Enumerations of theta-curves and handcuff graphs
(θ-曲線と手錠グラフの表について)

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Abstract

We enumerate all the θ-curves and handcuff graphs with up to seven crossings by using algebraic tangles and prime basic θ-polyhedra. Here, a θ-polyhedron is a connected graph embedded in 2-sphere, whose two vertices are 3-valent, and the others are 4-valent. There exist twenty-four prime basic θ-polyhedra with up to seven 4-valent vertices. We can obtain a θ-curve or handcuff graph diagram from a prime basic θ-polyhedron by substituting algebraic tangles for their 4-valent vertices.

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Contents

Introduction

A link is n circles $S^1 \cup \cdots \cup S^1$ embedded in a 3-sphere S^3. As a generalization of classical knot theory, we can consider other objects embed in S^3. A spatial graph is a graph embedded in S^3, and two spatial graphs G, G' are *equivalent* if there is a homeomorphism $h : (S^3, G) \to (S^3, G')$. If the graph consists of two vertices and three edges such that each edge joins the vertices, then it is called a θ-curve. Moreover, if the graph consists of two loops and an edge jointing the vertices of each loop, then it is called a handcuff graph.

In knot theory, there exists the study of creating a prime knot table. Similarly, there exists the study of making a table of spatial graphs. There exist earlier studies on tabulation of spatial graphs as follows: In **[32]**, J. Simon enumerated θ-curves with up to five crossings, and K_4 graphs with up to four crossings. In **[20]**, R. Litherland announced a table of prime θ-curves with up to seven crossings. However, we can suppose that their θ-curves obtained by adding an edge to some knot. Moreover, there is no published proof of the completeness of Litherland's table. Then we would like to complete the table of prime θ-curves with up to seven crossings.

We applied Conway's method to enumerate θ-curves. In **[5]**, J. H. Conway made an enumeration of prime knots and links by introducing the concept of a *tangle* and a *basic polyhedron*. Here, a tangle is a disjoint union of two arcs and some or no loops properly embedded in a 3-ball B^3, and a basic polyhedron is the 4-regular planar graph which has no bigon. We can obtain knots from basic polyhedra by substituting tangles for their vertices.

In order to apply Conway's method, we need the following works. First, we enumerated algebraic tangles with up to seven crossings ([**22**]). Second, we constructed a prime basic θ-polyhedron to enumerate prime θ-curves. Here, a θ-polyhedron is a connected graph embedded in 2-sphere, whose two vertices are 3-valent, and the others are 4-valent. Then our θ-polyhedron is different from Conway's polyhedron. Since we would like to make a prime θ-curve table, we omitted a non-prime θ-polyhedron. Then there exist twenty-four prime basic θ-polyhedra with up to seven 4-valent vertices. We can obtain θ-curves from prime basic θ-polyhedra by substituting *algebraic tangles* for their 4-valent vertices.

In **[21]**, the author obtained all the prime θ-curves with up to seven crossings, which are the exactly same table as Litherland's. Litherland classified these θ-curves by constituent knots and the Alexander polynomial, but we classified them by the *Yamada polynomial.* Moreover, we first enumerated all the prime handcuff graphs with up to seven crossings in the same way ([**22**]). We also classified these handcuff graphs by using the Yamada polynomial.

This paper is organized as follows: In Chapter 1, we give some definitions for a θ-curve and a handcuff graph, and Main Theorems. In Chapter 2, we give a table of algebraic tangles. In Chapter 3, we introduce the concept of a prime basic θ-polyhedron. In Chapter 4, we classify the θ-curves and handcuff graphs in Tables 1.1 and 1.2 by the Yamada polynomials and consider the chirality of these spatial graphs. In Chapter 5, we give some open problems.

CHAPTER 1

θ-curves and handcuff graphs

A spatial graph is a graph embedded in S^3, and two spatial graphs G, G' are *equivalent* if there is a homeomorphism $h : (S^3, G) \to (S^3, G')$. A θ-curve is a spatial graph consisting of two vertices and three edges, each edge joining the two vertices. In a letter **[20]**, Litherland announced a table of prime θ-curves with up to seven crossings (Table 1.1). However, there is no published proof of the completeness of his table. We enumerate prime θ-curves with up to seven crossings by extending Conway's method in order to confirm his table. A handcuff graph is a spatial graph consisting of two loops and an edge jointing the vertices of each loop. In the same way, we enumerate prime handcuff graphs with up to seven crossings.

This chapter is organized as follows: In Section 1, we give some definitions for a θ-curve, and Main Theorem 1. In Section 2, we give some definitions for a handcuff graph, and Main Theorem 2.

1. θ-curve

A *θ-curve* Θ is a graph embedded in S^3, which consists of two vertices $(v_, v_2)$ and three edges $(e_, e_2, e_3)$, such that each edge joins the vertices. A *constituent knot* Θ_{ij}, $1 \leq i < j \leq 3$, is a subgraph of Θ that consists of two vertices $(v_, v_2)$ and two edges (e_i, e_j). θ-curves are roughly classified by comparing the triples of constituent knots. A θ-curve is said to be *trivial* if it can be embedded in a 2-sphere in S^3.

DEFINITION 1.1 (Litherland **[20]**). A θ-curve is said to be *prime* if it satisfies the following conditions:
(C1) it is non-trivial;
(C2) it is not the order-2 vertex connected sum of non-trivial knot and
(possibly trivial) θ-curve;
(C3) it is not the order-3 vertex connected sum of two non-trivial θ-curves.

Here an *order-n vertex connected sum* of spatial graphs is defined as follows; cf. **[37]**. If graphs G_1 and G_2 each have a vertex of order n, remove regular neighborhoods of each of these vertices and glue the remaining 3-balls together in such a way that the n points of G_1 and the n points of G_2 in the boundary of each 3-ball are identified (see Fig. 1.1).

A non-trivial knot K, such as in Fig. 1.1(a), is called a *local knot* of a θ-curve.

REMARK 1.2. We may consider that a spatial graph G exists in $\mathbf{R}^3$, and present G by a *diagram* g in an obvious manner, which consists of two vertices and three edges immersed in $\mathbf{R}^2$ such that
(D1) each intersection is a transverse double point, and
(D2) in a neighborhood of each double point one of the two segments is removed.

We present a θ-curve Θ by a diagram θ, and its constituent knot Θ_{ij} by a diagram θ_{ij}.
In **[13]**, Kauffman gave the following lemma.

LEMMA 1.3. *If two spatial 3-valent graphs are ambient isotopic, then any diagrams of them are related by a finite sequence of the moves of Fig.* 1.2.

We often use Lemma 1.3 to conclude that two spatial 3-valent graphs are equivalent.

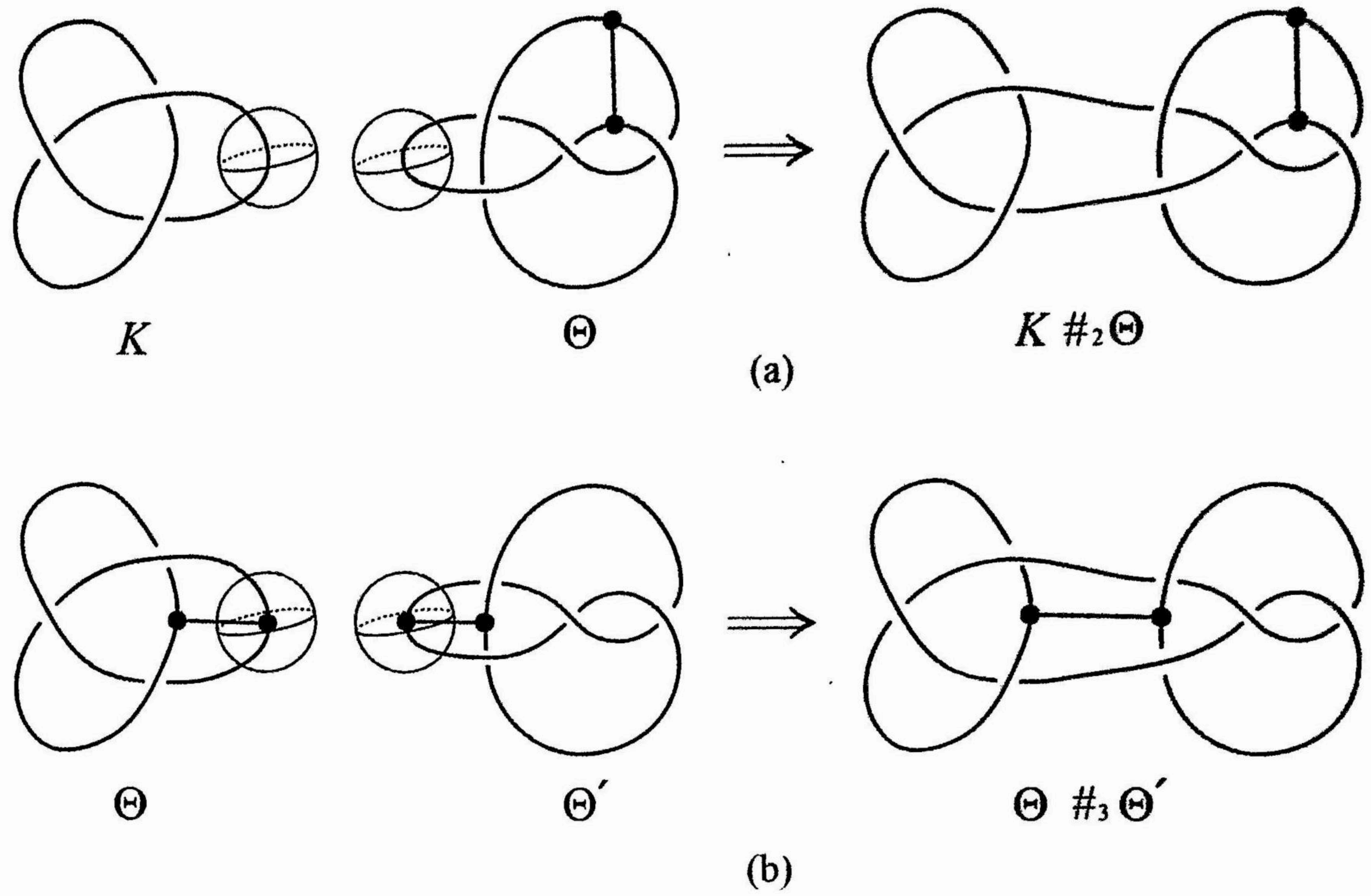


FIG. 1.1. (a) An order-2 vertex connected sum of spatial graphs,
(b) An order-3 vertex connected sum of spatial graphs.

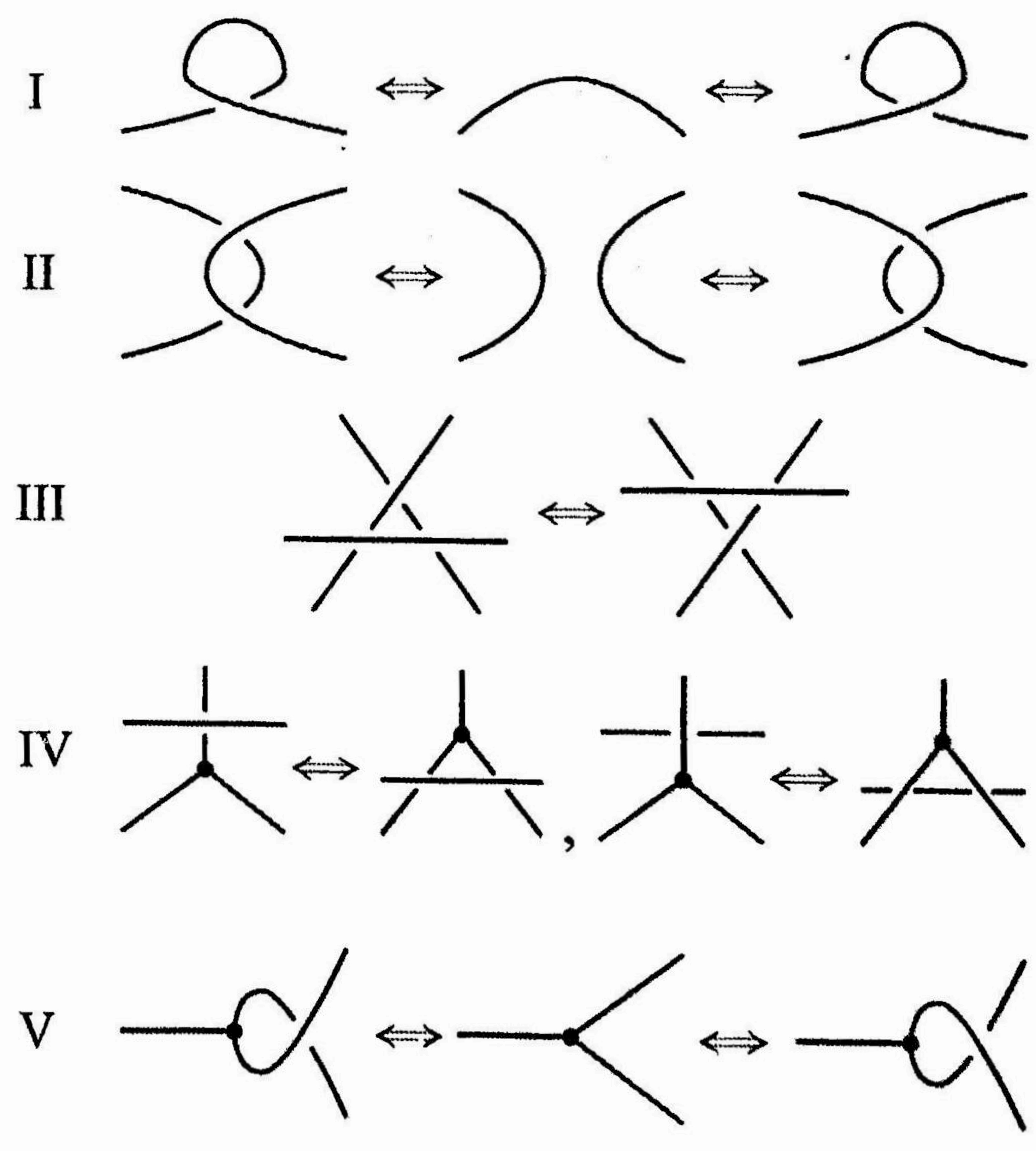
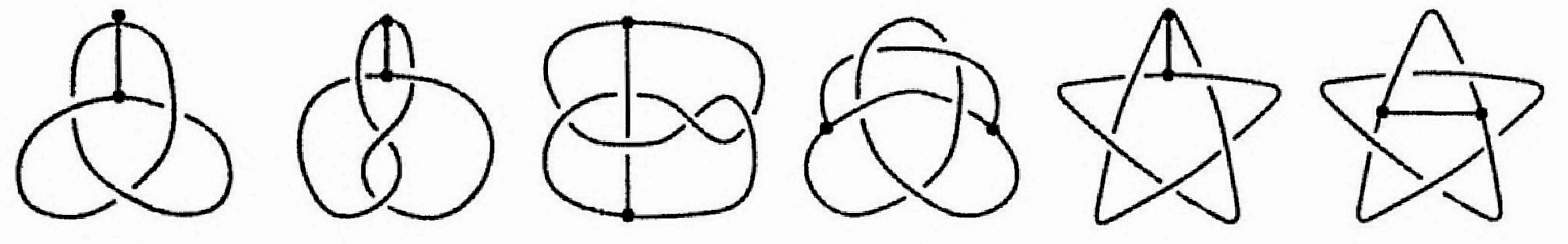


FIG. 1.2. Reidemeister moves for 3-regular spatial graphs.

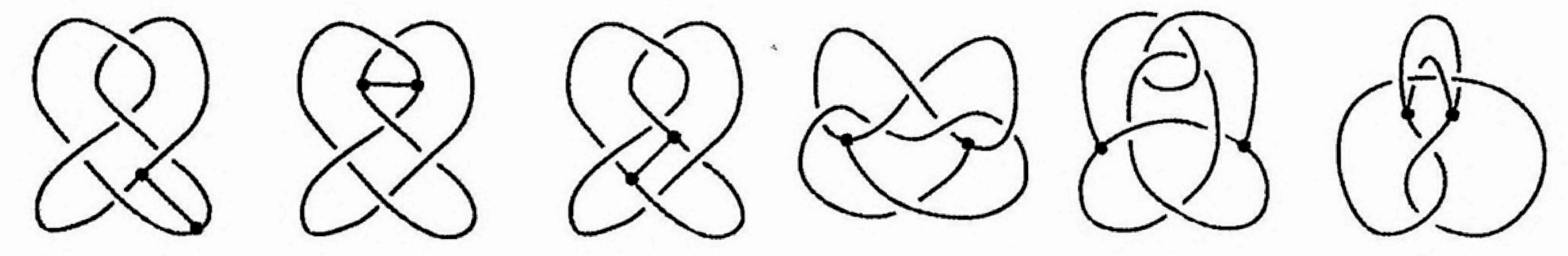
We obtain the following result:

MAIN THEOREM 1. Table 1.1 lists all the prime θ-curves with up to seven crossings.

