is of type* $(S)_+$.

We also recall (cf., e.g., [22]) the following result relating local minimizers in $C_n^1(\overline{\Omega})$ and in $W_n^{1,p}(\Omega)$ (see [17], [20] for a corresponding result for Dirichlet problems.)

So, let $f_0\colon \Omega \times \mathbb{R} \to \mathbb{R}$ be a function such that:

(i) for all $x \in \mathbb{R}$, $z \to f_0(z, x)$ is measurable;
(ii) for almost all $z \in \Omega$, $x \to f_0(z, x)$ is continuous;
(iii) for almost all $z \in \Omega$ and all $x \in \mathbb{R}$

$$|f_0(z, x)| \leq a_0(z) + c_0|x|^{r-1},$$

with $a_0 \in L^\infty(\Omega)_+$, $c_0 > 0$ and $1 < r < p^*$, where

$$p^* = \begin{cases} \dfrac{Np}{N-p} & \text{if} \quad p < N, \\ +\infty & \text{if} \quad p \geq N. \end{cases} \tag{2.2}$$

We set $F_0(z,x)=\int_0^x f_0(z,s)\,ds$ and consider the functional $\varphi_0\colon W_n^{1,p}(\Omega)\to\mathbb{R}$ defined by

$$\varphi_0(u)=\frac{1}{p}\|Du\|_p^p-\int_\Omega F_0(z,u(z))\,dz \quad \text{for all} \quad u\in W_n^{1,p}(\Omega).$$

Evidently $\varphi_0\in C^1(W_n^{1,p}(\Omega))$.

Proposition 6. *If $u_0\in W_n^{1,p}(\Omega)$ $(1<p<\infty)$ is a local $C_n^1(\overline{\Omega})$-minimizer of φ_0, i.e., there exists $r_0>0$ such that*

$$\varphi_0(u_0)\le\varphi_0(u_0+u) \quad \textit{for all} \quad u\in C_n^1(\overline{\Omega}), \quad \|u\|_{C_n^1(\overline{\Omega})}\le r_0,$$

then $u_0\in C_n^1(\overline{\Omega})$ and it is a local $W_n^{1,p}(\Omega)$-minimizer of φ_0, i.e., there exists $r_1>0$ such that

$$\varphi_0(u_0)\le\varphi_0(u_0+u) \quad \textit{for all} \quad u\in W_n^{1,p}(\Omega), \quad \|u\|\le r_1.$$

Next, let us recall a few basic facts about the spectrum of the negative Neumann p-Laplacian $(1<p<\infty)$. So, we consider the following nonlinear eigenvalue problem:

$$\begin{cases} -\triangle_p u(z)=\lambda|u(z)|^{p-2}u(z) & \text{in} \quad \Omega,\\ \dfrac{\partial u}{\partial n}=0 & \text{on} \quad \partial\Omega. \end{cases} \tag{2.3}$$

We say that $\lambda\in\mathbb{R}$ is an *eigenvalue* of $(-\triangle_p, W_n^{1,p}(\Omega))$, if problem (2.3), has a nontrivial solution $u\in W_n^{1,p}(\Omega)$. In fact, nonlinear regularity theory implies that $u\in C_n^1(\overline{\Omega})$ (see for example, Gasinski–Papageorgiou [18, pp. 737–738], and Lieberman [24].) It is easy to see that an eigenvalue $\lambda\in\mathbb{R}$ satisfies $\lambda\ge 0$. Moreover, $\lambda_0=0$ is an eigenvalue with corresponding eigenspace $\mathbb{R}$ (i.e., the space of constant functions) and λ_0 is isolated. By $\hat{u}_0$ we denote the corresponding L^p-normalized eigenfunction (principal eigenfunction). We have

$$\hat{u}_0(z)=\frac{1}{|\Omega|_N^{1/p}} \quad \text{for all} \quad z\in\overline{\Omega},$$

(where by $|\cdot|_N$ we denote the Lebesgue measure on $\mathbb{R}^N$). By virtue of the Ljusternik–Schnirelmann theory, we have a whole strictly increasing sequence $\{\lambda_k\}_{k\ge 0}$ of eigenvalues (known as the LS-eigenvalues of $(-\triangle_p, W_n^{1,p}(\Omega))$) such that $\lambda_k\to\infty$ as $k\to\infty$. If $p=2$ (linear eigenvalue problem), then these are all the eigenvalues of $(-\triangle, W_n^{1,2}(\Omega))$. If $p\ne 2$ (nonlinear eigenvalue problem), then we do not know if this is the case.

However, since $\lambda_0=0$ is isolated and the spectrum $\sigma(p)$ of $(-\triangle_p, W_n^{1,p}(\Omega))$ is closed, then

$$\lambda_1^*=\inf\{\lambda\colon \lambda\in\sigma(p),\ \lambda>0\}>0.$$

Evidently, $\lambda_1^* > 0$ is the second eigenvalue (the first nonzero eigenvalue) of $(-\triangle_p, W_n^{1,p}(\Omega))$ and $\lambda_1^* = \lambda_1$ (i.e., the second eigenvalue of $(-\triangle_p, W_n^{1,p}(\Omega))$ and the second LS-eigenvalue coincide).

A similar spectral theory is valid for the weighted eigenvalue problem

$$\begin{cases} -\triangle_p u(z) = \hat{\lambda} m(z)|u(z)|^{p-2}u(z) & \text{in} \quad \Omega, \\ \dfrac{\partial u}{\partial n} = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{2.4}$$

where $\hat{\lambda} \in \mathbb{R}$ and $m \in L^\infty(\Omega)_+$, $m \neq 0$. The eigenvalues of (2.4) will be denoted by $\hat{\lambda}(m)$. In particular, $\hat{\lambda}_k(1) = \lambda_k$ $(k = 0, 1, 2, \ldots)$. See, e.g., [2].

The Ljusternik–Schnirelmann theory provides a minimax characterization of $\lambda_1 > 0$. However, for our purposes that characterization is not convenient. Instead, we will use an alternative one, due to Aizicovici–Papageorgiou–Staicu ([4], Proposition 2).

So, let

$$\partial B_1^{L^p} = \{x \in L^p(\Omega) \colon \|x\|_p = 1\}$$

and set

$$S = W_n^{1,p}(\Omega) \cap \partial B_1^{L^p}.$$

Then we have (see [4]):

Proposition 7. *If*

$$\Gamma_0 = \{\gamma \in C([-1, 1], S) \colon \gamma(-1) = -\hat{u}_0,\ \gamma(1) = \hat{u}_0\},$$

then

$$\lambda_1 = \inf_{\gamma \in \Gamma_0} \max_{t \in [-1,1]} \|D\gamma(t)\|_p^p.$$

We will also use the notions of *upper and lower solutions*, which we recall next.

DEFINITION 8. (a) A function $u \in W^{1,p}(\Omega)$ is said to be an *upper solution* for problem (1.5) if

$$\int_\Omega \|Du\|_{\mathbb{R}^N}^{p-2}(Du, Dh)_{\mathbb{R}^N}\, dz \geq \int_\Omega f(z, u)h\, dz \quad \text{for all} \quad h \in W_n^{1,p}(\Omega),\ h \geq 0.$$

An upper solution is a *strict upper solution* for problem (1.5), if it is not a solution of (1.5).

(b) A function $u \in W^{1,p}(\Omega)$ is said to be a *lower solution* for problem (1.5) if

$$\int_\Omega \|Du\|_{\mathbb{R}^N}^{p-2}(Du, Dh)_{\mathbb{R}^N}\, dz \leq \int_\Omega f(z, u)h\, dz \quad \text{for all} \quad h \in W_n^{1,p}(\Omega),\ h \geq 0.$$

A lower solution is a *strict lower solution* for problem (1.5), if it is not a solution of (1.5).

Finally let us recall some basic definitions and facts from Morse theory, which we will use in the sequel.

The critical groups of $\varphi \in C^1(X)$ at an isolated critical point $x_0 \in X$ with $\varphi(x_0) = c$ are defined by

$$C_k(\varphi, x_0) = H_k(\varphi^c \cap U, (\varphi^c \cap U) \setminus \{x_0\}) \quad \text{for all integers} \quad k \geq 0.$$

Here U is a neighborhood of x_0 such that $K_\varphi \cap \varphi^c \cap U = \{x_0\}$ and $H_k(V, W)$ denotes the k^{th}-singular homology group with coefficients in $\mathbb{Z}$ for the topological pair (U, W) (cf. Mawhin–Willem [25]). The excision property of singular homology implies that the above definition of critical groups is independent of the choice of the neighborhood U.

Suppose that φ satisfies the C-condition and $\inf\{\varphi(x) : x \in K_\varphi\} > -\infty$. Let $c < \inf\{\varphi(x) : x \in K_\varphi\}$. The *critical groups of φ at infinity* are defined by

$$C_k(\varphi, \infty) = H_k(X, \varphi^c) \quad \text{for all integers} \quad k \geq 0.$$

The deformation theorem (see, for example, Gasinski–Papageorgiou [18, p. 626]) implies that this definition is independent of the choice of $c < \inf\{\varphi(x) : x \in K_\varphi\}$.

If K_φ is finite, then the *Morse type numbers of φ* are defined by

$$M_k = \sum_{x \in K_\varphi} \operatorname{rank} C_k(\varphi, x)$$

and the *Betti-type numbers of φ* are defined by

$$\beta_k = \operatorname{rank} C_k(\varphi, \infty)$$

for all integers $k \geq 0$. Then the *Poincaré–Hopf formula* holds, namely

$$\sum_{k \geq 0} (-1)^k M_k = \sum_{k \geq 0} (-1)^k \beta_k. \tag{2.5}$$

In what follows we use the notation $r^{\pm} = \max\{\pm r, 0\}$ for all $r \in \mathbb{R}$. Also, by $\|\cdot\|$ we denote the norm of $W_n^{1,p}(\Omega)$. Finally, $\|\cdot\|_p$ denotes the norm in $L^p(\Omega)$ or $L^p(\Omega, \mathbb{R}^N)$.

3. Solutions of constant sign

In this section, we produce four nontrivial smooth solutions of constant sign for problem (1.5) (two positive and two negative). Here and throughout the remainder of the paper we let $p \in (1, \infty)$.

The hypotheses on the nonlinearity $f(z, x)$ are the following:
$\mathbf{H}(f)_1$: The function $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is such that:
(i) for every $x \in \mathbb{R}$, $z \to f(z, x)$ is measurable;
(ii) for almost all $z \in \Omega$, $x \to f(z, x)$ is continuous and $f(z, 0) = 0$;
(iii) for almost all $z \in \Omega$ and all $x \in \mathbb{R}$ we have

$$|f(z, x)| \leq a(z) + c|x|^{r-1},$$

where $a \in L^\infty(\Omega)_+$, $c > 0$ and $p < r < p^*$, where p^* is given by (2.2);
(iv) if $F(z, x) = \int_0^x f(z, s)\, ds$, then

$$\lim_{|x|\to\infty} \frac{F(z, x)}{|x|^p} = +\infty, \quad \text{uniformly for a.a. } z \in \Omega$$

and there exist $\mu \in ((r - p)\max\{1, N/p\}, p^*)$ and $\beta_0 > 0$ such that

$$\liminf_{|x|\to\infty} \frac{f(z, x)x - pF(z, x)}{|x|^\mu} \geq \beta_0 \quad \text{uniformly for a.a. } z \in \Omega; \tag{3.1}$$

(v) there exist $\eta, \eta_1 \in L^\infty(\Omega)_+$, $\eta \neq 0$ such that

$$\eta(z) \leq \liminf_{x\to 0} \frac{f(z, x)}{|x|^{p-2}x} \leq \limsup_{x\to 0} \frac{f(z, x)}{|x|^{p-2}x} \leq \eta_1(z) \quad \text{uniformly for a.a. } z \in \Omega;$$

(vi) there exist $\xi_- < 0 < \xi_+$, and $\theta, \beta > 0$ such that for a.a. $z \in \Omega$

$$f(z, \xi_+) \leq -\theta < 0 < \theta \leq f(z, \xi_-)$$

and the function $x \to f(z, x) + \beta|x|^{p-2}x$ is nondecreasing on $[\xi_-, \xi_+]$.

REMARK. In hypothesis $\mathbf{H}(f)_1$ (iv) we have assumed condition (1.4) which dictates a p-superlinear growth for $F(z, \cdot)$ for a.a. $z \in \Omega$, and we have also imposed condition (3.1) which is weaker than the AR-condition (see (1.2)). Similar conditions were used by Costa–Magalhães [12] (for Dirichlet elliptic equations) and Fei [15] (for Hamiltonian systems).

EXAMPLE. The following function $f : \mathbb{R} \to \mathbb{R}$, $f = f(x)$ satisfies $\mathbf{H}(f)_1$. (For the sake of simplicity we drop the z-dependence.):

$$f(x) = \begin{cases} |x|^{p-2}x - \xi|x|^{r-2}x & \text{if} \quad |x| \leq 1, \\ |x|^{p-2}x\left(\ln|x| + \dfrac{1}{p}\right) + c & \text{if} \quad |x| > 1 \end{cases}$$

with $\xi > 1$, $c = (p - 1)/p - \xi$, $r > p$. Note that this $f(\cdot)$ does not satisfy the AR-condition.

First we will produce a strict lower solution $\underline{u} \in \operatorname{int} C_+$ and a strict upper solution $\overline{v} \in -\operatorname{int} C_+$ (see Definition 8). To this end, we prove two auxiliary lemmata which are of independent interest. So, let

$$V = \left\{ u \in W_n^{1,p}(\Omega) \colon \int_\Omega u(z)\,dz = 0 \right\}.$$

We have the following direct sum decomposition

$$W_n^{1,p}(\Omega) = \mathbb{R} \oplus V.$$

We set

$$\lambda_V = \inf \left\{ \frac{\|Du\|_p^p}{\|u\|_p^p} : u \in V,\ u \neq 0 \right\}. \tag{3.2}$$

Lemma 9. $0 < \lambda_V \leq \lambda_1$.

Proof. Evidently $\lambda_V \geq 0$ (see (3.2)). If $\lambda_V = 0$, then we can find $\{v_n\}_{n \geq 1} \subset V$ such that

$$\|v_n\|_p = 1 \quad \text{and} \quad \|Dv_n\|_p \to \lambda_V = 0 \quad \text{as} \quad n \to \infty.$$

Hence we may assume that

$$v_n \to \xi \quad \text{in} \quad W_n^{1,p}(\Omega) \quad \text{with} \quad \xi \in \mathbb{R},\ \|\xi\|_p = \xi |\Omega|_N^{1/p} = 1.$$

Since $v_n \in V$ for all $n \geq 1$ and V is a closed subspace of $W_n^{1,p}(\Omega)$, we have $\xi \in V$ and so $\xi = 0$, a contradiction to the fact that $\|\xi\|_p = 1$. Therefore $\lambda_V > 0$.

