Bujalance, E. and Gromadzki, G.
Osaka J. Math.
50 (2013), 251–269

ON AUTOMORPHISMS OF KLEIN SURFACES WITH INVARIANT SUBSETS

E. BUJALANCE and G. GROMADZKI

(Received June 30, 2010, revised July 1, 2011)

Abstract

It is well known that a group of automorphisms G of an unbordered Klein surface X of topological genus $g \geq 2$ in the orientable case and $g \geq 3$ otherwise has at most $84(g-\varepsilon)$ elements, where $\varepsilon = 1$ or 2 respectively. In the middle of the fifties, Oikawa used the cardinality k of a G-invariant subset to introduce the bound $|G| \leq 12(g-1) + 6k$ in the orientable case. Much later, T. Arakawa has generalized this result, involving $s = 2$ or 3 such subsets and showing in addition that the bound for $s = 3$ is sharp for infinitely many configurations. Here we improve the bound of Arakawa for $s = 2$, showing in particular that the last is never attained. In both orientable and non-orientable case, we also find bounds for arbitrary s and show their sharpness for infinitely many topological configurations. Using another well known theorem of Oikawa and the canonical Riemann double cover, we get similar results for bordered Klein surfaces.

1. Introduction

Let X be an unbordered Klein surface of topological genus $g \geq 2$ or $g \geq 3$, according to if X is orientable or not. It is well known that a group of automorphisms G of X has at most $84(g-\varepsilon)$ elements [13, 19], where correspondingly $\varepsilon = 1$ or 2. Here, following Singerman [19], a non-orientable unbordered Klein surface will be called a non-orientable Riemann surface. In [16] Oikawa took into account the cardinality k of a G-invariant subset and found a bound $|G| \leq 12(g-1) + 6k$ in the orientable case. Fifty years later, at the beginning of this century, T. Arakawa [2] obtained similar bounds in the orientable case for cardinalities of $s = 2$ and $s = 3$ G-invariant subsets. Here we improve the bound of Arakawa for $s = 2$, showing in particular that the last is never attained. Our bound is particularly useful in the proofs of some results concerning the orders of the groups of automorphisms of q-hyperelliptic and cyclic q-trigonal non-orientable unbordered Klein surfaces. We also find bounds for arbitrary s for both orientable and non-orientable Riemann surfaces, showing in addition that they are sharp for infinitely many topological configurations. Using another result from the

2000 Mathematics Subject Classification. Primary 30F10, 30F35, 30F50; Secondary 14H37, 14H45, 14H55.

E. Bujalance supported by MTM2008-00250 of the Spanish Ministery of Sciences and Innovation and G. Gromadzki by the Research Grant NN 201 366436 of the Polish Ministry of Science and Higher Education.

mentioned Oikawa's paper [16] and the canonical Riemann double cover, we get similar new results for bordered Klein surfaces. Moreover, we obtain immediate proofs of the number of well known results of other authors.

2. Some preliminaries

2.1. Fuchsian and non-euclidean crystallographic groups. We shall use combinatorial approach, based on Riemann uniformization theorem, Fuchsian groups and non-euclidean crystallographic groups (NEC groups in short), which was described extensively in [1] and [4].

An *NEC group* is a discrete and cocompact subgroup of the group $\mathcal{G}$ of isometries of the hyperbolic plane $\mathcal{H}$, including those which reverse orientation, and if such a subgroup contains only orientation preserving isometries, then it is called a Fuchsian group. Macbeath and Wilkie [14, 20] associated to every NEC group Λ a signature, which determines its algebraic structure. It has the form

$$(g; \pm; [m_1, \dots, m_r]; \{C_1, \dots, C_k\}). \tag{1}$$

The numbers $m_i \geq 2$ are called the *proper periods*, the brackets $C_i = (n_{i1}, \dots, n_{is_i})$ are the *period cycles*, the numbers $n_{ij} \geq 2$ are the *link periods* and $g \geq 0$ is said to be the *orbit genus* of Λ. The orbit space $\mathcal{H}/\Lambda$ is a surface of topological genus g, having k boundary components, and it is orientable or not according to the sign being $+$ or $-$.

A Fuchsian group can be regarded as an NEC with the signature

$$(g; +; [m_1, \dots, m_r]; \{-\}),$$

which shortly will be denoted by $(g; m_1, \dots, m_r)$; a Fuchsian group without periods will be denoted by $(g; -)$ and called a *Fuchsian surface group*; an NEC group which is not a Fuchsian group will be refered to as a *proper NEC group*. The group with the signature (1) has a presentation given by generators:
(a) x_i, $i = 1, \dots, r$, (hyperbolic rotations)
(b) c_{ij}, $i = 1, \dots, k$; $j = 0, \dots, s_i$, (hyperbolic reflections)
(c) e_i, $i = 1, \dots, k$, (connecting generators)
(d) a_i, b_i, $i = 1, \dots, g$ if the sign is $+$, (hyperbolic translations), d_i, $i = 1, \dots, g$ if the sign is $-$, (hyperbolic glide reflections)
and relations:
(1) $x_i^{m_i} = 1$, $i = 1, \dots, r$,
(2) $c_{is_i} = e_i^{-1} c_{i0} e_i$, $i = 1, \dots, k$,
(3) $c_{ij-1}^2 = c_{ij}^2 = (c_{ij-1} c_{ij})^{n_{ij}} = 1$, $i = 1, \dots, k$; $j = 1, \dots, s_i$,
(4) $x_1 \cdots x_r e_1 \cdots e_k a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1} = 1$, if the sign is $+$, $x_1 \cdots x_r e_1 \cdots e_k d_i^2 \cdots d_g^2 = 1$ if the sign is $-$.
Any set of generators of an NEC group satisfying the above relations will be called *a canonical set of generators* and reflections c_{ij-1}, c_{ij} will be said to be *consecutive*.

For the convenience, we shall call the products $c_{ij-1}c_{ij}$ the *canonical decomposable elliptic elements of* Λ. Connecting generators are usually hyperbolic translations but if the orbit genus is zero, the signature has only one proper period and one period cycle, then they are elliptic i.e. they are hyperbolic rotations.

