THE MAGNETIC FLOW ON THE MANIFOLD OF
ORIENTED GEODESICS OF
A THREE DIMENSIONAL SPACE FORM

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Abstract

Let $M$ be the three dimensional complete simply connected manifold of constant sectional curvature 0, 1 or $-1$. Let $\mathcal{L}$ be the manifold of all (unparametrized) complete oriented geodesics of $M$, endowed with its canonical pseudo-Riemannian metric of signature $(2, 2)$ and Kähler structure $J$. A smooth curve in $\mathcal{L}$ determines a ruled surface in $M$.

We characterize the ruled surfaces of $M$ associated with the magnetic geodesics of $\mathcal{L}$, that is, those curves $\sigma$ in $\mathcal{L}$ satisfying $\nabla_{\dot{\sigma}} \dot{\sigma} = J\dot{\sigma}$. More precisely: a timelike (spacelike) magnetic geodesic determines the ruled surface in $M$ given by the binormal vector field along a helix with positive (negative) torsion. Null magnetic geodesics describe cones, cylinders or, in the hyperbolic case, also cones with vertices at infinity. This provides a relationship between the geometries of $\mathcal{L}$ and $M$.

1. Introduction

For $\kappa = 0, 1, -1$, let $M_{\kappa}$ be the three dimensional complete simply connected manifold of constant sectional curvature $\kappa$, that is, $\mathbb{R}^3$, $S^3$ and the hyperbolic space $\mathbb{H}^3$. Let $\mathcal{L}_{\kappa}$ be the manifold of all (unparametrized) complete oriented geodesics of $M_{\kappa}$. We may think of an element $c$ in $\mathcal{L}_{\kappa}$ as the equivalence class of unit speed geodesics $\gamma : \mathbb{R} \to M_{\kappa}$ with image $c$ such that $\{\dot{\gamma}(s)\}$ is a positive basis of $T_{\gamma(s)}c$ for all $s$.

Let $\gamma$ be a complete unit speed geodesic of $M_{\kappa}$ and let $\mathcal{J}_{\gamma}$ be the space of all Jacobi fields along $\gamma$ which are orthogonal to $\gamma$. There exists a well-defined canonical isomorphism

$$T_{\gamma} : \mathcal{J}_{\gamma} \to T_{[\gamma]}\mathcal{L}_{\kappa}, \quad T_{\gamma}(J) = \left. \frac{d}{dt} \right|_0 [\gamma_t],$$

where $\gamma_t$ is any variation of $\gamma$ by unit speed geodesics associated with $J$ (see [11]).

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A pseudo-Riemannian metric of signature (2, 2) can be defined on $\mathcal{L}_\kappa$ as follows [12]: For $X \in T_{[\gamma]} \mathcal{L}_\kappa$, the square norm $\|X\| = \langle X, X \rangle$ is well defined by

$$\|X\| = \langle \dot{\gamma} \times J, J' \rangle,$$

where $X = T_\gamma(J)$, the cross product $\times$ is induced by a fixed orientation of $M_\kappa$ and $J'$ denotes the covariant derivative of $J$ along $\gamma$. Indeed, the right hand side of (2) is a constant function. In the following, for any vector $X$, we will denote $\|X\| = \langle X, X \rangle$ and $|X| = \sqrt{\langle X, X \rangle}$. Recall that $X$ is null, time-like or space-like if $\|X\| = 0$, $\|X\| < 0$ or $\|X\| > 0$, respectively.

Let $[\gamma] \in \mathcal{L}_\kappa$ and let $R_\gamma$ be the rotation in $M_\kappa$ fixing $\gamma$ through an angle of $\pi/2$. This rotation induces an isometry $\tilde{R}_\gamma$ of $\mathcal{L}_\kappa$ whose differential at $[\gamma]$ is a linear isometry of $T_{[\gamma]} \mathcal{L}_\kappa$ squaring to $-\text{id}$. This yields a complex structure $J$ on $\mathcal{L}_\kappa$. With the metric defined above, $\mathcal{L}_\kappa$ is Kähler. Recent generalizations of this fact can be found in [1, 3].

A magnetic geodesic $\sigma$ of $\mathcal{L}_\kappa$ is a curve satisfying $\nabla_\sigma \dot{\sigma} = J \dot{\sigma}$. These curves have constant speed, so they will be null, time-like or space-like.

A smooth curve in $\mathcal{L}_\kappa$ determines a ruled surface in $M_\kappa$. For $\kappa = 0, -1$, a generic geodesic of $\mathcal{L}_\kappa$ describes a helicoid in $M_\kappa$ [7, 6, respectively]. Our purpose is to characterize the ruled surfaces in $M_\kappa$ associated with the magnetic geodesics of $\mathcal{L}_\kappa$. For $v \in T M_\kappa$, $\gamma_v$ denotes the geodesic of $M_\kappa$ with initial velocity $v$.

**Theorem 1.** A generic magnetic geodesic $\sigma$ of $\mathcal{L}_\kappa$ describes the ruled surface in $M_\kappa$ given by the binormal vector field of a helix. More precisely, $\sigma$ is a time-like (space-like) magnetic geodesic of $\mathcal{L}_\kappa$ if and only if $\sigma$ has the form

$$\sigma(t) = [\gamma_{B(t)}],$$

where $B$ is the binormal vector field of a helix in $M_\kappa$ with curvature $k$, speed $1/k$ and positive (negative) torsion, for some $k > 0$.

Now we study null magnetic geodesics in $\mathcal{L}_{-1} = \mathcal{L}(\mathbb{H}^3)$. We recall some concepts related with the hyperbolic space (see for instance [5]).

Two unit speed geodesics $\gamma$ and $\sigma$ of $\mathbb{H}^3$ are said to be asymptotic if there exists a positive constant $C$ such that $d(\gamma(s), \sigma(s)) \leq C$, $\forall s \geq 0$. Two unit vectors $v, w \in T^1 \mathbb{H}^3$ are said to be asymptotic if the corresponding geodesics $\gamma_v$ and $\gamma_w$ have this property.

A point at infinity for $\mathbb{H}^3$ is an equivalence class of asymptotic geodesics of $\mathbb{H}^3$. The set of all points at infinity for $\mathbb{H}^3$ is denoted by $\mathbb{H}^3(\infty)$ and has a canonical differentiable structure diffeomorphic to the 2-sphere. The equivalence class represented by a geodesic $\gamma$ is denoted by $\gamma(\infty)$, and the equivalence class represented by the oppositely oriented geodesic $s \mapsto \gamma(-s)$ is denoted by $\gamma(-\infty)$. 

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The content is presented in a clear and readable format, with proper formatting for mathematical expressions and theorems. The text is divided into logical sections, making it easier to follow and understand the mathematical concepts discussed.
Given \( v \in T^1\mathbb{H}^3 \), the horosphere \( H(v) \) is the limit of metric spheres \( \{ S_n \} \) in \( \mathbb{H}^3 \) that pass through the foot point of \( v \) as the centers \( \{ p_n \} \) of \( \{ S_n \} \) converge to \( \gamma_0(\infty) \).