Next let $\gamma_0 \in \Gamma_0 = \{\gamma \in C([-1,1], S) \colon \gamma(-1) = -\hat{u}_0,\ \gamma(1) = \hat{u}_0\}$ and consider the function $\sigma_0 \colon [-1,1] \to \mathbb{R}$ defined by

$$\sigma_0(t) = \int_\Omega \gamma_0(t)(z)\,dz \quad \text{for all} \quad t \in [-1,1].$$

Evidently $\sigma_0(\,\cdot\,)$ is continuous and $\sigma_0(-1) = -\hat{u}_0|\Omega|_N < 0 < \sigma_0(1) = \hat{u}_0|\Omega|_N$. So, by Bolzano's theorem, we can find $t_0 \in (-1,1)$ such that

$$\sigma_0(t_0) = \int_\Omega \gamma_0(t_0)(z)\,dz = 0,$$

hence $\sigma_0(t_0) \in V$. Consequently from (3.2) we infer that

$$\lambda_V \leq \|D\gamma_0(t_0)\|_p^p \leq \max_{-1 \leq t \leq 1} \|D\gamma_0(t)\|_p^p. \tag{3.3}$$

Because $\gamma_0 \in \Gamma_0$ was arbitrary, from (3.3) and Proposition 7, we conclude that $\lambda_V \leq \lambda_1$. $\square$

Recall that the antimaximum principle says that, if $m, h \in L^{\infty}(\Omega)_+$, $h \neq 0$ and we consider the nonlinear Neumann problem

$$-\triangle_p u(z) = \hat{\lambda} m(z)|u(z)|^{p-2}u(z) - h(z) \quad \text{in} \quad \Omega, \quad \frac{\partial u}{\partial n} = 0 \quad \text{on} \quad \partial\Omega, \tag{3.4}$$

then there exists $\delta = \delta(m, h) > 0$ such that if $\hat{\lambda} \in (0, \delta)$, any solution $u \in W_n^{1,p}(\Omega)$ of (3.4) satisfies $u \in \operatorname{int} C_+$ (see Godoy–Gossez–Paczka [19]).

In general, no such $\delta > 0$ independent of h can be found. In the next lemma, we show that the antimaximum principle for the Neumann p-Laplacian holds L^{∞}-locally uniformly with respect to the weight function m (i.e., $\delta > 0$ can be chosen independent of m locally).

Lemma 10. *If $m, h \in L^{\infty}(\Omega)_+$, $h \neq 0$, then there exists $\delta > 0$ such that for $\xi \in L^{\infty}(\Omega)_+$ with $\|\xi - m\|_{\infty} \leq \delta$ and for $\hat{\lambda} \in (0, \delta)$, any solution $u \in W_n^{1,p}(\Omega)$ of the nonlinear Neumann problem*

$$-\triangle_p u(z) = \hat{\lambda}\xi(z)|u(z)|^{p-2}u(z) - h(z) \quad \textit{in} \quad \Omega, \quad \frac{\partial u}{\partial n} = 0 \quad \textit{on} \quad \partial\Omega,$$

satisfies $u \in \operatorname{int} C_+$,

Proof. We argue by contradiction. So, suppose we can find $\xi_n \in L^{\infty}(\Omega)_+ \setminus \{0\}$, $\hat{\lambda}_n > 0$ and $u_n \in C_n^1(\overline{\Omega})$, $n \geq 1$ such that $\xi_n \to m$ in $L^{\infty}(\Omega)_+$, $\hat{\lambda}_n \to 0$ as $n \to \infty$, and for all $n \geq 1$, $u_n \notin \operatorname{int} C_+$,

$$-\triangle_p u_n(z) = \hat{\lambda}_n \xi_n(z)|u_n(z)|^{p-2}u_n(z) - h(z) \quad \text{in} \quad \Omega, \quad \frac{\partial u_n}{\partial n} = 0 \quad \text{on} \quad \partial\Omega. \tag{3.5}$$

First assume that $\{u_n\}_{n\in\mathbb{N}} \subset L^{\infty}(\Omega)$ is bounded. Invoking Theorem 2 of Lieberman [24], we can find $\gamma \in (0,1)$ such that $u_n \in C_n^{1,\gamma}(\overline{\Omega})$ and $\{u_n\}_{n\in\mathbb{N}} \subset C_n^{1,\gamma}(\overline{\Omega})$ is bounded.

Recalling that $C_n^{1,\gamma}(\overline{\Omega})$ is embedded compactly in $C_n^1(\overline{\Omega})$, by passing to a suitable subsequence if necessary, we may assume that $u_n \to u$ in $C_n^1(\overline{\Omega})$ as $n \to \infty$.Therefore $u \in C_n^1(\overline{\Omega})$ satisfies

$$-\triangle_p u(z) = -h(z) \quad \text{in} \quad \Omega, \quad \frac{\partial u}{\partial n} = 0 \quad \text{on} \quad \partial\Omega, \tag{3.6}$$

(see (3.5)).

But problem (3.6) cannot have a solution (just take $\theta \equiv 1$ as test function in (3.6). So, we may assume (at least for a subsequence) that $\|u_n\|_{\infty} \to \infty$ as $n \to \infty$. Let $y_n = u_n/\|u_n\|_{\infty}$, $n \geq 1$. Then from (3.5) we see that

$$-\triangle_p y_n(z) = \hat{\lambda}_n \xi_n(z)|y_n(z)|^{p-2}y_n(z) - \frac{h(z)}{\|u_n\|_{\infty}^{p-1}} \quad \text{in} \quad \Omega, \quad \frac{\partial y_n}{\partial n} = 0 \quad \text{on} \quad \partial\Omega. \tag{3.7}$$

As above, using Theorem 2 of Lieberman [24] and by passing to a further subsequence if necessary, we may assume that

$$(3.8)\qquad y_n \to y \quad \text{in} \quad C_n^1(\overline{\Omega}) \quad \text{as} \quad n \to \infty, \quad \|y\|_\infty = 1 \quad (\text{hence } y \neq 0).$$

Then $y \in C_n^1(\overline{\Omega}) \setminus \{0\}$ satisfies

$$-\triangle_p y(z) = 0 \quad \text{in} \quad \Omega, \quad \frac{\partial y}{\partial n} = 0 \quad \text{on} \quad \partial\Omega,$$

hence

$$y = \beta \in \mathbb{R} \setminus \{0\}.$$

Note that $y_n \notin \operatorname{int} C_+$ for all $n \geq 1$, hence $y \notin \operatorname{int} C_+$ (see (3.8) and so $y(z) = \beta < 0$ for all $z \in \overline{\Omega}$. Therefore $y_n \in -\operatorname{int} C_+$ for all $n \geq n_0$ (see (3.8)) and so $u_n \in -\operatorname{int} C_+$ for all $n \geq n_0$ (for a sufficiently large n_0). But then from (3.5) we have a contradiction of the antimaximum principle (see Godoy–Gossez–Paczka [19], Theorem 3.2 and Remark 3.7). $\square$

Using the above two lemmata, we can produce a strict lower solution $\underline{u} \in \operatorname{int} C_+$ for problem (1.5).

Proposition 11. *If hypotheses* $\mathbf{H}(f)_1$ *hold, then problem* (1.5) *has a strict lower solution* $\underline{u} \in \operatorname{int} C_+$, $\underline{u}(z) \leq \xi_+$ *for all* $z \in \overline{\Omega}$, *and for every* $\varepsilon \in (0, 1]$, $\varepsilon\underline{u} \in \operatorname{int} C_+$ *is a strict lower solution, too.*

Proof. Let $m = 0$, $h = \hat{u}_0^{p-1} \in \mathbb{R}$ and consider $\delta > 0$ as postulated by Lemma 10. We can always assume that $\delta \in (0, \lambda_V)$. Let $\xi \in L^\infty(\Omega)_+ \setminus \{0\}$ with $0 \leq \xi(z) \leq \min\{\delta^2/2, \eta(z)\}$ a.e. in Ω. We consider the following auxiliary nonlinear Neumann problem

$$(3.9)\qquad -\triangle_p u(z) = \xi(z)|u(z)|^{p-2}u(z) - \hat{u}_0^{p-1} \quad \text{in} \quad \Omega, \quad \frac{\partial u}{\partial n} = 0 \quad \text{on} \quad \partial\Omega.$$

Let $\varphi_0 \colon W_n^{1,p}(\Omega) \to \mathbb{R}$ be the energy (Euler) functional for problem (3.9) defined by

$$\varphi_0(u) = \frac{1}{p}\|Du\|_p^p - \frac{1}{p}\int_\Omega \xi(z)|u(z)|^p\,dz + \hat{u}_0^{p-1}\int_\Omega u(z)\,dz \quad \text{for all} \quad u \in W_n^{1,p}(\Omega).$$

Evidently $\varphi_0 \in C^1(W_n^{1,p}(\Omega))$.

Claim 1. φ_0 *satisfies the* PS-*condition.*

Let $\{u_n\}_{n\geq 1} \subset W_n^{1,p}(\Omega)$ be a sequence such that

$$|\varphi_0(u_n)| \leq M_1 \quad \text{for some} \quad M_1 > 0, \quad \text{all} \quad n \geq 1, \tag{3.10}$$

and

$$\varphi_0'(u_n) \to 0 \quad \text{in} \quad W_n^{1,p}(\Omega)^* \quad \text{as} \quad n \to \infty. \tag{3.11}$$

We show that the sequence $\{u_n\}_{n\geq 1} \subset W_n^{1,p}(\Omega)$ is bounded. Arguing indirectly, suppose that $\|u_n\| \to \infty$ as $n \to \infty$ and set

$$y_n = \frac{u_n}{\|u_n\|}, \quad n \geq 1.$$

Then $\|y_n\| = 1$ for all $n \geq 1$ and so we may assume that

$$y_n \xrightarrow{w} y \quad \text{in} \quad W_n^{1,p}(\Omega) \quad \text{and} \quad y_n \to y \quad \text{in} \quad L^p(\Omega) \quad \text{as} \quad n \to \infty. \tag{3.12}$$

From (3.11) we have

$$\left| \langle A(y_n), y_n - y\rangle - \int_\Omega \xi |y_n|^{p-2} y_n (y_n - y)\, dz + \frac{\hat{u}_0^{p-1}}{\|u_n\|^{p-1}} \int_\Omega (y_n - y)\, dz \right|$$
$$\leq \varepsilon_n \|y_n - y\| \quad \text{for all} \quad n \geq 1 \quad \text{with} \quad \varepsilon_n \to 0^+.$$

Evidently

$$\int_\Omega \xi |y_n|^{p-2} y_n (y_n - y)\, dz, \quad \int_\Omega (y_n - y)\, dz \to 0 \quad \text{as} \quad n \to \infty$$

(see (3.12)). Hence

$$\lim_{n\to\infty} \langle A(y_n), y_n - y\rangle = 0$$

therefore

$$y_n \to y \quad \text{in} \quad W_n^{1,p}(\Omega) \quad \text{as} \quad n \to \infty, \quad \text{and so} \quad \|y\| = 1 \tag{3.13}$$

(see Proposition 5). From (3.11) we have

$$\left| \langle A(y_n), h\rangle - \int_\Omega \xi |y_n|^{p-2} y_n h\, dz + \frac{\hat{u}_0^{p-1}}{\|u_n\|^{p-1}} \int_\Omega h\, dz \right| \leq \varepsilon_n \|h\| \quad \text{for all} \quad h \in W_n^{1,p}(\Omega).$$

Passing to the limit as $n \to \infty$ and using (3.13), we obtain

$$\langle A(y), h\rangle = \int_\Omega \xi |y|^{p-2} y h\, dz \quad \text{for all} \quad h \in W_n^{1,p}(\Omega),$$

which, by an argument similar to that used in [27, pp. 24–25], for a Neumann problem with a p-normal derivative, yields

$$-\triangle_p y(z) = \xi(z)|y(z)|^{p-2}y(z) \quad \text{a.e. in} \quad \Omega, \quad \frac{\partial y}{\partial n} = 0 \quad \text{on} \quad \partial\Omega. \tag{3.14}$$

From the choice of the weight function $\xi \in L^\infty(\Omega)_+ \setminus \{0\}$ we have

$$\xi(z) < \lambda_V \le \lambda_1 \quad \text{a.e. in} \quad \Omega$$

(see Lemma 9). Exploiting the monotonicity property of the weighted eigenvalues of the Neumann p-Laplacian, we have $\hat{\lambda}_1(\lambda_1) = 1 < \hat{\lambda}_1(\xi)$ (see Barletta–Papageorgiou [6, Proposition 4.3]). Using this fact in (3.14)), we infer that $y = 0$, a contradiction (see (3.13)). This proves that the sequence $\{u_n\}_{n\ge1} \subset W_n^{1,p}(\Omega)$ is bounded. So, we may assume that

$$u_n \xrightarrow{w} u \quad \text{in} \quad W_n^{1,p}(\Omega) \quad \text{and} \quad u_n \to u \quad \text{in} \quad L^p(\Omega) \quad \text{as} \quad n \to \infty. \tag{3.15}$$

Again from (3.11) it follows

$$\left|\langle A(u_n), u_n - u\rangle - \int_\Omega \xi|u_n|^{p-2}u_n(u_n-u)\,dz + \hat{u}_0^{p-1}\int_\Omega (u_n-u)\,dz\right|$$
$$\le \varepsilon_n\|u_n-u\| \quad \text{for all} \quad n \ge 1.$$

We have

$$\int_\Omega \xi|u_n|^{p-2}u_n(u_n-u)\,dz, \quad \int_\Omega (u_n-u)\,dz \to 0 \quad \text{as} \quad n \to \infty$$

(see (3.15)), and so

$$\lim_{n\to\infty}\langle A(u_n), u_n-u\rangle = 0,$$

therefore

$$u_n \to u \quad \text{in} \quad W_n^{1,p}(\Omega) \quad \text{as} \quad n \to \infty.$$

This proves Claim 1.

Claim 2. $\varphi_0|_V \ge 0$.

Let $v \in V$. Then

$$\begin{aligned}\varphi_0(v) &= \frac{1}{p}\|Dv\|_p^p - \frac{1}{p}\int_\Omega \xi(z)|v(z)|^p\,dz + \hat{u}_0^{p-1}\int_\Omega v(z)\,dz \\ &= \frac{1}{p}\|Dv\|_p^p - \frac{1}{p}\int_\Omega \xi(z)|v(z)|^p\,dz \quad (\text{since } v \in V)\end{aligned}$$

$$\geq \frac{1}{p}\|Dv\|_p^p - \frac{\lambda_V}{p}\|v\|_p^p \quad (\text{since } \xi(z) < \lambda_V \text{ a.e. in } \Omega)$$
$$\geq 0 \quad (\text{see } (3.2)).$$

This proves Claim 2.