Every NEC group has a fundamental region associated, whose hyperbolic area $\mu(\Lambda)$, for an NEC group Λ with signature (1), is given by

$$2\pi\left(\alpha g + k - 2 + \sum_{i=1}^{r}\left(1 - \frac{1}{m_i}\right) + \frac{1}{2}\sum_{i=1}^{k}\sum_{i=1}^{s_i}\left(1 - \frac{1}{n_{ij}}\right)\right),$$

where $\alpha = 2$ if the sign is $+$ and $\varepsilon = 1$ otherwise. It is known that an abstract group with the presentation given by the generators (a)–(d) and the relations (1)–(4) can be realized as an NEC group with the signature (1) if and only if the above expression is positive. Finally, if Γ is a subgroup of finite index in an NEC group Λ then it is an NEC group itself and there is a Hurwitz–Riemann formula which says that

$$[\Lambda : \Gamma] = \frac{\mu(\Gamma)}{\mu(\Lambda)}.$$

2.2. Riemann and Klein surfaces and their group of automorphisms. Now, by the Riemann uniformization theorem, a compact Riemann surface of genus $g \geq 2$ can be represented as the orbit space $\mathcal{H}/\Gamma$, for some Fuchsian surface group Γ with the signature $(g; -)$. A conformal automorphism or shortly an automorphism of Riemann surface is an auto-homeomorphism whose local forms are analytic or they composed with the complex conjugation are analytic. Clearly an automorphism preserve orientation if and only if no maps of the later type as local forms appear. Furthermore, a group of conformal automorphisms G of a surface given in such a way, can be represented as the factor group Λ/Γ, where Λ is a proper NEC or a Fuchsian group according to if G contains orientation reversing automorphisms or not. Now an unbordered Klein surface is a compact surface, possibly non-orientable, with a dianalytic structure which, roughly speaking, differs from the classical analytic structure by the fact that the complex reflection $z \to -\bar{z}$ is allowed for transition maps [1]. Combinatorial study of groups of automorphisms of unbordered non-orientable Klein surfaces is possible essentially due to the same facts. This time, however, a group Γ uniformizing X is an NEC surface group with the signature $(g; -; [-]; \{-\})$ and so, in particular, Λ necessarily is a proper NEC group.

Throughout the paper, an automorphism of an a Riemann surface will mean an an automorphism preserving orientation, unless otherwise stated, $\mathrm{Aut}^{\pm}(X)$ will denote the group of all, including orientation reversing, automorphisms and $\mathrm{Aut}(X)$ its subgroup of index ≤ 2 containing orientation preserving ones.

2.3. Some abstract group theory. Finally we will need the following simple result of abstract group theory, being a kind of mathematical folklore, known as a Poincare lemma.

Lemma 2.1. *Given an abstract group G and a subgroup K of finite index there exist a subgroup H of K of a finite index which is normal in G.*

Proof. Let G/K be the set of left cosets and let $\mathrm{S}(G/K)$ be the group of its permutations. Then the mapping $\varphi\colon G \to \mathrm{S}(G/K)$ given by $\varphi(g)(xK) = (gx)K$ is a homomorphism. Furthermore for $g \in \ker\varphi$, $K = \varphi(g)(K) = gK$ which means that $g \in K$. So $H = \ker\varphi$ is a subgroup we are looking for. □

3. Classical Riemann surfaces

Let X be a Riemann surface with a group of automorphisms G. A G-invariant subset B of X is said to be *irreducible* if it has no G-invariant proper subsets. Obviously, irreducible G-invariant subset has no more than $|G|$ elements and those subsets whose cardinalities are strictly smaller than $|G|$, are said to be *proper*. Clearly, the mentioned Oikawa and Arakawa bounds are not attained if one of the subsets is not proper. This is one of the reasons which allow us to restrict ourselves to studying formulae for proper irreducible G-invariant subsets only. Another such reason is provided by the lemma below, from which it follows that the whole surface, except a finite number of points, is covered by the improper irreducible G-invariant subsets which actually means that, up to certain extent, the fact that the surface has improper G-invariant subsets actually does not impose restrictions on the order of its group of automorphisms.

3.1. On proper irreducible *G*-invariant subsets. The next lemma follows from the basic geometrical properties of the canonical projection $X \to X/G$, which is ramified covering, but for the sake of convenience, the consistency with algebraic character of the paper and completeness of our exposition we give an alternative algebraic proof.

Lemma 3.1. *Let $X = \mathcal{H}/\Gamma$, let $G = \Delta/\Gamma \subseteq \mathrm{Aut}(X)$. Then each proper irreducible G-invariant subset of X is equal to $\{\pi(\delta h_i) \mid \delta \in \Delta\}$ and has $|G|/m_i$ elements, where h_i runs over the points fixed by the canonical elliptic generators x_i of Δ.*

Proof. Let $\theta\colon \Delta \to G$ and $\pi : \mathcal{H} \to X$ be the canonical projections. Then, given $x \in X$ and $g \in G$,

$$gx = \pi(\delta h) \quad \text{for} \quad g = \theta(\delta) \quad \text{and} \quad x = \pi(h). \tag{2}$$

Now let B denote a proper irreducible G-invariant subset and let $x = \pi(h) \in B$. First we shall show that $\pi(h_i) \in B$ for some i. Since B is irreducible, it equals the orbit Gx

and since it is proper, x is a fixed point of some $g = \theta(\delta)$. But then $\gamma\delta h = h$ for some $\gamma \in \Gamma$. Hence $\gamma\delta$ is elliptic and therefore is conjugate to a power of some of elliptic generator, say $\gamma\delta = \delta_i x_i^{k_i} \delta_i^{-1}$, and so $h = \delta_i h_i$. But then $\pi(h_i) = \theta(\delta_i)^{-1}x \in B$. Now assume that $\pi(h_j) \in B$. Then $\pi(h_j) = \theta(\delta)\pi(h_i)$, which gives $\gamma\delta h_i = h_j$. It follows that x_j and $(\gamma\delta)x_i(\gamma\delta)^{-1}$ have common fixed point, which in turn means that they are conjugate, a contradiction which completes the first part of the proof.

For the second part, observe we already know that $B = \{\pi(\delta h_i) \mid \delta \in \Delta\}$ for some i. Now $\pi(\delta h_i) = \pi(\delta' h_i)$ if and only if $\delta^{-1}\gamma\delta' = x_i^{k_i}$ for some i and $\gamma \in \Gamma$ and so $\delta^{-1}\delta' \in \Gamma\langle x_i\rangle = \langle \Gamma, x_i\rangle$. Hence B has

$$[\Delta : \Gamma\langle x_i\rangle] = \frac{[\Delta : \Gamma]}{[\Gamma\langle x_i\rangle : \Gamma]} = \frac{|G|}{m_i}$$

elements. □

3.2. On Oikawa and Arakawa results. As we mentioned in the introduction, Oikawa [16] and Arakawa [2] found the bounds for the order of a group G of automorphisms of a compact Riemann surface X, taking into account the genus $g \geq 2$ and the orders k, l, m of $s \leq 3$, G-invariant subsets. It is easy to see that the Oikawa bound $12(g-1) + 6k$ is attained, for a Riemann surface X admitting a G-invariant subset of cardinality k, if and only if the group of automorphisms of X is generated by two elements of order 2 and 3. The Arakawa bound $2(g-1) + k + l + m$ is attained just for the Riemann surfaces X for which the canonical projection $X \to X/G$ is ramified over exactly three points. Here we show that the Arakawa bound $8(g-1) + k + 4l$ is never attained and we shall find more precise bound, which is attained for infinitely many genera.