Below we present a more precise definition.

Let \( \psi^\pm : \mathcal{L}(\mathbb{H}^3) \to \mathbb{H}^3(\infty) \) be the smooth functions given by \( \psi^\pm([\gamma]) = \gamma(\pm\infty) \) and let \( \mathcal{D}^\pm \) be the distributions on \( \mathcal{L}(\mathbb{H}^3) \) given by \( \mathcal{D}^\pm_{[\gamma]} = \text{Ker}(d\psi^\pm_{[\gamma]}) \). These distributions are called the horospherical distributions on \( \mathcal{L}(\mathbb{H}^3) \).

**Cones with vertices at infinity:** Let \( x \in \mathbb{H}^3(\infty) \) and let \( v_o \in T^1\mathbb{H}^3 \) such that \( \gamma_{v_o}(\pm\infty) \in x \). Let \( t \mapsto v(t) \) be a curve in \( T^1\mathbb{H}^3 \) such that \( v(0) = \pm v_o \), \( v(t) \) is asymptotic to \( \pm v_o \) for all \( t \in \mathbb{R} \) and the foot points of \( v(t) \) lie on a circle of geodesic curvature \( \pm k \) (with \( k > 0 \)) and speed \( 1/k \) in the horosphere determined by \( \pm v_o \). Under these conditions we say that the curve in \( \mathcal{L}(\mathbb{H}^3) \) given by \( t \mapsto [\gamma_{v(t)}^\pm] \) describes a forward cone with vertex at \( x \) (for \( + \)) or a backward cone with vertex at \( x \) (for \( - \)). These cones can be better visualized in the upper half space model of \( \mathbb{H}^3 \) (in particular \( \mathbb{H}^3(\infty) = \{ z = 0 \} \cup \{ \infty \} \)):

- Let \( \gamma_{v_o}(s) = \left( (1/k) \cos(t), \pm(1/k) \sin(t), e^{\pm t} \right) \). A curve \( \sigma \) in \( \mathcal{L}(\mathbb{H}^3) \) describes a cone with forward (respectively, backward) vertex at \( \infty \) if it is \( SL(2, \mathbb{C}) \)-congruent to \( t \mapsto [\gamma_{v_o}^+] \) (respectively, \( t \mapsto [\gamma_{v_o}^-] \)).

**Theorem 2.** A null magnetic geodesic of \( \mathcal{L}(\mathbb{H}^3) \) describes in \( \mathbb{H}^3 \) a cylinder, a cone with vertex at \( p \in \mathbb{H}^3 \) or a cone with vertex at infinity. More precisely, if \( \sigma \) is a curve in \( \mathcal{L}(\mathbb{H}^3) \), then

a) \( \sigma \) is a null magnetic geodesic with \( \dot{\sigma}(0) \in \mathcal{D}_{\sigma(0)}^\pm \) if and only if \( \sigma \) describes a cone with vertex at \( \sigma(0)(\pm\infty) \) (forward for \( + \) and backward for \( - \));

b) \( \sigma \) is a null magnetic geodesic with \( \dot{\sigma}(0) \notin \mathcal{D}_{\sigma(0)}^\pm \) if and only if \( \sigma \) either has the form

\[
\sigma(t) = [\gamma_{v(t)}^B],
\]

where \( B \) is the binormal vector field of a helix \( h \) in \( \mathbb{H}^3 \) with curvature \( k \), speed \( 1/k \) and zero torsion (in particular, \( h \) is contained in a totally geodesic surface \( S \) and \( B \) is normal to \( S \) and parallel along \( h \)), or \( \sigma \) has the form

\[
\sigma(t) = [\gamma_{v(t)}^s],
\]

where \( v \) is a curve with geodesic curvature \( k \) and speed \( 1/k \) in \( T^1_p\mathbb{H}^3 \), for some \( p \in \mathbb{H}^3 \), for certain \( k > 0 \).

**Theorem 3.** The ruled surfaces associated with null magnetic geodesics of \( \mathcal{L}_\kappa \) for \( \kappa = 0, 1 \) are described in an analogous manner as in the previous theorem, except that case a) is empty. Besides, for \( \kappa = 1 \), a null magnetic geodesic has simultaneously the forms (4) and (5).
2. Preliminaries

For the simultaneous analysis of the three cases \( \kappa = 0,1,-1 \), we consider the standard presentation of \( M_\kappa \) as a submanifold of \( \mathbb{R}^4 \). That is, \( \mathbb{R}^3 = \{ (1, x) \in \mathbb{R}^4 \mid x \in \mathbb{R}^3 \} \), \( \mathbb{S}^3 = \{ x \in \mathbb{R}^4 \mid |x|^2 = 1 \} \) and \( \mathbb{H}^3 = \{ x \in \mathbb{R}^4 \mid -x_0^2 + x_1^2 + x_2^2 + x_3^2 = -1 \text{ and } x_0 > 0 \} \).

Let \( G_\kappa \) be the identity component of the isometry group of \( M_\kappa \), that is, \( G_0 = SO_3 \ltimes \mathbb{R}^3 \), \( G_1 = SO_4 \) and \( G_{-1} = O_0(1,3) \). We consider the usual presentation of \( G_0 \) as a subgroup of \( GL_4(\mathbb{R}) \). The group \( G_\kappa \) acts on \( \mathcal{L}_\kappa \) as follows: \( g \cdot [y] = [g \circ y] \). This action is transitive and smooth.

If we denote by \( \mathfrak{g}_\kappa \) the Lie algebra of \( G_\kappa \) we have that

\[
\mathfrak{g}_\kappa = \left\{ \begin{pmatrix} 0 & -\kappa x^t \\ x & B \end{pmatrix} \mid x \in \mathbb{R}^3, B \in SO_3 \right\}.
\]

Let \( \gamma_0 \) be the geodesic in \( M_\kappa \) with \( \gamma_0(0) = e_0 \) and initial velocity \( e_1 \in T_{e_0}M_\kappa \), where \( \{ e_0, e_1, e_2, e_3 \} \) is the canonical basis of \( \mathbb{R}^4 \). For \( A, B \in \mathbb{R}^{2 \times 2} \), let \( \text{diag}(A, B) = \begin{pmatrix} A & 0_2 \\ 0_2 & B \end{pmatrix} \), where \( 0_2 \) denotes the \( 2 \times 2 \) zero matrix. Then the isotropy subgroup of \( G_\kappa \) at \( [\gamma_0] \) is

\[
H_\kappa = \{ \text{diag}(R_\kappa(t), B) \mid t \in \mathbb{R}, B \in SO_2 \},
\]

where

\[
R_0(t) = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}, \quad R_1(t) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}, \quad R_{-1}(t) = \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}.
\]

Let \( j = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \). The Lie algebra of \( H_\kappa \) is

\[
\mathfrak{h}_\kappa = \{ \text{diag}(r_\kappa(t), sj) \mid s, t \in \mathbb{R} \},
\]

where \( r_\kappa(t) = \begin{pmatrix} 0 & -\kappa t \\ t & 0 \end{pmatrix} \). We may identify \( \mathcal{L}_\kappa \) with \( G_\kappa/H_\kappa \) via the diffeomorphism

\[
\phi: G_\kappa/H_\kappa \to \mathcal{L}_\kappa, \quad \phi(gH_\kappa) = g \cdot [\gamma_0].
\]

For \( x, y \in \mathbb{R}^2 \) we denote \( Z(x, y) = \begin{pmatrix} 0_2 & (-\kappa x, -y)^t \\ (x, y) & 0_2 \end{pmatrix} \). Let

\[
\mathfrak{p}_\kappa = \{ Z(x, y) \in \mathfrak{g}_\kappa \mid x, y \in \mathbb{R}^2 \},
\]

which is an \( \text{Ad}(H_\kappa) \)-invariant complement of \( \mathfrak{h}_\kappa \).