Claim 3. *For $t > 0$ large, we have $\varphi_0(\pm t\hat{u}_0) < 0$.*

For every $t \in \mathbb{R}$, we have

$$\varphi_0(t\hat{u}_0) = -\frac{|t|^p}{p}\hat{u}_0^p \int_\Omega \xi(z)\,dz + t\hat{u}_0^p|\Omega|_N. \tag{3.16}$$

Since $p > 1$, it is clear from (3.16) that for $t > 0$ large, we have

$$\varphi_0(\pm t\hat{u}_0) < 0.$$

Claims 1, 2 and 3 permit the use of Theorem 2 (the saddle point theorem) and so we obtain $\tilde{u} \in W_n^{1,p}(\Omega)$, $\tilde{u} \neq 0$ such that

$$\varphi_0'(\tilde{u}) = A(\tilde{u}) - \xi|\tilde{u}|^{p-2}\tilde{u} + \hat{u}_0^{p-1} = 0,$$

hence

$$A(\tilde{u}) = \xi|\tilde{u}|^{p-2}\tilde{u} - \hat{u}_0^{p-1},$$

therefore

$$-\triangle_p\tilde{u}(z) = \xi(z)|\tilde{u}(z)|^{p-2}\tilde{u}(z) - \hat{u}_0^{p-1} \quad \text{a.e. in} \quad \Omega, \quad \frac{\partial\tilde{u}}{\partial n} = 0 \quad \text{on} \quad \partial\Omega, \tag{3.17}$$

(see [27]). From the choice of the weight function $\xi \in L^\infty(\Omega)_+ \setminus \{0\}$ and Lemma 10, it follows that $\tilde{u} \in \operatorname{int} C_+$. Since $\hat{u}_0 \in \operatorname{int} C_+$, we can find $\varepsilon_0 \in (0, 1)$ small such that

$$\hat{u}_0^{p-1} - \varepsilon_0\tilde{u}^{p-1} \in \operatorname{int} C_+. \tag{3.18}$$

By virtue of hypothesis $\mathbf{H}(f)_1(v)$, given $\varepsilon \in (0, \varepsilon_0)$, there exists $\hat{\delta} = \hat{\delta}(\varepsilon) > 0$ such that

$$f(z, x) \geq (\eta(z) - \varepsilon)x^{p-1} \quad \text{for a.a.} \quad z \in \Omega, \quad \text{all} \quad x \in [0, \hat{\delta}]. \tag{3.19}$$

Since $\tilde{u} \in \operatorname{int} C_+$, we can choose $\tilde{\beta} \in (0, 1)$ small such that

$$0 < \underline{u}(z) := \tilde{\beta}\tilde{u}(z) \leq \min\{\xi_+, \hat{\delta}\} \quad \text{for all} \quad z \in \overline{\Omega}. \tag{3.20}$$

Then for a.a. $z \in \Omega$, we have

$$\begin{aligned}
-\triangle_p \underline{u}(z) &= \tilde{\beta}^{p-1}(-\triangle_p \tilde{u}(z)) = \tilde{\beta}^{p-1}[\xi(z)\tilde{u}(z)^{p-1} - \hat{u}_0^{p-1}] \quad \text{(see (3.17))} \\
&= \xi(z)\underline{u}(z)^{p-1} - (\tilde{\beta}\hat{u}_0)^{p-1} \\
&< \xi(z)\underline{u}(z)^{p-1} - \varepsilon_0 \underline{u}(z)^{p-1} \quad \text{(see (3.18))} \\
&< (\xi(z) - \varepsilon)\underline{u}(z)^{p-1} \quad (\text{since } \varepsilon \in (0, \varepsilon_0)) \\
&\le (\eta(z) - \varepsilon)\underline{u}(z)^{p-1} \quad (\text{since } \xi(z) \le \eta(z) \text{ a.e. in } \Omega) \\
&\le f(z, \underline{u}(z)) \quad \text{(see (3.19) and (3.20))},
\end{aligned}$$

hence $\underline{u} \in \operatorname{int} C_+$ is a strict lower solution for problem (1.5). Moreover, from the above argument it is clear that for every $\varepsilon \in (0, 1)$, $\varepsilon\underline{u} \in \operatorname{int} C_+$ is also a strict lower solution for problem (1.5). □

In a similar way, working on the negative half-axis, we obtain

Proposition 12. *If hypotheses* $\mathbf{H}(f)_1$ *hold, then problem* (1.5) *has a strict upper solution* $\overline{v} \in -\operatorname{int} C_+$, $\xi_- \le \overline{v}(z)$ *for all* $z \in \overline{\Omega}$, *and for every* $\varepsilon \in (0, 1]$, $\varepsilon\overline{v} \in -\operatorname{int} C_+$ *is a strict upper solution, too.*

Next using $\underline{u} \in \operatorname{int} C_+$ and $\overline{v} \in -\operatorname{int} C_+$, we will produce the first two nontrivial, smooth constant sign solutions of (1.5).

Proposition 13. *If hypotheses* $\mathbf{H}(f)_1$ *hold, then problem* (1.5) *has at least two nontrivial smooth constant sign solutions* $u_0 \in \operatorname{int} C_+$ *with* $u_0 - \underline{u} \in \operatorname{int} C_+$, $\xi_+ - u_0 \in \operatorname{int} C_+$, *and* $v_0 \in -\operatorname{int} C_+$ *with* $\overline{v} - v_0 \in \operatorname{int} C_+$, $v_0 - \xi_- \in \operatorname{int} C_+$

Proof. Let $\tau_+ \colon W_n^{1,p}(\Omega) \to W_n^{1,p}(\Omega)$ be the continuous map defined by

$$\tau_+(u)(z) = \begin{cases} \underline{u}(z) & \text{if} \quad u(z) \le \underline{u}(z), \\ u(z) & \text{if} \quad \underline{u}(z) < u(z) < \xi_+, \\ \xi_+ & \text{if} \quad \xi_+ \le u(z). \end{cases} \tag{3.21}$$

Then, for $\varepsilon \in (0, 1)$ we consider the functional $\varphi_+^\varepsilon \colon W_n^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\varphi_+^\varepsilon(u) = \frac{1}{p}\|Du\|_p^p + \frac{\varepsilon}{p}\|u\|_p^p - \int_\Omega F(z, u(z))\,dz - \frac{\varepsilon}{p}\|\tau_+(u)\|_p^p \quad \text{for all} \quad u \in W_n^{1,p}(\Omega).$$

Note that $\varphi_+^\varepsilon \in C^1(W_n^{1,p}(\Omega))$. Moreover, exploiting the compact embedding of $W_n^{1,p}(\Omega)$ into $L^r(\Omega)$ $(p < r < p^*)$, we can easily verify that φ_+^ε is sequentially weakly lower semicontinuous. Then we conclude that there exists $u_0 \in [\underline{u}, \xi_+] := \{u \in W_n^{1,p}(\Omega) \colon \underline{u}(z) \le$

$u(z) \le \xi_+$ a.e. in $\Omega\}$ such that

$$\varphi_+^{\varepsilon}(u_0) = \inf\{\varphi_+^{\varepsilon}(u) \colon u \in [\underline{u}, \xi_+]\}. \tag{3.22}$$

For any $y \in [\underline{u}, \xi_+]$ let

$$\sigma_+(t) = \varphi_+^{\varepsilon}(ty + (1-t)u_0) \quad \text{for all} \quad t \in [0,1].$$

From (3.22) it follows that

$$0 \le \sigma_+'(0)$$

hence

$$0 \le \langle A(u_0), y - u_0\rangle - \int_\Omega f(z, u_0)(y - u_0)\, dz. \tag{3.23}$$

Let $h \in W_n^{1,p}(\Omega)$, $\delta > 0$ and consider

$$y(z) = \begin{cases} \underline{u}(z) & \text{if} \quad z \in \{u_0 + \delta h \le \underline{u}\}, \\ u_0(z) + \delta h(z) & \text{if} \quad z \in \{\underline{u} < u_0 + \delta h < \xi_+\}, \\ \xi_+ & \text{if} \quad z \in \{\xi_+ \le u_0 + \delta h\}. \end{cases}$$

We have $y \in W_n^{1,p}(\Omega)$ and $\underline{u}(z) \le y(z) \le \xi_+$ for all $z \in \Omega$. We use y as a test function in (3.23). We obtain:

$$\begin{aligned} 0 \le{}& \delta \int_\Omega \|Du_0\|_{\mathbb{R}^N}^{p-2} \langle Du_0, Dh\rangle_{\mathbb{R}^N}\, dz - \delta \int_\Omega f(z, u_0) h\, dz \\ &+ \int_{\{u_0 + \delta h \le \underline{u}\}} \Big[\|D\underline{u}\|_{\mathbb{R}^N}^{p-2} \langle D\underline{u}, D(\underline{u} - u_0 - \delta h)\rangle_{\mathbb{R}^N} - f(z, \underline{u})(\underline{u} - u_0 - \delta h)\Big]\, dz \\ &+ \int_{\{\xi_+ \le u_0 + \delta h\}} f(z, \xi_+)(u_0 + \delta h - \xi_+)\, dz \\ &+ \int_{\{u_0 + \delta h \le \underline{u}\}} (f(z, \underline{u}) - f(z, u_0))(\underline{u} - u_0 - \delta h)\, dz \\ &+ \int_{\{\xi_+ \le u_0 + \delta h\}} (f(z, \xi_+) - f(z, u_0))(\xi_+ - u_0 - \delta h)\, dz \\ &- \int_{\{u_0 + \delta h \le \underline{u}\}} \Big\langle \|Du_0\|_{\mathbb{R}^N}^{p-2} Du_0 - \|D\underline{u}\|_{\mathbb{R}^N}^{p-2} D\underline{u}, Du_0 - D\underline{u}\Big\rangle_{\mathbb{R}^N}\, dz \\ &- \delta \int_{\{u_0 + \delta h \le \underline{u}\}} \Big\langle \|Du_0\|_{\mathbb{R}^N}^{p-2} Du_0 - \|D\underline{u}\|_{\mathbb{R}^N}^{p-2} D\underline{u}, Dh\Big\rangle_{\mathbb{R}^N}\, dz \\ &- \int_{\{\xi_+ \le u_0 + \delta h\}} \|Du_0\|_{\mathbb{R}^N}^{p}\, dz - \delta \int_{\{\xi_+ \le u_0 + \delta h\}} \|Du_0\|_{\mathbb{R}^N}^{p-2} \langle Du_0, Dh\rangle_{\mathbb{R}^N}\, dz. \end{aligned} \tag{3.24}$$

Recall that $\underline{u} \in \operatorname{int} C_+$ is a lower solution for problem (1.5) (see Proposition 11). Hence

$$\int_{\{u_0+\delta h \le \underline{u}\}} \left[\|D\underline{u}\|_{\mathbb{R}^N}^{p-2} \langle D\underline{u}, D(\underline{u} - u_0 - \delta h)\rangle_{\mathbb{R}^N} - f(z, \underline{u})(\underline{u} - u_0 - \delta h) \right] dz \le 0. \tag{3.25}$$

Also, hypothesis $\mathbf{H}(f)_1$ (vi), implies that

$$\int_{\{\xi_+ \le u_0+\delta h\}} f(z, \xi_+)(u_0 + \delta h - \xi_+)\, dz \le 0. \tag{3.26}$$

Due to the monotonicity of the map $\xi \to \|\xi\|_{\mathbb{R}^N}^{p-2}\xi$, $\xi \in \mathbb{R}^N$, we have

$$-\int_{\{u_0+\delta h \le \underline{u}\}} \left\langle \|Du_0\|_{\mathbb{R}^N}^{p-2} Du_0 - \|D\underline{u}\|_{\mathbb{R}^N}^{p-2} D\underline{u}, Du_0 - D\underline{u} \right\rangle_{\mathbb{R}^N} dz \le 0. \tag{3.27}$$

Hypothesis $\mathbf{H}(f)_1$ (iii) implies that

$$\begin{aligned} &\int_{\{u_0+\delta h \le \underline{u}\}} (f(z, \underline{u}) - f(z, u_0))(\underline{u} - u_0 - \delta h)\, dz \\ &\le -c_1 \delta \int_{\{u_0+\delta h \le \underline{u} < u_0\}} h\, dz \quad \text{for some} \quad c_1 > 0 \end{aligned} \tag{3.28}$$

and

$$\begin{aligned} &\int_{\{\xi_+ \le u_0+\delta h\}} (f(z, \xi_+) - f(z, u_0))(\xi_+ - u_0 - \delta h)\, dz \\ &\le c_2 \delta \int_{\{u_0 < \xi_+ \le u_0+\delta h\}} h\, dz \quad \text{for some} \quad c_2 > 0. \end{aligned} \tag{3.29}$$

We return to (3.24) and use (3.25)–(3.29). We obtain

$$\begin{aligned} 0 \le &\int_\Omega \|Du_0\|_{\mathbb{R}^N}^{p-2} \langle Du_0, Dh\rangle_{\mathbb{R}^N}\, dz - \int_\Omega f(z, u_0) h\, dz \\ &- c_1 \int_{\{u_0+\delta h \le \underline{u} < u_0\}} h\, dz + c_2 \int_{\{u_0 < \xi_+ \le u_0+\delta h\}} h\, dz \\ &- \int_{\{u_0+\delta h \le \underline{u}\}} \langle \|Du_0\|_{\mathbb{R}^N}^{p-2} Du_0 - \|D\underline{u}\|_{\mathbb{R}^N}^{p-2} D\underline{u}, Dh\rangle_{\mathbb{R}^N}\, dz \\ &- \int_{\{\xi_+ \le u_0+\delta h\}} \|Du_0\|_{\mathbb{R}^N}^{p-2} \langle Du_0, Dh\rangle_{\mathbb{R}^N}\, dz. \end{aligned} \tag{3.30}$$

In (3.30) we pass to the limit as $\delta \to 0^+$. Using Stampacchia's theorem (see, for example Gasinski–Papageorgiou [18, p. 195]) we obtain

$$0 \le \int_\Omega \|Du_0\|_{\mathbb{R}^N}^{p-2} \langle Du_0, Dh\rangle_{\mathbb{R}^N}\, dz - \int_\Omega f(z, u_0) h\, dz \quad \text{for all} \quad h \in W_n^{1,p}(\Omega),$$

hence $A(u_0) = N(u_0)$, where $N(u)(\cdot) = f(\cdot, u(\cdot))$ for all $u \in W_n^{1,p}(\Omega)$, therefore

$$-\triangle_p u_0(z) = f(z, u_0(z)) \quad \text{a.e. in} \quad \Omega, \quad \frac{\partial u_0}{\partial n} = 0 \quad \text{on} \quad \partial\Omega$$

(see [27]), and we conclude that u_0 solves (1.5), $u_0 \in [\underline{u}, \xi_+]$ and $u_0 \in \operatorname{int} C_+$ (by Theorem 2 of Lieberman [24]).