Theorem 3.2. *Let X be a Riemann surface of genus $g \geq 2$ with two proper irreducible G-invariant subsets of cardinalities k and l. Then either $|G| \leq 2(g-1) + k + l$ or G has order precisely*

$$\frac{m}{m-1}(2(g-1) + k + l) \tag{3}$$

for some $m \geq 2$. Furthermore, given $m \geq 2$ there are infinitely many values of g for which there exist Riemann surfaces of genus g, admitting two proper irreducible invariant subsets of cardinalities k and l, and the group g of automorphisms of order (3) *and in such case necessarily the orbit space X/G is the sphere with exactly three cone points of orders $|G|/k$, $|G|/l$ and m.*

Proof. Let $X = \mathcal{H}/\Gamma$ and let $G = \Delta/\Gamma$. Observe that by Lemma 3.1, Δ has at least two periods m_1, m_2 and we have $k = |G|/m_1$ and $l = |G|/m_2$. If the orbit genus of Δ is nonzero, then $\mu(\Delta) \geq 2\pi(2 - 1/m_1 - 1/m_2)$. Thus, by the Hurwitz–Riemann formula $2(g-1)/|G| \geq 2-1/m_1-1/m_2$ and in turn $|G| \leq g-1+k/2+l/2$, which is strictly smaller than $2(g-1)+k+l$. Therefore we may assume that the orbit genus of Δ equals 0. But now, if its signature has more than four periods, then $\mu(\Delta) \geq 2\pi(1-1/m_1-1/m_2)$ for some m_1, m_2. Hence again from the Hurwitz–Riemann formula, $2(g-1)/|G| \geq 1-1/m_1-1/m_2$, which in turn gives $|G| \leq 2(g-1)+k+l$. Thus we can assume that Δ has at exactly three periods $m_1, m_2, m_3 = m$. But then, $\mu(\Delta) = 2\pi(1-1/m_1-1/m_2-1/m_3)$ and so by the Hurwitz–Riemann formula $2(g-1)/|G| = ((m-1)/m)-1/m_1-1/m_2$, which in turn implies that G has the order (3).

Now, given m_1, m_2 and $m_3 = m \geq 2$ such that $1/m_1+1/m_2+1/m_3 < 1$, $(0; m_1, m_2, m_3)$ is a signature of a Fuchsian group Δ. Then the Fenchel "conjecture" [6, 11] guarantees that there exists a Fuchsian surface subgroup Γ of Δ of finite index and the Poincare lemma allows us to assume that Γ is normal in Δ which means that the group, $G = \Delta/\Gamma$ is generated by two elements a and b of orders m_1 and m_2, whose product has order m_3. In this way we obtain a Riemann surface $X = \mathcal{H}/\Gamma$ of some genus g, admitting two proper irreducible G-invariant subsets of cardinalities $k = |G|/m_1$ and $l = |G|/m_2$, and a group of automorphisms of order (3). Finally, given a prime p let, in the above construction, Γ_p be a p-Frattini subgroup of Γ, i.e. the subgroup generated by all commutators and all p-powers. Then Γ_p is a characteristic subgroup of Γ and hence is normal in Δ and $|\Gamma|/\Gamma_p$ is elementary abelian group of order p^{2g} and hence $\mathcal{H}/\Gamma_p$ has genus $(g-1)p^{2g}+1$. So taking Γ_p for all primes p or iterating this construction for given p by taking an infinite series of descending subgroups $\Gamma_0 = \Gamma$, $\Gamma_n = (\Gamma_{n-1})^p$ we obtain surfaces and subsets in question for infinitely many genera. □

Corollary 3.3. *Let X be a Riemann surface of genus $g \geq 2$ with two proper irreducible G-invariant subsets of cardinalities k and l. Then $|G| \leq 4(g-1)+2k+2l$ and this bound is attained for infinitely many values of g.*

Corollary 3.4. *The bound of Arakawa $8(g-1)+k+4l$ for the order of the group of automorphisms of a Riemann surface of genus $g \geq 2$ having two invariant subsets of cardinalities k and l is never attained.*

Proof. Indeed, by Theorem 3.2, $|G| \leq 2(g-1)+k+l < 8(g-1)+k+4l$ since $8(g-1)+k+4l$ is not of the form (3). □

For an automorphism of a Riemann surface having 2 fixed points we have the following result of Szemberg [17].

Corollary 3.5. *The order of an automorphism φ of a compact Riemann surface X of genus $g \geq 2$, having two fixed points, is at most $4g$.*

3.3. The bound for $s \geq 4$ and its attainment. As we mentioned in the introduction, Oikawa and Arakawa have found the bounds for the order of a group G of automorphisms of a compact Riemann surface X involving the genus and the cardinalities of $s \leq 3$ G-invariant subsets. Here we shall find such bounds for arbitrary s.

Theorem 3.6. *Let X be a Riemann surface of genus $g \geq 2$ with a group of automorphisms G and with G-invariant irreducible subsets $B_1, \ldots, B_n$ of cardinalities $q_1 \leq \cdots \leq q_n$, and assume that $s \geq 4$ of them are proper. Then each q_i divides $|G|$ and*

$$|G| \leq \frac{2}{s-2}(g-1) + \frac{q_1 + \cdots + q_s}{s-2}.$$

Conversely, for each s these bounds are attained for infinitely many values of g.

Proof. Let $X = \mathcal{H}/\Gamma$, $G = \Delta/\Gamma$ and let $\pi : X \to X/G$ be the canonical projection. By Lemma 3.1, Δ has signature $(\gamma; m_1, \ldots, m_r)$, where $q_i = |G|/m_i$ for $i \leq s$ and so, in particular, $r \geq s$. Thus

$$\frac{2(g-1)}{|G|} \geq s - 2 - \left(\frac{1}{m_1} + \cdots + \frac{1}{m_s}\right),$$

by the Hurwitz–Riemann formula, which in turn gives our bound.

Now, given $s \geq 4$ and $q_1, \ldots, q_s$ being divisors of $|G|$, where some q_i is strictly smaller than $|G|/2$ if $s = 4$, let $m_i = |G|/q_i$ for $i = 1, \ldots, s$. Consider a Fuchsian group Δ with signature $(0; m_1, \ldots, m_s)$. Then, as before, the Fenchel "conjecture" guarantees that there exists a Fuchsian surface subgroup Γ of Δ of finite index and the Poincare lemma allows us to assume that Γ is normal in Δ. Therefore this configuration provides a Riemann surface $\mathcal{H}/\Gamma$ of some genus g, having s proper irreducible G-invariant subsets of cardinalities, say, $q_1, \ldots, q_s$ and a group $G = \Delta/\Gamma$ of automorphisms of order

$$\frac{2}{s-2}(g-1) + \frac{q_1 + \cdots + q_s}{s-2}.$$

Clearly our surface has also $n - s$ improper G-invariant subsets. Finally, the Frattini construction for primes p allows us to produce surfaces and subsets as above for infinitely many genera. □

As an immediate Corollary we have the following bound of Farkas and Kra [9].