For \( \kappa = 0,1 \), we consider on \( \mathfrak{g}_\kappa \) the inner product such that \( \mathfrak{h}_\kappa \perp \mathfrak{p}_\kappa \), \( \| Z(x, y) \| = \det(x, y) \) and

\[
\| \text{diag}(r_\kappa(t), sj) \| = -ts.
\]
(for $\kappa = 0$, we have learnt of this inner product from [9, p.499]). On $\mathfrak{g}_{-\kappa}$ we consider the Killing form ($\mathfrak{h}_{\kappa} \perp \mathfrak{p}_{\kappa}$ also holds). For $\kappa = 0,1,-1$, this inner product on $\mathfrak{g}_{\kappa}$ induces on $G_\kappa$ a bi-invariant metric. Thus, there exists an unique pseudo-Riemannian metric on $\mathcal{L}_\kappa \simeq G_\kappa/H_\kappa$ such that $\pi: G_\kappa \to G_\kappa/H_\kappa$ is a pseudo-Riemannian submersion. For $\kappa = 0,1$, this metric on $\mathcal{L}_\kappa$ coincides with the given in (2), see Lemma 5 b). For $\kappa = -1$, the metric on $\mathcal{L}_{-1}$ associated with the Killing form is different from the one in (2). However, the magnetic geodesics of either metric on $\mathcal{L}_{-1}$ are the same. This follows since the geodesics are the same (see [11]), so the Levi-Civita connections coincide.

Let us call $A = \text{diag}(0_2, j)$, which is in the center of $\mathfrak{h}_{\kappa}$. We have that $\text{ad}_A$ is orthogonal and $\text{ad}_A^2 = -\text{id}$ in $\mathfrak{p}_{\kappa}$. Hence, $\text{ad}_A$ induces a complex structure on $G_\kappa/H_\kappa$.

A straightforward computation shows that it coincides, via $\phi$ in (7), with the complex structure given in the introduction. With the metric above and this complex structure, $\mathcal{L}_\kappa$ is a Hermitian symmetric space.

As a direct application of a result by Adachi, Maeda and Udagawa in [2] (see also [8] and Remark 1 in [4]) we have

**Theorem 4.** Let $\sigma$ be a magnetic geodesic of $G_\kappa/H_\kappa$ with initial conditions $\sigma(0) = H_\kappa$ and $\dot{\sigma}(0) = X \in \mathfrak{p}_\kappa$. Then $\sigma(t) = \pi(\exp t(X + A))$.

As we saw in (1), $\mathcal{J}_{\gamma_0}$ is isomorphic to $T_{[\gamma_0]}\mathcal{L}_\kappa \cong \mathfrak{p}_\kappa$. In the next Lemma we relate $\mathfrak{p}_\kappa$ and $\mathcal{J}_{\gamma_0}$ explicitly, involving the matrix $A$.

**Lemma 5.** Let $Z = Z(x, y) \in \mathfrak{p}_\kappa$.

a) The Jacobi field $J(s) = (d/dt)\big|_0 \exp t(Z + A) \cdot \gamma_0(s)$ in $\mathcal{J}_{\gamma_0}$ is the unique one that satisfies $J(0) = (0, 0, x)'$ and $J'(0) = (0, 0, y)'$.

b) $T_{\gamma_0}(J) = d(\phi \circ \pi)Z$ and its norm is $\|d(\phi \circ \pi)Z\| = \det(x, y)$.

**Proof.** For each $\kappa$, we consider the following parameterization of $\gamma_0$:

$$\gamma_0(s) = (1, s, 0, 0), \quad \text{if} \quad \kappa = 0;$$

$$\gamma_0(s) = (\cos s, \sin s, 0, 0), \quad \text{if} \quad \kappa = 1;$$

$$\gamma_0(s) = (\cosh s, \sinh s, 0, 0), \quad \text{if} \quad \kappa = -1.$$

Given $Z = Z(x, y) \in \mathfrak{p}_\kappa$, the Jacobi field along $\gamma_0$ defined by $J(s) = (d/dt)\big|_0 \exp t(Z + A) \cdot \gamma_0(s)$ belongs to $\mathcal{J}_{\gamma_0}$, because for all $s \in \mathbb{R}$,

$$\langle J(s), \dot{\gamma_0}(s) \rangle = \langle (Z + A)(\gamma_0(s)), \dot{\gamma_0}(s) \rangle = 0,$$

since $(Z + A)(\gamma_0(s))$ is orthogonal to $e_0$ and $e_1$, while $\dot{\gamma_0}(s)$ has non zero components only in these two directions.
One verifies easily that $J(0) = (Z + A)(e_0) = (0, 0, x)'$. On the other hand,

\[
J'(0) = \left. \frac{D}{ds} \right|_{s=0} \left. \frac{\partial}{\partial t} \right|_{t=0} \exp t(Z + A) \cdot \gamma_o(s) = \frac{D}{dt} \left. \exp t(Z + A)(e_1) = (Z + A)(e_1) = (0, 0, y)'. \right.
\]

Besides,

\[
T_{\gamma_o}(J) = \left. \frac{d}{dt} \right|_{t=0} \left[ \exp t(Z + A) \cdot \gamma_o \right] = \frac{d}{dt} \left|_{t=0} \phi(\exp t(Z + A)H_c) \right.
\]

\[
= \frac{d}{dt} \left|_{t=0} \phi(\pi(\exp t(Z + A))) = d\phi \circ d\pi Z, \right.
\]

where the last equality holds since $A \in \mathfrak{h}_c$. Finally, the norm (2) of $d(\phi \circ \pi)Z$ equals

\[
\|d(\phi \circ \pi)Z\| = \langle \dot{\gamma}_o(0) \times J(0), J'(0) \rangle = \det(x, y)
\]

and the assertions of b) are verified. \qed

Let $Z(x, y) \in \mathfrak{p}_c$ and let $h = \text{diag}(R_c(t), B) \in H_\kappa$, where $B \in SO_2$ and

\[
R_c(t) = \begin{pmatrix} c_c(t) & -\kappa s_c(t) \\ s_c(t) & c_c(t) \end{pmatrix}
\]

is as in (6). Then $\text{Ad}(h)Z(x, y) = Z(Bx_1, By_1)$, where

\[
x_1 = c_c(t)x - s_c(t)y, \quad y_1 = \kappa s_c(t)x + c_c(t)y.
\]

We denote by $\epsilon_1$ and $\epsilon_2$ the vectors of the canonical basis of $\mathbb{R}^2$. 