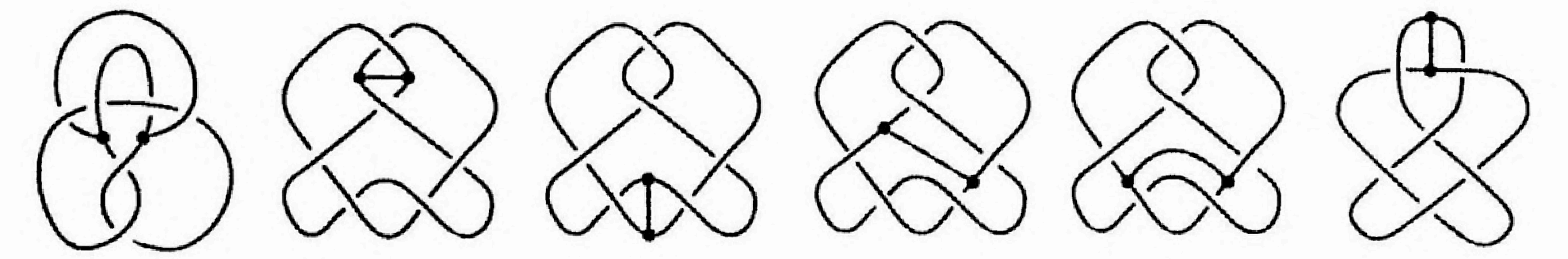
TABLE 1.1. Prime θ-curves with up to seven crossings.



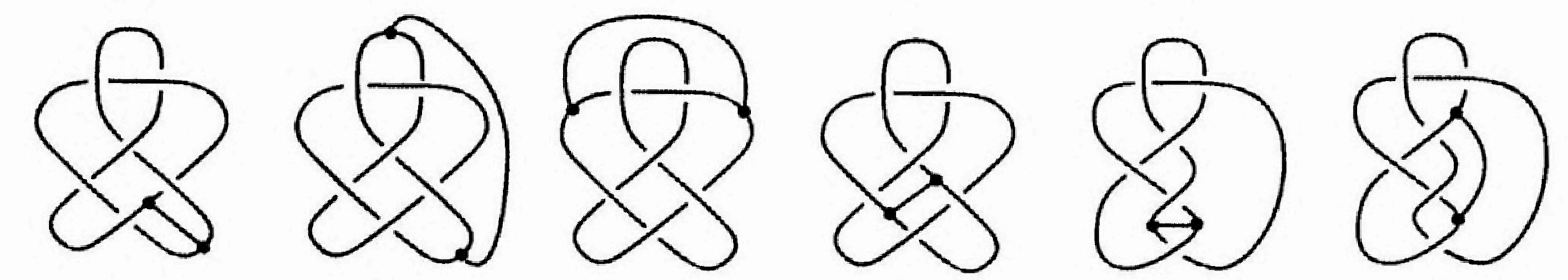
3_1 4_1 5_1 5_2 5_3 5_4



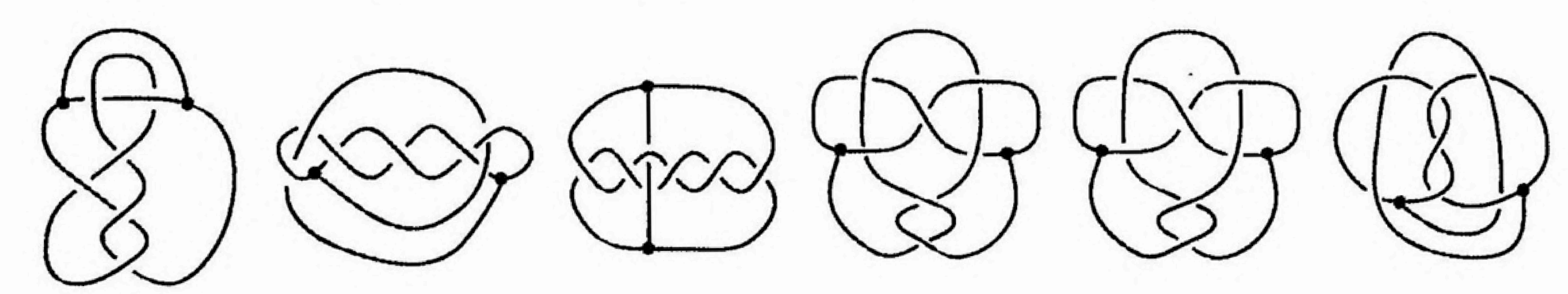
5_5 5_6 5_7 6_1 6_2 6_3



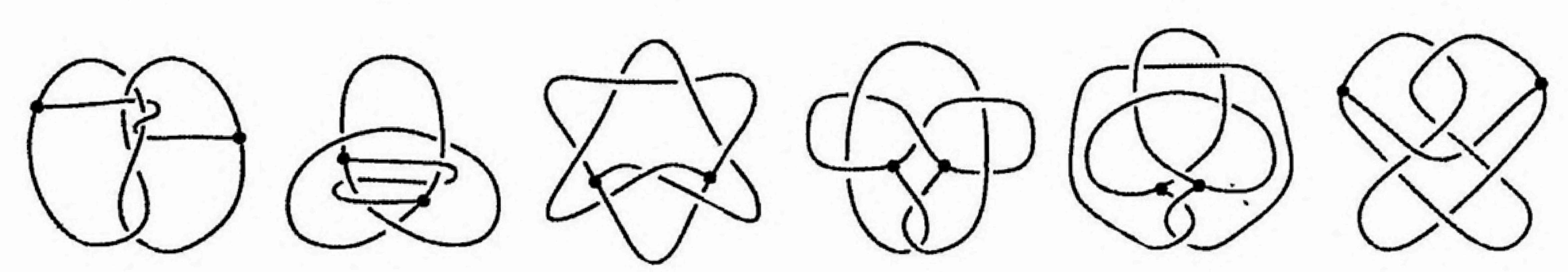
6_4 6_5 6_6 6_7 6_8 6_9



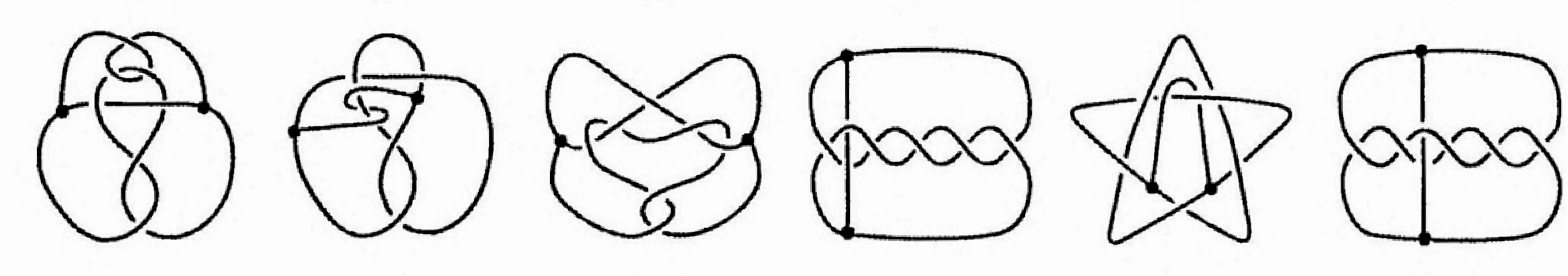
6_{10} 6_{11} 6_{12} 6_{13} 6_{14} 6_{15}



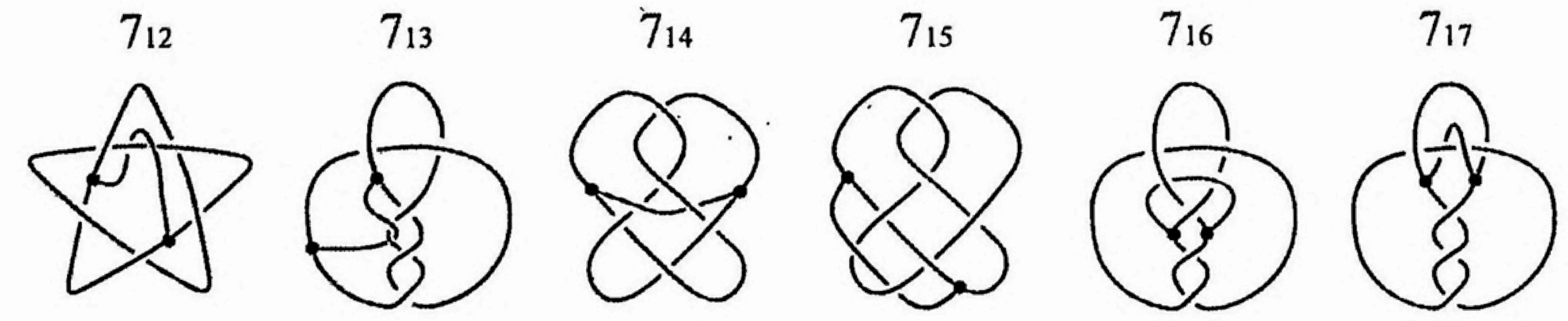
6_{16} 7_1 7_2 7_3 7_4 7_5



7_6 7_7 7_8 7_9 7_{10} 7_{11}

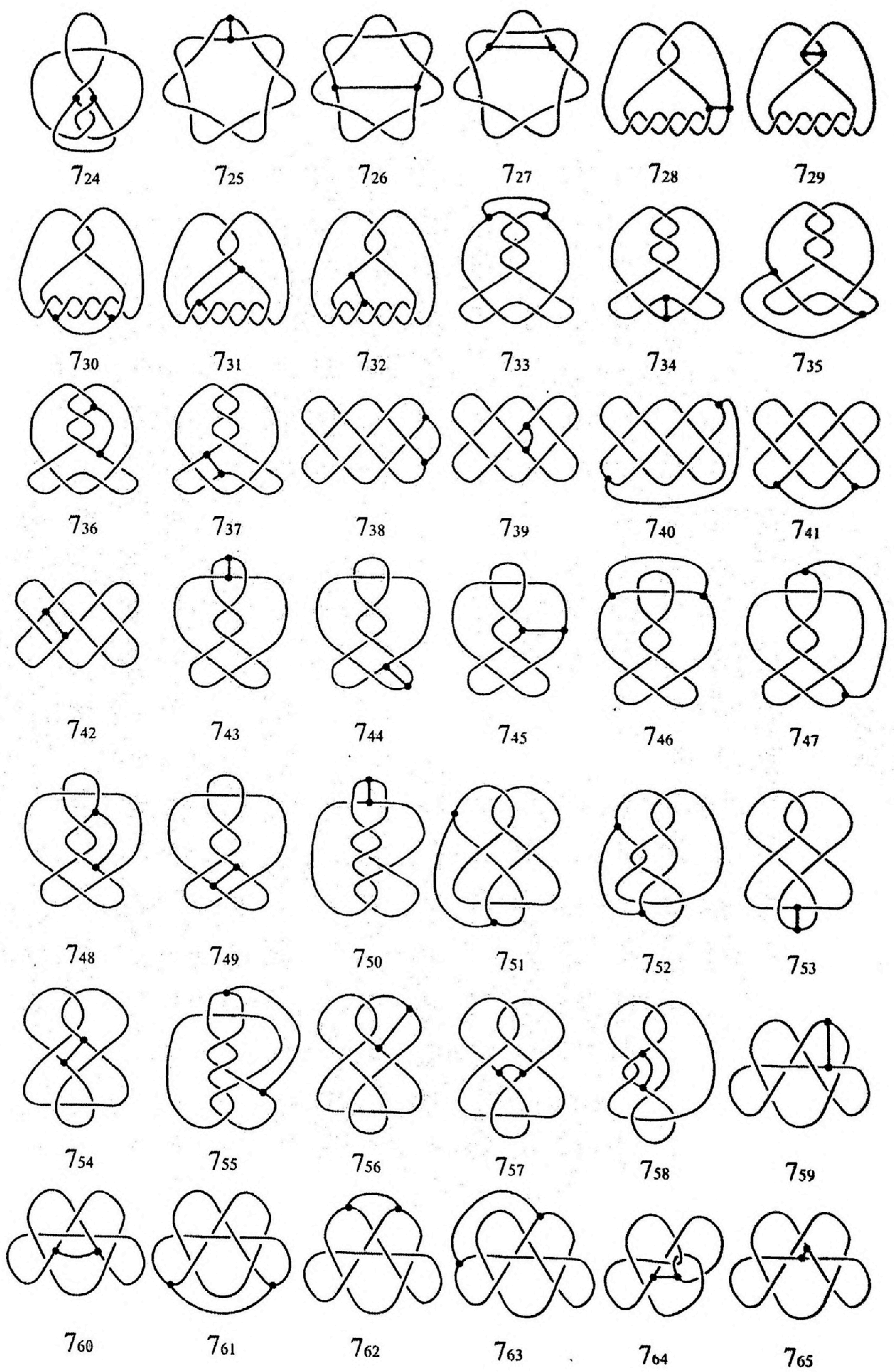


7_{12} 7_{13} 7_{14} 7_{15} 7_{16} 7_{17}



7_{18} 7_{19} 7_{20} 7_{21} 7_{22} 7_{23}

TABLE 1.1. Prime θ-curves with up to seven crossings (continued).



The θ-curves in Table 1.1 are listed in order of increasing crossing number and their constituent knots.

2. Handcuff graph

A *handcuff graph* Φ is a graph embedded in S^3 consisting of two vertices (v_1, v_2) and three edges (e_1, e_2, e_3), where e_3 has distinct endpoints v_1 and v_2, and e_1 and e_2 are loops based at v_1 and v_2, respectively. A *constituent link* Φ_{12} is a subgraph of Φ that consists of two vertices (v_1, v_2) and two edges (e_1, e_2). A handcuff graph is said to be *trivial* if it can be embedded in a 2-sphere in S^3.

Let Φ be a handcuff graph and Σ a 2-sphere which decomposes S^3 into 3-balls B_1, B_2. If $\Sigma \cap \Phi$ consists of a single point w, and neither $(\Phi - w) \cap B_1$ nor $(\Phi - w) \cap B_2$ is empty, then Σ is called an *admissible sphere of type* I for Φ. If $\Sigma \cap \Phi$ consists of two points, and the annulus $A = \Sigma \setminus \mathrm{Int}N(\Phi; S^3)$ is essential in $S^3 \setminus \mathrm{Int}N(\Phi; S^3)$, then Σ is called an *admissible sphere of type* II for Φ; cf. [33]. If $\Sigma \cap \Phi$ consists of three points, and neither $\Phi \cap B_1$ nor $\Phi \cap B_2$ is an unknotted bouquet (Fig. 1.3), then Σ is called an *admissible sphere of type* III for Φ; cf. [6]. By an *admissible sphere*, we mean either an admissible sphere of type I, II, or III.

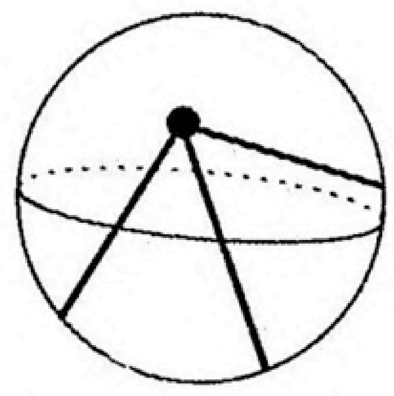
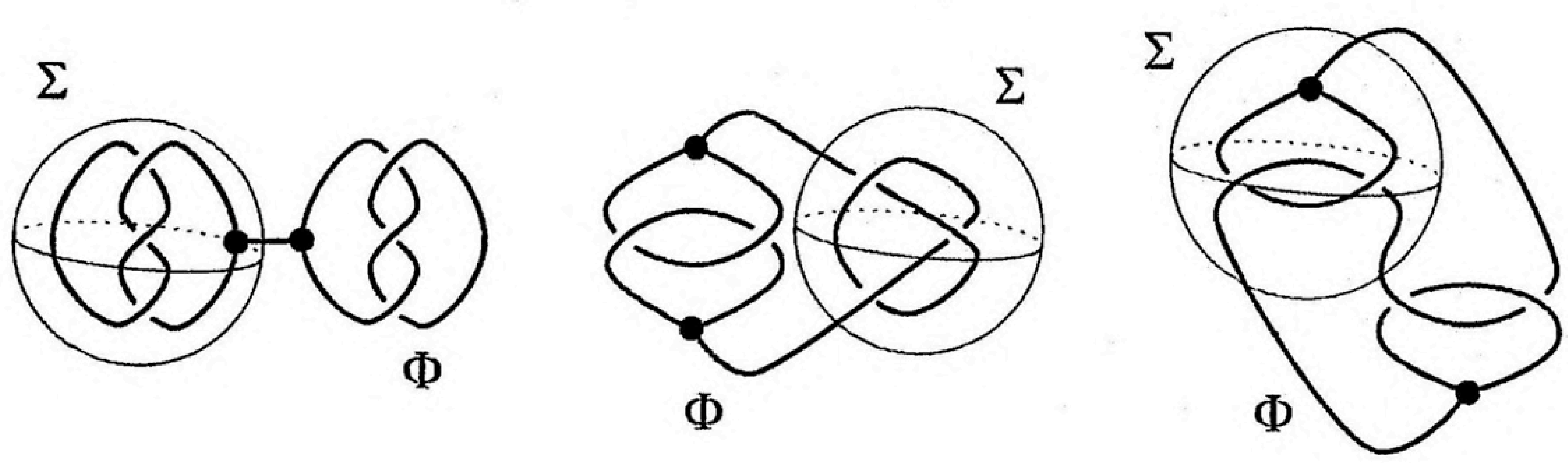


FIG. 1.3. An unknotted bouquet.

DEFINITION 1.4. A handcuff graph Φ is said to be *prime* if Φ is nontrivial and does not have an admissible sphere.

For example, handcuff graphs in Fig. 1.4 (a),(b),(c) have admissible spheres of types I, II, III, respectively. Therefore, these handcuff graphs are nonprime.