Corollary 3.7. *A single automorphism of a compact Riemann surface of genus $g \geq 2$, having $q \geq 3$ fixed points, has order not exceeding $(2/(q-2))g + 1$.*

3.4. Automorphisms of bordered orientable Klein surfaces. There is another important result in the mentioned already paper of Oikawa [16]. Namely, a bordered orientable Klein surface X with a group of automorphisms G and k boundary components, can be embedded in an unbordered Riemann surface X' so that $X' \setminus X$ is composed of k disjoint open discs and the action of G can be extended to X'. Furthermore, G preserves the centers of the discs attached in X'. Later on, it was remarked by Greenleaf and May [10], that the same holds true for non-orientable Riemann surfaces. On the other hand, there is another tool which allows to relate bordered and unbordered Klein surfaces—the Riemann double cover. These two tools will allow us to derive immediately some known results on the groups of automorphism of bordered Klein surfaces and to prove some new results.

We start with the following famous bound for bordered orientable Klein surfaces of topological genus $g \geq 2$, introduced by May in [15].

Corollary 3.8. *Let X be a bordered orientable Klein surface of topological genus $g \geq 2$ and having k boundary components. Then* $\mathrm{Aut}^{\pm}(X) \leq 12(p-1)$, *where p stands for the algebraic genus of X.*

Proof. Here $p = 2g + k - 1$ and, on the other hand, G can be considered as a group of automorphisms of a compact Riemann surface of the genus g with an invariant subset of cardinality k. From the Oikawa result, we know that the group of orientation preserving automorphisms has at most $12(g-1) + 6k$ elements. So for the group G of all automorphisms, including the orientation reversing ones, we have $|G| \leq 2(12(g-1) + 6k) = 12(p-1)$. □

Next we obtain results of Pozo from [18]

Corollary 3.9. *Let G be a group of automorphisms of an orientable bordered Klein surface X of topological genus $g \geq 2$, including orienation reversing ones, and having* 2 *or* 3 *invariant subsets of the set of boundary components. Then, respectively,* $|G| \leq 4(p-1)$ *or* $|G| \leq 2(p-1)$, *where p stands for the algebraic genus of X.*

Proof. Let X' be a Riemann surface obtainned by mentioned theorem of Oikawa and let Let $k_1, \dots, k_s$, where $s = 2$ or $s = 3$, be the cardinalities of s invariant subsets in question. Then for their sum k, $p \geq 2g + k - 1$. So by our Corollary 3.3 for $s = 2$ and by the bound of Arakawa for $s = 3$ mentioned at the begining of Subsection 3.2 the group of orientation preserving automorphisms of X' has no more than $4(g-1) + 2k$ and $2(g-1) + k$ respectively elements. So, $|G| \leq 8(g-1) + 4k \leq 4(p-1)$ for $s = 2$ and $|G| \leq 4(g-1) + 2k \leq 2(p-1)$ for $s = 3$, which completes the proof. □

Similarly, for more than three invariant subsets of the set of boundary components we obtain at once the following new generalization of Corollary 3.9.

Corollary 3.10. *Let G be a group of automorphisms, including the orientation reversing ones, of an orientable bordered Klein surface X of topological genus $g \geq 2$ and algebraic genus p, having $s \geq 4$ invariant subsets of the set of boundary components. Then $|G| \leq 2(p-1)/(s-2)$.*

Proof. The group G can be considered as a group of automorphisms of a compact Riemann surface X' of genus g having s invariant subsets, say of cardinalities $k_1, \ldots, k_s$ and let $k = k_1 + \cdots + k_s$. Then on the one hand, for the subgroup G^+ of G consisting of orientation preserving automorphisms of X', $|G^+| \leq 2(g-1)/(s-2) + k/(s-2)$, by Theorem 3.6, while on the other hand for the algebraic genus p of X we have $p \geq 2g + k - 1$ and hence the result. □

We again back to results of Pozo from [18].

Corollary 3.11. *Let G be a group of automorphisms, including the orientation reversing ones, of a bordered orientable Klein surface X of algebraic genus $p \geq 2$ having an invariant subset of interior points of cardinality q. Then $|G| \leq 4(p-1) + 4q$.*

Proof. The canonical Riemann double cover $\tilde{X}$ is a Riemann surface of genus p having the two subsets of cardinalities q. Let $\tilde{G}$ be the group of automorphisms of $\tilde{X}$. Then, for the subgroup $\tilde{G}^+$ of orientation preserving automorphisms of $\tilde{X}$, these two subsets are invariant. So, by Corollary 3.3, $|\tilde{G}^+| \leq 4(p-1) + 4q$ and, as $|\tilde{G}| = 2|G|$, the result follows. □

The next result is new and in particular it strengthens the result of Pozo for $s = 2$

Corollary 3.12. *Let G be a group of automorphisms, including the orientation reversing ones, of a bordered orientable Klein surface X of algebraic genus $p \geq 2$ having s invariant subsets of interior points of cardinalities $q_1, \ldots, q_s$. Then*

$$|G| \leq \frac{1}{s-1}(p-1) + \frac{1}{s-1}(q_1 + \cdots + q_s).$$

Corollary 3.13. *Let G be a group of automorphisms, including orientation reversing ones, of an orientable bordered orientable Klein surface X of topological genus $g \geq 2$ having s invariant subsets of the set of boundary components of cardinalities $p_1, \ldots, p_s$ and t invariant subsets of interior points of cardinalities $q_1, \ldots, q_t$, where $t + s \geq 4$. Then*

$$|G| \leq \frac{4}{s+t-2}(g-1) + \frac{2}{s+t-2}(p_1 + \cdots + p_s + q_1 + \cdots + q_t).$$

Proof. By Oikawa theorem, the subgroup G^+ of G acts as a group of automorphisms of a closed Riemann surface of genus g with $s+t$ invariant subsets of cardinalities $p_1, \ldots, p_s, q_1, \ldots, q_s$ and hence the result follows by Theorem 3.6. □

4. Non-orientable Riemann surfaces

Each classical Riemann surface can be viewed as an orientable unbordered Klein surface and, following Singerman [19], we shall refer to a non-orientable unbordered Klein surface as to a *non-orientable Riemann surface*. It is well known [19], that a non-orientable Riemann surface of topological genus $g \geq 3$ (which is the number of cross caps attached to the sphere), has at most $84(g-2)$ elements and this bound is both attained and not attained for infinitely many values of g. The groups for which it is attained, are known in the literature as H-groups.