**Lemma 6.** Let $Z(x, y) \neq 0$ in $\mathfrak{p}_c$.

a) If $\{x, y\}$ is a linearly independent set of $\mathbb{R}^2$, then there exists $h \in H_\kappa$ such that $\text{Ad}(h)Z(x, y) = Z(a\epsilon_1, b\epsilon_2)$, with $a > 0$ and $b \neq 0$, for $\kappa = 0, \pm 1$.

b) If $\kappa = 0, 1$ and $\{x, y\}$ is a linearly dependent set of $\mathbb{R}^2$, then there exists $h \in H_\kappa$ such that either $\text{Ad}(h)Z(x, y) = Z(0, b\epsilon_2)$, with $b \neq 0$, or $\text{Ad}(h)Z(x, y) = Z(a\epsilon_1, 0)$, with $a > 0$. This is true for $\kappa = -1$ if in addition $|x| \neq |y|$.

c) For $\kappa = 1$, there exists $h \in H_\kappa$ such that $\text{Ad}(h)Z(\epsilon_1, 0) = Z(0, \epsilon_2)$.

Proof. For the proof of a), as $\{x, y\}$ is a linearly independent set, then for $\kappa = 0, \pm 1$ there exists $t \in \mathbb{R}$ such that $\langle x_t, y_t \rangle = 0$. Indeed, for each $\kappa$, this is equivalent
to the fact that the equation
\[ c_3 - c_2t = 0 \quad \text{if} \quad \kappa = 0; \]
\[ \frac{1}{2}(c_1 - c_2) \sin(2t) + c_3 \cos(2t) = 0 \quad \text{if} \quad \kappa = 1; \]
\[ -\frac{1}{2}(c_1 + c_2) \sinh(2t) + c_3 \cosh(2t) = 0 \quad \text{if} \quad \kappa = -1 \]
has a real solution, where \( c_1 = \langle x, x \rangle, \ c_2 = \langle y, y \rangle \) and \( c_3 = \langle x, y \rangle \). But the linear independence of \( x \) and \( y \) determines the existence of the solution in each case. Then, we can take \( B \in SO_2 \) such that \( Bx_i = a\epsilon_1 \), with \( a > 0 \) and \( By_i = b\epsilon_2 \), with \( b \neq 0 \). Therefore the isometry \( h = \text{diag}(R_z(t), B) \in H_\kappa \) satisfies \( \text{Ad}(h)Z(x, y) = Z(a\epsilon_1, b\epsilon_2) \).

For the proof of \( b \), first we suppose that \( x = 0 \) or \( y = 0 \) (but not both zero since \( Z(x, y) \neq 0 \)). Let \( B \in SO_2 \) such that \( Bx = a\epsilon_1 \) with \( a > 0 \), if \( x \neq 0 \), and in the case that \( y \neq 0 \), let \( B \in SO_2 \) such that \( By = b\epsilon_2 \), with \( b \neq 0 \). Then we can take \( h = \text{diag}(I, B) \in H_\kappa \).

Now, let \( x \neq 0 \) and \( y \neq 0 \). So \( x = \lambda y \) or \( y = \lambda x \), with \( \lambda \neq 0 \). We suppose that \( y = \lambda x \) (for \( x = \lambda y \) the argument is similar). In the cases \( \kappa = 0, 1 \) there exists \( t \in \mathbb{R} \) such that \( x_t = 0 \). In fact, from the hypothesis and some computations, \( t \in \mathbb{R} \) is obtained by solving
\[ 1 - \lambda t = 0, \quad \text{if} \quad \kappa = 0 \quad \text{and} \quad \cos t - \lambda \sin t = 0, \quad \text{if} \quad \kappa = 1. \]
Thus, taking \( B \in SO_2 \) such that \( By_i = b\epsilon_2 \) (with \( b \neq 0 \) as \( y_i \neq 0 \)), we have that \( h = \text{diag}(R_z(t), B) \in H_\kappa \) satisfies \( \text{Ad}(h)Z(x, y) = Z(0, b\epsilon_2) \).

For \( \kappa = -1 \), as in the cases \( \kappa = 0, 1 \), we find \( t \in \mathbb{R} \) such that either \( x_t = 0 \) or \( y_t = 0 \) by solving
\[ \cosh t - \lambda \sinh t = 0, \quad \text{and} \quad -\sinh t + \lambda \cosh t = 0, \]
respectively. But these equations have a solution if and only if \( \lambda \neq \pm 1 \). That is, if and only if \( |x| \neq |y| \). Hence, taking \( B \in SO_2 \) such that either \( By_i = b\epsilon_2 \) or \( Bx_i = a\epsilon_1 \) (with \( a > 0 \); here again we have that \( x_i \neq 0 \), as appropriate. Then \( h = \text{diag}(R_{-1}(t), B) \in H_{-1} \) is as desired in this case.

For part \( c \), we observe that \( h = \text{diag}(R_L(\pi/2), B) \in H_1 \), where \( B \in SO_2 \) takes \( \epsilon_1 \) to \( \epsilon_2 \), satisfies \( \text{Ad}(h)Z(\epsilon_1, 0) = Z(0, \epsilon_2) \).

Remark. The previous lemma corresponds, geometrically, with the fact of finding \( s \in \mathbb{R} \) at which the Jacobi field associated with \( Z(x, y) \) (given by Lemma 5) and its covariant derivative are orthogonal.
Recall that if $h$ is a regular curve in $M_{\kappa}$ of constant speed $a$, then the Frenet frame of $h$ is

$$T(t) = \frac{1}{a} \dot{h}(t), \quad N(t) = \frac{\dot{h}'(t)}{|\dot{h}'(t)|}, \quad B(t) = T(t) \times N(t)$$

(here the prime denotes the covariant derivative along $h$), and its curvature and torsion are given by

$$k(t) = \frac{1}{a^2} |\dot{h}'(t)|, \quad \tau(t) = -\frac{1}{a} \langle B'(t), N(t) \rangle.$$

For each $g \in G_{\kappa}$ we have that $g$ is an isometry of $L_{\kappa}$ and preserves the Hermitian structure. Hence, $g$ takes magnetic geodesics to magnetic geodesics.

3. Time- and space-like magnetic geodesics

Proof of Theorem 1. Let $Z \in \mathfrak{p}_{\kappa}$ be the initial velocity of $\sigma$, with $\|Z\| \neq 0$. First, we consider the case $Z = Z(ae_1, be_2)$, with $a > 0$ and $b \neq 0$.