(a) (b) (c)

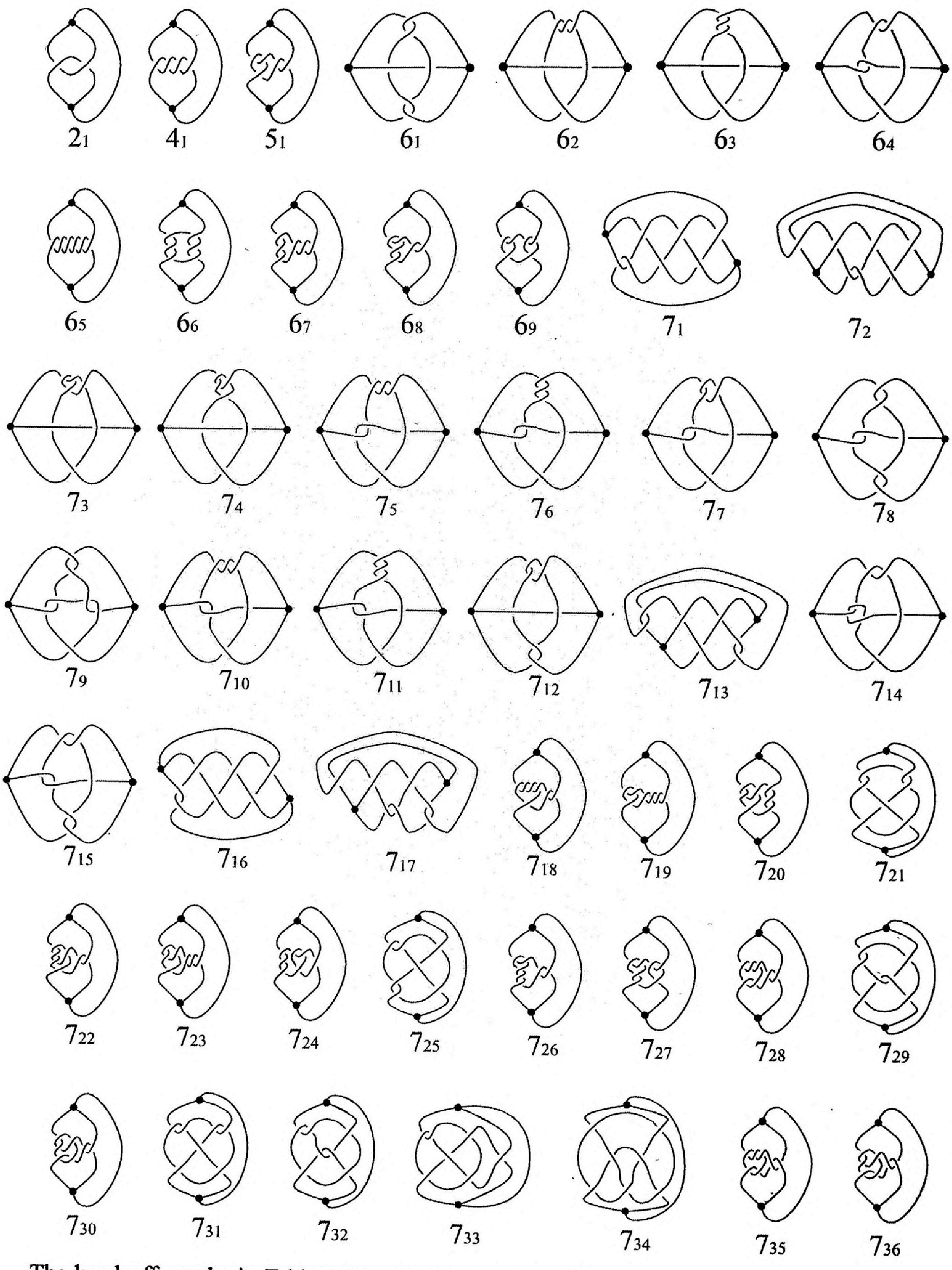
FIG. 1.4. Nonprime handcuff graphs.

REMARK 1.5. We can redefine the primeness of a θ-curve by using an admissible sphere, that is, a θ-curve Θ is said to be prime if Θ is nontrivial and has neither an admissible sphere of type II nor III.

The following is also our main result.

MAIN THEOREM 2. Table 1.2 lists all prime handcuff graphs with up to seven crossings.

TABLE 1.2. Prime handcuff graphs with up to seven crossings.



The handcuff graphs in Table 1.2 are listed in order of increasing crossing number and their constituent links.

CHAPTER 2

Algebraic tangles

In **[5]** J. H. Conway introduced the concept of a *tangle* in order to enumerate knots and links. A tangle is a disjoint union of two arcs and some or no loops properly embedded in a 3-ball B^3. Two tangles T and S are *isotopic* if there is an isotopy of the 3-ball B^3 that takes one tangle to the other while fixing each point of the boundary, and *freely equivalent* if there is a homeomorphism of B^3 which takes T to S without restriction that the endpoints stay fixed. In **[29]** Y. Nakanishi listed a table of algebraic tangles of five crossings or less up to isotopy by using Conway's method. In **[40]** H. Yamano gave a table of prime 2-string tangles of seven crossings or less up to free equivalence by using Conway's method. In **[12]** T. Kanenobu, H. Saito and S. Satoh classified 2-string tangles of seven crossings or less up to free equivalence by using disk-graphs. In this chapter, we classify algebraic tangles of seven crossings or less up to *equivalence*, which is weaker than isotopy, but stronger than free equivalence (Definition 2.1). This chapter is organized as follows: In Section 1, we give some definitions. In Section 2, we list a table of algebraic tangles.

1. Definitions

We review Conway's method **[5]**. We define a *tangle* as a pair (B^3, t), where t is a 1-manifold properly embedded in a unit 3-ball
$B^3 = \{(x, y, z) \in R^3 \mid x^2 + y^2 + z^2 \leq 1\}$ with four boundary components

$$\text{NE} = \left(1/\sqrt{2}, 1/\sqrt{2}, 0\right), \quad \text{SE} = \left(1/\sqrt{2}, -1/\sqrt{2}, 0\right),$$
$$\text{SW} = \left(-1/\sqrt{2}, -1/\sqrt{2}, 0\right), \quad \text{NW} = \left(-1/\sqrt{2}, 1/\sqrt{2}, 0\right);$$

see Fig. 2.1.

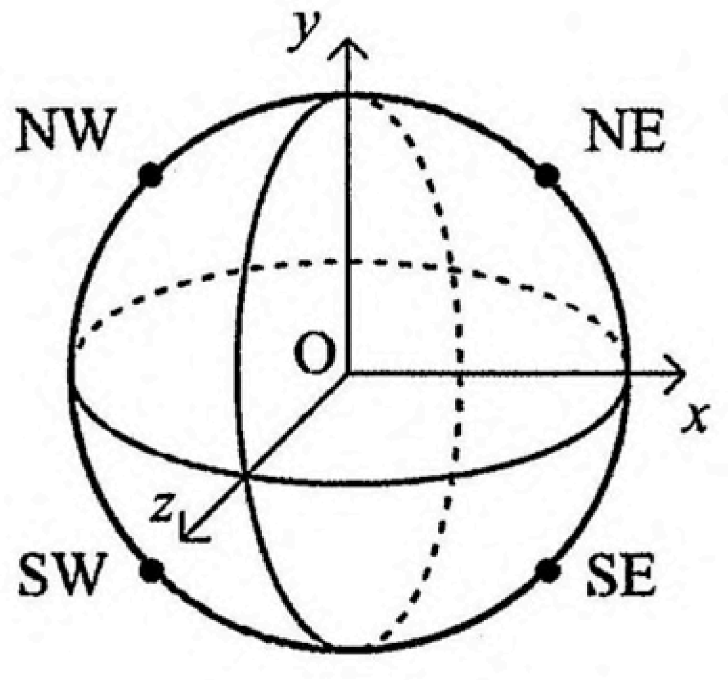


FIG. 2.1. A 3-ball and its 4 boundary components.

Let $T = (B^3, t)$ be a tangle such that t consists of two arcs and n circles. We say T is a *V^n-tangle* (resp. an *H^n-tangle*, an *X^n-tangle*) if T has an arc connecting NE and SE (resp. NW, SW).

We present a tangle by a regular diagram as in Fig. 2.2(a), where we use the projection $(x, y, z) \mapsto (x, y)$. Let R be a tangle. We denote by μR, νR, $\rho_x R$, $\rho_y R$, $\rho_z R$ the tangles obtained from R by reflecting with regard to the xy-plane; $\mu(x, y, z) = (x, y, -z)$, by turning it counter-clockwise by $\pi/2$; $\nu(x, y, z) = (-y, x, z)$, by rotating it through angle π; $\rho_x(x, y, z) = (x, -y, -z)$, $\rho_y(x, y, z) =$

$(-x, y, -z)$, and $\rho_z(x, y, z) = (-x, -y, z)$, respectively. We present these tangles diagrammatically as shown in Fig. 2.2. We call μR the *mirror image* of R.

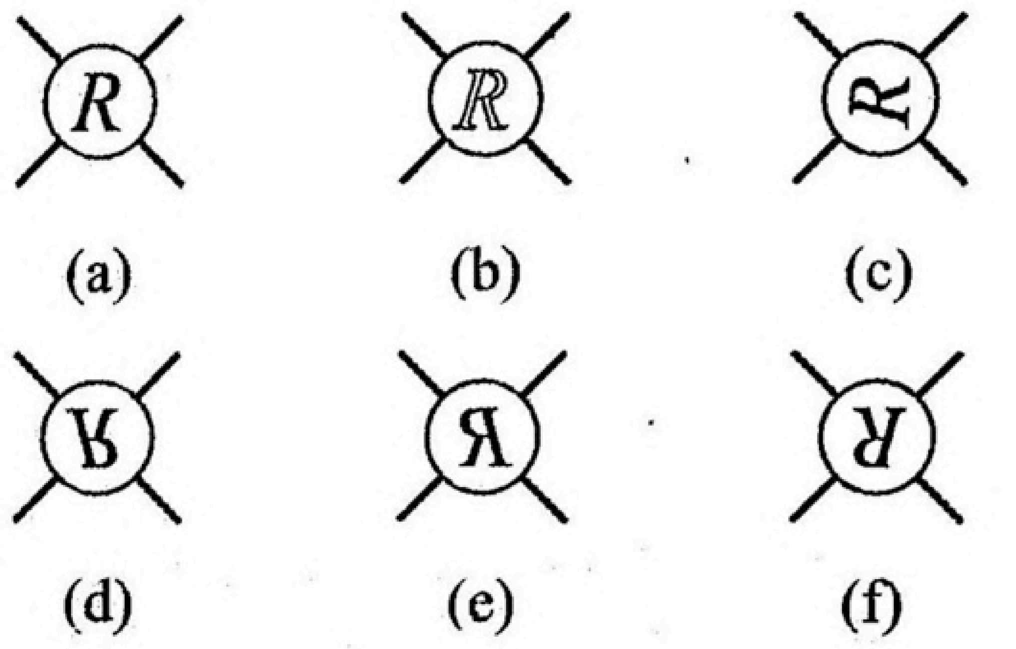


FIG. 2.2. (a) A tangle R. (b) The tangle μR. (c) The tangle νR. (d) The tangle $\rho_x R$. (e) The tangle $\rho_y R$. (f) The tangle $\rho_z R$.

We say that two tangles are *isotopic* if there is an isotopy of the 3-ball B^3 that takes one tangle to the other while fixing each point of the boundary, that is, their diagrams are related by a finite sequence of *Reidemeister moves* as shown in Fig. 2.3 inside the circle defining the tangle while the endpoints of the strings remain fixed.

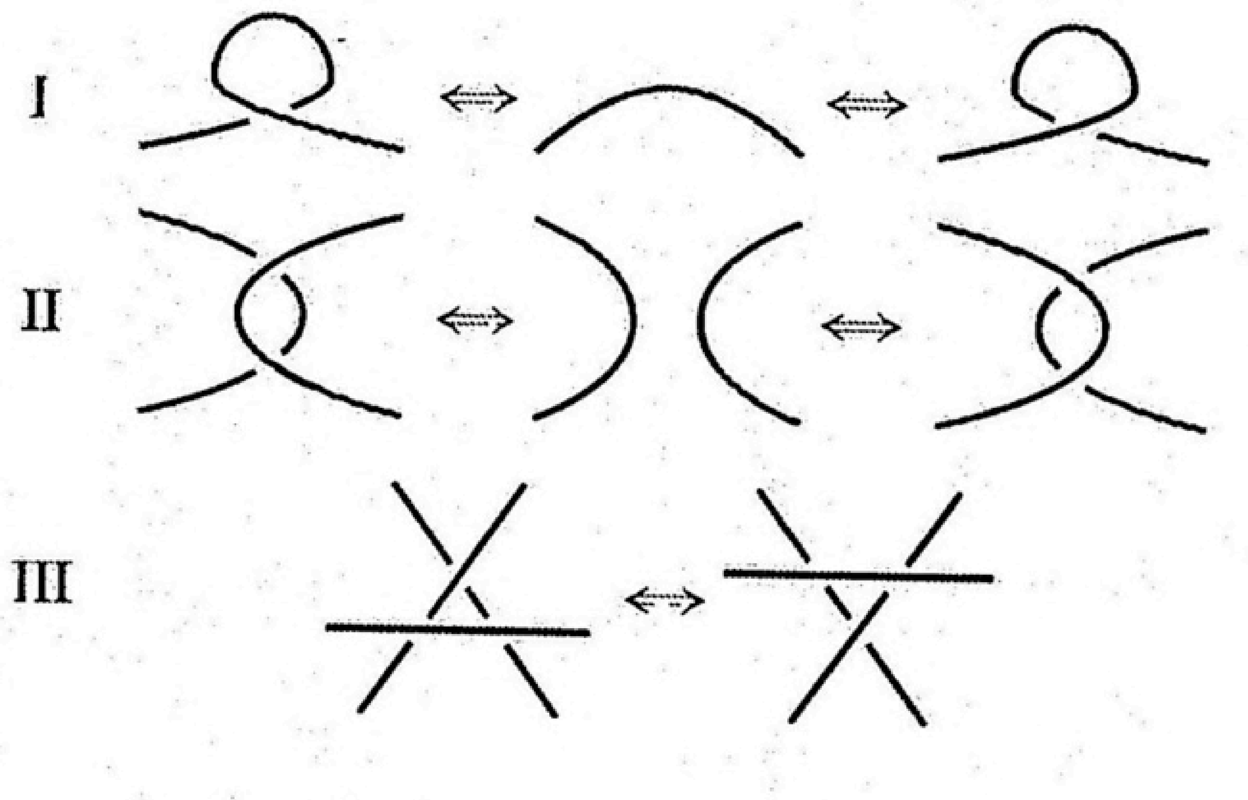


FIG. 2.3. Reidemeister moves.

DEFINITION 2.1. We say that two tangles T and T' are *equivalent* if T is isotopic to one of the following eight tangles:

$$T',\ \rho_x T', \rho_y T', \rho_z T', \nu T', \nu\rho_x T', \nu\rho_y T', \nu\rho_z T'.$$

For a tangle diagram D, we denote by $c(D)$ the number of crossings of D. The *crossing number* of a tangle T, denoted by $c(T)$, is the minimal number of $c(D)$'s for all the diagrams D which present the equivalence class of T.

Given two tangles T and S, we define new tangles $T + S$, TS, $T+$ and $T-$ as shown in Fig. 2.4; $T + S$ and TS are the *sum* and *product* of T and S respectively. Notice that $TS = \rho_x\mu\nu(T) + S$, where $\rho_x\mu\nu(T)$ is the tangle obtained from T by reflecting across the NW and SE diagonal line.

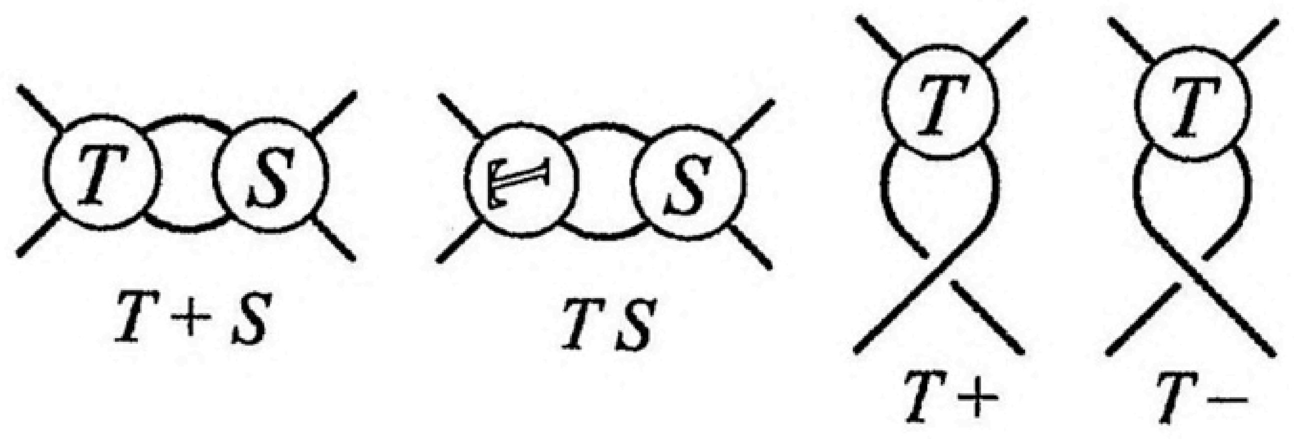


FIG. 2.4. The operations.