4.1. On proper irreducible *G*-invariant subsets. Let $X = \mathcal{H}/\Gamma$ be a non-orientable Riemann surface, let $G = \Lambda/\Gamma$ be its group of automorphisms and let $\pi\colon \mathcal{H} \to X$ and $\theta\colon \Lambda \to G$ be the canonical projections. Given a canonical system of generators for Λ, let $h_1, \ldots, h_t$ be the set of fixed points of all canonical elliptic generators and canonical decomposable elliptic elements $x_1, \ldots, x_t$ and let $l_1, \ldots, l_m$ be the axes of canonical reflections $c_1, \ldots, c_m$ of Λ. With these notations we have the following

Lemma 4.1. *Each proper irreducible G-invariant subset of X equals either $\{\pi(\lambda h_i) \mid \lambda \in \Lambda\}$, and has $|G|/2m_i$ or $|G|/m_i$ elements depending on if x_i is decomposable or not, or $\{\pi(\lambda k_i) \mid \lambda \in \Lambda\}$ for some $k_i \in l_i \setminus \{h_1, \ldots, h_r\}$ and has $|G|/2$ elements.*

Proof. The proof is similar to that of Lemma 3.1. Also here, given $x \in X$ and $g \in G$

$$gx = \pi(\lambda h) \quad \text{if} \quad g = \theta(\lambda) \quad \text{and} \quad x = \pi(h).$$

Now let B be a proper irreducible G-invariant subset and let $x = \pi(h) \in B$. Since B is irreducible, it equals the orbit Gx and since it is proper, x is a fixed point for some $g = \theta(\lambda)$. But then $\gamma\lambda h = h$ for some $\gamma \in \Gamma$. So $\gamma\lambda$ is either elliptic or a reflection and therefore is conjugate to a power of some canonical elliptic generator or canonical decomposable elliptic element, say $\gamma\lambda = \lambda_i x_i^{n_i} \lambda_i^{-1}$, or to some c_i, say $\gamma\lambda = \lambda_i c_i \lambda_i^{-1}$. Hence $h = \lambda_i h_i$ or $h = \lambda_i k_i$ for some $k_i \in l_i \setminus \{h_1, \ldots, h_t\}$. In the first case, $\pi(h_i) = \theta(\lambda_i)^{-1} x \in B$ and in the second one $\pi(k_i) = \theta(\lambda_i)^{-1} x \in B$.

Finally, as before, $\pi(\lambda h_i) = \pi(\lambda' h_i)$ if and only if $\lambda^{-1}\gamma\lambda' = x_i^{\alpha_i}$ for some integer α_i and $\gamma \in \Gamma$ and so $\lambda^{-1}\lambda'$ belongs to $\Gamma\langle x_i \rangle$ if x_i is not a product of two reflections or to $\Gamma\langle c, c'\rangle$ if $x_i = cc'$ and hence B has respectively

$$[\Lambda : \Gamma\langle x_i\rangle] = \frac{[\Lambda : \Gamma]}{[\Gamma\langle x_i\rangle : \Gamma]} = \frac{|G|}{m_i}$$

or

$$[\Lambda : \Gamma\langle c, c'\rangle] = \frac{[\Lambda : \Gamma]}{[\Gamma\langle c, c'\rangle : \Gamma]} = \frac{|G|}{2m_i}$$

elements.

Similarly, $\pi(\lambda k_i) = \pi(\lambda' k_i)$ if and only if $\lambda^{-1}\gamma\lambda' = c_i$ for some $\gamma \in \Gamma$ and so if and only if $\lambda^{-1}\lambda'$ belongs to $\Gamma\langle c_i\rangle$. Hence B has

$$[\Lambda : \Gamma\langle c_i\rangle] = \frac{[\Lambda : \Gamma]}{[\Gamma\langle c_i\rangle : \Gamma]} = \frac{|G|}{2}$$

elements. □

DEFINITION 4.2. The two types of G-invariant subsets from Lemma 4.1, corresponding respectively to canonical elliptic generators and canonical decomposable elliptic elements or canonical reflections, are called respectively of the *first* or the *second type*.

4.2. The bounds and their attainments. Observe that if a surface has a G-invariant subset of the second type, then it has infinitely many such subsets. So, first we shall see that these play rather limited role in our studies.

Corollary 4.3. *Let G be a group of automorphisms of a non-orientable Riemann surface X of topological genus $g \geq 3$ not having invariant subsets of the first type. Then $|G| \leq g - 2$.*

Proof. If $G = \Lambda/\Gamma$, then Λ has no elliptic elements. Thus $\mu(\Lambda)$ is a multiplicity of 2π and therefore $|G| \leq g - 2$. □

In the remainder of the subsection we shall deal with actions allowing invariant subsets of the first type. We start with the following theorem concerning one invariant subset.

Theorem 4.4. *Let X be a non-orientable Riemann surface of topological genus $g \geq 3$ with a group of automorphisms G and let B be a G-invariant irreducible subset of the first type of cardinality k. Then $|G| \leq 12(g - 2 + k)$ and this bound is attained for infinitely many g.*

Proof. Let $X = \mathcal{H}/\Gamma$ and let $G = \Lambda/\Gamma$ be a group of automorphisms of X. Notice that since Λ is a proper NEC group, either its orbit genus is non zero and the sign is $-$ or it has a period cycle. The set B in question comes from a proper period m or a link period n of Λ, i.e. $|B| = |G|/m$ or $|B| = |G|/2n$

In the first case, if there exists another proper period m', then $\mu(\Lambda) \geq 2\pi(1-1/m-1/m') \geq 2\pi(1/2-1/m)$ and so $|G| \leq 2(g-2+k)$ by the Hurwitz–Riemann formula. Now if there is a period cycle with a link period n, then $\mu(\Lambda) \geq 2\pi(1/4-1/m)$ which by the Hurwitz–Riemann formula gives $|G| \leq 4(g-2+k)$. Finally, if m is the unique proper period and there are no link periods, then $\mu(\Lambda) \geq 2\pi(1-1/m)$, which gives $|G| \leq g-2+k$.

In the second case, if there are at least two proper periods, then $\mu(\Lambda) \geq 2\pi(1/2-1/2n)$ and so $|G| \leq 2(g-2+k)$. So assume that Λ has just one proper period m. If $m \geq 3$, then $\mu(\Lambda) \geq 2\pi(1/6-1/2n)$ and so $|G| \leq 6(g-2+k)$. If $m = 2$ then either there is another link period and $\mu(\Lambda) \geq 2\pi(1/4-1/2n)$, which gives $|G| \leq 4(g-2+k)$ or the orbit genus is nonzero, or else there is another period cycle. But in the last two cases $\mu(\Lambda) \geq 2\pi(1-1/2n)$ and so $|G| \leq g-2+k$. Therefore we may assume that there are no proper periods in Λ. If the orbit genus is nonzero or there are two period cycles, then $\mu(\Lambda) \geq 2\pi(1/2-1/2n)$ and so $|G| \leq 2(g-2+k)$. Thus we can assume that Λ has signature $(0;+;[-];\{(n,n_2,\ldots,n_s)\})$. If $s \geq 4$, then $\mu(\Lambda) \geq 2\pi(1/4-1/2n)$ and so $|G| \leq 4(g-2+k)$. So assume that $s = 3$. Then $\mu(\Lambda) \geq 2\pi(1/12-1/2n)$ and therefore $|G| \leq 12(g-2+k)$.