Let $s > 0$ and set $y_s(z) = \underline{u}(z) + s$. Note that $y_s \in \operatorname{int} C_+$. Let $\beta > 0$ be as in hypothesis $\mathbf{H}(f)_1$ (vi). We have

$$\begin{aligned} &-\triangle_p y_s(z) + \beta y_s(z)^{p-1} \\ &= -\triangle_p \underline{u}(z) + \beta \underline{u}(z)^{p-1} + \rho(s) \quad \text{with} \quad \rho(s) \to 0^+ \quad \text{as} \quad s \to 0^+ \\ &= \xi(z)\underline{u}(z)^{p-1} - (\tilde{\beta}\hat{u}_0)^{p-1} + \beta\underline{u}(z)^{p-1} + \rho(s) \end{aligned} \tag{3.31}$$

(see the proof of Proposition 11). Recalling that $\underline{u}(z) \in [0, \hat{\delta}]$ for all $z \in \overline{\Omega}$ and using (3.19), we have

$$\begin{aligned} &f(z, \underline{u}(z)) - \xi(z)\underline{u}(z)^{p-1} + (\tilde{\beta}\hat{u}_0)^{p-1} \\ &\geq (\eta(z) - \varepsilon)\underline{u}(z)^{p-1} - \xi(z)\underline{u}(z)^{p-1} + (\tilde{\beta}\hat{u}_0)^{p-1} \\ &\geq (\tilde{\beta}\hat{u}_0)^{p-1} - \varepsilon\underline{u}(z)^{p-1} \quad (\text{since } \xi \leq \eta). \end{aligned} \tag{3.32}$$

We choose $\varepsilon \in (0, \varepsilon_0)$ close to ε_0 so that

$$(\tilde{\beta}\hat{u}_0)^{p-1} - \varepsilon\underline{u}^{p-1} \in \operatorname{int} C_+ \quad (\text{see (3.18)}),$$

hence

$$(\tilde{\beta}\hat{u}_0)^{p-1} - \varepsilon\underline{u}(z)^{p-1} \geq \tilde{\xi} > 0 \quad \text{for all} \quad z \in \overline{\Omega}. \tag{3.33}$$

We choose $s > 0$ small such that $\rho(s) < \tilde{\xi}$ (recall $\rho(s) \to 0^+$ as $s \to 0^+$). Then

$$\begin{aligned} &\xi(z)\underline{u}(z)^{p-1} - (\tilde{\beta}\hat{u}_0)^{p-1} + \beta\underline{u}(z)^{p-1} + \rho(s) \\ &< f(z, \underline{u}(z)) + \beta\underline{u}(z)^{p-1} \quad (\text{by (3.32), (3.33) and since } \rho(s) < \tilde{\xi} \text{ for } s > 0 \text{ small}) \\ &\leq f(z, u_0(z)) + \beta u_0(z)^{p-1} \quad (\text{see } \mathbf{H}(f)_1 \text{ (vi) and recall that } \underline{u} \leq u_0) \\ &= -\triangle_p u_0(z) + \beta u_0(z)^{p-1} \quad \text{a.e. on } \Omega, \end{aligned}$$

hence

$$-\triangle_p y_s(z) + \beta y_s(z)^{p-1} \leq -\triangle_p u_0(z) + \beta u_0(z)^{p-1} \quad \text{in} \quad \Omega \quad (\text{see (3.31)}),$$

therefore

$$y_s(z) \leq u_0(z),$$

and we conclude that

$$u_0 - \underline{u} \in \operatorname{int} C_+ \quad \text{(recall that } y_s = \underline{u} + s,\ s > 0.) \tag{3.34}$$

Also, if $\tau > 0$ and $y_\tau = u_0 + \tau \in \operatorname{int} C_+$, then

$$\begin{aligned} &-\triangle_p y_\tau(z) + \beta y_\tau(z)^{p-1} \\ &= -\triangle_p u_0(z) + \beta u_0(z)^{p-1} + \hat{\rho}(\tau) \quad \text{with} \quad \hat{\rho}(\tau) \to 0^+ \quad \text{as} \quad \tau \to 0^+ \\ &= f(z, u_0(z)) + \beta u_0(z)^{p-1} + \hat{\rho}(\tau) \\ &\le f(z, \xi_+) + \beta \xi_+^{p-1} + \hat{\rho}(\tau) \quad \text{a.e. in} \quad \Omega \end{aligned}$$

(see $\mathbf{H}(f)_1$ (vi) and recall that $u_0 \le \xi_+$). But we know that $f(z, \xi_+) \le -\theta < 0$ for a.a. $z \in \Omega$ (see $\mathbf{H}(f)_1$ (vi)). Since $\hat{\rho}(\tau) \to 0^+$ as $\tau \to 0^+$, we choose $\tau > 0$ small such that $\hat{\rho}(\tau) \le \theta$. Then

$$-\triangle_p y_\tau(z) + \beta y_\tau(z)^{p-1} \le \beta \xi_+^{p-1} = -\triangle_p \xi_+ + \beta u_0(z)^{p-1} \quad \text{a.e. in } \Omega,$$

hence $y_\tau \le \xi_+$, therefore

$$\xi_+ - u_0 \in \operatorname{int} C_+ \quad \text{(recall that } y_\tau = u_0 + \tau,\ \tau > 0). \tag{3.35}$$

As a remark of independent interest, we note that by virtue of (3.34) and (3.35), we can find $r_0 > 0$ small such that

$$\overline{B}_{r_0}^{C_n^1(\overline{\Omega})} := \left\{ u \in C_n^1(\overline{\Omega}) \colon \|u - u_0\|_{C_n^1(\overline{\Omega})} \le r_0 \right\} \subseteq [\underline{u}, \xi_+]. \tag{3.36}$$

So, if $\varphi \colon W_n^{1,p}(\Omega) \to \mathbb{R}$ is the C^1-energy functional for problem (1.5) defined by

$$\varphi(u) = \frac{1}{p} \|Du\|_p^p - \int_\Omega F(z, u(z))\, dz \quad \text{for all} \quad u \in W_n^{1,p}(\Omega),$$

then

$$\varphi|_{\overline{B}_{r_0}^{C_n^1(\overline{\Omega})}} = \varphi_+^\varepsilon|_{\overline{B}_{r_0}^{C_n^1(\overline{\Omega})}}$$

(see (3.21) and (3.36)). This means that $u_0 \in [\underline{u}, \xi_+]$ is a local $C_n^1(\overline{\Omega})$-minimizer of φ, and so from Proposition 6 we infer that u_0 is also a local $W_n^{1,p}(\Omega)$-minimizer of φ

Similarly, let $\tau_- \colon W_n^{1,p}(\Omega) \to W_n^{1,p}(\Omega)$ be the truncation map defined by

$$\tau_-(u)(z) = \begin{cases} \xi_- & \text{if} \quad u(z) \le \xi_-, \\ u(z) & \text{if} \quad \xi_- < u(z) < \overline{v}(z), \\ \overline{v}(z) & \text{if} \quad \overline{v}(z) \le u(z). \end{cases} \tag{3.37}$$

Then for $\varepsilon \in (0, 1)$ we consider the C^1-functional $\varphi_-^\varepsilon \colon W_n^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\varphi_-^\varepsilon(u) = \frac{1}{p}\|Du\|_p^p + \frac{\varepsilon}{p}\|u\|_p^p - \int_\Omega F(z, u(z))\,dz - \frac{\varepsilon}{p}\|\tau_-(u)\|_p^p \quad \text{for all } u \in W_n^{1,p}(\Omega).$$

Working now with φ_-^ε and using (3.37), as above we obtain a second nontrivial smooth constant sign solution $v_0 \in -\operatorname{int} C_+$. $\square$

Next, using u_0 and v_0, we will generate two additional smooth constant sign solutions for problem (1.5).

Proposition 14. *If hypothesis* $\mathbf{H}(f)_1$ *hold, then problem* (1.5) *has two additional nontrivial smooth constant sign solutions* $\hat{u} \in \operatorname{int} C_+$, $u_0 \le \hat{u}$, $\hat{u} \ne u_0$ *and* $\hat{v} \in -\operatorname{int} C_+$, $\hat{v} \le v_0$, $\hat{v} \ne v_0$.

Proof. Let $\beta > 0$ be as in hypothesis $\mathbf{H}(f)_1$ (vi) and consider the following Caratheodory function

$$\overline{f}_+^\beta(z, x) = \begin{cases} f(z, u_0(z)) + \beta u_0^{p-1}(z) & \text{if} \quad x \le u_0(z), \\ f(z, x) + \beta x^{p-1} & \text{if} \quad u_0(z) < x. \end{cases} \tag{3.38}$$

We set $\overline{F}_+^\beta(z, x) = \int_0^x \overline{f}_+^\beta(z, s)\,ds$ and introduce the C^1-functional $\overline{\varphi}_+^\beta \colon W_n^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\overline{\varphi}_+^\beta(u) = \frac{1}{p}\|Du\|_p^p + \frac{\beta}{p}\|u\|_p^p - \int_\Omega \overline{F}_+^\beta(z, u(z))\,dz \quad \text{for all} \quad u \in W_n^{1,p}(\Omega).$$

We consider the following auxiliary nonlinear Neumann problem

$$-\triangle_p u(z) + \beta|u(z)|^{p-2}u(z) = \overline{f}_+^\beta(z, u(z)) \quad \text{a.e. in} \quad \Omega, \quad \partial u/\partial n = 0 \quad \text{on} \quad \partial\Omega. \tag{3.39}$$

The critical points of $\overline{\varphi}_+^\beta$ are the solutions of (3.39). By virtue of hypothesis $\mathbf{H}(f)_1(vi)$ and (3.38), $u = 0$ is a lower solution for problem (3.39). Also

$$\overline{f}_+^\beta(z, \xi_+) = f(z, \xi_+) + \beta\xi_+^{p-1} < \beta\xi_+^{p-1} \quad \text{for a.a.} \quad z \in \Omega$$

(see hypothesis $\mathbf{H}(f)_1$ (vi)), hence $\xi_+ \in \operatorname{int} C_+$ is a (strict) upper solution for problem (3.39). We introduce the following truncation of the nonlinearity $\overline{f}_+^\beta(z, x)$:

$$g_+^\beta(z, x) = \begin{cases} \overline{f}_+^\beta(z, 0) & \text{if} \quad x \le 0, \\ \overline{f}_+^\beta(z, x) & \text{if} \quad 0 < x < \xi_+, \\ \overline{f}_+^\beta(z, \xi_+) & \text{if} \quad \xi_+ \le x. \end{cases} \tag{3.40}$$

This is a Caratheodory function. We set $G_+^{\beta}(z,x) = \int_0^x g_+^{\beta}(z,s)\,ds$ and consider the C^1-functional $\psi_\beta colon W_n^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\psi_\beta(u) = \frac{1}{p}\|Du\|_p^p + \frac{\beta}{p}\|u\|_p^p - \int_\Omega G_+^{\beta}(z,u(z))\,dz \quad \text{for all} \quad u \in W_n^{1,p}(\Omega).$$

It is clear from (3.40) that ψ_β is coercive and it is also sequentially weakly lower semicontinuous. Hence, we can find $\tilde{u}_0 \in W_n^{1,p}(\Omega)$ such that

$$\psi_\beta(\tilde{u}_0) = \{\psi_\beta(u) \colon u \in W_n^{1,p}(\Omega)\},$$

hence

$$\psi_\beta'(\tilde{u}_0) = 0,$$

therefore

$$A(\tilde{u}_0) + \beta|\tilde{u}_0|^{p-2}\tilde{u}_0 = N_\beta(\tilde{u}_0) \tag{3.41}$$

where

$$N_\beta(u)(\,\cdot\,) = g_+^{\beta}(\,\cdot\,, u(\,\cdot\,)) \quad \text{for all} \quad u \in W_n^{1,p}(\Omega).$$

On (3.41) we act with $(u_0 - \tilde{u}_0)^+ \in W_n^{1,p}(\Omega)$ and obtain

$$\begin{aligned}
&\langle A(\tilde{u}_0), (u_0 - \tilde{u}_0)^+\rangle + \beta \int_{\{u_0 > \tilde{u}_0\}} |\tilde{u}_0|^{p-2}\tilde{u}_0(u_0 - \tilde{u}_0)\,dz \\
&= \int_{\{u_0 > \tilde{u}_0\}} g_+^{\beta}(z,\tilde{u}_0)(u_0 - \tilde{u}_0)\,dz \\
&= \int_{\{u_0 > \tilde{u}_0\}} (f(z,u_0) + \beta u_0^{p-1})(u_0 - \tilde{u}_0)\,dz \quad \text{(see (3.40) and (3.38))} \\
&= \langle A(u_0), (u_0 - \tilde{u}_0)^+\rangle + \beta \int_{\{u_0 > \tilde{u}_0\}} u_0^{p-1}(u_0 - \tilde{u}_0)\,dz,
\end{aligned}$$

hence

$$\langle A(\tilde{u}_0) - A(u_0), (u_0 - \tilde{u}_0)^+\rangle + \beta \int_\Omega (|\tilde{u}_0|^{p-2}\tilde{u}_0 - |u_0|^{p-2}u_0)(u_0 - \tilde{u}_0)^+\,dz = 0,$$

which implies

$$|\{u_0 > \tilde{u}_0\}|_N = 0, \quad \text{hence} \quad u_0 \le \tilde{u}_0.$$

In a similar fashion we also show that $\tilde{u}_0(z) \le \xi_+$ for a.a. $z \in \overline{\Omega}$, i.e., $\tilde{u}_0 \in [u_0, \xi_+]$. Hence from (3.41), (3.40) and (3.38), we infer that $\tilde{u}_0$ is a solution of (1.5), and nonlinear regularity implies that $\tilde{u}_0 \in \operatorname{int} C_+$ (see Lieberman [24]).

If $\tilde{u}_0 \neq u_0$, then we are done, because this is the desired second nontrivial smooth positive solution of (1.5) and $u_0 \leq \tilde{u}_0$, $u_0 \neq \tilde{u}_0$.