The simplest tangles are the 0 and ∞ tangles as shown in Figs. 2.5 (a) and (b). Further, for a positive integer n we define the n tangle and the $-n$ tangle as shown in Figs. 2.5 (c) and 2.5 (d), which are called *integral tangles*.

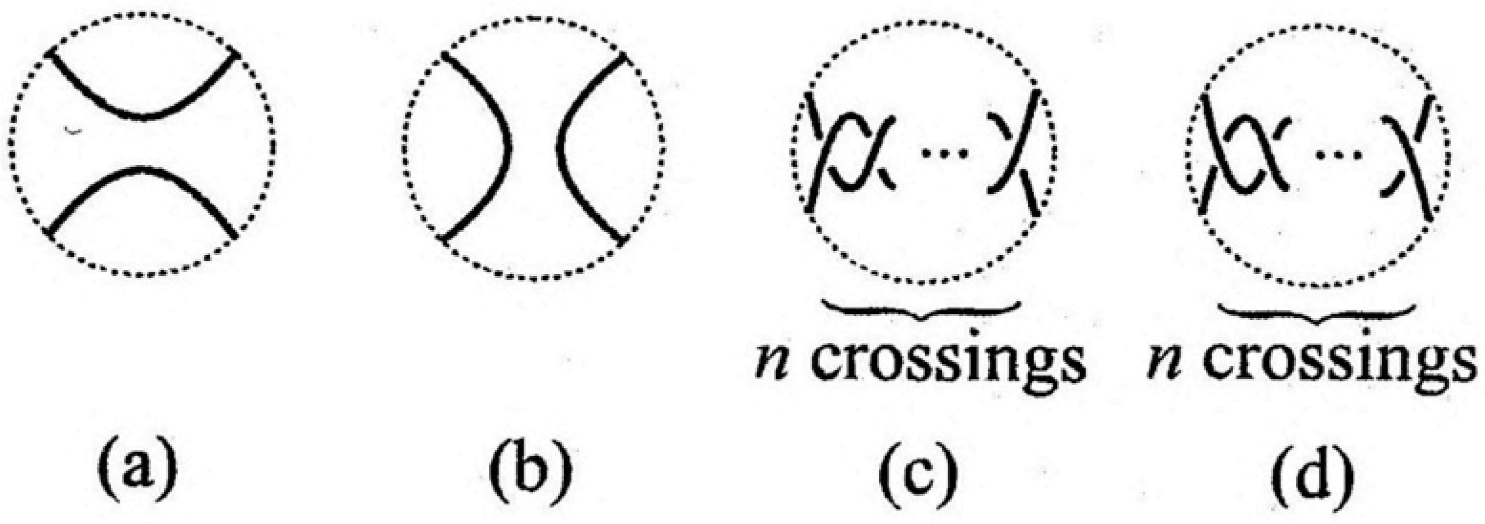


FIG. 2.5. (a) The 0 tangle. (b) The ∞ tangle. (c) The n tangle. (d) The $-n$ tangle.

A tangle T is said to be *algebraic* if T is obtained from the 0 and ∞ tangles by a finite sequence of the operations given in Fig. 2.4. Thus, an algebraic tangle is obtained from the 0, ∞, and integral tangles by the operations of addition and multiplication. We denote the n tangle simply by n, and the $-n$ tangle by $\bar{n}$. For integral tangles $a_1, a_2, a_3, \ldots, a_{i-1}, a_i$, the tangle $a_1a_2a_3\cdots a_{i-1}a_i$, abbreviating $((\cdots(a_1\ a_2)a_3\cdots a_{i-1})a_i)$, is called a *rational tangle*. Two rational tangles $a_1a_2\ldots a_{i-1}a_i$ and $b_1b_2\ldots b_{j-1}b_j$ are isotopic if and only if the corresponding rational numbers (including $1/0 = \infty$)

$$a_i + \cfrac{1}{a_{i-1} + \cfrac{1}{\ddots + \cfrac{1}{a_2 + \cfrac{1}{a_1}}}} \quad \text{and} \quad b_j + \cfrac{1}{b_{j-1} + \cfrac{1}{\ddots + \cfrac{1}{b_2 + \cfrac{1}{b_1}}}}$$

are the same.

REMARK 2.2. For the above continued fraction, we can assume that each a_m $(1 \leq m \leq i)$ has the same sign.

The comma notation $(a_1, a_2, \ldots, a_i) = (a_1\,0) + (a_2\,0) + \cdots + (a_i\,0)$ is preferred to the sum notation, but is only used with two or more terms in the bracket. Fig. 2.6 shows the step-by-step formation of two algebraic tangles 2 1 1 1 and 2.1, 2 as examples.

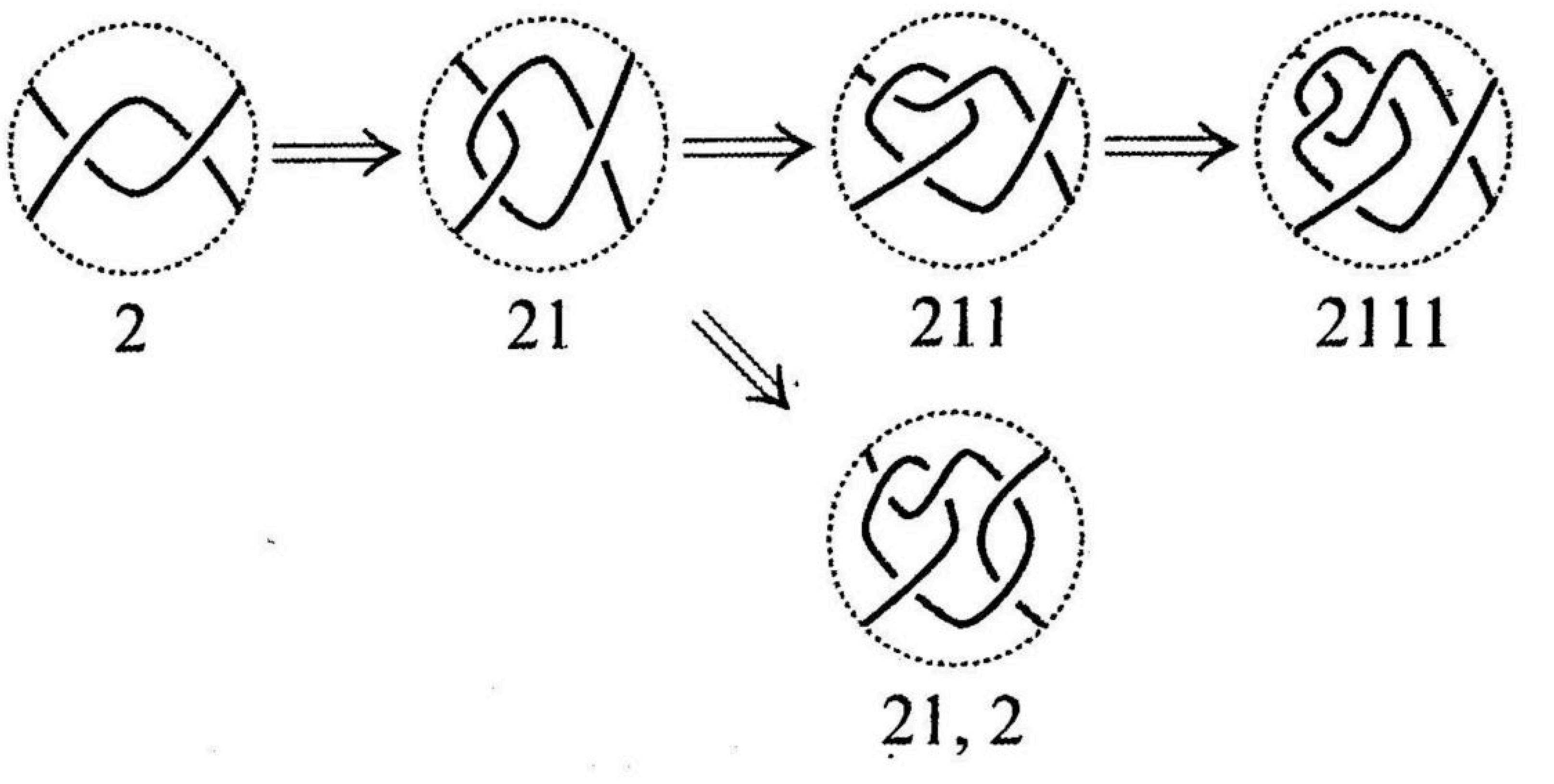


FIG. 2.6. Algebraic tangles.

A tangle $T = (B^3, t)$ is said to be *splittable* if there exists a disk Δ such that Δ does not meet t, but splits two arcs of t in B^3.

Let T be a tangle. We define the *numerator*, $N(T)$, and *denominator*, $D(T)$, the links as shown in Fig. 2.7. We call the set of links $\{N(T), D(T)\}$ the *corresponding links* for T.

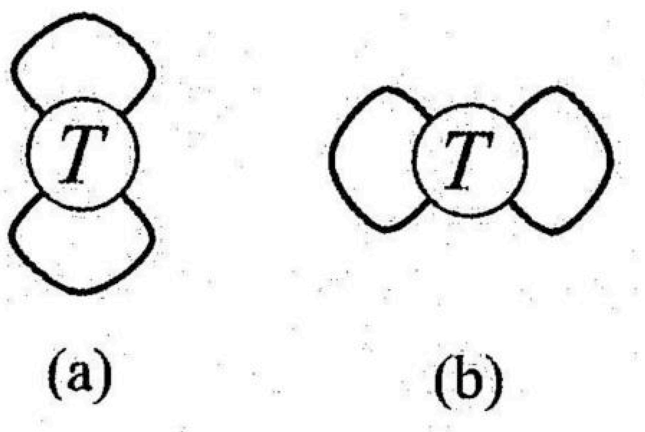


FIG. 2.7. (a) The numerator $N(T)$. (b) The denominator $D(T)$.

Clearly, we have

PROPOSITION 2.3. *Suppose that T and S are equivalent tangles. Then their corresponding links $\{N(T), D(T)\}$ and $\{N(S), D(S)\}$ present the same set of isotopic links.*

Let R be a tangle. We define the *double*, $W(R)$ by the link as shown in Fig. 2.8;

$$W(R) = N(R + \rho_y \mu R).$$

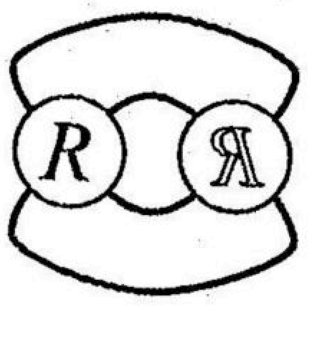


FIG. 2.8. The double $W(R)$.

Clearly, we have

PROPOSITION 2.4. *Suppose that T and S are equivalent tangles. Then, their doubles $W(T)$ and $W(S)$ are isotopic.*

2. Table of algebraic tangles

We list a table of unsplittable algebraic tangles with seven crossings or less up to equivalence in tion 2.1. However, we list either T or μT for each tangle T, even if they are not equivalent.

THEOREM 2.5. *Table* 2.1 *exhibits diagrams of unsplittable algebraic tangles with up to seven crossings.*

Links in the second column correspond to Rolfsen's knot table [31]. Specifically, 0 is the trivial knot, $L_1 \# L_2$ is a connected sum of links L_1 and L_2, and $\overline{L}$ is the mirror image of L. The last column gives the type of the algebraic tangle, where $X = X^0$, $H = H^0$, $V = V^0$.

TABLE 2.1. Algebraic tangles with up to seven crossings.

T	$(N(T), D(T))$	type
1	$(0,\ 0)$	X
2	$(2_1^2,\ 0)$	H
3	$(3_1,\ 0)$	X
2 1	$(\overline{3_1},\ 2_1^2)$	V
4	$(4_1^2,\ 0)$	H
3 1	$(\overline{4_1^2},\ \overline{3_1})$	H
2 2	$(4_1,\ 2_1^2)$	V
2 1 1	$(4_1,\ 3_1)$	X
$2, 2$	$(\overline{4_1^2},\ 2_1^2 \# 2_1^2)$	V^1
$2, \bar{2}$	$(0_1^2,\ 2_1^2 \# 2_1^2)$	V^1
5	$(5_1,\ 0)$	X
4 1	$(\overline{5_1},\ \overline{4_1^2})$	V
3 2	$(5_2,\ \overline{3_1})$	X
3 1 1	$(\overline{5_2},\ 4_1^2)$	V
2 3	$(\overline{5_2},\ 2_1^2)$	V
2 2 1	$(5_2, 4_1)$	X
2 1 2	$(5_1^2,\ \overline{3_1})$	H
2 1 1 1	$(\overline{5_1^2},\ 4_1)$	H
$2, 2+$	$(\overline{5_1^2},\ 2_1^2 \# 2_1^2)$	V^1
$(2, 2)1$	$(5_1^2,\ 4_1^2)$	X^1
$(2, 2)\bar{1}$	$(0_1^2,\ 4_1^2)$	X^1
$3, 2$	$(\overline{5_1},\ \overline{3_1} \# 2_1^2)$	V
$3, \bar{2}$	$(0,\ \overline{3_1} \# 2_1^2)$	V
$2\,1, 2$	$(5_2,\ 3_1 \# 2_1^2)$	V

TABLE 2.1. Algebraic tangles with up to seven crossings (continued).

T	$(N(T), D(T))$	type
6	$(6^2_1,\ 0)$	H
$5\,1$	$(\overline{6^2_1},\ \overline{5_1})$	H
$4\,2$	$(6_1,\ \overline{4^2_1})$	V
$4\,1\,1$	$(\overline{6_1},\ 5_1)$	X
$3\,3$	$(6^2_2,\ \overline{3_1})$	H
$3\,2\,1$	$(\overline{6^2_2},\ \overline{5_2})$	H
$3\,1\,2$	$(6_2,\ 4^2_1)$	V
$3\,1\,1\,1$	$(\overline{6_2},\ 5_2)$	X
$2\,4$	$(\overline{6_1}, 2^2_1)$	V
$2\,3\,1$	$(6_1,\ 5_2)$	X
$2\,2\,2$	$(6^2_3, 4_1)$	H
$2\,2\,1\,1$	$(\overline{6^2_3},\ \overline{5_2})$	H
$2\,1\,3$	$(\overline{6_2},\ 3_1)$	X
$2\,1\,2\,1$	$(6_2,\ \overline{5^2_1})$	V
$2\,1\,1\,2$	$(6_3,\ 4_1)$	X
$2\,1\,1\,1\,1$	$(6_3,\ 5^2_1)$	V
$2, 2++$	$(\overline{6^2_3},\ 2^2_1\#2^2_1)$	V^1
$(2,2)2$	$(\overline{6^3_1},\ 4^2_1)$	H^1
$(2,2)\bar{2}$	$(\overline{6^3_3},\ 4^2_1)$	H^1
$(2,\bar{2})2$	$(\overline{6^3_3},\ 0^2_1)$	H^1
$3, 2+$	$(6_2,\ \overline{3_1}\#2^2_1)$	V
$2\,1, 2+$	$(6_3,\ 3_1\#2^2_1)$	V
$(3,2)1$	$(6_2,\ 5_1)$	X
$(3,2)\bar{1}$	$(0,\ 5_1)$	X
$(3,\bar{2})\bar{1}$	$(0,\ 5_2)$	X
$(2\,1,2)1$	$(6_3,\ \overline{5_2})$	X
$(2,2+)1$	$(\overline{6^2_3},\ 5^2_1)$	X^1
$(2,2)1\,1$	$(6^3_1,\ \overline{5^2_1})$	H^1
$4, 2$	$(\overline{6^2_1},\ \overline{4^2_1}\#2^2_1)$	V^1
$4, \bar{2}$	$(2^2_1,\ \overline{4^2_1}\#2^2_1)$	V^1
$3\,1, 2$	$(6^2_2,\ 4^2_1\#2^2_1)$	V^1
$2\,2, 2$	$(6_1,\ 4_1\#2^2_1)$	V
$2\,2, \bar{2}$	$(0,\ 4_1\#2^2_1)$	V
$2\,1\,1, 2$	$(\overline{6_2},\ 4_1\#2^2_1)$	V
$(2,2), 2$	$(\overline{6^2_3},\ 4^2_1\#2^2_1)$	V^1
$(2,2), \bar{2}$	$(\overline{4^2_1},\ 4^2_1\#2^2_1)$	V^1
$(2,\bar{2}), 2$	$(5^2_1,\ 0^2_1\#2^2_1)$	V^1
$3, 3$	$(\overline{6^2_1},\ \overline{3_1}\#\overline{3_1})$	H
$3, \bar{3}$	$(0^2_1,\ 3_1\#\overline{3_1})$	H
$3, 2\,1$	$(6_1,\ 3_1\#\overline{3_1})$	X
$3, \bar{2}\,\bar{1}$	$(3_1,\ \overline{3_1}\#\overline{3_1})$	X
$2\,1, 2\,1$	$(\overline{6^2_3},\ 3_1\#3_1)$	H
$2, 2, 2$	$(6^3_1,\ 2^2_1\#2^2_1\#2^2_1)$	V^2
$2, 2, \bar{2}$	$(6^3_3,\ 2^2_1\#2^2_1\#2^2_1)$	V^2

TABLE 2.1. Algebraic tangles with up to seven crossings (continued).