Observe that for $n \geq 7$, $\mu(\Lambda) = 2\pi(1/12-1/2n)$ if and only if Λ has signature $(0;+;[-];\{(2,3,n)\})$. Now it is known [19] that for $n = 7$ (for $n > 7$ it seems to be a folklore), there are normal subgroups Γ of Λ with signatures $(g;-;[-];\{-\})$ for infinitely many values of g and so the bound $12(g-2+k)$ is attained for infinitely many configurations. □

REMARK 4.5. Observe that the bound $12(g-2+k)$ for the order of a group of automorphisms of a non-orientable Riemann surface of genus g is attained for an NEC group Λ with signature $(0;+;[-];\{(2,3,n)\})$. Therefore necessarily Λ has reflections and so the corresponding surfaces have infinitely many irreducible G-invariant subsets of the second type.

Theorem 4.6. *Let X be a non-orientable Riemann surface of topological genus $g \geq 3$ and let B_1 and B_2 be proper, G-invariant, irreducible subsets of the first type of cardinalities k and l. Then $|G| \leq 4(g-2+k+l)$ and this bound is attained for infinitely many genera.*

Proof. Let $X = \mathcal{H}/\Gamma$ be a non-orientable Riemann surface with a group of automorphisms $G = \Lambda/\Gamma$. Three cases are possible

(i) $|B_1| = |G|/m_1$, $|B_2| = |G|/m_2$ for some proper periods m_1, m_2 of Λ,
(ii) $|B_1| = |G|/2n_1$, $|B_2| = |G|/2n_2$ for some link periods n_1, n_2 of Λ,
(iii) $|B_1| = |G|/m$, $|B_2| = |G|/2n$ for some proper period m and a link period n of Λ,

In the case (i), $\mu(\Lambda) \geq 2\pi(1 - 1/m_1 - 1/m_2)$ and so by the Hurwitz–Riemann formula, $|G| \leq g - 2 + k + l$.

Now consider the case (ii). Here either Λ has nonzero orbit genus or a proper period, or two period cycles, or else it has a signature $(0; +; [-]; \{(n_1, n_2, \ldots, n_s)\})$. In the first case, $\mu(\Lambda) \geq 2\pi(1 - 1/2n_i - 1/2n_2)$, which gives $|G| \leq g - 2 + k + l$. In the second case, $\mu(\Lambda) \geq 2\pi(1/2 - 1/2n_1 - 1/2n_2)$, which by the Hurwitz–Riemann formula gives $|G| \leq 2(g - 2 + k + l)$. In the third case, $\mu(\Lambda) \geq 2\pi(1 - 1/2n_1 - 1/2n_2)$ which by the Hurwitz–Riemann formula gives again $|G| \leq g - 2 + k + l$. So let Λ have a signature $(0; +; [-]; \{(n_1, n_2, \ldots, n_s)\})$. If $s \geq 4$ then $\mu(\Lambda) \geq 2\pi(1/2 - 1/2n_1 - 1/2n_2)$, which by the Hurwitz–Riemann formula gives $|G| \leq 2(g - 2 + k + l)$. So assume that $s = 3$. Then $\mu(\Lambda) \geq 2\pi(1/4 - 1/2n_1 - 1/2n_2)$, which gives $|G| \leq 4(g - 2 + k + l)$.

Finally in the case (iii), $\mu(\Lambda) \geq 2\pi(1/2 - 1/m - 1/2n)$, which gives $|G| \leq 2(g - 2 + k + l)$.

To prove that the bound $4(g - 2 + k)$ is attained for infinitely many g, let G be an arbitrary H-group, say of order N. Then the corresponding surface has two G-invariant subsets of cardinalities $k = N/6$ and $l = N/14$. Now since $N = 84(g - 2)$ we have $g = N/84 + 2$ and therefore $4(g - 2 + k + l) = N$, which completes the proof. □

REMARK 4.7. Observe that the bound $4(g - 2 + k + l)$ for the order of a group of automorphisms of a non-orientable Riemann surface of genus g having two invariant subsets of points of the first type, is attained for H-groups and so for an NEC group Λ with signature $(0; +; [-]; \{(2, 3, 7)\})$. Hence Λ has reflections and again the corresponding surfaces have infinitely many G-invariant subsets of the second type.

Theorem 4.8. *Let X be a non-orientable Riemann surface of topological genus $g \geq 3$ with a group of automorphisms G and with G-invariant irreducible subsets $B_1, \ldots, B_n$ of cardinalities $q_1 \leq \cdots \leq q_n$ and assume that $B_1, \ldots, B_s$, where $s \geq 3$, are of the first type. Then*

$$|G| \leq \frac{2}{s-2}(g - 2 + q_1 + \cdots + q_s)$$

and this bound is attained for infinitely many g.

Proof. Let, as always, $X = \mathcal{H}/\Gamma$ and $G = \Lambda/\Gamma$. Let $B_1, \ldots, B_t$ come from the proper periods $m_1, \ldots, m_t$ and let $B_{t+1}, \ldots, B_s$ come from the link periods $n_1, \ldots, n_{s-t}$. Then $|B_i| = |G|/m_i$ for $i = 1, \ldots, t$ $|B_i| = |G|/2n_{i-t}$ for $i = t + 1, \ldots s$ and so

$$\begin{aligned}\mu(\Lambda) &\geq 2\pi\left(-1 + t - \sum_{i=1}^{t}\frac{1}{m_i} + \frac{s-t}{2} - \sum_{i=1}^{s-t}\frac{1}{2n_i}\right)\\ &\geq 2\pi\left(\frac{s-2}{2} - \sum_{i=1}^{t}\frac{1}{m_i} - \sum_{i=1}^{s-t}\frac{1}{2n_i}\right),\end{aligned}$$

which by the Hurwitz–Riemann formula gives

$$|G| \leq \frac{2}{s-2}(g-2+q_1+\cdots+q_s).$$

Now we shall show that these bounds are attained for infinitely many g and all $s \geq 3$.

For $s \geq 5$, let $G = Z_2 \oplus \overset{s-1}{\cdots} \oplus Z_2 = \langle a_1, \ldots, a_{s-1}\rangle$, let Λ be an NEC group with signature $(0; +; [-]; \{(2, \overset{s}{\ldots}, 2)\})$ and let $\theta\colon \Lambda \to G$ be an epimorphism defined by

$$\theta(c_i) = \begin{cases} a_{i+1} & \text{for} \quad i = 0, \ldots, s-2, \\ a_1 a_2 & \text{for} \quad i = s-1, \\ a_1 & \text{for} \quad i = s. \end{cases}$$

Now, neither a canonical reflection nor a canonical elliptic element belongs to $\Gamma = \ker\theta$. So Γ is torsion free. But since orientation reversing $c_0c_1c_{s-1}$ isometry belongs to Γ, the last has signature $(g; -; [-]; \{-\})$, where by the Hurwitz–Riemann formula $g = 2^{s-3}(s-4)+2$ and so $X = \mathcal{H}/\Gamma$ is a non-orientable Riemann surface of genus g having the group of automorphisms G and G-invariant subsets $B_1, \ldots, B_s$ of cardinalities $q_1, \ldots, q_s$, for which

$$|G| = \frac{2}{s-2}(g-2+q_1+\cdots+q_s).$$

Finally, given an odd prime p, let for the above Γ, Γ_p be its p-Frattini subgroup i.e. the subgroup generated by all commutators and all p-powers. Now Γ_p is a characteristic subgroup of Γ and hence is normal in Λ, which produces surfaces X_p and subsets as above for infinitely many genera $g_p = p^{g-1}(2^{s-3}(s-4)+2)$.