If $\tilde{u}_0 = u_0$, then because $u_0 \in \operatorname{int} C_+$ and $\xi_+ - u_0 \in \operatorname{int} C_+$ (see (3.35)), we see that $\tilde{u}_0 = u_0$ is a $C^1_n(\overline{\Omega})$-local minimizer of $\overline{\varphi}^\beta_+$, hence by Proposition 6 it is also a local $W^{1,p}_n(\Omega)$-local minimizer of $\overline{\varphi}^\beta_+$. Then reasoning as in Aizicovici–Papageorgiou–Staicu [1] (see the proof of Proposition 29), we can find $\rho \in (0,1)$ small such that

$$\overline{\varphi}^\beta_+(u_0) < \inf\{\overline{\varphi}^\beta_+(u) \colon \|u - u_0\| = \rho\} = \overline{\eta}^+_\rho. \tag{3.42}$$

Claim 1. $\overline{\varphi}^\beta_+$ *satisfies the* C-*condition.*

Let $\{u_n\}_{n\geq 1} \subset W^{1,p}_n(\Omega)$ be a sequence such that

$$|\overline{\varphi}^\beta_+(u_n)| \leq M_2 \quad \text{for some} \quad M_2 > 0, \quad \text{all} \quad n \geq 1, \tag{3.43}$$

and

$$(1 + \|u_n\|)(\overline{\varphi}^\beta_+)'(u_n) \to 0 \quad \text{in} \quad W^{1,p}_n(\Omega)^* \quad \text{as} \quad n \to \infty. \tag{3.44}$$

We show that $\{u_n\}_{n\geq 1} \subset W^{1,p}_n(\Omega)$ is bounded. From (3.44) we have

$$\begin{aligned} &\left| \langle A(u_n), h\rangle + \beta \int_\Omega |u_n|^{p-2} u_n h \, dz - \int_\Omega \overline{f}^\beta_+(z, u_n) h \, dz \right| \\ &\leq \frac{\varepsilon_n}{1 + \|u_n\|} \|h\| \quad \text{for all} \quad h \in W^{1,p}_n(\Omega) \quad \text{with} \quad \varepsilon_n \to 0^+. \end{aligned} \tag{3.45}$$

In (3.45) we first choose $h = -u_n^- \in W^{1,p}_n(\Omega)$ to obtain

$$\begin{aligned} &\left| \|Du_n^-\|_p^p + \beta \|u_n^-\|_p^p - \int_\Omega (f(z, u_0) + \beta u_0^{p-1})(-u_n^-) \, dz \right| \\ &\leq \varepsilon_n \quad \text{for all } n \geq 1 \end{aligned}$$

(see (3.38)), hence

$$\|u_n^-\|^p \leq c_3 \|u_n^-\| \quad \text{for some } c_3 > 0, \text{ all } n \geq 1,$$

therefore

$$\{u_n^-\}_{n\geq 1} \subset W^{1,p}_n(\Omega) \quad \text{is bounded.} \tag{3.46}$$

Next, in (3.45) we choose $h = u_n^+ \in W_n^{1,p}(\Omega)$. Then

$$\begin{aligned} &-\|Du_n^+\|_p^p - \beta\|u_n^+\|_p^p + \int_{\{u_n^+ \le u_0\}} f(z,u_0)u_n^+\,dz \\ &+ \int_{\{u_n^+ > u_0\}} f(z,u_n^+)u_n^+\,dz + \beta\int_{\{u_n^+ \le u_0\}} u_0^{p-1}u_n^+\,dz + \beta\int_{\{u_n^+ > u_0\}} (u_n^+)^p\,dz \\ &\le \varepsilon_n \quad \text{for all } n \ge 1 \text{ (see (3.38)).} \end{aligned} \tag{3.47}$$

On the other hand, from (3.43) and (3.47), we have

$$\begin{aligned} &\|Du_n^+\|_p^p + \beta\|u_n^+\|_p^p - p\int_{\{u_n^+ \le u_0\}} f(z,u_0)u_n^+\,dz \\ &- p\int_{\{u_n^+ > u_0\}} (F(z,u_n^+) - F(z,u_0))\,dz - \beta p\int_{\{u_n^+ \le u_0\}} u_0^{p-1}u_n^+\,dz \\ &- \beta\int_{\{u_n^+ > u_0\}} ((u_n^+)^p - u_0^p)\,dz \\ &\le M_3 \quad \text{for some } M_3 > 0, \text{ all } n \ge 1. \end{aligned} \tag{3.48}$$

Adding (3.47) and (3.48), we obtain

$$\begin{aligned} &\int_{\{u_n^+ > u_0\}} (f(z,u_n^+)u_n^+ - pF(z,u_n^+))\,dz \\ &\le c_4 + (p-1)\int_{\{u_n^+ \le u_0\}} (f(z,u_0) + \beta u_0^{p-1})u_n^+\,dz \\ &\le c_5 \quad \text{for some } c_4, c_5 > 0, \text{ all } n \ge 1. \end{aligned} \tag{3.49}$$

By virtue of hypothesis $\mathbf{H}(f)_1$ (iv), we can find $\beta_1 \in (0, \beta_0)$ and $M_4 > 0$ such that

$$0 < \beta_1 x^\mu \le f(z,x)x - pF(z,x) \quad \text{for a.a. } z \in \Omega, \quad \text{all } x \ge M_4. \tag{3.50}$$

On the other hand, hypothesis $\mathbf{H}(f)_1$ (iii) implies that

$$\begin{aligned} &|f(z,x)x - pF(z,x)| \le M_5 \quad \text{for some } M_5 > 0, \\ &\text{for a.a. } z \in \Omega, \quad \text{all } 0 \le x < M_4. \end{aligned} \tag{3.51}$$

Combining (3.50) and (3.51), we have

$$\begin{aligned} &\beta_1(u_n^+(z))^\mu - M_5 \le f(z,u_n^+(z))u_n^+(z) - pF(z,u_n^+(z)) \\ &\text{for a.a. } z \in \{u_n^+ > u_0\}, \quad \text{all } n \ge 1 \quad \text{and some } M_6 > 0. \end{aligned} \tag{3.52}$$

Returning to (3.49) and using (3.52), we obtain

$$\beta_1 \|u_n^+\|_\mu^\mu \le M_7, \quad \text{for some a.a.} \quad M_7 > 0, \quad \text{all} \quad n \ge 1,$$

hence

$$\{u_n^+\}_{n\ge1} \subset L^\mu(\Omega) \quad \text{is bounded.} \tag{3.53}$$

We can always assume that $\mu \le r < p^*$ (see hypothesis $\mathbf{H}(f)_1$ (iv)). So we can find $t \in [0, 1)$ such that

$$\frac{1}{r} = \frac{1-t}{\mu} + \frac{t}{p^*}. \tag{3.54}$$

Invoking the interpolation inequality (see, for example, Gasinski–Papageorgiou [18, p. 905]), we have

$$\begin{aligned} \|u_n^+\|_r &\le \|u_n^+\|_\mu^{1-t} \|u_n^+\|_{p^*}^t \\ &\le M_8 \|u_n^+\|^t \quad \text{for some} \quad M_8 > 0, \quad \text{all} \quad n \ge 1, \quad \text{(see (3.53)).} \end{aligned} \tag{3.55}$$

In case $p = N$, hence $p^* = \infty$, we replace p^* in (3.54) and (3.55) by $\hat{p}$ where $\hat{p} > r$ is large enough.

From (3.45) with $h = u_n^+ \in W_n^{1,p}(\Omega)$ and (3.46) we have

$$\|Du_n^+\|_p^p - \int_{\{u_n^+ > u_0\}} f(z, u_n^+) u_n^+ \, dz \le M_9 \quad \text{for some} \quad M_9 > 0, \quad \text{all} \quad n \ge 1. \tag{3.56}$$

Hypothesis $\mathbf{H}(f)_1$ (iii) implies that

$$\begin{aligned} &f(z, u_n^+(z)) u_n^+(z) \le c_6 (1 + |u_n^+(z)|^r) \\ &\text{for a.a.} \quad z \in \Omega, \quad \text{all} \quad n \ge 1 \quad \text{and some} \quad c_6 > 0. \end{aligned} \tag{3.57}$$

We use (3.57) in (3.56) and we obtain

$$\begin{aligned} \|Du_n^+\|_p^p &\le c_7 (1 + \|u_n^+(z)\|_r^r) \\ &\le c_8 (1 + \|u_n^+(z)\|^{tr}) \quad \text{for some} \quad c_7, c_8 > 0, \quad \text{all} \quad n \ge 1 \end{aligned} \tag{3.58}$$

(see (3.55)). Suppose that $\|u_n^+\| \to \infty$ as $n \to \infty$. We set

$$y_n = \frac{u_n^+}{\|u_n\|}, \quad \text{for all} \quad n \ge 1.$$

Then $\|y_n\| = 1$ for all $n \ge 1$, and so we may assume that

$$y_n \xrightarrow{w} y \quad \text{in} \quad W_n^{1,p}(\Omega) \quad \text{and} \quad y_n \to y \quad \text{in} \quad L^r(\Omega) \quad \text{as} \quad n \to \infty. \tag{3.59}$$

From (3.58) we have

$$\|Dy_n\|_p^p \le \frac{c_8}{\|u_n^+\|^p} + \frac{c_8}{\|u_n^+\|^{p-tr}}. \tag{3.60}$$

The condition on μ (see hypothesis $\mathbf{H}(f)_1$ (iv)) is equivalent to saying that $tr < p$. So, if in (3.60) we pass to the limit as $n \to \infty$, then $\|Dy\|_p = 0$ (see (3.59)), hence $y = \alpha \in \mathbb{R}$.

If $\alpha = 0$, then $y_n \to 0$ in $W_n^{1,p}(\Omega)$ as $n \to \infty$, a contradiction to the fact that $\|y_n\| = 1$ for all $n \ge 1$.

If $\alpha > 0$ (recall that $y \ge 0$), then $u_n^+(z) \to \infty$ for a.a. $z \in \Omega$, as $n \to \infty$. From (3.43) and (3.46) we have

$$\int_{\{u_n^+ > u_0\}} \frac{F(z, u_n^+)}{\|u_n^+\|^p}\, dz \le M_{10}\left(1 + \frac{1}{\|u_n^+\|^p}\right) \quad \text{for some} \quad M_{10} > 0, \quad \text{all} \quad n \ge 1. \tag{3.61}$$

From hypothesis $\mathbf{H}(f)_1$ (iv), we know that given $\gamma > 0$, we can find $M_{11} = M_{11}(\gamma) > 0$ such that

$$\frac{F(z, x)}{x^p} \ge \gamma > 0 \quad \text{for a.a.} \quad z \in \Omega, \quad \text{all} \quad x \ge M_{11}. \tag{3.62}$$

Returning to (3.61) and using (3.62), we have

$$\begin{aligned}
&\int_{\{u_n^+ > u_0\}} \frac{F(z, u_n^+)}{\|u_n^+\|^p}\, dz \\
&= \int_{\{u_n^+ > u_0\} \cap \{u_n^+ \ge M_{11}\}} \frac{F(z, u_n^+)}{\|u_n^+\|^p}\, dz + \int_{\{u_n^+ > u_0\} \cap \{u_n^+ < M_{11}\}} \frac{F(z, u_n^+)}{\|u_n^+\|^p}\, dz \\
&\ge \int_{\{u_n^+ > u_0\} \cap \{u_n^+ \ge M_{11}\}} \gamma y_n(z)^p\, dz - \frac{M_{12}}{\|u_n^+\|^p} \quad \text{for some} \quad M_{12} > 0, \quad \text{all} \quad n \ge 1.
\end{aligned}$$

Since $u_n^+(z) \to \infty$ for a.a. $z \in \Omega$ as $n \to \infty$, we have

$$\chi_{\{u_n^+ > u_0\} \cap \{u_n^+ \ge M_{11}\}}(z) \to \chi_\Omega(z) \quad \text{for a.a.} \quad z \in \Omega,$$

hence

$$\liminf_{n\to\infty} \int_{\{u_n^+ > u_0\}} \frac{F(z, u_n^+)}{\|u_n^+\|^p}\, dz \ge \gamma \alpha^p |\Omega|_N. \tag{3.63}$$

Because $\gamma > 0$ was arbitrary, from (3.63) we infer that

$$\lim_{n\to\infty} \int_{\{u_n^+ > u_0\}} \frac{F(z, u_n^+)}{\|u_n^+\|^p}\, dz = +\infty. \tag{3.64}$$

Comparing (3.61) and (3.64), we reach a contradiction. This proves that $\{u_n^+\}_{n\geq 1} \subset W_n^{1,p}(\Omega)$ is bounded, which combined with (3.46) implies that $\{u_n\}_{n\geq 1} \subset W_n^{1,p}(\Omega)$ is bounded. Hence we may assume that

$$u_n \xrightarrow{w} u \quad \text{in} \quad W_n^{1,p}(\Omega) \quad \text{and} \quad u_n \to u \quad \text{in} \quad L^r(\Omega) \quad \text{as} \quad n \to \infty. \tag{3.65}$$

If in (3.45) we choose $h = u_n - u \in W_n^{1,p}(\Omega)$ and then pass to the limit as $n \to \infty$, then

$$\lim_{n\to\infty} \langle A(u_n), u_n - u\rangle = 0 \quad \text{(see (3.65))}$$

hence

$$u_n \to u \quad \text{in} \quad W_n^{1,p}(\Omega) \quad \text{as} \quad n \to \infty \quad \text{(see Proposition 5),}$$

therefore $\overline{\varphi}_+^\beta$ satisfies the C-condition. This proves Claim 1.

Claim 2. $\overline{\varphi}_+^\beta(\xi) \to -\infty$ *as* $\xi \to +\infty$, $\xi \in \mathbb{R}$.

We may assume that $\xi \in \mathbb{R}$, $\xi > |u_0|_\infty$. Then

$$\begin{aligned}\overline{\varphi}_+^\beta(\xi) = \frac{\beta}{p}\xi^p|\Omega|_N - \int_\Omega f(z,u_0)u_0\,dz - \int_\Omega (F(z,\xi) - F(z,u_0))\,dz \\ - \beta\int_\Omega u_0^p\,dz - \frac{\beta}{p}\int_\Omega (\xi^p - u_0^p)\,dz \quad \text{(see (3.38)),}\end{aligned}$$

hence $\overline{\varphi}_+^\beta(\xi) \to -\infty$ as $\xi \to +\infty$ (see hypothesis $\mathbf{H}(f)_1$ (iv)). This proves Claim 2.

Then (3.42) and Claims 1 and 2 permit the use of Theorem 1 (the mountain pass theorem), which yields $\hat{u} \in W_n^{1,p}(\Omega)$ such that

$$\overline{\varphi}_+^\beta(u_0) < \overline{\eta}_\rho^+ \leq \overline{\varphi}_+^\beta(\hat{u}) \tag{3.66}$$

and

$$(\overline{\varphi}_+^\beta)'(\hat{u}) = 0. \tag{3.67}$$

From (3.66) it follows that $\hat{u} \neq u_0$. From (3.67) we see that

$$A(\hat{u}) + \beta|\hat{u}|^{p-2}\hat{u} = \overline{N}_\beta(\hat{u}) \tag{3.68}$$

where

$$\overline{N}_\beta(u)(\cdot) = \overline{f}_+^\beta(\cdot, u(\cdot)) \quad \text{for all} \quad u \in W_n^{1,p}(\Omega).$$

Acting on (3.68) with $(u_0 - \hat{u})^+ \in W_n^{1,p}(\Omega)$ and using (3.38), as before we show that $\hat{u} \geq u_0$. Hence (3.68) becomes

$$A(\hat{u}) = N(\hat{u}),$$

where $N(u)(\,\cdot\,) = f(\,\cdot\,, u(\,\cdot\,))$ for all $u \in W_n^{1,p}(\Omega)$, hence $\hat{u} \in \operatorname{int} C_+$, $\hat{u} \geq u_0$, $\hat{u} \neq u_0$ and $\hat{u}$ is a solution of (1.5) (see [27]).