T	$(N(T), D(T))$	type
7	$(7_1,\ 0)$	X
6 1	$(\overline{7_1},\ \overline{6^2_1})$	V
5 2	$(7_2,\ \overline{5_1})$	X
5 1 1	$(\overline{7_2},\ 6^2_1)$	V
4 3	$(7_3,\ \overline{4^2_1})$	V
4 2 1	$(\overline{7_3},\ \overline{6_1})$	X
4 1 2	$(7^2_1,\ 5_1)$	H
4 1 1 1	$(\overline{7^2_1},\ 6_1)$	H
3 4	$(\overline{7_3},\ \overline{3_1})$	X
3 3 1	$(7_3,\ 6^2_2)$	V
3 2 2	$(7_5,\ \overline{5_2})$	X
3 2 1 1	$(\overline{7_5},\ 6^2_2)$	V
3 1 3	$(7_4,\ 4^2_1)$	V
3 1 2 1	$(\overline{7_4},\ \overline{6_2})$	X
3 1 1 2	$(7^2_2,\ 5_2)$	H
3 1 1 1 1	$(\overline{7^2_2},\ 6_2)$	H
2 5	$(\overline{7_2},\ 2^2_1)$	V
2 4 1	$(7_2,\ 6_1)$	X
2 3 2	$(7^2_3,\ 5_2)$	H
2 3 1 1	$(\overline{7^2_3},\ \overline{6_1})$	H
2 2 3	$(\overline{7_5},\ 4_1)$	X
2 2 2 1	$(7_5,\ \overline{6^2_3})$	V
2 2 1 2	$(7_6,\ \overline{5_2})$	X
2 2 1 1 1	$(\overline{7_6},\ \overline{6^2_3})$	V
2 1 4	$(\overline{7^2_1},\ 3_1)$	H
2 1 3 1	$(7^2_1,\ 6_2)$	H
2 1 2 2	$(\overline{7_6},\ \overline{5^2_1})$	V
2 1 2 1 1	$(7_6,\ \overline{6_2})$	X
2 1 1 3	$(\overline{7^2_2},\ 4_1)$	H
2 1 1 2 1	$(7^2_2,\ 6_3)$	H
2 1 1 1 2	$(7_7,\ 5^2_1)$	V
2 1 1 1 1 1	$(\overline{7_7},\ 6_3)$	X
2, 2 + ++	$(\overline{7^2_3},\ 2^2_1\#2^2_1)$	V^1
(2, 2)3	$(\overline{7^2_4},\ 4^2_1)$	X^1
(2, 2)$\bar{3}$	$(\overline{7^2_8},\ 4^2_1)$	X^1
(2, $\bar{2}$)3	$(\overline{7^2_7},\ 0^2_1)$	X^1
3, 2 + +	$(\overline{7_5},\ \overline{3_1}\#2^2_1)$	V
2 1, 2 + +	$(7_6,\ 3_1\#2^2_1)$	V
(3, 2)2	$(\overline{7^2_4},\ 5_1)$	H
(3, 2)$\bar{2}$	$(\overline{7^2_7},\ 5_1)$	H
(3, $\bar{2}$)2	$(\overline{7^2_7},\ 0)$	H
(3, $\bar{2}$)$\bar{2}$	$(7^2_8,\ 0)$	H

TABLE 2.1. Algebraic tangles with up to seven crossings (continued).

T	$(N(T), D(T))$	type
$(2\,1,2)2$	$(\overline{7^2_5},\ \overline{5_2})$	H
$(2\,1,2)\bar{2}$	$(\overline{7^2_8},\ \overline{5_2})$	H
$(2,2+)2$	$(\overline{7^3_1},\ \overline{5^2_1})$	H^1
$(2,2)1\,2$	$(7^2_5,\ \overline{5^2_1})$	X^1
$4,2+$	$(\overline{7^2_1},\ \overline{4^2_1}\#2^2_1)$	V^1
$3\,1,2+$	$(\overline{7^2_2},\ 4^2_1\#2^2_1)$	V^1
$2\,2,2+$	$(\overline{7_6},\ 4_1\#2^2_1)$	V
$2\,1\,1,2+$	$(\overline{7_7},\ 4_1\#2^2_1)$	V
$(2,2),2+$	$(\overline{7^2_5},\ 4^2_1\#2^2_1)$	V^1
$3,3+$	$(\overline{7_4},\ \overline{3_1}\#\overline{3_1})$	X
$3,2\,1+$	$(7^2_2,\ 3_1\#\overline{3_1})$	H
$2\,1,2\,1+$	$(7_7,\ 3_1\#3_1)$	X
$2,2,2+$	$(7^3_1,\ 2^2_1\#2^2_1\#2^2_1)$	V^2
$(2,2++)1$	$(7^2_3,\ 6^2_3)$	X^1
$(2,2)2\,1$	$(7^2_4,\ 6^3_1)$	V^1
$(2,2)\bar{2}\,\bar{1}$	$(7^2_8,\ 6^3_3)$	V^1
$(2,\bar{2})2\,1$	$(7^2_7,\ 6^3_3)$	V^1
$(3,2+)1$	$(7_5,\ 6_2)$	X
$(2\,1,2+)1$	$(\overline{7_6},\ 6_3)$	X
$(3,2)1\,1$	$(7^2_4,\ \overline{6_2})$	H
$(2\,1,2)1\,1$	$(7^2_5,\ 6_3)$	H
$(2,2+)1\,1$	$(7^3_1,\ 6^2_3)$	H^1
$(2,2)1\,1\,1$	$(\overline{7^2_5},\ \overline{6^3_1})$	V^1
$(4,2)1$	$(7^2_1,\ 6^2_1)$	X^1
$(4,2)\bar{1}$	$(2^2_1,\ 6^2_1)$	X^1
$(4,\bar{2})\bar{1}$	$(6^2_2,\ 2^2_1)$	X^1
$(3\,1,2)1$	$(7^2_2,\ 6^2_2)$	X^1
$(2\,2,2)1$	$(7_6,\ \overline{6_1})$	X
$(2\,2,2)\bar{1}$	$(0,\ \overline{6_1})$	X
$(2\,2,\bar{2})\bar{1}$	$(\overline{6_2},\ 0)$	X
$(2\,1\,1,2)1$	$(7_7,\ 6_2)$	X
$((2,2),2)1$	$(7^2_5,\ 6^2_3)$	X^1
$((2,2),2)\bar{1}$	$(4^2_1,\ 6^2_3)$	X^1
$((2,2),\bar{2})\bar{1}$	$(7^2_7,\ 4^2_1)$	X^1
$((2,\bar{2}),2)1$	$(7^2_8,\ \overline{5^2_1})$	X^1
$((2,\bar{2}),2)\bar{1}$	$(5^2_1,\ \overline{5^2_1})$	X^1
$(3,3)1$	$(7_4,\ 6^2_1)$	V
$(3,3)\bar{1}$	$(\overline{3_1},\ 6^2_1)$	V
$(3,\bar{3})1$	$(\overline{6_1},\ 0^2_1)$	V
$(3,2\,1)1$	$(7^2_2,\ \overline{6_1})$	H
$(3,\bar{2}\,\bar{1})\bar{1}$	$(6^2_3,\ \overline{3_1})$	H
$(2\,1,2\,1)1$	$(\overline{7_7},\ 6^2_3)$	V
$(2,2,2)1$	$(\overline{7^3_1},\ \overline{6^3_1})$	X^2
$(2,2,2)\bar{1}$	$(\overline{6^3_3},\ \overline{6^3_1})$	X^2
$(2,2,\bar{2})\bar{1}$	$(6^3_3,\ \overline{6^3_3})$	X^2

TABLE 2.1. Algebraic tangles with up to seven crossings (continued).

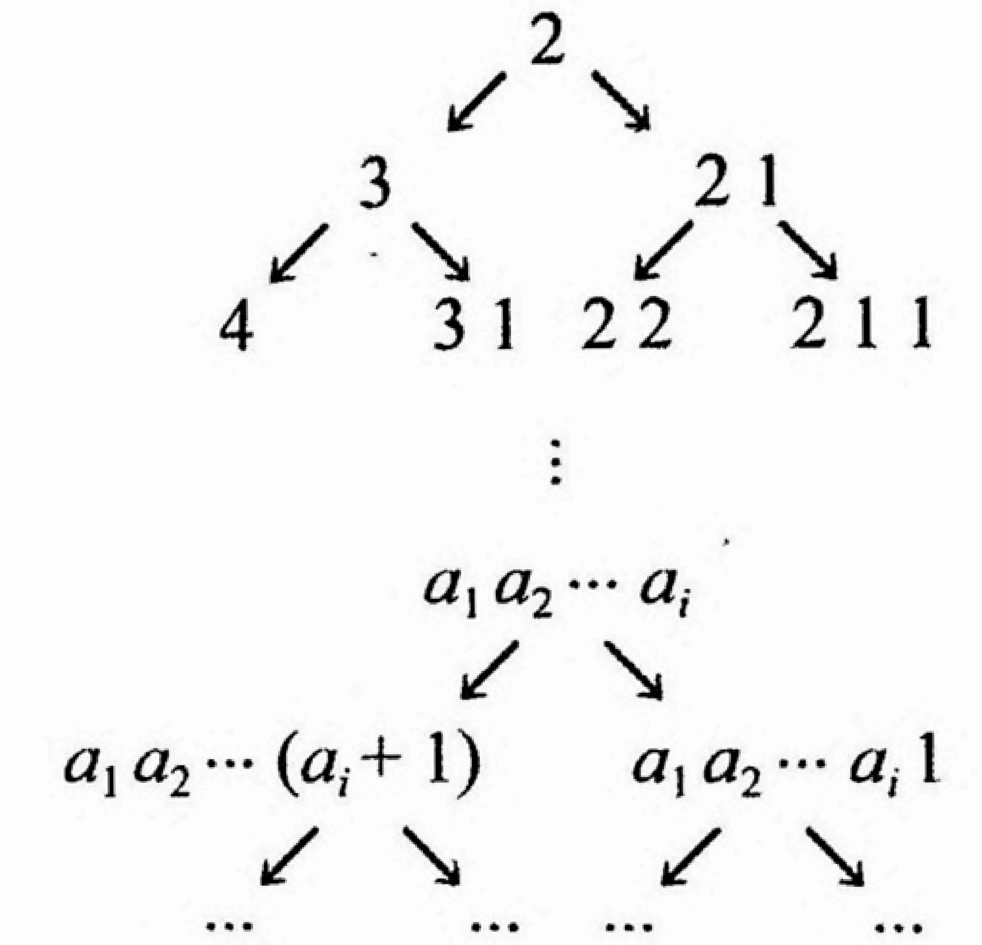
T	$(N(T), D(T))$	type
$5, 2$	$(\overline{7_1},\ \overline{5_1}\#2_1^2)$	V
$5, \bar{2}$	$(\overline{3_1},\ \overline{5_1}\#2_1^2)$	V
$4\,1, 2$	$(\overline{7_3},\ 5_1\#2_1^2)$	V
$3\,2, 2$	$(7_3,\ \overline{5_2}\#2_1^2)$	V
$3\,2, \bar{2}$	$(0,\ \overline{5_2}\#2_1^2)$	V
$3\,1\,1, 2$	$(\overline{7_4},\ 5_2\#2_1^2)$	V
$2\,3, 2$	$(7_2,\ 5_2\#2_1^2)$	V
$2\,3, \bar{2}$	$(3_1,\ 5_2\#2_1^2)$	V
$2\,2\,1, 2$	$(\overline{7_5},\ \overline{5_2}\#2_1^2)$	V
$2\,1\,2, 2$	$(7_1^2,\ \overline{5_1^2}\#2_1^2)$	V^1
$2\,1\,2, \bar{2}$	$(2_1^2,\ \overline{5_1^2}\#2_1^2)$	V^1
$2\,1\,1\,1, 2$	$(7_2^2,\ 5_1^2\#2_1^2)$	V^1
$(3, 2), 2$	$(7_5,\ 5_1\#2_1^2)$	V
$(3, 2), \bar{2}$	$(5_2,\ 5_1\#2_1^2)$	V
$(3, \bar{2}), 2$	$(6_2,\ 2_1^2)$	V
$(3, \bar{2}), \bar{2}$	$(6_3,\ 2_1^2)$	V
$(2\,1, 2), 2$	$(\overline{7_6},\ \overline{5_2}\#2_1^2)$	V
$(2\,1, 2), \bar{2}$	$(\overline{5_1},\ \overline{5_2}\#2_1^2)$	V
$(2, 2+), 2$	$(7_3^2,\ 5_1^2\#2_1^2)$	V^1
$(2, 2+), \bar{2}$	$(0_1^2,\ 5_1^2\#2_1^2)$	V^1
$((2, 2)1), 2$	$(7_4^2,\ \overline{5_1^2}\#2_1^2)$	V^1
$((2, 2)\bar{1}), \bar{2}$	$(7_8^2,\ 0_1^2\#2_1^2)$	V^1
$((2, \bar{2})1), 2$	$(7_7^2,\ \overline{4_1^2}\#2_1^2)$	V^1
$4, 3$	$(\overline{7_1},\ \overline{4_1^2}\#3_1)$	V
$4, \bar{3}$	$(0,\ \overline{4_1^2}\#3_1)$	V
$3\,1, 3$	$(7_3,\ 4_1^2\#\overline{3_1})$	V
$3\,1, \bar{3}$	$(4_1,\ \overline{4_1^2}\#3_1)$	V
$2\,2, 3$	$(7_2,\ 4_1\#\overline{3_1})$	X
$2\,2, \bar{3}$	$(0,\ 4_1\#3_1)$	X
$2\,1\,1, 3$	$(\overline{7_1^2},\ 4_1\#\overline{3_1})$	H
$2\,1\,1, \bar{3}$	$(4_1^2,\ 4_1\#3_1)$	H
$(2, 2), 3$	$(7_3^2,\ 4_1^2\#\overline{3_1})$	X^1
$(2, 2), \bar{3}$	$(\overline{5_1^2},\ 4_1^2\#3_1)$	X^1
$(2, \bar{2}), 3$	$(6_3^2,\ 0_1^2\#\overline{3_1})$	X^1
$4, 2\,1$	$(7_2,\ \overline{4_1^2}\#3_1)$	V
$3\,1, 2\,1$	$(7_5,\ 4_1^2\#3_1)$	V
$2\,2, 2\,1$	$(\overline{7_3^2},\ 4_1\#3_1)$	H
$2\,1\,1, 2\,1$	$(\overline{7_6},\ 4_1\#3_1)$	X
$(2, 2), 2\,1$	$(\overline{7_1^3},\ 4_1^2\#3_1)$	H^1
$(2, 2), \bar{2}\,\bar{1}$	$(6_3^3,\ 4_1^2\#\overline{3_1})$	H^1
$(2, \bar{2}), 2\,1$	$(\overline{6_1^3},\ 0_1^2\#3_1)$	H^1
$3, 2, 2$	$(7_4^2,\ \overline{3_1}\#2_1^2\#2_1^2)$	V^1
$3, 2, \bar{2}$	$(7_7^2,\ \overline{3_1}\#2_1^2\#2_1^2)$	V^1
$3, \bar{2}, \bar{2}$	$(\overline{7_8^2},\ \overline{3_1}\#2_1^2\#2_1^2)$	V^1
$2\,1, 2, 2$	$(7_5^2,\ 3_1\#2_1^2\#2_1^2)$	V^1

SKETCH OF PROOF OF THEOREM 2.5. First, we enumerate the rational tangles with up to crossings. Second, we give algebraic (but not rational) tangle diagrams with up to seven cro by using operations in Fig. 2.4. Third, we classify the tangles up to equivalence.

In order to enumerate all the rational tangles of n crossings, by Definition 2.1 and Remark 2.2, we produce the sequences of positive integers $a_1 a_2 \cdots a_i$ satisfying

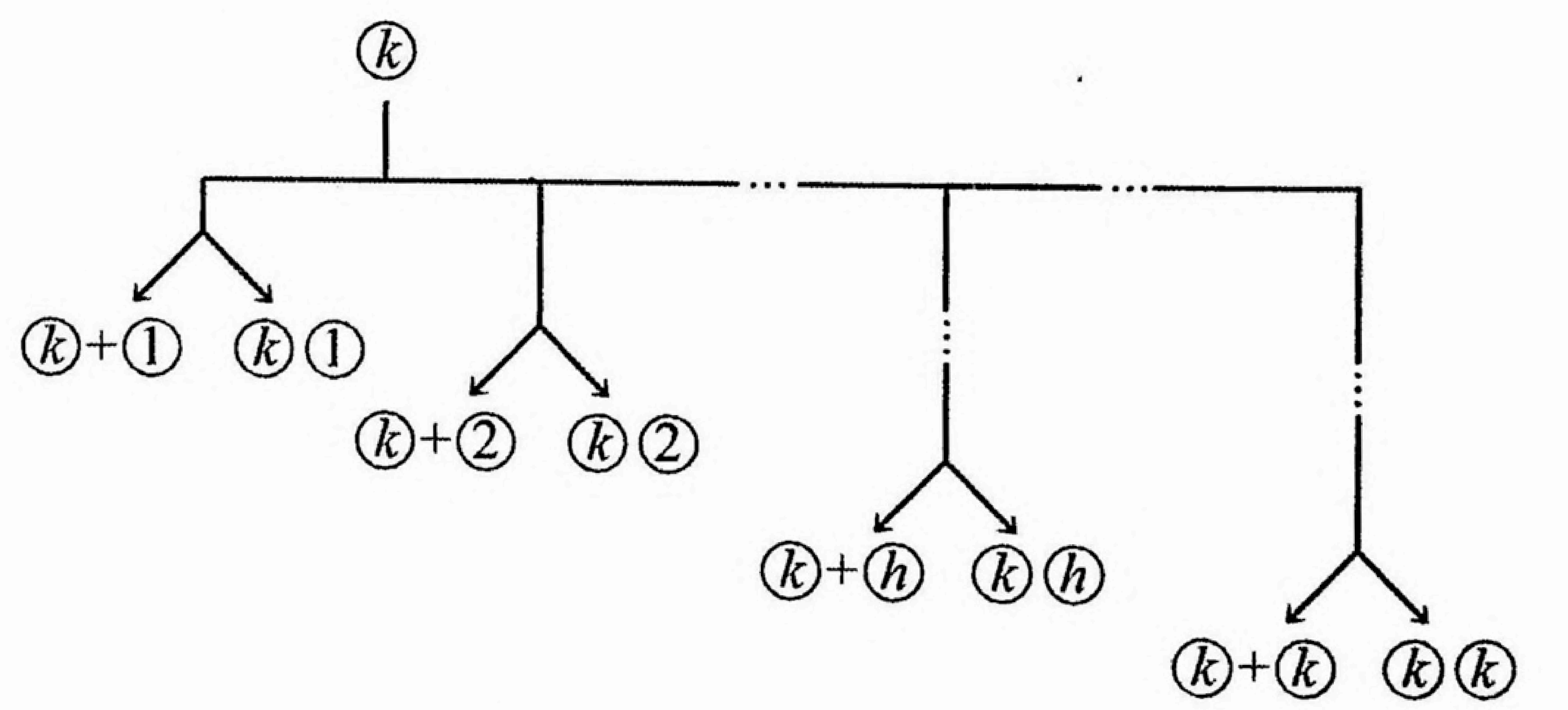
$$a_1 + a_2 + \cdots + a_i = n.$$

Specifically, we construct a binary tree as follows.