The case $s = 4$ must be treated separately. Let $G = Z_2 \oplus D_3 = \langle x \mid x^2\rangle \oplus \langle a, b \mid a^2, b^2, (ab)^3\rangle$, let Λ be an NEC group with signature $(0; -; [-]; \{(2, 2, 2, 3)\})$ and let $\theta\colon \Lambda \to G$ be an epimorphism which maps consecutive canonical reflections into a, x, xb, b, a. Then, as for $s > 4$, we argue that $\Gamma = \ker\theta$ has signature $(g; -; [-]; \{-\})$ and so we obtain a non-orientable Riemann surface for which the above bound is attained. Next, using the above Frattini arguments, we produce such surfaces and subsets for infinitely many genera g.

Finally for $s = 3$, the bound $2(g-2+k+l+m)$ for the order of a group of automorphisms of a non-orientable Riemann surface of genus g having three invariant subsets of points of cardinalities k, l and m, is attained, for example, for surfaces with H-groups G of automorphisms. □

REMARK 4.9. Observe that the formulae from the above theorem does not involve explicitly the cardinality of the G-invariant subsets of the second type if $s \neq t$. However this can be done writing the bound from our theorem as

$$|G| \leq \frac{2}{s-2}(g-2+q_1+\cdots+q_s) + 0\cdot(q_{s+1}+\cdots+q_t).$$

and obtaining in this way a function properly involving all q_i.

The following Corollary generalizes the principal result from [7], where it was proved for bordered non-orientable Klein surfaces, and the result from [3], where the bound was found for $q = 1, 2$.

Corollary 4.10. *Let φ be an automorphism of a non-orientable Riemann surface of topological genus $g \geq 3$ having $q \geq 3$ fixed points. Then*

$$|\varphi| \leq \frac{2}{q-2}(g+q-2).$$

5. Bordered non-orientable Klein surfaces

Now we shall derive some results concerning automorphisms of bordered non-orientable Klein surfaces with fixed points. Here, as in subsection 3.4, we shall use two tools: an analogue of Oikawa theorem mentioned by Greenleaf and May in [10] for non-orientable bordered Klein surfaces, and the canonical Riemann double cover described in [1]. The symbol p will stand for the algebraic genus of a non-orientable bordered Klein surface.

Corollary 5.1. *Let G be a group of automorphisms of a bordered non-orientable Klein surface of topological genus $g \geq 3$ having k boundary components. Then $|G| \leq 12(p-1)$.*

Proof. G can be considered as a group of automorphisms of a non-orientable Riemann surface having invariant subset of cardinality k. But then, by Theorem 4.4, $|G| \leq 12(g-2+k) = 12(p-1)$. □

The next two corollaries can be obtained using, for bordered non-orientable Klein surface X, the canonical double cover $\tilde{X}$ as in Section 3.4

Corollary 5.2. *Let G be a group of automorphisms of a bordered, non-orientable Klein surface X of algebraic genus $p \geq 2$ having an invariant subset of points of cardinality q. Then $|G| \leq 4(p-1) + 4q$.*

Proof. The canonical Riemann double cover $\tilde{X}$ of X has genus p and let $\tilde{G}$ be the lifting of G. Then $\tilde{G}^+$ has two invariant subsets of points on $\tilde{X}$ of cardinalities q and so $|G| = |\tilde{G}^+| \leq 4(p-1) + 4q$, by Corollary 3.3. □

Corollary 5.3. *Let G be a group of automorphisms of a bordered, non-orientable orientable Klein surface X of algebraic genus $p \geq 2$, having $s \geq 2$ invariant subsets* *of interior points of cardinalities* $q_1, \ldots, q_s$. *Then*

$$|G| \leq \frac{1}{s-1}(p-1) + \frac{1}{s-1}(q_1 + \cdots + q_s).$$

Proof. Let $\tilde{X}$ be the canonical Riemann double cover and let $\tilde{G}$ be the lifting of G. Then $\tilde{G}^+$ has 2s invariant subsets of points on $\tilde{X}$ of cardinalities $q_1, q_1, \ldots, q_s, q_s$ and so

$$|\tilde{G}^+| \leq \frac{2}{2s-2}(p-1) + \frac{2}{2s-2}(q_1 + \cdots + q_s)$$

by Theorem 3.6. Hence the result since $|G| = |\tilde{G}^+|$. □

Similarly, using Oikawa theorem and Theorem 4.8, we obtain

Corollary 5.4. *Let* G *be a group of a bordered, non-orientable Klein surface* X *of topological genus* $g \geq 3$ *having* s *invariant subsets of the set of boundary components of cardinalities* $p_1, \ldots, p_s$ *and* t *invariant subsets of interior points of cardinalities* $q_1, \ldots, q_t$, *where* $t + s \geq 3$. *Then*

$$|G| \leq \frac{2}{s+t-2}(g-2) + \frac{2}{s+t-2}(p_1 + \cdots + p_s + q_1 + \cdots q_t).$$

Using our results one can also obtain, at once, the bounds for the group of automorphisms of q-hyperelliptic and cyclic q-trigonal bordered Klein surfaces of genus large enough which were found in [18] by Pérez del Pozo. We finish the paper by considering similar problem for non-orientable unbordered Klein surfaces.

Recall that a Klein surface X which admit an automorphism φ of order p so that the orbit space $X/\langle\varphi\rangle$ has algebraic genus q is said to be a (p, q)-gonal Klein surface and for $p = 2$ and $p = 3$ we obtain the concepts of q-hyperellipticity and q-trigonality respectively.

Theorem 5.5. *Let* X *be a non-orientable* q*-hyperelliptic* $(q \geq 2)$ *Riemann surface of algebraic genus* $p > q + 1$ *and let* φ *be the* q*-hyperelliptic involution. Then*

$$|\mathrm{Aut}(X)| \leq \begin{cases} 8(p-q) & \textit{if } X/\varphi \textit{ has nonempty boundary,} \\ 24(p-q) & \textit{otherwise.} \end{cases}$$

Proof. Let $X = \mathcal{H}/\Gamma$, $\mathrm{Aut}(X) = \Lambda/\Gamma$ and $\langle\varphi\rangle = \Gamma'/\Gamma$ for some NEC groups Γ, Γ', Λ. By [5], Γ' has signature

$$(h; \pm; [2, \overset{p+1-2q}{\ldots}, 2]; \{(-), \overset{q+1-\eta h}{\ldots}, (-)\}),$$

for some h in range $0 \le h \le q/2$ or $1 \le h \le q$ respectively, where $\eta = 2$ or 1 if the sign is $+$ or $-$, or else $h = q + 1$ for $\eta = 1$ and p odd.