For each $t \in \mathbb{R}$, let $\alpha(t) = \exp t(Z + A)$. By Theorem 4 and the diffeomorphism $\phi$ in (7), we know that $\sigma(t) = \alpha(t) \cdot [\gamma_0]$, that is, $\sigma(t) = [\alpha(t) \cdot \gamma_0]$.

Let $h$ be the curve in $M_{\kappa}$ given by $h(t) = \alpha(t)(e_0)$. As $\alpha$ is a one-parameter subgroup of isometries of $M_{\kappa}$, we have that $h$ is a curve with constant curvature and torsion, thus $h$ is a helix in $M_{\kappa}$.

Let us see that $\sigma(t) = [\gamma_{B(t)}]$, where $B(t)$ is the binormal field of $h$. For each $t \in \mathbb{R}$, the initial velocity of the geodesic $\alpha(t) \cdot \gamma_0$ is $d(\sigma(t))(e_1)$, hence $\sigma(t) = [\gamma_{d(\alpha(t))}(e_1)]$. Then, we have to verify that $B(t) = d(\alpha(t))(e_1)$, for all $t \in \mathbb{R}$. Since $\alpha(t)$ is an isometry that preserves the helix and takes the Frenet frame at $t = 0$ to the Frenet frame at $t$, is suffices to show that $B(0) = e_1$.

By the usual identifications, since $\alpha(t)$ is a linear transformation, we can write $d(\alpha(t))(e_1) = \alpha(t)(e_1)$, so

$$\dot{h}(t) = \alpha(t)((Z + A)e_0) \quad \text{and} \quad \dot{h}'(t) = [\alpha(t)((Z + A)^2 e_0)]^T,$$

where $T$ denotes the tangent projection. Since

$$\dot{h}(0) = (Z + A)e_0 = ae_2,$$
$$\dot{h}'(0) = [(Z + A)^2 e_0]^T = [-\kappa a^2 e_0 + ae_3]^T = ae_3$$

and $\alpha(t)$ is an isometry, we have $|\dot{h}(t)| = a = |\dot{h}'(t)|$. By the computation before and (8) we obtain

$$B(0) = \frac{1}{a^2} \dot{h}(0) \times \dot{h}'(0) = e_1.$$
Consequently, \( B(t) = \alpha(t)(e_1) \). Then \( B'(t) = [\alpha(t)((Z + A)e_1)]^T \) and \( B'(0) = be_3 \). Besides, using (8) and the previous computations, it follows that \( N(0) = e_3 \). Therefore, by (9) we have that the curvature and torsion of \( h \) are equal to

\[
(10) \quad k = \frac{1}{a}, \quad \tau = \frac{b}{a}.
\]

The assertion regarding the sign of the torsion is immediate from Lemma 5 b) and (10). Thus, the theorem is proved in this particular case.

Now, let \( \sigma \) be a magnetic geodesic with \( \sigma(0) = [\gamma] \) and initial velocity with non zero norm. Since \( G_3 \) acts transitively on \( L_k \), there is an isometry \( g \) such that \( g \cdot [\gamma] = [\gamma_0] \). So, the magnetic geodesic \( g \cdot \sigma \) also has initial velocity with non zero norm and \( g \cdot \sigma(0) = [\gamma_0] \). By Lemma 5 b), if \( d(\phi \circ \pi)Z(x, y) \) is the initial velocity of \( g \cdot \sigma \), we have that the vectors \( \{x, y\} \) are linearly independent. Then, by Lemma 6 a), there exists \( h \in H_k \) such that \( \text{Ad}(h)Z(x, y) = Z(ae_1, be_2) \), with \( a > 0 \) and \( b \neq 0 \). Since \((h \circ g) \cdot \sigma)'(0) = d(\phi \circ \pi)(\text{Ad}(h)Z(x, y))\), the curve \((h \circ g) \cdot \sigma\) is a magnetic geodesic of the type studied above. Therefore, \( \sigma \) has the form (3).

Conversely, let \( h \) be a helix in \( M_k \) with curvature \( k > 0 \), non zero torsion \( \tau \) and speed \( 1/k \). Let \( \{T, B, N\} \) be the Frenet frame of \( h \). As \( M_k \) is a simply connected manifold of constant curvature, we have that there exists an isometry \( g \) of \( M_k \) preserving the orientation such that \( g(h(0)) = e_0 \) and its differential at \( h(0) \) takes \( B(0) \) to \( e_1 \), \( T(0) \) to \( e_2 \) and \( N(0) \) to \( e_3 \).

Let \( a = 1/k \) and \( b = -\tau/k \). Let \( Z = Z(ae_1, be_2) \in p_k \). We consider, for each \( t \in \mathbb{R} \), \( \alpha(t) = \exp(t(Z + A)) \). According to computations from the first part of the proof, both helices have initial position \( e_0 \), curvature \( k \), torsion \( \tau \), speed \( 1/k \) and the same Frenet frame at \( t = 0 \). Hence \( (g \circ h)(t) = \alpha(t)e_0 \). So, if we call \( \tilde{B} \) the binormal field of \( g \circ h \), we have that \( \tilde{B}(t) = d(\alpha(t))e_1 \), for all \( t \). Finally, since the curve \([\gamma_{\tilde{B}(t)}]\) is a magnetic geodesic in \( L_k \) and

\[
[\gamma_{\tilde{B}(t)}] = [\gamma_{d_{g^{-1}}\tilde{B}(t)}] = g^{-1} \cdot [\gamma_{\tilde{B}(t)}],
\]

we obtain that \([\gamma_{\tilde{B}(t)}]\) is a magnetic geodesic. \( \square \)

4. Null magnetic geodesics

We deal first with the hyperbolic case. We use the notation given in the introduction and we recall from [5] certain properties of horospheres and related concepts. To simplify the notation we omit the subindex \( \kappa = -1 \).

Let \( \gamma \) be a geodesic of \( \mathbb{H}^3 \). Then, for each \( p \in \mathbb{H}^3 \) there exists a unique unit speed geodesic \( \alpha \) of \( \mathbb{H}^3 \) such that \( \alpha(0) = p \) and \( \alpha \) is asymptotic to \( \gamma \). Let \( v \in T^1\mathbb{H}^3 \). If \( p \) is any point of \( \mathbb{H}^3 \), then \( v(p) \) denotes the unique unit tangent vector at \( p \) that is asymptotic to \( v \). The Busemann function \( f_v: \mathbb{H}^3 \to \mathbb{R} \) is defined by

\[
f_v(p) = \lim_{s \to +\infty} d(p, \gamma_v(s)) - s,
\]
and satisfies \( \text{grad}_p (f_v) = -v(p) \). The horosphere determined by \( v \) is given by
\[
H(v) = \{ q \in M \mid f_v(q) = 0 \}.
\]

The Jacobi vector fields orthogonal to \( \gamma_o \) have the form
\[
J(s) = e^s U(s) + e^{-s} V(s),
\]
where \( U \) and \( V \) are parallel vector fields along \( \gamma_o \) and orthogonal to \( \gamma_o \).