The sequence corresponds to the rational tangle $a_1 a_2 \cdots a_i$ (cf. [11]).

In order to enumerate all the algebraic (but not rational) tangle diagrams of n crossings, we construct the following tree.



Here, ⓚ denotes an algebraic tangle with k crossings, ⓚ+ⓗ and ⓚⓗ are new algebraic tangles with $(k+h)$ crossings (see Fig. 2.9). By Definition 2.1, we may assume $1 \leq h \leq k$. If ⓚ is the rational tangle $a_1 a_2 \cdots a_i$ $(a_m > 0)$, ⓚ+① corresponds to $a_1 a_2 \cdots (a_i + 1)$ and ⓚ① corresponds to $a_1 a_2 \cdots a_i\, 1$.

EXAMPLE 2.6. We give the case where $k = h = 2$ in Fig. 2.9. Note that the algebraic tangles with two crossings are $2, \bar{2}, 2\,0, \bar{2}\,0$.

Then, for each tangle T, we investigate the corresponding links $\{N(T), D(T)\}$ and compare them. Except for the tangles 5 and $(3,2)\bar{1}$, we show these tangles are mutually distinct by the corresponding links. In fact, $\{N(5), D(5)\} = \{N((3,2)\bar{1}), D((3,2)\bar{1})\} = \{0, 5_1\}$. However, their doubles $W(5)$ and $W((3,2)\bar{1})$ are not isotopic. So, the tangles 5 and $(3,2)\bar{1}$ are not equivalent. □

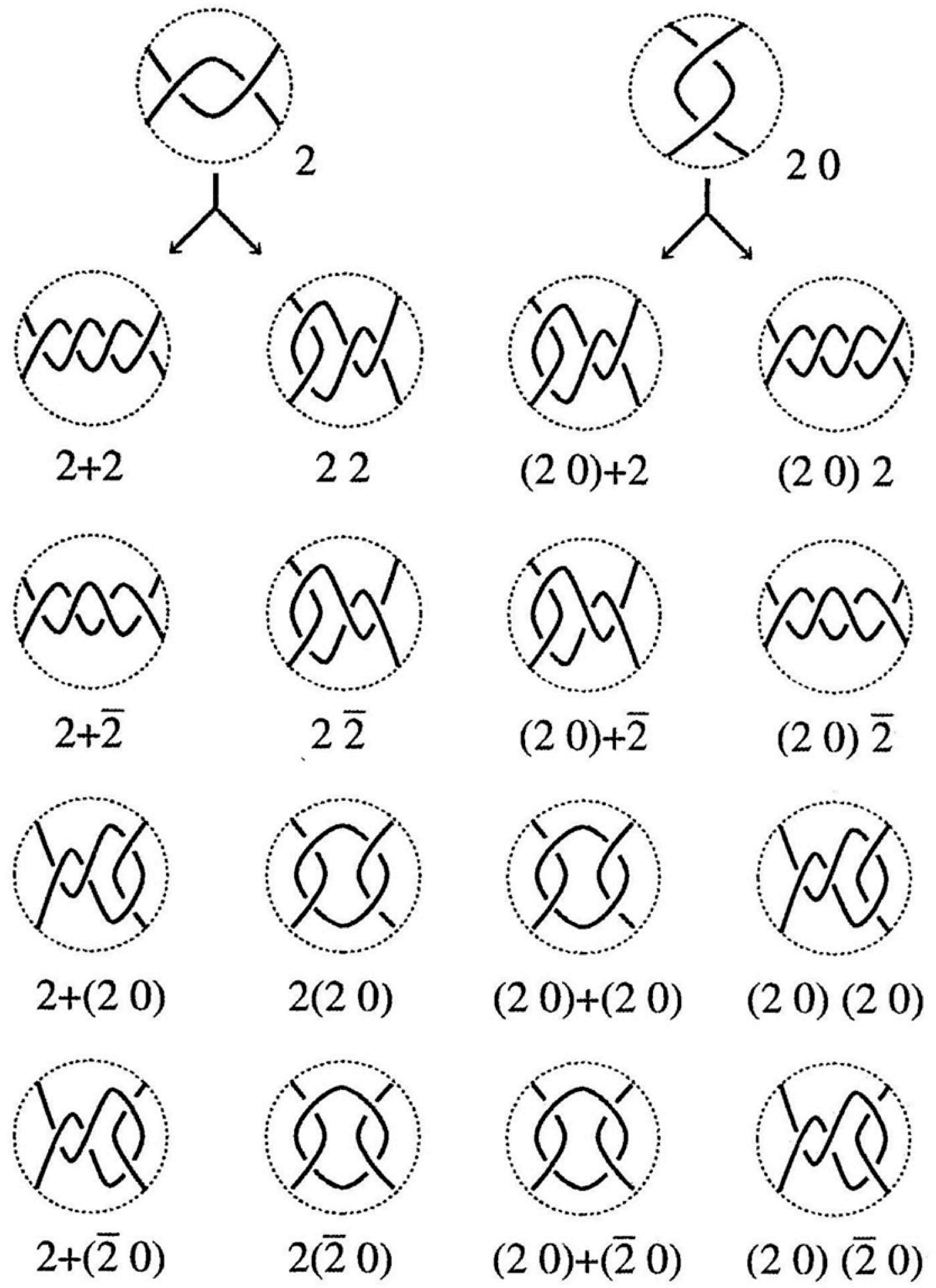


FIG. 2.9. The case $k = h = 2$.

CHAPTER 3

Prime basic θ-polyhedron

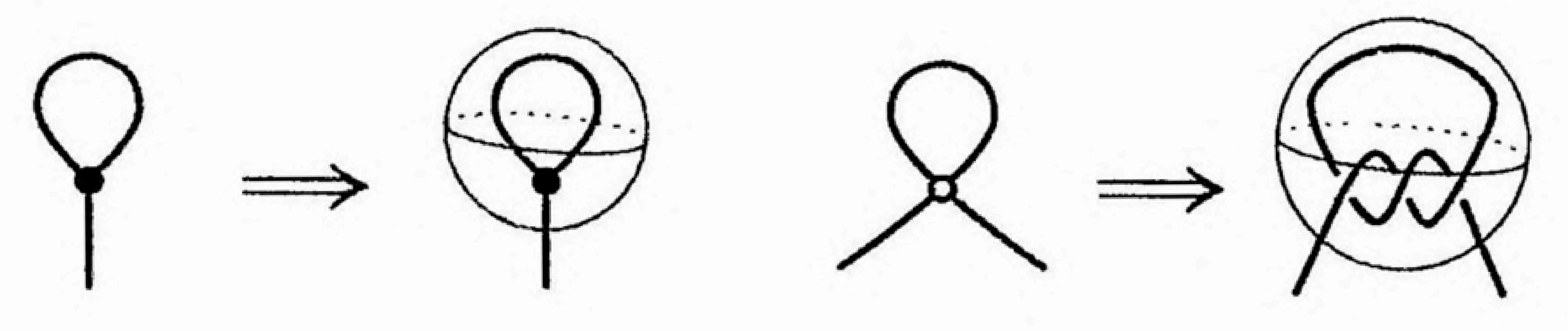
In [5], Conway enumerated prime knots with up to eleven crossings, and links with up to ten crossings by introducing the concept of a tangle and a *basic polyhedron.* A polyhedron is a connected 4-regular graph embedded in 2-sphere. In this chapter, we construct a *prime basic θ-polyhedron* to enumerate prime θ-curves and handcuff graph. A θ-polyhedron is a connected graph embedded in 2-sphere, whose two vertices are 3-valent, and the others are 4-valent. Thus our θ-polyhedron is different from Conway's polyhedron.

This chapter is organized as follows: In Section 1, we introduce the concept of a prime basic θ-polyhedron. In Section 2, we give prime basic θ-polyhedra with up to seven 4-valent vertices. In Section 3, we enumerate type-$*$ prime basic θ-polyhedra (cf. [21]).

1. Definition

Let P_Θ be a connected planar graph. Then P_Θ is called a *θ-polyhedron* if its two vertices are 3-valent and the other vertices are 4-valent. A θ-polyhedron P_Θ is said to be *basic* if it contains no loop and no bigon.

REMARK 3.1. If P_Θ contains a loop, then P_Θ produces a spatial 3-valent graph diagram which has an admissible sphere of type I or II (Fig. 3.1).



(a) Type I (b) Type II

FIG. 3.1. Admissible spheres.

REMARK 3.2. If P_Θ contains a bigon, then a spatial 3-valent graph diagram obtained from P_Θ is also obtained from another polyhedron P'_Θ with fewer 4-valent vertices than P_Θ. In fact, adding two algebraic tangles, we obtain another algebraic tangle (Fig. 3.2).

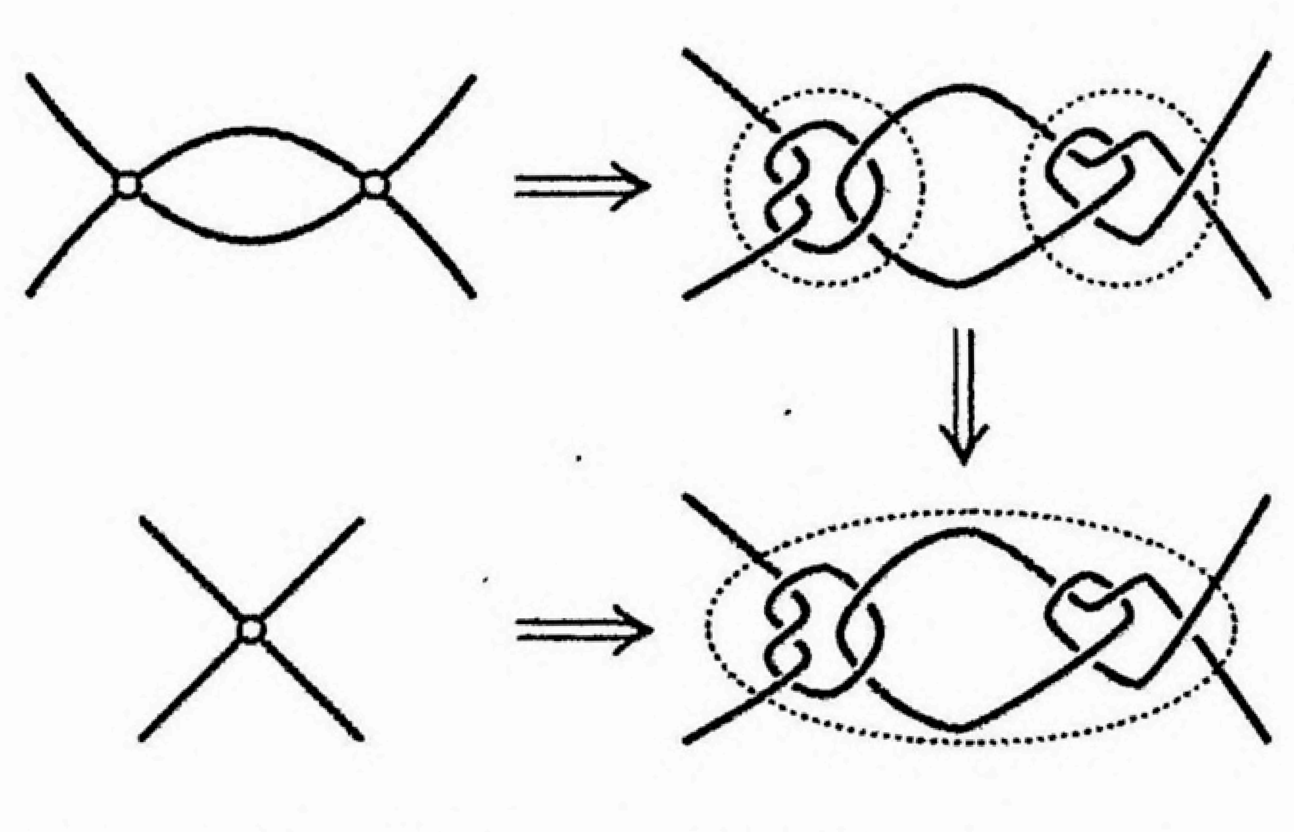


FIG. 3.2. The sum of algebraic tangles.

Let C be a circle which decomposes S^2 into 2-disks D_1, D_2. If C meets P_Θ in less than or equa to three points, and both $P_\Theta \cap D_1$ and $P_\Theta \cap D_2$ contain 4-valent vertices, then C is called a *cuttin circle* for P_Θ (Fig. 3.3).

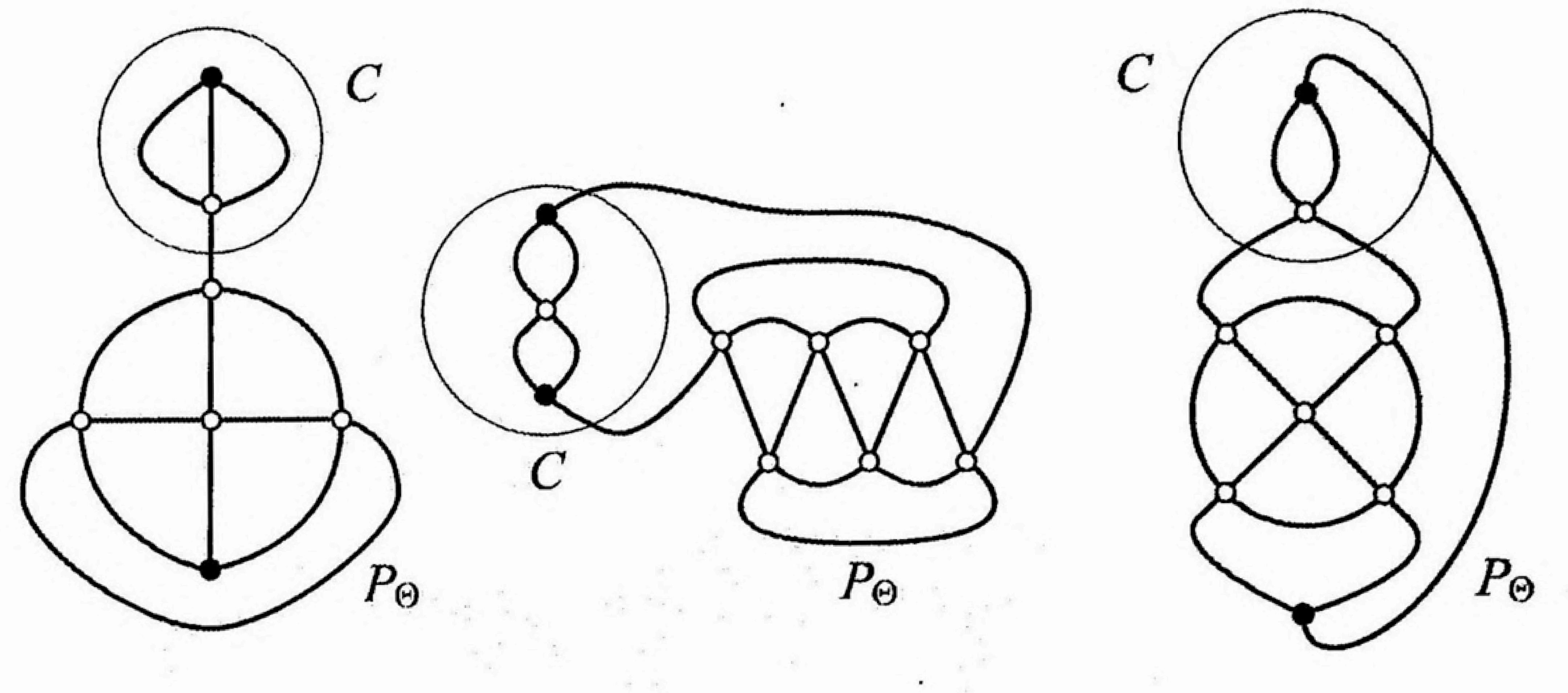


FIG. 3.3. Nonprime θ-polyhedra.