In a similar fashion, using this time the Caratheodory function

$$\overline{f}_-^{\beta}(z,x) = \begin{cases} f(z,x) + \beta|x|^{p-2}x & \text{if} \quad x < v_0(z), \\ f(z, v_0(z)) + \beta|v_0(z)|^{p-2}v_0(z) & \text{if} \quad x \geq v_0(z) \end{cases}$$

we obtain a second nontrivial smooth negative solution $\hat{v} \in -\operatorname{int} C_+$, $\hat{v} \leq v_0$, $\hat{v} \neq v_0$. $\square$

Next we will produce extremal nontrivial smooth solutions of constant sign (i.e., the smallest positive and the biggest negative solutions). To do this, we will need the following lemma from Aizicovici–Papageorgiou–Staicu [2].

Lemma 15. *If hypotheses* $\mathbf{H}(f)_1$ (i), (ii), (iii) *hold, then*:
(a) *If* $y_1, y_2 \in W^{1,p}(\Omega)$ *are upper solutions for problem* (1.5), *then* $y = \min\{y_1, y_2\} \in W^{1,p}(\Omega)$ *is an upper solution, too*;
(b) *If* $w_1, w_2 \in W^{1,p}(\Omega)$ *are lower solutions for problem* (1.5), *then* $w = \max\{w_1, w_2\} \in W_n^{1,p}(\Omega)$ *is a lower solution, too.*

Using this lattice-type structure of the sets of upper and lower solutions of problem (1.5), we can produce extremal nontrivial smooth solutions of constant sign.

Proposition 16. *If hypotheses* $\mathbf{H}(f)_1$ *hold, then problem* (1.5) *has a smallest positive solution* $u_+ \in \operatorname{int} C_+$ *and a biggest negative solution* $v_- \in -\operatorname{int} C_+$.

Proof. Let $\underline{u} \in \operatorname{int} C_+$ be the lower solution produced in Proposition 11. We first show that problem (1.5) has a smallest solution bigger than $\underline{u}$. To this end let

$$\mathcal{S}_{\underline{u}} = \{u \in C_+ : \underline{u} \leq u,\ u \text{ is a solution of (1.5)}\}.$$

From Proposition 13 we see that $\mathcal{S}_{\underline{u}} \neq \emptyset$. We show that $\mathcal{S}_{\underline{u}}$ is downward directed, i.e., if $u_1, u_2 \in \mathcal{S}_{\underline{u}}$, then there exists $u \in \mathcal{S}_{\underline{u}}$ such that $u \leq \min\{u_1, u_2\}$.

By virtue of Lemma 15 (a), $\tilde{u} = \min\{u_1, u_2\} \in W_n^{1,p}(\Omega) \cap C(\overline{\Omega})$ is an upper solution for problem (1.5). For $\varepsilon \in (0,1)$ we introduce the following Caratheodory function

$$\tilde{f}_+^{\varepsilon}(z,x) = \begin{cases} f(z, \underline{u}(z)) + \varepsilon \underline{u}(z)^{p-1} & \text{if} \quad x \leq \underline{u}(z), \\ f(z,x) + \varepsilon x^{p-1} & \text{if} \quad \underline{u}(z) < x < \tilde{u}(z), \\ f(z, \tilde{u}(z)) + \varepsilon \tilde{u}(z)^{p-1} & \text{if} \quad \tilde{u}(z) \leq x. \end{cases} \tag{3.69}$$

We set $\tilde{F}_+^{\varepsilon}(z,x) = \int_0^x \tilde{f}_+^{\varepsilon}(z,s)\,ds$ and consider the C^1-functional $\tilde{\varphi}_+^{\varepsilon}\colon W_n^{1,p}(\Omega) \to \mathbb{R}$

defined by

$$\tilde{\varphi}_+^{\varepsilon}(u) = \frac{1}{p}\|Du\|_p^p + \frac{\varepsilon}{p}\|u\|_p^p - \int_\Omega \tilde{F}_+^{\varepsilon}(z, u(z))\,dz \quad \text{for all} \quad u \in W_n^{1,p}(\Omega).$$

The functional $\tilde{\varphi}_+^{\varepsilon}$ is coercive (see (3.69)) and sequentially weakly lower semicontinuous. So, we can find $\tilde{u}_0 \in [\underline{u}, \tilde{u}]$ such that

$$\tilde{\varphi}_+^{\varepsilon}(\tilde{u}_0) = \inf\{\tilde{\varphi}_+^{\varepsilon}(u) \colon u \in [\underline{u}, \tilde{u}]\}. \tag{3.70}$$

For any $y \in [\underline{u}, \tilde{u}]$, let

$$\tilde{\sigma}_+(t) = \tilde{\varphi}_+^{\varepsilon}(ty + (1-t)\tilde{u}_0), \quad t \in [0, 1].$$

From (3.70) we have

$$0 \le \tilde{\sigma}'_+(0),$$

hence

$$0 \le \langle A(\tilde{u}_0), y - \tilde{u}_0\rangle + \varepsilon \int_\Omega \tilde{u}_0^{p-1}(y - \tilde{u}_0)\,dz - \int_\Omega \tilde{f}_+^{\varepsilon}(z, \tilde{u}_0)(y - \tilde{u}_0)\,dz. \tag{3.71}$$

Let $h \in W_n^{1,p}(\Omega)$, $\delta > 0$ and consider

$$y(z) = \begin{cases} \underline{u}(z) & \text{if} \quad z \in \{\tilde{u}_0 + \delta h \le \underline{u}\}, \\ \tilde{u}_0(z) + \delta h(z) & \text{if} \quad z \in \{\underline{u} < \tilde{u}_0 + \delta h < \tilde{u}\}, \\ \tilde{u}(z) & \text{if} \quad z \in \{\tilde{u} \le \tilde{u}_0 + \delta h\}. \end{cases}$$

Evidently $y \in [\underline{u}, \tilde{u}]$. Using this y in (3.71) and reasoning as in the proof of Proposition 13, we obtain

$$0 \le \int_\Omega \|D\tilde{u}_0\|^{p-2}\langle D\tilde{u}_0, Dh\rangle_{\mathbb{R}^N}\,dz + \varepsilon \int_\Omega \tilde{u}_0^{p-1}h\,dz - \int_\Omega \tilde{f}_+^{\varepsilon}(z, \tilde{u}_0)h\,dz. \tag{3.72}$$

Since $h \in W_n^{1,p}(\Omega)$ is arbitrary, from (3.72) we infer that

$$A(\tilde{u}_0) + \varepsilon\tilde{u}_0^{p-1} = \tilde{N}_\varepsilon(\tilde{u}_0) \quad \text{where} \quad \tilde{N}_\varepsilon(u)(\,\cdot\,) = \tilde{f}_+^{\varepsilon}(\,\cdot\,, \tilde{u}_0(\,\cdot\,)) \quad \text{for all} \quad h \in W_n^{1,p}(\Omega),$$

hence

$$A(\tilde{u}_0) = N(\tilde{u}_0) \quad (\text{since } \tilde{u}_0 \in [\underline{u}, \tilde{u}], \text{ see (3.69)}),$$

therefore

$$-\triangle_p\tilde{u}_0(z) = f(z, \tilde{u}_0(z)) \quad \text{a.e. in} \quad \Omega, \quad \frac{\partial\tilde{u}_0}{\partial n} = 0 \quad \text{on} \quad \partial\Omega \quad (\text{see (3.17)}).$$

We conclude that $\tilde{u}_0 \in \operatorname{int} C_+$ (by nonlinear regularity theory, see [24]) and it solves problem (1.5). Therefore, $\tilde{u}_0 \in \mathcal{S}_{\underline{u}}$, $\tilde{u}_0 \le \tilde{u} = \min\{u_1, u_2\}$ which proves that $\mathcal{S}_{\underline{u}}$ is downward directed.

Let $C \subseteq \mathcal{S}_{\underline{u}}$ be a chain (i.e., a totally (linearly) ordered subset of $\mathcal{S}_{\underline{u}}$). From Dunford–Schwartz [14, p. 336]), we have

$$\inf C = \inf_{n \in N} u_n \quad \text{with} \quad \{u_n\}_{n \ge 1} \subseteq C.$$

We may assume that $\{u_n\}_{n\ge1}$ is decreasing (see for example, Heikkilä–Lakshmikantham [21, p.15]). So,

$$A(u_n) = N(u_n) \quad \text{and} \quad \underline{u} \le u_n \le u_1 \quad \text{for all} \quad n \ge 1. \tag{3.73}$$

From (3.73) it follows that $\{u_n\}_{n\ge1} \subseteq W_n^{1,p}(\Omega)$ is bounded. Hence, we may assume that

$$u_n \xrightarrow{w} u' \quad \text{in} \quad W_n^{1,p}(\Omega) \quad \text{and} \quad u_n \to u' \quad \text{in} \quad L^r(\Omega) \quad \text{as} \quad n \to \infty.$$

From (3.73) we have

$$\langle A(u_n), u_n - u' \rangle = \int_\Omega f(z, u_n)(u_n - u')\, dz \to 0 \quad \text{as} \quad n \to \infty,$$

hence

$$u_n \to u' \quad \text{in} \quad W_n^{1,p}(\Omega) \quad \text{as} \quad n \to \infty \quad \text{(see Proposition 5)}. \tag{3.74}$$

Passing to the limit as $n \to \infty$ in (3.73) and using (3.74), we obtain

$$A(u') = N(u')$$

hence

$$u' = \inf C \in \mathcal{S}_{\underline{u}}.$$

Invoking the Kuratowski–Zorn lemma, we infer that $\mathcal{S}_{\underline{u}}$ has a minimal element u_*. Since $\mathcal{S}_{\underline{u}}$ is downward directed, we conclude that u_* is the smallest element of $\mathcal{S}_{\underline{u}}$.

Now, let $\varepsilon_n \to 0^+$ as $n \to \infty$ with $\varepsilon_n \in (0, 1]$ for all $n \ge 1$ and set $\underline{u}_n = \varepsilon_n \underline{u}$, for $n \ge 1$. From Proposition 11 we know that for every $n \ge 1$, $\underline{u}_n \in \operatorname{int} C_+$ is a lower solution for problem (1.5). From the first part of the proof, it follows that the set $\mathcal{S}_{\underline{u}_n}$ has a smallest element $u_*^n \in \operatorname{int} C_+$. We have

$$A(u_*^n) = N(u_*^n) \quad \text{with} \quad u_*^n \le u_*^1 \quad \text{for all} \quad n \ge 1, \tag{3.75}$$

hence $\{u_*^n\}_{n\ge1} \subseteq W_n^{1,p}(\Omega)$ is bounded. Therefore, we may assume that

$$u_*^n \xrightarrow{w} u_+ \quad \text{in} \quad W_n^{1,p}(\Omega) \quad \text{and} \quad u_*^n \to u_+ \quad \text{in} \quad L^r(\Omega) \quad \text{as} \quad n \to \infty. \tag{3.76}$$

As before, acting on (3.75) with $u_*^n - u_+ \in W_n^{1,p}(\Omega)$, passing to the limit as $n \to \infty$ and using (3.76) and Proposition 5, we obtain $u_*^n \to u_+$ in $W_n^{1,p}(\Omega)$ as $n \to \infty$. So, if in (3.75) we pass to the limit as $n \to \infty$, then $A(u_+) = N(u_+)$, hence $u_+ \in C_+$ is a solution of (1.5).

We show that $u_+ \neq 0$. Suppose $u_+ = 0$ and set $y_n = u_*^n / \|u_*^n\|$, $n \geq 1$. Then $\|y_n\| = 1$ for all $n \geq 1$, and so we may assume that

$$y_n \xrightarrow{w} y \quad \text{in} \quad W_n^{1,p}(\Omega) \quad \text{and} \quad y_n \to y \quad \text{in} \quad L^p(\Omega) \quad \text{as} \quad n \to \infty.$$

From (3.75) we have

$$A(y_n) = \frac{N(u_*^n)}{\|u_*^n\|^{p-1}} \quad \text{for all} \quad n \geq 1. \tag{3.77}$$

By virtue of hypotheses $\mathbf{H}(f)_1$ (iii), (iv), we have

$$|f(z,x)| \leq c_9 |x|^{p-1} \quad \text{for a.a.} \quad z \in \Omega, \quad \text{all} \quad 0 \leq x \leq \|u_*^1\|_\infty \quad \text{with} \quad c_9 > 0. \tag{3.78}$$

From (3.78) it follows that $\{N(u_*^n)/\|u_*^n\|^{p-1}\}_{n\geq 1} \subseteq L^{p'}(\Omega)$ is bounded and so we may assume that

$$\frac{N(u_*^n)}{\|u_*^n\|^{p-1}} \xrightarrow{w} h \quad \text{in} \quad L^{p'}(\Omega) \quad \text{as} \quad n \to \infty. \tag{3.79}$$

As before, acting on (3.77) with $y_n - y \in W_n^{1,p}(\Omega)$, passing to the limit as $n \to \infty$ and using Proposition 5, we obtain

$$y_n \to y \quad \text{in} \quad W_n^{1,p}(\Omega) \quad \text{as} \quad n \to \infty, \quad \text{and so} \quad \|y\| = 1. \tag{3.80}$$

Note that $u_*^n(z) \to 0$ a.e. in Ω as $n \to \infty$. Then, using hypothesis $\mathbf{H}(f)_1$ (v) and reasoning as in the proof of Proposition 14 of Aizicovici–Papageorgiou–Staicu [1], we show that

$$h = \tilde{\eta} y^{p-1} \quad \text{with} \quad \eta \leq \tilde{\eta} \leq \eta_1. \tag{3.81}$$

So, if in (3.77) we pass to the limit as $n \to \infty$ and we use (3.79), (3.80) and (3.81), we obtain

$$A(y) = \tilde{\eta} y^{p-1}, \quad y \geq 0, \quad y \neq 0,$$

hence

$$-\triangle_p y(z) = \tilde{\eta}(z) y(z)^{p-1} \quad \text{a.e. in} \quad \Omega, \quad \frac{\partial y}{\partial n} = 0 \quad \text{on} \quad \partial\Omega, \quad y \geq 0, \ y \neq 0,$$

a contradiction, since y must be nodal. This proves that $u_+ \in C_+ \setminus \{0\}$. We have

$$-\triangle_p u_+(z) = f(z, u_+(z)) \quad \text{a.e. in} \quad \Omega, \quad \frac{\partial u_+}{\partial n} = 0 \quad \text{on} \quad \partial\Omega$$

hence

$$\triangle_p u_+(z) \le c_9 u_+(z)^{p-1} \quad \text{a.e. in} \quad \Omega \quad \text{(see (3.78))},$$

therefore

$$u_+ \in \operatorname{int} C_+ \quad \text{(see Vázquez [31])}.$$

Note that if x is another positive solution of (1.5), then $x \in \operatorname{int} C_+$ (cf. [31]), so for large n, it follows that $\underline{u}_n \le x$ and consequently $u_*^n \le x$. As a result, $u_+ \le x$, so u_+ is the smallest positive solution of (1.5).