A Jacobi vector field \( Y \) along a geodesic \( \gamma \) of \( \mathbb{H}^3 \) is said to be stable (unstable) if there exists a constant \( c > 0 \) such that
\[
|Y(s)| \leq c \quad \forall s \geq 0 \quad (\forall s \leq 0).
\]

In what follows we shall denote by \( \tilde{\pi} \) the canonical projection from \( T\mathbb{H}^3 \) onto \( \mathbb{H}^3 \). We recall that in the introduction we have defined the smooth maps \( \psi^\pm: \mathcal{L}(\mathbb{H}^3) \to \mathbb{H}^3(\infty) \) by \( \psi^\pm[\gamma] = \gamma(\pm \infty) \) and the distributions \( \mathcal{D}^\pm \) in \( \mathcal{L}(\mathbb{H}^3) \) given by \( \mathcal{D}^\pm_{[\gamma]} = \text{Ker}(d\psi^\pm_{[\gamma]}) \).

We need to relate the distributions \( \mathcal{D}^\pm \) with distributions \( \tilde{\mathcal{E}}^\pm \) and \( \mathcal{E}^\pm \) on \( G \) and \( T^1\mathbb{H}^3 \), respectively.

Let \( \tilde{\mathcal{E}}^\pm \) be the left invariant distribution on \( G \) defined at \( I \in G \) by
\[
\tilde{\mathcal{E}}^\pm_I = \{ Z(u, \mp u) \in p \mid u \in \mathbb{R}^2 \}.
\]
As the canonical action of \( G \) on \( T^1\mathbb{H}^3 \) is transitive, the projection \( \tilde{p}: G \to T^1\mathbb{H}^3 \) given by \( \tilde{p}(g) = dg_{e_0} e_1 \) is a submersion. Since given \( v \in T^1\mathbb{H}^3 \) there exists \( g \in G \) such that \( \tilde{p}(g) = v \), we define:
\[
\mathcal{E}^\pm(v) = (d\tilde{p} \tilde{\mathcal{E}}^\pm)(\tilde{p}(g)) = d\tilde{p}_g(\tilde{\mathcal{E}}^\pm_g).
\]
We have that \( \mathcal{E}^\pm \) determines a well defined distribution on \( T^1\mathbb{H}^3 \), which is called the horospherical distribution on \( T^1\mathbb{H}^3 \). This distribution has the following property: if \( t \mapsto v(t) \) is a curve in \( T^1\mathbb{H}^3 \) tangent to the distribution \( \mathcal{E}^\pm \), then \( \tilde{\pi}(v(t)) \) is in the horosphere \( H(\pm v(0)) \).

**Lemma 7.** Let \( Z \in \tilde{\mathcal{E}}^\pm_I \). For each \( t \in \mathbb{R} \), let \( \gamma_t^\pm(s) = \exp t(Z + A) \cdot \gamma_o(\pm s) \). Then the geodesics \( \gamma_t^\pm \) are asymptotic to each other for all \( t \in \mathbb{R} \).

**Proof.** Let \( J \) be the Jacobi vector field associated with the variation by geodesics \( t \mapsto \gamma_t^\pm \). By Lemma 5 a), \( J(0) = -J'(0) \). Hence, by (11) we have that \( J(s) = e^{-s} U(s) \), where \( U \) is a parallel vector field along \( \gamma_o \) orthogonal to \( \gamma_o \). Thus, \( J \) is a stable vector field, that is, there exists \( c > 0 \) such that \( |J(s)| \leq c \forall s \geq 0 \).

We have to show that given \( t_0, t_1 \in \mathbb{R} \) with \( t_0 < t_1 \), there exists \( N > 0 \) such that
\[
d(\gamma_{t_0}^\pm(s), \gamma_{t_1}^\pm(s)) \leq N \quad \forall s \geq 0.
\]
For fixed $s$,
\[ d(\gamma_{t_0}^\pm(s), \gamma_{t_1}^\pm(s)) \leq \text{length}([t_0, t_1]) = \int_{t_0}^{t_1} \left| \frac{d}{dt} \gamma_{t}^\pm(s) \right| dt. \]

For each $t \in \mathbb{R}$, let $J_t(s) = (d/dt)\gamma_{t}^\pm(s)$. We observe that $J_{t'}(s) = d \exp(t'Z) J_t(s)$ for all $t, t'$. Since $\exp(t'Z)$ is an isometry, we have $|J_t(s) - J(s)|$. Therefore,
\[ \int_{t_0}^{t_1} |J_t(s)| dt = \int_{t_0}^{t_1} |J(s)| dt \leq c(t_1 - t_0) \]
for all $s \geq 0$. Then, we may take $N = c(t_1 - t_0) > 0$.

We consider the projection $p : T^1 \mathbb{H}^3 \to \mathcal{L}(\mathbb{H}^3)\), $p(v) = [\gamma_v]$. We call $\mathcal{D}^\pm$ the distribution on $\mathcal{L}(\mathbb{H}^3)$ $p$-related with $\mathcal{E}^\pm$ (well defined). More specifically, given $[\gamma] \in \mathcal{L}(\mathbb{H}^3)$ and $v \in T^1 \mathbb{H}^3$ such that $p(v) = [\gamma]$. $\mathcal{D}^\pm$.

**Proposition 8.** Let $\mathcal{D}^\pm$ and $\mathcal{D}^\pm$ be the distributions on $\mathcal{L}(\mathbb{H}^3)$ defined above. Then $\mathcal{D}^\pm = \mathcal{D}^\pm$.

Proof. Since $\mathcal{D}^\pm$ and $\mathcal{D}^\pm$ are $G$-invariant, it is enough to show $\mathcal{D}^\pm_{[\gamma_v]} = dp_{(0, e_1)}(\mathcal{E}^\pm_{(0, e_1)})$ (we observe that $\mathcal{D}^\pm_{[\gamma_v]} = dp_{(0, e_1)}(\mathcal{E}^\pm_{(0, e_1)})$).

Let $Z \in \mathcal{E}^\pm_{[\gamma_v]}$. We take the curve in $\mathcal{L}(\mathbb{H}^3)$ given by $\alpha(t) = \exp(tZ) \cdot [\gamma_v]$. As $\alpha(t) = p \circ \bar{p}(\exp(tZ))$, we have that $\alpha(0) = [\gamma_v]$ and $\dot{\alpha}(0) = d(p \circ \bar{p})_I Z$. That is,
\[ \dot{\alpha}(0) \in dp_{(0, e_1)}(\mathcal{E}^\pm_{(0, e_1)}). \]

Besides,
\begin{equation}
\frac{d}{dt} \bigg|_0 \exp(tZ) \cdot \gamma_v(s) = \frac{d}{dt} \bigg|_0 \exp(t(Z + A) \cdot \gamma_v(s)),
\end{equation}

since both Jacobi fields have the same initial conditions. Hence, Lemma 7 applies to the geodesics $\gamma_{t_0}^\pm([\gamma]) = \exp(tZ) \cdot \gamma_v(\pm s)$. Thus, $\psi^\pm \circ \alpha$ is constant. Then $(d \psi^\pm)_{[\gamma_v]}(\dot{\alpha}(0)) = 0$, that is, $\dot{\alpha}(0) \in \mathcal{D}^\pm_{[\gamma_v]}$.