DEFINITION 3.3. A θ-polyhedron P_Θ is said to be *prime* if P_Θ does not have a cutting circle.

REMARK 3.4. A nonprime P_Θ produces a nonprime θ-curve or handcuff graph. However, a prime basic P_Θ may produce a nonprime θ-curve or handcuff graph; see Tables 4.1 and 4.2.

A graph is said to be 3-*connected* if we must remove at least three vertices to disconnect it. Using the following lemma due to Whitney([35],[36]), the uniqueness of planar embedding of a prime basic θ-polyhedron with more than three vertices are guaranteed.

LEMMA 3.5. *A 3-connected planar graph has a unique planar embedding.*

Prime basic θ-polyhedra are classified into two types, according as if their 3-valent vertices are adjacent or not. We call the former *type-$\times$ prime basic θ-polyhedra*, and the latter *type-$*$ prime basic θ-polyhedra*. First, we give type-$\times$ prime basic θ-polyhedra with up to seven 4-valent vertices.

THEOREM 3.6. *There exist seven type-$\times$ prime basic θ-polyhedra with up to seven 4-valent vertices as in Fig.* 3.4.

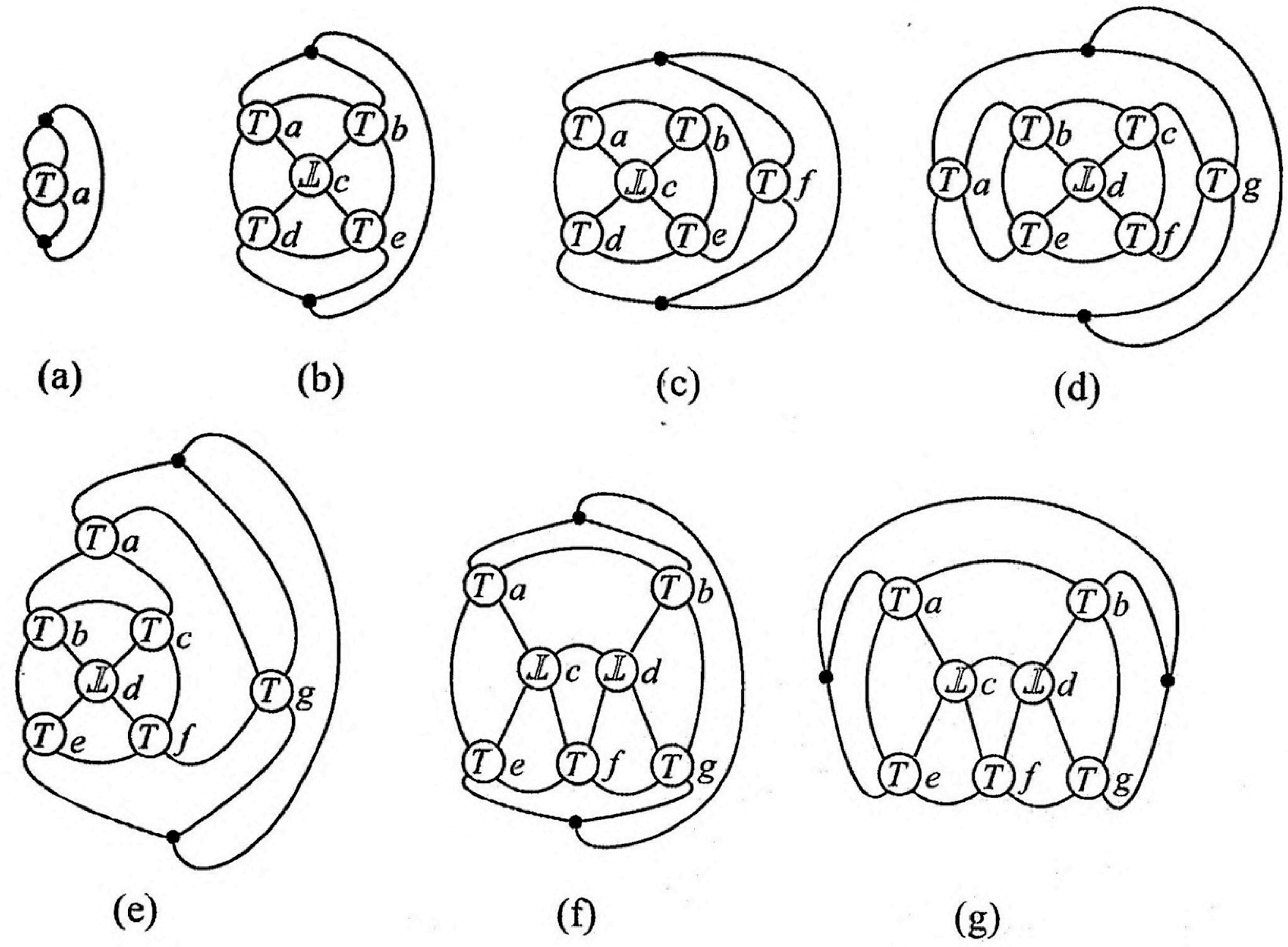


FIG. 3.4.
(a) $1^1_\times a$ (b) $5^1_\times a.b.c.d.e$ (c) $6^1_\times a.b.c.d.e.f$ (d) $7^1_\times a.b.c.d.e.f.g$
(e) $7^2_\times a.b.c.d.e.f.g$ (f) $7^3_\times a.b.c.d.e.f.g$ (g) $7^4_\times a.b.c.d.e.f.g$

REMARK 3.7. From now on, 4-valent vertices in a θ-polyhedron will be denoted by $\textcircled{T}$ or $\textcircled{\mathbb{T}}$ in the figures, because we substitute 4-valent vertices by tangles later. The vertex $\textcircled{T}$ corresponds to a tangle T, and $\textcircled{\mathbb{T}}$ corresponds to a tangle μT; see Fig.2.2.

To prove Theorem 3.6, we use the following result. In **[40]**, Yamano classified tangles of seven crossings or less in his master's thesis, where he used the concept of *prime basic* 4-*regular disk graphs*. Let Q be a connected 4-regular planar graph, and v be a vertex of Q. We denote the neighborhood of v by $N(v)$, and the interior of $N(v)$ by Int$N(v)$. A pair (B^2, P) is called a 4-*regular disk graph* if (B^2, P) and $\left(S^2 \setminus \text{Int}N(v), Q \setminus \text{Int}N(v)\right)$ are homeomorphic; see Fig. 3.5.

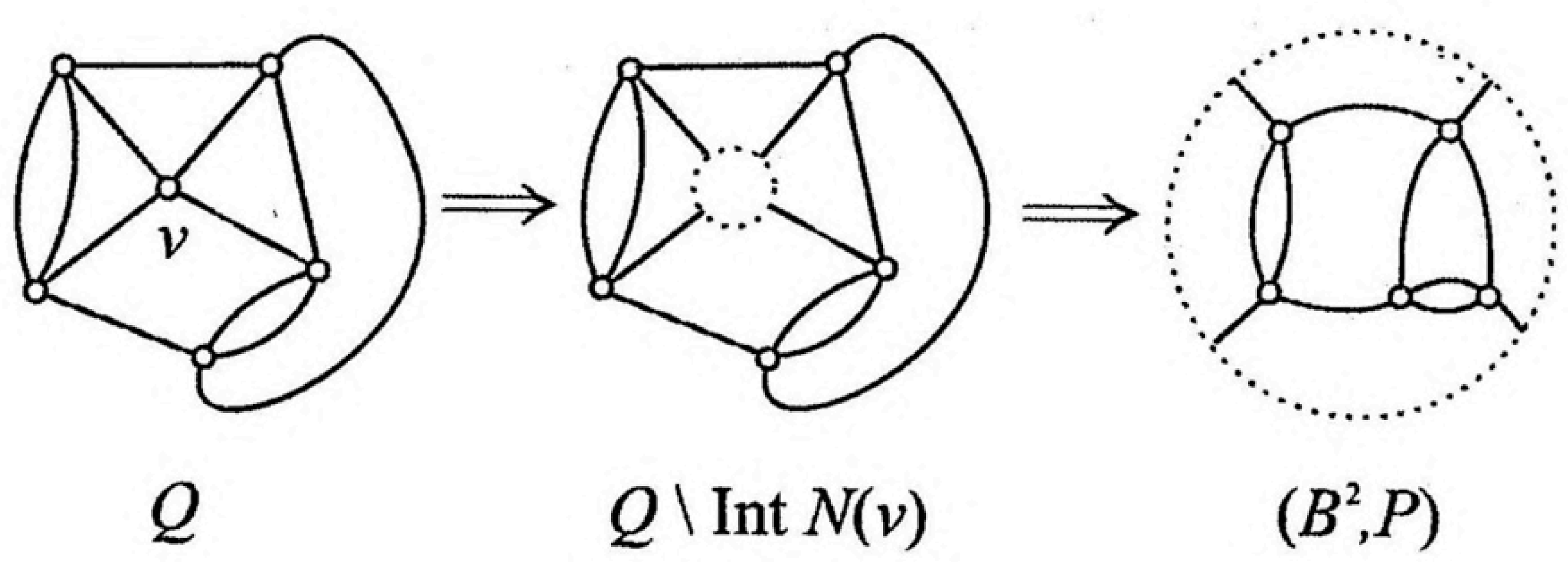


FIG. 3.5. The making of a 4-regular disk graph.

A 4-regular disk graph (B^2, P) is said to be *basic* if P contains no loop and no bigon, and be *prime* if for any disk D in B^2, such that ∂D meets P transversely in two points, D contains no vertex. Yamano (**[40]**) gave the following lemma:

LEMMA 3.8. *There exist six prime basic* 4*-regular disk graphs with up to seven vertices as in Fig.* 3.6.

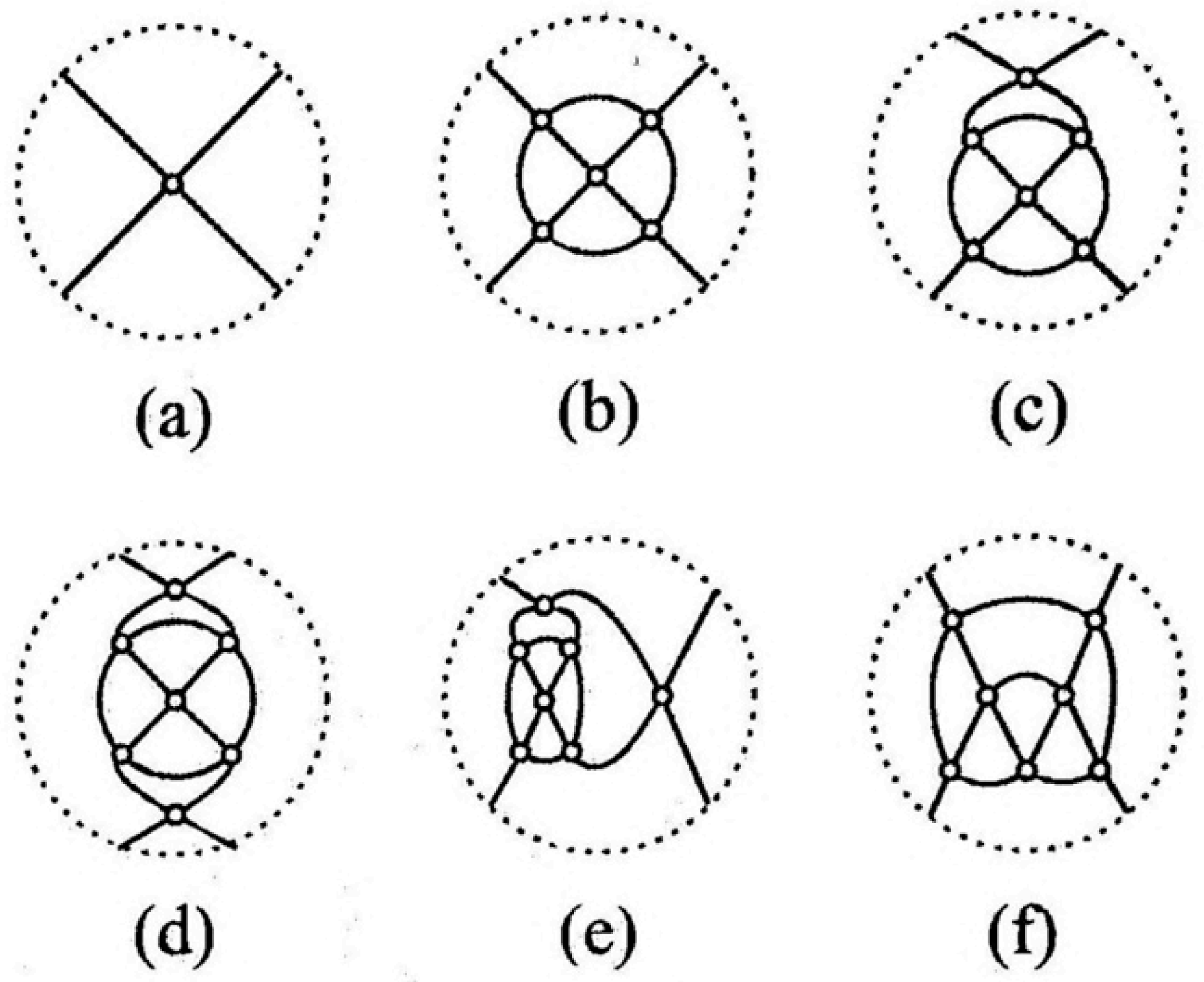


FIG. 3.6. (a) P_1 (b) P_5 (c) P_6 (d) P_{7-1} (e) P_{7-2} (f) P_{7-3}

Yamano enumerated a tangle diagram from a prime basic 4-regular disk graph by substituting algebraic tangles for their vertices.

PROOF OF THEOREM 3.6. By using Lemma 3.8, we construct type-$\times$ prime basic θ-polyhedra. First, we construct the "numerator" and the "denominator" of a prime basic 4-regular disk graph. Second, we add a vertex on each of the new edges. Third, we join these two vertices by an edge; see Fig. 3.7. Finally, we check whether it is 3-connected. From Lemma 3.5, we obtain Theorem 3.6. □

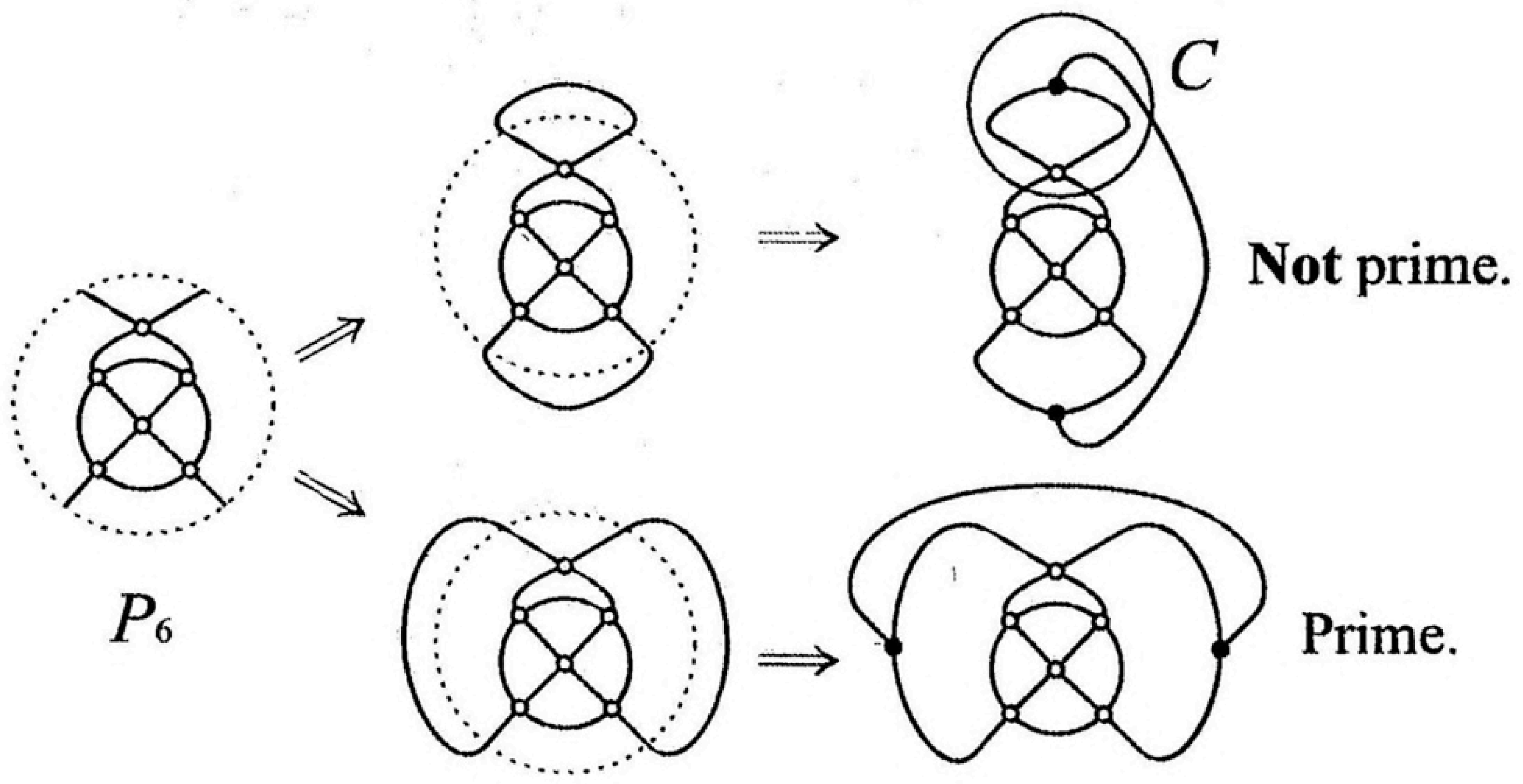


FIG. 3.7. The making of a θ-polyhedron.