Now $Y = X/\varphi = \mathcal{H}/\Gamma'$ is a Klein surface of algebraic genus q, orientable or not according to the sign of Γ', having $p + 1 - 2q$ distinguished interior points. But since $p > q + 1$, Γ' is unique, by [9], and so in particular this set of interior points is invariant under the action of Λ/Γ'.

Now if Y has nonempty boundary, then using Corollaries 3.11 or 5.2 we obtain $|\mathrm{Aut}(Y)| \le 4(p-q)$ in the orientable and non-orientable cases. Therefore $|\mathrm{Aut}(X)| = 2|\mathrm{Aut}(Y)| \le 8(p-q)$.

If Y is unbordered then, by Oikawa theorem in the orientable case (observe that here we allow also antianalytic automorphisms) and by Theorem 4.4 in the nonorientable one, $|\mathrm{Aut}(Y)| \le 12(p-q)$ and so $|\mathrm{Aut}(X)| = 2|\mathrm{Aut}(Y)| \le 24(p-q)$. □

In a similar way, using characterization of cyclic q-trigonality from [8], we can prove

Theorem 5.6. *Let X be a non-orientable cyclic q-trigonal Riemann surface of algebraic genus $p \ge q + 1$, where $q \ge 2$, and let φ be automorphism of cyclic q-trigonality. Then*

$$|\mathrm{Aut}(X)| \le \begin{cases} 6(p-q) & \text{if } X/\varphi \text{ has nonempty boundary,} \\ 18(p-q) & \text{otherwise.} \end{cases}$$

Proof. Let, as before, $X = \mathcal{H}/\Gamma$, $G = \Lambda/\Gamma$ and $\langle\varphi\rangle = \Gamma'/\Gamma$ for some NEC groups Γ, Γ', Λ. By [8], Γ' has signature

$$(h; \pm; [3, \overset{(p+2-3q)/2}{\ldots}, 3]; \{(-), \overset{q+1-\eta h}{\ldots}, (-)\}),$$

where $\eta = 2$ or 1 according to if the sign is $+$ or $-$, h is an integer in range $0 \le h \le (q+1)/\eta$ and X/φ is unbordered only for $\eta = 1$ and $h = q+1$. Now $Y = X/\varphi = \mathcal{H}/\Gamma'$ is a Klein surface of algebraic genus q having $(p + 2 - 3q)/2$ distinguished interior points. But, since $p \ge q + 1$, Γ' is unique, by [9] and so in particular this set is invariant under the action of Λ/Γ'.

If Y has nonempty boundary then, using Corollaries 3.11 or 5.2, we obtain $|\mathrm{Aut}^{\pm}(Y)| \le 2(p-q)$ in the orientable and non-orientable case respectively and therefore $|\mathrm{Aut}^{\pm}(X)| = 3|\mathrm{Aut}^{\pm}(Y)| \le 6(p-q)$, while if Y is unbordered, then it is a non-orientable Riemann surface of topological genus $q + 1$. So $|\mathrm{Aut}(Y)| \le 6(p-q)$, by Theorem 4.4 and therefore $|\mathrm{Aut}^{\pm}(X)| = 3|\mathrm{Aut}(Y)| \le 18(p-q)$. □

ACKNOWLEDGEMENTS. The authors are very grateful to an anonymous referee for pointing out some gaps in the submitted version, for his or her accurate and valuable comments, suggestions and in particular for remarking that our construction of infinitely

many normal subgroup in a given Fuchsian group through the use of Poincare lemma and Fenchel "conjecture" can be recognized as a corollary of the Selberg lemma.

References

[1] N.L. Alling and N. Greenleaf: Foundations of the Theory of Klein Surfaces, Lecture Notes in Mathematics **219**, Springer, Berlin, 1971.

[2] T. Arakawa: *Automorphism groups of compact Riemann surfaces with invariant subsets*, Osaka J. Math. **37** (2000), 823–846.

[3] E. Bujalance: *Cyclic groups of automorphisms of compact nonorientable Klein surfaces without boundary*, Pacific J. Math. **109** (1983), 279–289.

[4] E. Bujalance, J.J. Etayo, J.M. Gamboa and G. Gromadzki: Automorphism Groups of Compact Bordered Klein Surfaces, Lecture Notes in Mathematics **1439**, Springer, Berlin, 1990.

[5] J.A. Bujalance and B. Estrada: *q-hyperelliptic compact nonorientable Klein surfaces without boundary*, Int. J. Math. Math. Sci. **31** (2002), 215–227.

[6] T.C. Chau: *A note concerning Fox's paper on Fenchel's conjecture*, Proc. Amer. Math. Soc. **88** (1983), 584–586.

[7] C. Corrales, J.M. Gamboa and G. Gromadzki: *Automorphisms of Klein surfaces with fixed points*, Glasg. Math. J. **41** (1999), 183–189.

[8] B. Estrada and E. Martínez: *q-trigonal Klein surfaces*, Israel J. Math. **131** (2002), 361–374.

[9] H.M. Farkas and I. Kra: Riemann Surfaces, Graduate Texts in Mathematics **71**, Springer, New York, 1980.

[10] N. Greenleaf and C.L. May: *Bordered Klein surfaces with maximal symmetry*, Trans. Amer. Math. Soc. **274** (1982), 265–283.

[11] R.H. Fox: *On Fenchel's conjecture about F-groups*, Mat. Tidsskr. B. **1952** (1952), 61–65.

[12] G. Gromadzki: *On conjugacy of p-gonal automorphisms of Riemann surfaces*, Rev. Mat. Complut. **21** (2008), 83–87.

[13] A. Hurwitz: *Über algebraische Gebilde mit eindeutigen Transformationen in sich*, Math. Ann. **41** (1893), 402–442.

[14] A.M. Macbeath: *The classification of non-euclidean plane crystallographic groups*, Canad. J. Math. **19** (1967), 1192–1205.

[15] C.L. May: *Automorphisms of compact Klein surfaces with boundary*, Pacific J. Math. **59** (1975), 199–210.

[16] K. Oikawa: *Notes on conformal mappings of a Riemann surface onto itself*, Kōdai Math. Sem. Rep. **8** (1956), 23–30.

[17] T. Szemberg: *Automorphisms of Riemann surfaces with two fixed points*, Ann. Polon. Math. **55** (1991), 343–347.

[18] Á.L. Pérez del Pozo: *Automorphism groups of compact bordered Klein surfaces with invariant subsets*, Manuscripta Math. **122** (2007), 163–172.

[19] D. Singerman: *Automorphisms of compact non-orientable Riemann surfaces*, Glasgow Math. J. **12** (1971), 50–59.

[20] H.C. Wilkie: *On non-Euclidean crystallographic groups*, Math. Z. **91** (1966), 87–102.

E. Bujalance
Departamento de Matemáticas Fundamentales
UNED
Paseo Senda del Rey 9. 28040-Madrid
Spain
e-mail: ebujalan@mat.uned.es

G. Gromadzki
Institute of Mathematics
Gdańsk University
Wita Stwosza 57, 80-952 Gdańsk
Poland
e-mail: greggrom@math.univ.gda.pl