On the other hand, let $\varphi : T^1 \mathbb{H}^3 \to \mathcal{L}(\mathbb{H}^3), \varphi(v) = [\gamma_v]$, be the submanifold whose image $\mathcal{L}_v(\mathbb{H}^3)$ consists of all the oriented geodesics passing through $e_0$. Besides, $H(\infty)$ is a manifold with the differentiable structure (well defined) such that $F_{e_0} : T^1 \mathbb{H}^3 \to H(\infty)$ given by $F_{e_0}(v) = [\gamma_v]$. We take the curve in $\mathcal{E}^\pm_{[\gamma_v]} \circ \varphi = F_{e_0}$, we have that $(d \psi^\pm)_{[\gamma_v]}$ is surjective. Now, $(d \psi^-)_{[\gamma_v]}$ is also surjective because $\psi^-$ is the composition of $\psi^+$ with the diffeomorphism of $\mathcal{L}(\mathbb{H}^3)$ assigning $[\gamma^{-1}]$ to $[\gamma]$. Therefore, $\dim \mathcal{D}^\pm_{[\gamma_v]} = \dim \mathcal{D}^\pm_{[\gamma_v]}$ and equality follows.
The word cylinder in the statement of Theorem 2 refers to a ruled surface determined by a parallel vector field along a curve $c$ of constant geodesic curvature $k$ contained in a totally geodesic surface in $M_k$ (and normal to it), as explained. For $k = -1$, this ruled surface is diffeomorphic to $S^1 \times \mathbb{R}$ if $|k| > 1$; otherwise it is diffeomorphic to a plane.

Proof of Theorem 2 a). By Lemma 5 b), we have that every element of $\mathcal{D}_{\gamma_0}^{\pm}$ is null. As $G$ acts transitively on $\mathcal{L}(\mathbb{H}^3)$ and by the $G$-invariance of the horospherical distributions, we may suppose without loss of generality that $\sigma(0) = \gamma_0$, hence $\dot{\sigma}(0) \in \mathcal{D}_{\gamma_0}^{\pm}$. By Proposition 8, there exists $Z \in \tilde{E}_1^{\pm}$ such that $\dot{\sigma}(0) = (dp_{e_1,e_1}(d\tilde{p}))Z$. Thus, by Theorem 4, $\sigma(t) = [\exp t(Z + A) \cdot \gamma_0]$.

We assume that $Z \in \tilde{E}_1^{+}$. Let us show that $\sigma$ describes a forward cone with vertex at $\gamma_0(\pm \infty)$. In a similar way, if $Z \in \tilde{E}_1^{-}$, then $\sigma$ describes a backward cone with vertex at $\gamma_0(-\infty)$.

We consider the geodesics $\gamma(t) = \exp t(Z + A) \cdot \gamma_0(s)$ of $\mathbb{H}^3$. As $Z \in \tilde{E}_1^{+}$, by Lemma 7, we have that the geodesics $\gamma_1$ are asymptotic to each other for all $t$. Hence, $z(t) = \dot{\gamma}(0)$ is a curve in $T^1\mathbb{H}^3$ of asymptotic vectors to $e_1$.

Let $c(t) = \tilde{\pi}(z(t)) = \exp t(Z + A)(e_0)$. In order to see that $c(t) \in H(e_1)$ for all $t$, we observe that

$$
\frac{d}{dt} f_{e_1}(c(t)) = (df_{e_1})_{c(t)} \tilde{c}(t) = (\text{grad}_{c(t)} f_{e_1}) \cdot \tilde{c}(t).
$$

Since $\text{grad}_p(f_{e_1}) = -v(p)$ we have that

$$
\text{grad}_{c(t)}(f_{e_1}) = -z(t) = -d(\exp t(Z + A))e_1.
$$

On the other hand,

$$
\tilde{c}(t) = d(\exp t(Z + A))(Z + A)e_0.
$$

Since $\exp t(Z + A)$ is an isometry and observing that $(Z + A)e_0$ and $e_1$ are perpendicular ($Z \in \tilde{E}_1^{+}$), it follows that the expression in (13) is equal to $-\langle e_1, (Z + A)(e_0) \rangle = 0$. Then, $f_{e_1}(c(t)) = f_{e_1}(e_0) = 0$ for all $t$, that is, $c(t) \in H(e_1)$ for all $t$.

Now, as $c$ is the orbit through $e_0$ of a one-parameter subgroup of isometries of $G$ preserving $H(e_1)$, its geodesic curvature and speed are constant. If $Z = Z(u, -u)$ for certain $0 \neq u \in \mathbb{R}^2$, we obtain that the speed of $c$ is $|u|$. For each $v \in T^1\mathbb{H}^3$ we consider on $H(v)$ the orientation given by $-\text{grad} f_v$. The geodesic curvature of $c$ is then

$$
k = \langle -\text{grad}_{e_0}(f_{e_1}), \tilde{c}(0) \times \tilde{c}'(0) \rangle = \frac{1}{|u|},
$$

since $\tilde{c}(0) = (Z + A)e_0$ and $\tilde{c}'(0) = ((Z + A)^2e_0)^T$. As for each $v \in T^1\mathbb{H}^3$, $H(v)$, with the induced metric of $\mathbb{H}^3$, is isometric to $\mathbb{R}^2$, we have that $c(t)$ runs along a circle on $H(e_1)$ of geodesic curvature $k = 1/|u| > 0$ and speed $1/k = |u|$.
Besides, $\sigma(t) = [\gamma_{c(t)}]$. Thus we have that all conditions are satisfied in order to assert that $\sigma$ describes a forward cone with vertex at $\gamma_o(+\infty)$.

Conversely, let $\sigma$ be a curve in $L(\mathbb{H}^3)$ that describes a forward cone with vertex at infinity. As $G$ acts transitively on the positively oriented frame bundle, and also each element of $G$ takes horospheres to horospheres, preserving their orientation, we may suppose that $\sigma(t) = [\gamma_{\hat{c}(t)}]$, where $\gamma(t)$ is a curve in $T^1\mathbb{H}^3$ of asymptotic vectors to $\gamma(0) = e_1$ and $c(t) = \hat{\pi}(\gamma(t))$ is a curve of geodesic curvature $k$ and speed $1/k$ in $H(e_1)$ with $c(0) = (1/k)e_2$, for some $k > 0$. Let $Z = Z((1/k)e_1, -(1/k)e_1) \in \hat{E}^+_t$. We define

$$\bar{c}(t) = \exp t(Z + A)(e_0) \quad \text{and} \quad \bar{v}(t) = d(\exp t(Z + A))(e_1).$$

We showed above that $\bar{c}(t)$ is a curve of geodesic curvature $k$ and speed $1/k$ in $H(e_1)$. Moreover, $\bar{c}(0) = e_0$ and the initial velocity of $\bar{c}$ is $(1/k)e_2$. So, we obtain that $\bar{c} = c$. This implies, together with the identities $\pi \circ \bar{v} = \bar{c}$ and $\hat{\pi} \circ v = e$, that $\hat{\pi} \circ \bar{v} = \hat{\pi} \circ v$.