In a similar fashion, working on the negative half-axis, and using the upper solution $\overline{v} \in -\operatorname{int} C_+$ (see Proposition 13) and Lemma 15 (b), we produce the biggest negative solution $v_- \in -\operatorname{int} C_+$ for problem (1.5). □

4. A nodal solution

In this section, using the extremal nontrivial smooth constant sign solutions obtained in Proposition 16, we will produce a nodal solution. Our strategy is to introduce suitable truncations of the nonlinearity at $\{v_-, u_+\}$, and then obtain a solution of (1.5) in $[v_-, u_+]$, distinct from v_-, u_+. Evidently, if we show that this solution is nontrivial, then it must be nodal. To show the nontriviality of this solution, we rely on Proposition 7 and Theorem 3.

To produce a nodal solution we need to strengthen the hypothesis on $f(z, \cdot)$ near zero (see $\mathbf{H}(f)_1$ (v)). So, the new hypotheses on $f(t, z)$ are the following:
$\mathbf{H}(f)_2$: The function $f : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies conditions (i), (ii), (iii), (iv) and (vi) of $\mathbf{H}(f)_1$, and
(v) there exist $\eta > \lambda_1$ and $\eta_1 \in L^\infty(\Omega)_+$ such that

$$\eta \le \liminf_{x \to 0} \frac{f(z, x)}{|x|^{p-2}x} \le \limsup_{x \to 0} \frac{f(z, x)}{|x|^{p-2}x} \le \eta_1(z) \quad \text{uniformly for a.a.} \quad z \in \Omega;$$

In other words, assumptions $\mathbf{H}(f)_2$ (i), (ii), (iii), (iv), (vi) are the same as $\mathbf{H}(f)_1$ (i), (ii), (iii), (iv), (vi), respectively, while $\mathbf{H}(f)_2$ (v) is stronger than $\mathbf{H}(f)_1$ (v).

Theorem 17. *If hypotheses* $\mathbf{H}(f)_2$ *hold, then problem* (1.5) *has at least five nontrivial smooth solutions* $u_0, \hat{u} \in \operatorname{int} C_+$, $u_0 \le \hat{u}$, $u_0 \ne \hat{u}$, $\xi_+ - u_0 \in \operatorname{int} C_+$, $v_0, \hat{v} \in -\operatorname{int} C_+$, $\hat{v} \le v_0$, $v_0 \ne \hat{v}$, $v_0 - \xi_- \in \operatorname{int} C_+$, *and* $x_0 \in C_n^1(\overline{\Omega}) \setminus \{0\}$ *nodal.*

Proof. Let $u_+ \in \operatorname{int} C_+$ and $v_- \in -\operatorname{int} C_+$ be the two extremal nontrivial smooth constant sign solutions of (1.5) obtained in Proposition 16. Let $\beta > 0$ as in hypothesis $\mathbf{H}(f)_2$ (vi) and introduce the following Caratheodory functions:

$$\hat{f}_+^\beta(z,x) = \begin{cases} 0 & \text{if} \quad x \le 0, \\ f(z,x) + \beta x^{p-1} & \text{if} \quad 0 < x < u_+(z), \\ f(z,u_+(z)) + \beta u_+(z)^{p-1} & \text{if} \quad u_+(z) \le x, \end{cases} \tag{4.1}$$

$$\hat{f}_-^\beta(z,x) = \begin{cases} f(z,v_-(z)) + \beta |v_-(z)|^{p-2} v_-(z) & \text{if} \quad x \le v_-(z), \\ f(z,x) + \beta |x|^{p-2} x & \text{if} \quad v_-(z) < x < 0, \\ 0 & \text{if} \quad 0 \le x, \end{cases} \tag{4.2}$$

$$\hat{f}^\beta(z,x) = \begin{cases} f(z,v_-(z)) + \beta |v_-(z)|^{p-2} v_-(z) & \text{if} \quad x \le v_-(z), \\ f(z,x) + \beta |x|^{p-2} x & \text{if} \quad v_-(z) < x < u_+(z), \\ f(z,u_+(z)) + \beta u_+(z)^{p-1} & \text{if} \quad u_+(z) \le x. \end{cases} \tag{4.3}$$

Let $\hat{F}_\pm^\beta(z,x) = \int_0^x \hat{f}_\pm^\beta(z,s)\,ds$ and $\hat{F}^\beta(z,x) = \int_0^x \hat{f}^\beta(z,s)\,ds$. We define the C^1-functionals $\hat{\varphi}_\pm^\beta,\ \hat{\varphi}\colon W_n^{1,p}(\Omega) \to \mathbb{R}$ by

$$\hat{\varphi}_\pm^\beta(u) = \frac{1}{p}\|Du\|_p^p + \frac{\beta}{p}\|u\|_p^p - \int_\Omega \hat{F}_\pm^\beta(z,u(z))\,dz \quad \text{for all} \quad u \in W_n^{1,p}(\Omega)$$

and

$$\hat{\varphi}^\beta(u) = \frac{1}{p}\|Du\|_p^p + \frac{\beta}{p}\|u\|_p^p - \int_\Omega \hat{F}^\beta(z,u(z))\,dz \quad \text{for all} \quad u \in W_n^{1,p}(\Omega).$$

In what follows

$$\begin{aligned} I_+ &= [0,u_+] = \{u \in W_n^{1,p}(\Omega) \colon 0 \le u(z) \le u_+(z) \text{ a.e. in } \Omega\}, \\ I_- &= [v_-,0] = \{u \in W_n^{1,p}(\Omega) \colon v_-(z) \le u(z) \le 0 \text{ a.e. in } \Omega\}, \\ I &= [v_-,u_+] = \{u \in W_n^{1,p}(\Omega) \colon v_-(z) \le u(z) \le u_+(z) \text{ a.e. in } \Omega\}. \end{aligned}$$

Claim 1. (a) *The critical points of $\hat{\varphi}_+^\beta$ are in I_+ (specifically in $\{0,u_+\}$).*
(b) *The critical points of $\hat{\varphi}_-^\beta$ are in I_- (specifically in $\{v_-,0\}$).*
(c) *The critical points of $\hat{\varphi}^\beta$ are in I.*

We do the proof for (c), the proofs of (a) and (b) being similar. So, let $u \in W_n^{1,p}(\Omega)$ be a critical point of $\hat{\varphi}^\beta$. Then

$$A(u) + \beta |u|^{p-2} u = \hat{N}_\beta(u) \quad \text{where} \quad \hat{N}_\beta(y)(\,\cdot\,) = \hat{f}^\beta(\,\cdot\,,y(\,\cdot\,)) \quad \forall y \in W_n^{1,p}(\Omega). \tag{4.4}$$

On (4.4) we act with $(u-u_+)^+ \in W_n^{1,p}(\Omega)$. Then

$$\begin{aligned}\tag{4.5} &\langle A(u),(u-u_+)^+\rangle + \beta\int_{\{u>u_+\}} |u|^{p-2}u(u-u_+)^+\,dz\\ &= \int_{\{u>u_+\}} f(z,u_+)(u-u_+)\,dz + \beta\int_{\{u>u_+\}} u_+^{p-1}(u-u_+)\,dz\end{aligned}$$

(see (4.3)). Since $u_+ \in \operatorname{int} C_+$ is a solution of problem (1.5), we have

$$\tag{4.6} -\langle A(u_+),(u-u_+)^+\rangle = -\int_{\{u>u_+\}} f(z,u_+)(u-u_+)\,dz.$$

Adding (4.5) and (4.6), we obtain

$$\begin{aligned}&\int_{\{u>u_+\}} \langle \|Du\|^{p-2}Du - \|Du_+\|^{p-2}Du_+, Du-Du_+\rangle_{\mathbb{R}^N}\,dz\\ &+ \beta\int_{\{u>u_+\}} (|u|^{p-2}u - |u_+|^{p-2}u_+)(u-u_+)\,dz = 0\end{aligned}$$

hence $|\{u>u_+\}|_N = 0$, i.e., $u \le u_+$. Similarly, acting on (4.4) with $(v_- - u)^+ \in W_n^{1,p}(\Omega)$, we show that $v_- \le u$, i.e., $u \in I$.

In a similar fashion, using this time (4.1), (4.2), and the extremality of u_+, v_- we show that the critical points of $\hat{\varphi}_+^\beta$ are in $\{u_+,0\}$ and the critical points of $\hat{\varphi}_-^\beta$ are in $\{v_-,0\}$. This proves Claim 1.

Claim 2. *u_+ and v_- are local minimizers of $\hat{\varphi}^\beta$.*

Note that $\hat{\varphi}_+^\beta$ is coercive and sequentially weakly lower semicontinuous. So, we can find $\hat{u}_0 \in W_n^{1,p}(\Omega)$ such that

$$\hat{\varphi}_+^\beta(\hat{u}_0) = \inf\big\{\hat{\varphi}_+^\beta(y) : y \in W_n^{1,p}(\Omega)\big\}.$$

By virtue of hypothesis $\mathbf{H}(f)_2$ (v) we can find $\hat{\eta} \in (\lambda_1,\eta)$ and $\delta \in (0,\hat{m})$, where $\hat{m} \in (0, \min_{\overline{\Omega}} u_+)$ (recall that $u_+ \in \operatorname{int} C_+$) such that

$$f(z,x) \ge \hat{\eta}x^{p-1} \quad \text{for a.a.} \quad z\in\Omega, \quad \text{all} \quad x\in[0,\delta],$$

hence

$$\hat{F}_+^\beta(z,x) \ge \frac{\hat{\eta}+\beta}{p}x^p \quad \text{for a.a.} \quad z\in\Omega, \quad \text{all} \quad x\in[0,\delta].$$

Therefore, if $\xi \in [0, \delta]$, then

$$\hat{\varphi}_+^\beta(\xi) \leq -\frac{\hat{\eta}}{p}\xi^p|\Omega|_N < 0,$$

hence

$$\hat{\varphi}_+^\beta(\hat{u}_0) < 0 = \hat{\varphi}_+^\beta(0), \quad \text{i.e.,} \quad \hat{u}_0 \neq 0.$$

From Claim 1 (a) we know that $\hat{u}_0 = u_+$. Therefore u_+ is the unique global minimizer of $\hat{\varphi}_+^\beta$. Since $u_+ \in \operatorname{int} C_+$ (see Proposition 16) and $\hat{\varphi}_+^\beta|_{C_+} = \hat{\varphi}^\beta|_{C_+}$, it follows that u_+ is a local $C_n^1(\Omega)$-minimizer of $\hat{\varphi}^\beta$. Invoking Proposition 6, we conclude that u_+ is a local $W_n^{1,p}(\Omega)$-minimizer of $\hat{\varphi}^\beta$.

Similarly, working this time with $\hat{\varphi}_-^\beta$, we show that $v_- \in -\operatorname{int} C_+$ is a local minimizer of $\hat{\varphi}^\beta$. This proves Claim 2.

Without any loss of generality, we may assume that $\hat{\varphi}^\beta(v_-) \leq \hat{\varphi}^\beta(u_+)$. Moreover, because of Claim 2, arguing as in the proof of Proposition 29 in Aizicovici–Papageorgiou–Staicu [1], we can find $\rho_0 \in (0, 1)$ small such that

$$\hat{\varphi}^\beta(v_-) \leq \hat{\varphi}^\beta(u_+) < \{\hat{\varphi}^\beta(u) \colon \|u - u_+\| = \rho_0\} = \hat{\eta}_0. \tag{4.7}$$

Note that $\hat{\varphi}^\beta$ is coercive (see (4.3)). So, it satisfies the PS-condition. This fact and ((4.7) enable us to apply Theorem 1 (the mountain pass theorem) and obtain $x_0 \in W_n^{1,p}(\Omega)$ such that

$$\hat{\varphi}^\beta(v_-) \leq \hat{\varphi}^\beta(u_+) < \hat{\eta}_0 \leq \hat{\varphi}^\beta(x_0) \tag{4.8}$$

and

$$(\hat{\varphi}^\beta)'(x_0) = 0. \tag{4.9}$$

From (4.8) it follows that $x_0 \notin \{v_-, u_+\}$. From (4.9) and Claim 1 (c), we have $x_0 \in I$. Then, from (4.9) and the extremality of v_- and u_+ we infer that if $x_0 \neq 0$, then $x_0 \in C_n^1(\overline{\Omega})$ (by nonlinear regularity, cf. Lieberman [24]) is a nodal solution of (1.5). Hence our goal next is to show the nontriviality of x_0. From the Mountain Pass theorem (see Theorem 1) we have

$$\hat{\varphi}^\beta(x_0) = \inf_{\gamma \in \Gamma} \max_{-1 \leq t \leq 1} \hat{\varphi}^\beta(\gamma(t)), \tag{4.10}$$

where

$$\Gamma = \{\gamma \in C([-1, 1], W_n^{1,p}(\Omega)) \colon \gamma(-1) = v_-,\ \gamma(1) = u_+\}.$$

Hence, if we can produce a path $\gamma_* \in \Gamma$ such that $\hat{\varphi}^\beta(\gamma_*(t)) < 0$ for all $t \in [-1, 1]$, then $\hat{\varphi}^\beta(x_0) < 0 = \hat{\varphi}^\beta(0)$ (see (4.10)) and so $x_0 \neq 0$. Therefore we focus on producing such a path $\gamma_* \in \Gamma$.

To this end, let $S = W_n^{1,p}(\Omega) \cap \partial B_1^{L^p}$ equipped with the $W_n^{1,p}(\Omega)$-topology and $S_c = C_n^1(\overline{\Omega}) \cap \partial B_1^{L^p}$ equipped with the $C_n^1(\overline{\Omega})$-topology. (Recall that $\partial B_1^{L^p} = \{x \in L^p(\Omega)\colon \|x\|_p = 1\}$).