According to the first part of the proof, $\bar{v}$ and $v$ are curves of asymptotic vectors to $e_1$. Hence, $-\bar{v}(t) = \text{grad}_{\bar{c}(t)}(f_{e_1}) = -v(t)$. Therefore, $[\gamma_{\hat{c}(t)}] = [\gamma_{\bar{c}(t)}]$, which is a null magnetic geodesic with initial velocity in the horospherical distribution since $[\gamma_{\bar{c}(t)}] = [\exp t(Z + A) \cdot \gamma_o]$. \hfill $\square$

**Proof of Theorem 2 b).** We suppose first that $\sigma$ is a null magnetic geodesic such that $\sigma(0) = [\gamma_o]$ and $\dot{\sigma}(0) = d(\phi \circ \pi)Z(a_1, 0)$, with $a > 0$. The expression (4) and the relation between the speed and curvature of $h$ are obtained as in the proof of Theorem 1. By (10) we know that the torsion of $h$ is $\tau = -b/a = 0$ (since $b = 0$). Thus $h$ is contained in a totally geodesic surface $S$ of $\mathbb{H}^3$ and $B$ is normal to $S$.

Now, we suppose that $\dot{\sigma}(0) = d(\phi \circ \pi)Z$, where $Z = Z(0, be_2)$ with $b \neq 0$. By Theorem 4 we have that $\sigma(t) = [\alpha(t) \cdot \gamma_o]$, where $\alpha(t) = \exp t(Z + A)$. Since $Z + A$ is in the Lie algebra of the isotropy subgroup of $G$ at $e_0 \in \mathbb{H}^3$, we get that $\alpha(t)$ fixes $e_0$. Moreover, if $v$ is the curve in $T^1_{e_0}\mathbb{H}^3$ given by $v(t) = d(\alpha(t))e_1$, then

$$\sigma(t) = [\alpha(t) \cdot \gamma_o] = [\gamma_{\bar{c}(t)}],$$

since the initial velocity of the geodesic $\alpha(t) \cdot \gamma_o$ is $v(t)$, for each $t \in \mathbb{R}$.

Furthermore, as $v$ is the orbit through $e_1$ of a one-parameter subgroup of $H$ (the canonical differential action of $G$ on $T^1_{e_0}\mathbb{H}^3$), then $v$ has constant speed and constant geodesic curvature in $T^1_{e_0}\mathbb{H}^3 \cong S^2$. Easy computations yield

$$\dot{v}(0) = (0, 0, b)' \quad \text{and} \quad \ddot{v}(0) = (-b^2, -b, 0)'.$$

So, the speed of $v$ is $|b|$ and its geodesic curvature is

$$k = \frac{\langle v(0), \dot{v}(0) \times \ddot{v}(0) \rangle}{|b|^3} \leq \frac{1}{|b|}.$$
(we consider the orientation of the sphere given by the unit normal field pointing outwards). Thus, $v$ is a curve in $T_{e_0}^1\mathbb{H}^3$ of geodesic curvature $k > 0$ and speed $1/k$. Consequently, $\sigma$ has the form (5).

Now, let $\sigma$ be a null magnetic geodesic such that $\sigma(0) = [y]$ and $\dot{\sigma}(0) \notin D^\pm_{[y]}$. As $G$ acts transitively on $L^\pm(\mathbb{H}^3)$ and by the $G$-invariance of the horospherical distributions, we may suppose that $\sigma(0) = [y_0]$ and $\dot{\sigma}(0) \notin D^\pm_{[y_0]}$. Let $Z = Z(x, y) \in \mathfrak{p}$ such that $\dot{\sigma}(0) = d(\phi \circ \pi)Z$. By Lemma 5 b), as the norm of the initial velocity of $\sigma$ is zero, we have that $x$ and $y$ are linearly dependent, and since $d(\phi \circ \pi)Z \notin D^\pm_{[y_0]}$, we also have $|x| \neq |y|$. Now, the isometries in Lemma 6 b) take $\sigma$ to magnetic geodesics of the particular types studied above. Therefore, $\sigma$ has the form (4) or has the form (5), as desired.

Conversely, given a helix $h$ in $\mathbb{H}^3$ with curvature $k$, speed $1/k$ and torsion $\tau = 0$, the proof that the expression (4) is a magnetic geodesic is identical to the proof of the converse of Theorem 1. As $h$ has zero torsion, the initial velocity of the magnetic geodesic in (4) is not in the distributions $D^\pm$.

Let $\nu$ be a curve in $T^1_\nu\mathbb{H}^3$ with geodesic curvature $k > 0$ and speed $1/k$. Let $g$ be the isometry of $\mathbb{H}^3$ preserving the orientation such that $g(p) = e_0$, $dg(\nu(0)) = e_1$ and $dg(\nu(0)) = be_3$, for certain $b > 0$. Hence, $g \cdot \nu$ is a curve in $T_{e_0}^1\mathbb{H}^3$ having the same geodesic curvature and the same speed as $\nu$, and also $b = 1/k$. As we showed above, $\tilde{\nu}$ is a curve in $T_{e_0}^1\mathbb{H}^3$ with $\tilde{\nu}(0) = g \cdot \nu(0)$ and with the same initial velocity and geodesic curvature that $g \cdot \nu$. By uniqueness, we have that $\tilde{\nu} = g \cdot \nu$. To complete the proof we observe that $g \cdot [\gamma_0(t)] = [\gamma_{g \cdot \nu(t)}] = [\gamma_{\nu(t)}]$.

Proof of Theorem 3. Lemma 6 b) implies that the analogue of Theorem 2 a) is empty for the cases $\kappa = 0, 1$. The proof of the fact that every curve $\sigma$ in $L^\pm_\kappa$ is a null magnetic geodesic if and only if $\sigma$ has the form (4) or (5) is similar to that of Theorem 2 b).

We check the last statement of the theorem. Without lost of generality, we consider only null magnetic geodesics passing through $[\gamma_0]$ at $t = 0$. We observe that if, in particular, $\sigma$ is a magnetic geodesic with initial velocity $d(\phi \circ \pi)Z(ae_1, 0)$, with $a > 0$, (that is, $\sigma$ has the form (4)), then by Lemma 6 c) there exists $h \in H_1$ such that $\text{Ad}(h)Z(ae_1, 0) = Z(0, ae_2)$. Hence, $h \cdot \sigma$ is a null magnetic geodesic with initial velocity $d(\phi \circ \pi)Z(0, ae_2)$, and then it has the form (5). So, $\sigma$ also has this form.

References


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