It is clear that S_c is dense in S for the $W_n^{1,p}(\Omega)$-topology. We consider the following sets of paths

$$\Gamma_0 = \{\gamma_0 \in C([-1,1], S)\colon \gamma_0(-1) = -\hat{u}_0,\ \gamma_0(1) = \hat{u}_0\}$$

and

$$\Gamma_0^c = \{\gamma_0 \in C([-1,1], S_c)\colon \gamma_0(-1) = -\hat{u}_0,\ \gamma_0(1) = \hat{u}_0\}.$$

Evidently, Γ_0^c is dense Γ_0 for the $C([-1,1], S)$ -topology. Recall that by virtue of hypothesis $\mathbf{H}(f)_2$ (v), we can find $\hat{\eta} \in (\lambda_1, \eta)$ and $\delta_0 \in (0, m_0)$ where

$$m_0 = \min\left\{\min_{\overline{\Omega}} u_+,\ \min_{\overline{\Omega}} |v_-|\right\}$$

such that

$$\frac{f(z,x)}{|x|^{p-2}x} \geq \hat{\eta} \quad \text{for a.a.} \quad z \in \Omega, \quad \text{all} \quad |x| \leq \delta_0,$$

hence

$$\hat{F}^{\beta}(z,x) \geq \frac{\hat{\eta} + \beta}{p}|x|^p\hat{\eta} \quad \text{for a.a.} \quad z \in \Omega, \quad \text{all} \quad |x| \leq \delta_0 \quad \text{(see (4.3)).} \tag{4.11}$$

From the density of Γ_0^c in Γ_0 and Proposition 7, we can find $\gamma_0 \in \Gamma_0^c$ such that

$$\|D\gamma_0(t)\|_p^p \leq \lambda_1 + \varepsilon \quad \text{for all} \quad t \in [-1,1], \quad \text{with} \quad \varepsilon \in (0, \hat{\eta} - \lambda_1). \tag{4.12}$$

Note that the set $\gamma_0([-1,1]) \subset C_n^1(\overline{\Omega})$ is compact and recall that $-v_-, u_+ \in \operatorname{int} C_+$ (see Proposition 16). Therefore we can find $\tilde{\xi} \in (0,1)$ small such that

$$|\tilde{\xi} u(z)| \leq \delta_0 \quad \text{for all} \quad z \in \overline{\Omega} \quad \text{and} \quad \tilde{\xi} u \in [v_-, u_+] \quad \text{for all} \quad u \in \gamma_0([-1,1]). \tag{4.13}$$

Hence, for all $u \in \gamma_0([-1,1])$ we have

$$\begin{aligned} \hat{\varphi}^{\beta}(\tilde{\xi} u) &= \frac{\tilde{\xi}^p}{p}\|Du\|_p^p + \frac{\beta\tilde{\xi}^p}{p}\|u\|_p^p - \int_\Omega \hat{F}^{\beta}(z, \tilde{\xi} u)\,dz \\ &\leq \frac{\tilde{\xi}^p}{p}(\lambda_1 + \varepsilon) - \frac{\tilde{\xi}^p}{p}\hat{\eta} < 0. \end{aligned} \tag{4.14}$$

So if we set $\hat{\gamma}_0 := \tilde{\xi}\gamma_0$, then $\hat{\gamma}_0$ is a continuous path in $W_n^{1,p}(\Omega)$, which connects $-\tilde{\xi}\hat{u}_0$ and $\tilde{\xi}\hat{u}_0$, and we have

$$\hat{\varphi}^{\beta}|_{\hat{\gamma}_0} < 0 \quad \text{(see (4.14)).} \tag{4.15}$$

Next, we infer from Claim 1 and the proof of Claim 2 that the only critical points of $\hat{\varphi}_+^\beta$ are 0 and u_+, and $\hat{\varphi}_+^\beta(u_+) < 0$. It also follows that $K^0_{\hat{\varphi}_+^\beta} = \{0\}$, $\hat{\varphi}_+^\beta$ has no critical values in $(a, 0)$, where $a = \hat{\varphi}_+^\beta(u_+) = \inf \hat{\varphi}_+^\beta$, and $(\hat{\varphi}_+^\beta)^{-1}(a) = \{u_+\}$. Moreover, since $\hat{\varphi}_+^\beta$ is coercive, it satisfies the PS-condition. So, we can apply Theorem 3 (the second deformation theorem) with $a = \hat{\varphi}_+^\beta(u_+)$ and $b = 0$, and obtain a homotopy $h\colon [0, 1] \times ((\hat{\varphi}_+^\beta)^0 \setminus \{0\}) \to (\hat{\varphi}_+^\beta)^0$ such that

$$h(1, ((\hat{\varphi}_+^\beta)^0 \setminus \{0\})) = \{u_+\} \tag{4.16}$$

and

$$\hat{\varphi}_+^\beta(h(t, u)) \le \hat{\varphi}_+^\beta(u) \quad \text{for all} \quad t \in [0, 1], \quad \text{all} \quad u \in (\hat{\varphi}_+^\beta)^0 \setminus \{0\}. \tag{4.17}$$

We set $\hat{\gamma}_+(t) = h(t, \tilde{\xi}\hat{u}_0)$ for all $t \in [0,1]$. Then $\hat{\gamma}_+$ is continuous and $\hat{\gamma}_+(0) = h(0, \tilde{\xi}\hat{u}_0) = \tilde{\xi}\hat{u}_0$ (since h is a homotopy), $\hat{\gamma}_+(1) = u_+$ (see (4.15), (4.16)). Moreover, due to (4.15), (4.17), we have

$$\hat{\varphi}_+^\beta|_{\hat{\gamma}_+} < 0.$$

Note that

$$\begin{aligned}\hat{\varphi}^\beta(u) &= \frac{1}{p}\|Du\|_p^p + \frac{\beta}{p}\|u\|_p^p - \int_\Omega \hat{F}^\beta(z, u)\, dz \\ &= \frac{1}{p}\|Du\|_p^p + \frac{\beta}{p}\|u\|_p^p - \int_\Omega [\hat{F}^\beta(z, u^+) + \hat{F}^\beta(z, -u^-)]\, dz.\end{aligned}$$

By virtue of hypothesis $\mathbf{H}(f)_2$ (vi), 0 is a global minimizer of $x \to F(z,x) + (\beta/p)|x|^p$ on $[\xi_-, \xi_+]$ and so, $\int_\Omega \hat{F}^\beta(z, -u^-)\, dz \ge 0$. Hence

$$\hat{\varphi}^\beta(u) \le \frac{1}{p}\|Du\|_p^p + \frac{\beta}{p}\|u\|_p^p - \int_\Omega \hat{F}_+^\beta(z, u)\, dz = \hat{\varphi}_+^\beta(u),$$

and so

$$\hat{\varphi}^\beta|_{\hat{\gamma}_+} < 0. \tag{4.18}$$

In a similar fashion, we produce another continuous path $\hat{\gamma}_-$ in $W_n^{1,p}(\Omega)$ which connects v_- and $-\tilde{\xi}\hat{u}_0$ such that

$$\hat{\varphi}^\beta|_{\hat{\gamma}_-} < 0. \tag{4.19}$$

We concatenate the paths $\hat{\gamma}_-$, $\hat{\gamma}_0$ and $\hat{\gamma}_+$ to produce a path $\gamma_* \in \Gamma$ such that (see (4.15), (4.18), (4.19))

$$\hat{\varphi}^\beta|_{\gamma_*} < 0,$$

hence $x_0 \neq 0$.

Therefore $x_0 \in C_n^1(\overline{\Omega})$ is a nodal solution for the problem (1.5). $\square$

5. The semilinear case

In this section, by strengthening the conditions on the nonlinearity $f(z,\cdot)$ and using Morse theory, we can improve the conclusion of Theorem 17 and produce a second nodal solution in the case when $p = 2$ in (1.5). See also Dancer–Du [13].

The problem under consideration is the following

$$-\triangle u(z) = f(z, u(z)) \quad \text{in} \quad \Omega, \quad \frac{\partial u}{\partial n} = 0 \quad \text{on} \quad \partial\Omega. \tag{5.1}$$

The new hypotheses on $f(z, x)$ are the following:

H$(f)_3$: The function $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is such that:

(i) for every $x \in \mathbb{R}$, $z \to f(z, x)$ is measurable;

(ii) for almost all $z \in \Omega$, $x \to f(z, x)$ is C^1 and $f(z, 0) = 0$;

(iii) for almost all $z \in \Omega$ and all $x \in \mathbb{R}$ we have

$$|f_x'(z, x)| \leq a(z) + c|x|^{r-2}, \quad 2 < r < 2^*,$$

where $a \in L^\infty(\Omega)_+$ and $c > 0$;

(iv) if $F(z, x) = \int_0^x f(z, s)\, ds$, then

$$\lim_{|x|\to\infty} \frac{F(z, x)}{x^2} = +\infty, \quad \text{uniformly for a.a.} \quad z \in \Omega$$

and there exists $\mu \in ((r-2)\max\{1, N/2\}, 2^*)$ such that

$$\liminf_{|x|\to\infty} \frac{f(z, x)x - 2F(z, x)}{|x|^\mu} > 0, \quad \text{uniformly for a.a.} \quad z \in \Omega;$$

(v) there exist an integer $m > 1$ and functions η, $\eta_1 \in L^\infty(\Omega)_+$ such that $\lambda_m \leq \eta(z)$ a.e. in Ω, $\lambda_m \neq \eta$, $\eta_1(z) \leq \lambda_{m+1}$ a.e. in Ω, $\eta_1 \neq \lambda_{m+1}$ and

$$\eta(z) \leq f_x'(z, 0) = \lim_{x\to 0} \frac{f(z, x)}{x} \leq \eta_1(z), \quad \text{uniformly for a.a.} \quad z \in \Omega;$$

(vi) there exist $\xi_- < 0 < \xi_+$, and $\theta > 0$ such that

$$f(z, \xi_+) \leq -\theta < 0 < \theta \leq f(z, \xi_-) \quad \text{for a.a.} \quad z \in \Omega.$$

REMARK. Hypotheses **H**$(f)_3$ (ii), (iii) imply that for every $\xi > 0$, we can find $\beta = \beta(\xi) > 0$ such that for a.a. $z \in \Omega$, the function $x \to f(z, x) + \beta x$ is nondecreasing on $[-\xi, \xi]$.

EXAMPLE. The following function satisfies $\mathbf{H}(f)_3$ (as before, for the sake of simplicity we drop the z-dependence:

$$f(x) = \begin{cases} cx - \xi|x|^{r-2}x & \text{if} \quad |x| \le 1, \\ x\left(\ln|x| + \dfrac{1}{2}\right) - \theta x + \beta & \text{if} \quad |x| > 1 \end{cases}$$

with $r > 2$, $c \in (\lambda_m, \lambda_{m+1})$, $m > 1$, $\xi > c$, $\beta = c + \theta - \xi - 1/2$, $c = \xi(r-1) + 3/2 - \theta$, $\theta > 3/2$.

Theorem 18. *If hypotheses* $\mathbf{H}(f)_3$ *hold, then problem* (5.1) *has at least six nontrivial smooth solutions* $u_0, \hat{u} \in \operatorname{int} C_+$, $\hat{u} - u_0 \in \operatorname{int} C_+, \xi_+ - u_0 \in \operatorname{int} C_+$, $v_0, \hat{v} \in -\operatorname{int} C_+$, $v_0 - \hat{v} \in \operatorname{int} C_+$, $v_0 - \xi_- \in \operatorname{int} C_+$, *and* $x_0, y_0 \in C_n^1(\overline{\Omega})$, *both nodal.*

Proof. From Theorem 17 we already have five nontrivial smooth solutions $u_0, \hat{u} \in \operatorname{int} C_+$, $u_0 \le \hat{u}$, $u_0 \ne \hat{u}$, $\xi_+ - u_0 \in \operatorname{int} C_+$, $v_0, \hat{v} \in -\operatorname{int} C_+$, $\hat{v} \le v_0$, $v_0 \ne \hat{v}$, $v_0 - \xi_- \in \operatorname{int} C_+$, and $x_0 \in C_n^1(\overline{\Omega}) \setminus \{0\}$ nodal.

Let $\hat{\xi} = \max\{\|\hat{u}\|_\infty, \|\hat{v}\|_\infty\}$. We know that we can find $\beta = \beta(\hat{\xi}) > 0$ such that for a.a. $z \in \Omega$, the function $x \to f(z, x) + \beta x$ is nondecreasing on $[-\hat{\xi}, \hat{\xi}]$. Then

$$\begin{aligned} &-\triangle(\hat{u} - u_0)(z) + \beta(\hat{u} - u_0)(z) \\ &= f(z, \hat{u}(z)) + \beta\hat{u}(z) - f(z, u_0(z)) - \beta u_0(z) \ge 0 \quad \text{a.e. in} \quad \Omega, \end{aligned}$$

hence $(\hat{u} - u_0) \in \operatorname{int} C_+$ (see Vázquez [31]). Similarly, we show that $v_0 - \hat{v} \in \operatorname{int} C_+$.

Consider the functional $\hat{\varphi}^\beta$ introduced in the proof of Theorem 17. Note that hypotheses $H(f)_3$ (i), (ii), (iii) imply that $\hat{\varphi}^\beta \in C^{2-0}(H_n^1(\Omega))$. Also recall that $x_0 \in C_n^1(\overline{\Omega})$ is a critical point of $\hat{\varphi}^\beta$ of mountain pass type (see the proof of Theorem 17). Hence, from Li–Li–Liu [23] (see also Mawhin–Willem [25, p. 195]), we have

$$C_k(\hat{\varphi}^\beta, x_0) = \delta_{k,1}\mathbb{Z} \quad \text{for all} \quad k \ge 0. \tag{5.2}$$

Also, we know that $u_+ \in \operatorname{int} C_+$, $v_- \in -\operatorname{int} C_+$ are local minimizers of $\hat{\varphi}^\beta$ (see Claim 2 in the proof of Theorem 17). Hence

$$C_k(\hat{\varphi}^\beta, u_+) = C_k(\hat{\varphi}^\beta, v_-) = \delta_{k,0}\mathbb{Z} \quad \text{for all} \quad k \ge 0. \tag{5.3}$$

Note that $\hat{\varphi}^\beta$ is C^2 in a neighborhood of $u = 0$, and by virtue of hypothesis $\mathbf{H}(f)_3$ (v) and the unique continuation property, $u = 0$ is a nondegenerate critical point of $\hat{\varphi}^\beta$ with Morse index $d_m = \dim \bigoplus_{i=0}^m E(\lambda_i)$ ($E(\lambda_i)$ being the eigenspace for the eigenvalue λ_i). Then

$$C_k(\hat{\varphi}^\beta, 0) = \delta_{k,d_m}\mathbb{Z} \quad \text{for all} \quad k \ge 0, \tag{5.4}$$

(see, for example, Mawhin–Willem [25, p. 188]).

Finally, recalling that $\hat{\varphi}^\beta$ is coercive (hence bounded below), directly from the definition of critical groups at infinity, we have

$$C_k(\hat{\varphi}^\beta, \infty) = \delta_{k,0}\mathbb{Z} \quad \text{for all} \quad k \geq 0. \tag{5.5}$$

Suppose that $\{0, v_-, u_+, x_0\}$ are the only critical points of $\hat{\varphi}^\beta$. Then from (5.2), (5.3), (5.4), (5.5) and the Poincaré–Hopf formula (see (2.5)), we have

$$2(-1)^0 + (-1)^1 + (-1)^{d_m} = (-1)^0,$$

hence $(-1)^{d_m} = 0$, a contradiction.

This means that $\hat{\varphi}^\beta$ has one more critical point $y_0 \notin \{0, v_-, u_+, x_0\}$. Note that $y_0 \in C^1_n(\overline{\Omega}) \setminus \{0\}$ (regularity theory) and is a nodal solution of (5.1) (see Claim 1 in the proof of Theorem 17). $\square$

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