REMARK 3.9. Since the θ-polyhedron $1^1_\times a$ has only three vertices, it is not 3-connected. However, there does not exist another planar embedding.

Next, we give type-$*$ prime basic θ-polyhedra with up to seven 4-valent vertices.

THEOREM 3.10. *There exist seventeen type-$*$ prime basic θ-polyhedra with up to seven* 4*-valent vertices as in Fig.* 3.8.

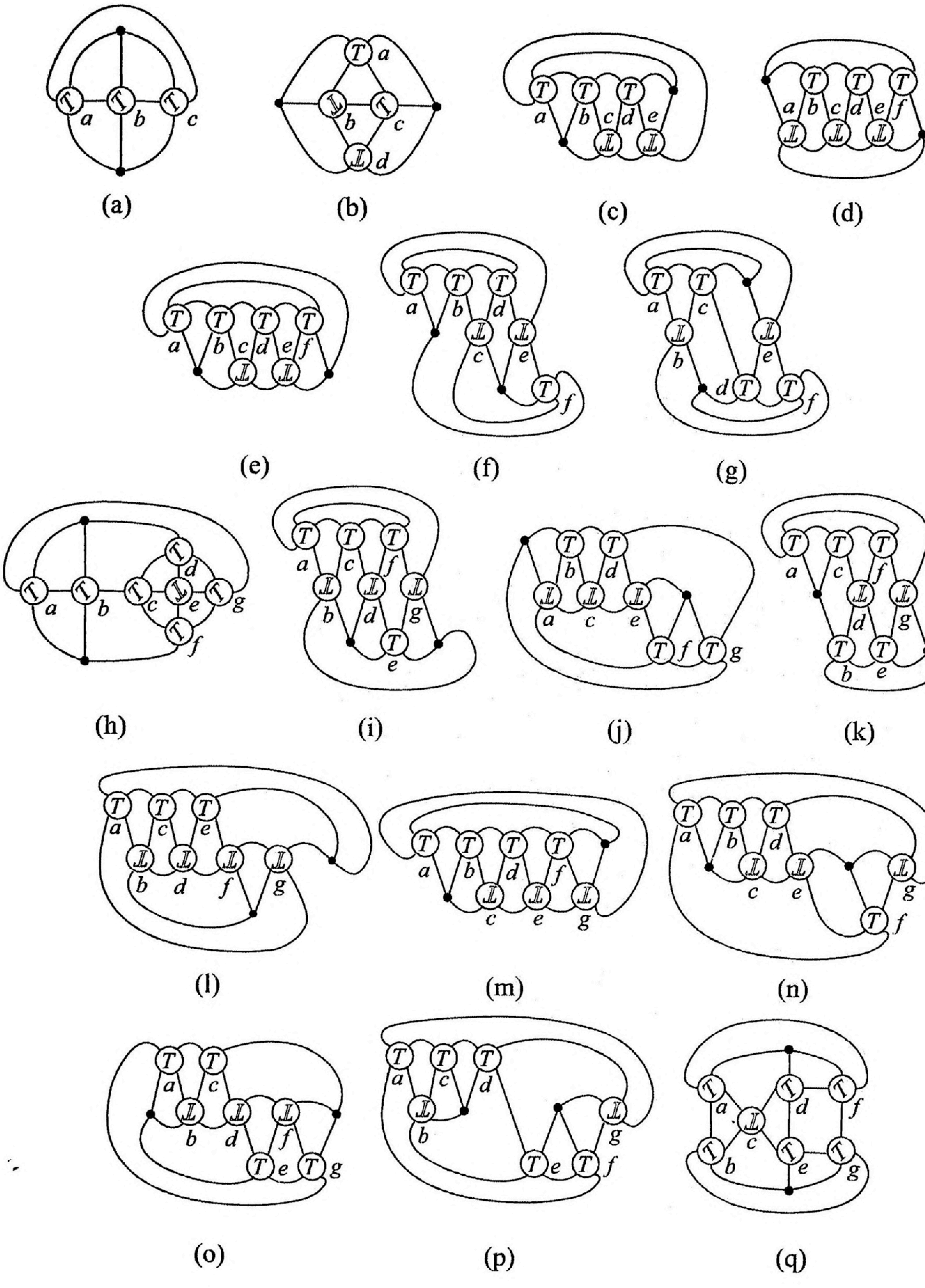


FIG. 3.8.
(a) $3^1_* a.b.c$ (b) $4^1_* a.b.c.d$ (c) $5^1_* a.b.c.d.e$ (d) $6^1_* a.b.c.d.e.f$
(e) $6^2_* a.b.c.d.e.f$ (f) $6^3_* a.b.c.d.e.f$ (g) $6^4_* a.b.c.d.e.f$
(h) $7^1_* a.b.c.d.e.f.g$ (i) $7^2_* a.b.c.d.e.f.g$ (j) $7^3_* a.b.c.d.e.f.g$ (k) $7^4_* a.b.c.d.e.f.g$
(l) $7^5_* a.b.c.d.e.f.g$ (m) $7^6_* a.b.c.d.e.f.g$ (n) $7^7_* a.b.c.d.e.f.g$
(o) $7^8_* a.b.c.d.e.f.g$ (p) $7^9_* a.b.c.d.e.f.g$ (q) $7^{10}_* a.b.c.d.e.f.g$

2. Proof of Theorem 3.10.

In this section, we will prove Theorem 3.10. To prove it, we need the following well-known theorem due to Kuratowski ([17]).

LEMMA 3.11 (Kuratowski's Theorem [17]). *A graph G is planar if and only if G does not have a subgraph which is subdivision of the complete graph K_5 or the complete bipartite graph $K_{3,3}$ (Fig. 3.9).*

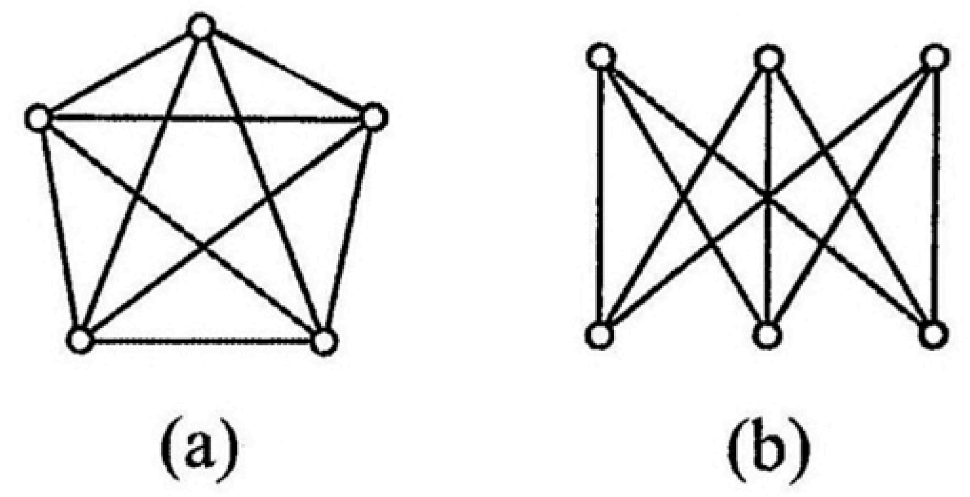


FIG. 3.9. (a) K_5, (b) $K_{3,3}$.

PROOF OF THEOREM 3.10. We enumerate type-$*$ prime basic θ-polyhedra according to the number of 4-valent vertices t, $3 \leq t \leq 7$. Such a prime basic θ-polyhedron satisfies the following conditions:
(P1) It contains no loop.
(P2) It contains no bigon.
(P3) Its 3-valent vertex and 4-valent vertex are not connected with two edges.
(P4) Its 3-valent vertex and 4-valent vertex are not connected with three edges.
(P5) It is a planar graph.

In fact, the conditions (P1), (P2), (P3), (P5) follow from the definition of a basic prime θ-polyhedron. If a polyhedron does not satisfy (P4), then we obtain a handcuff graph; see Fig. 3.10.

If 4-valent vertices X and Y are connected by an edge, we write $X \sim Y$.

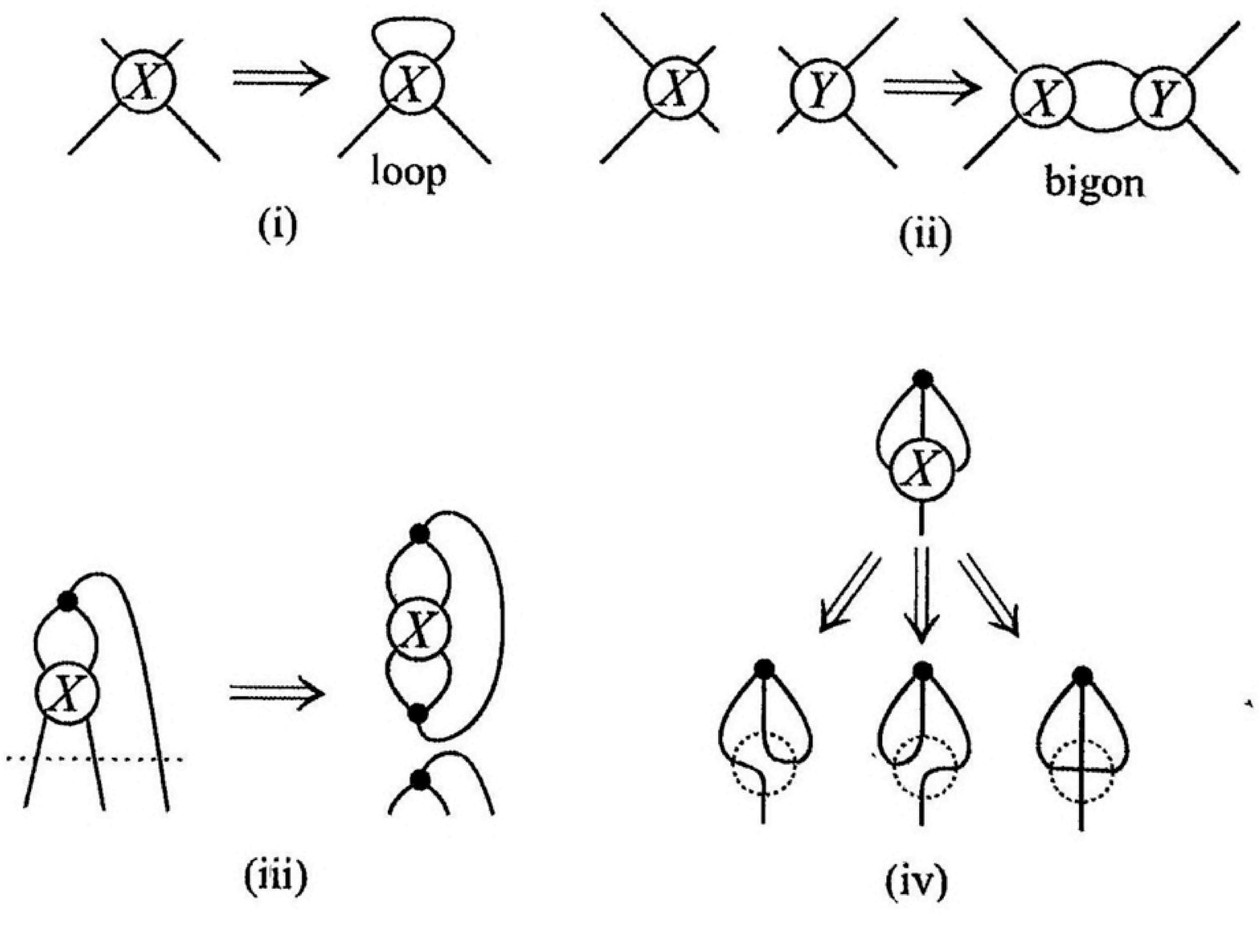


FIG. 3.10. Conditions.

The conditions (P3) and (P4) induce the following four patterns according to the number of 4-ent vertices which are adjacent to the both 3-valent vertices: (a) three, (b) two, (c) one, (d) zero; see Fig. 3.11. Conversely, all prime θ-curves are constructed from one of these patterns.

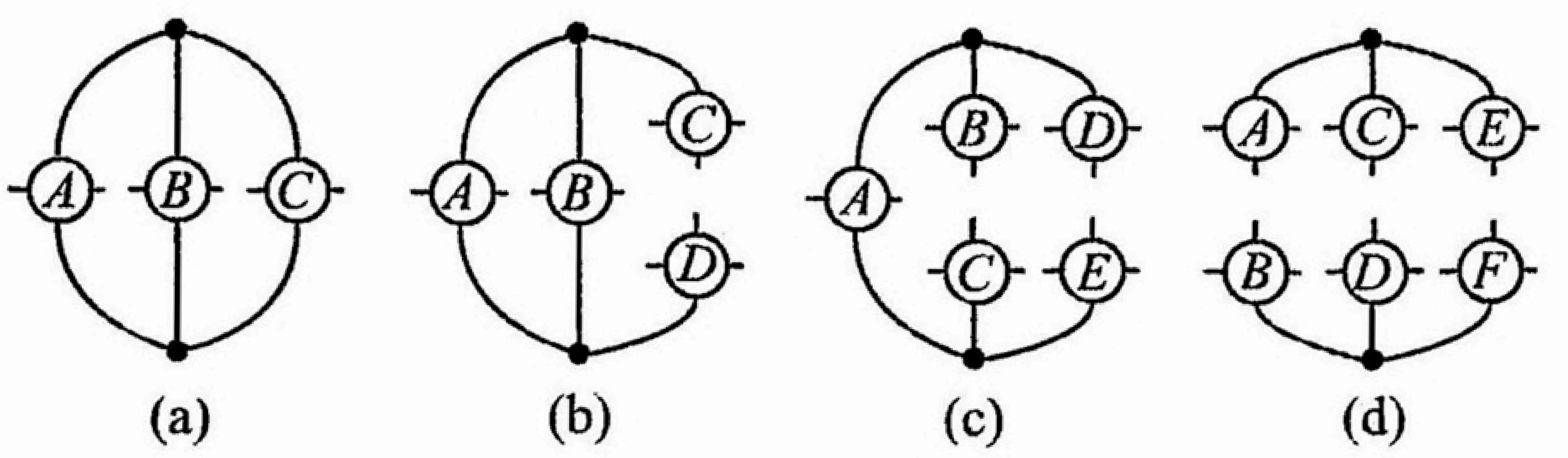


FIG. 3.11. Patterns.

From now on, we call small arcs which join a 4-valent vertex "hands".

2.1. $t = 3$. The pattern (a) only occurs. By the condition (P2), we obtain 3^1_*; see Fig. 3.12.

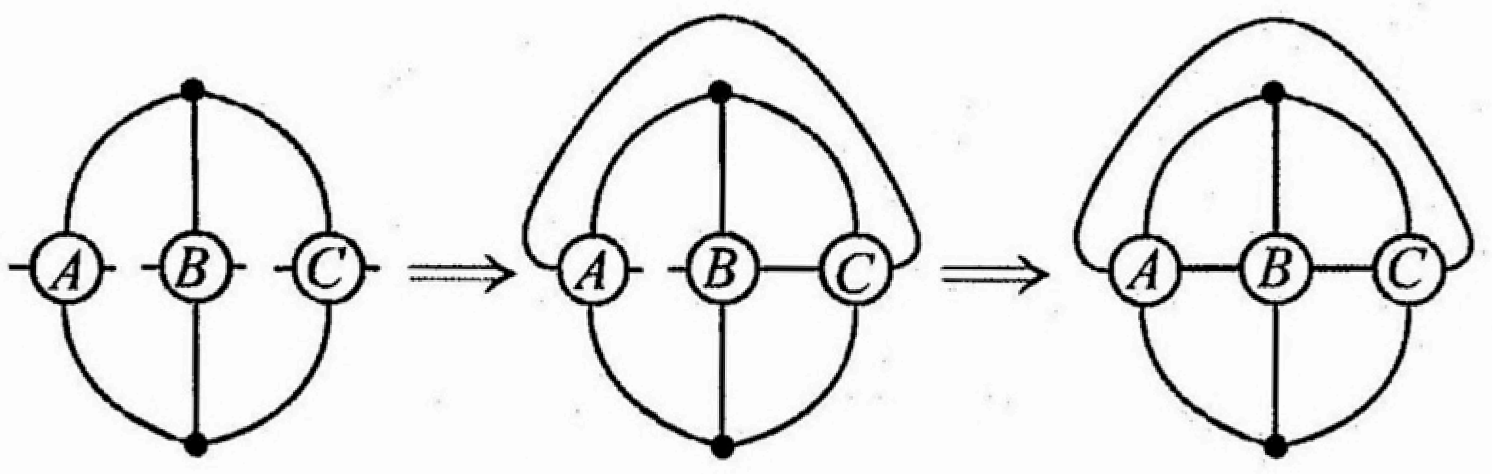


FIG. 3.12. $t = 3$.

2.2. $t = 4$. The patterns (a) and (b) occur.
Pattern (a). See Fig. 3.13. We consider how the hands of D connect. By the condition (P2), $D \sim A$, B, C, which gives a graph containing $K_{3,3}$, and it does not satisfy the condition (P5).

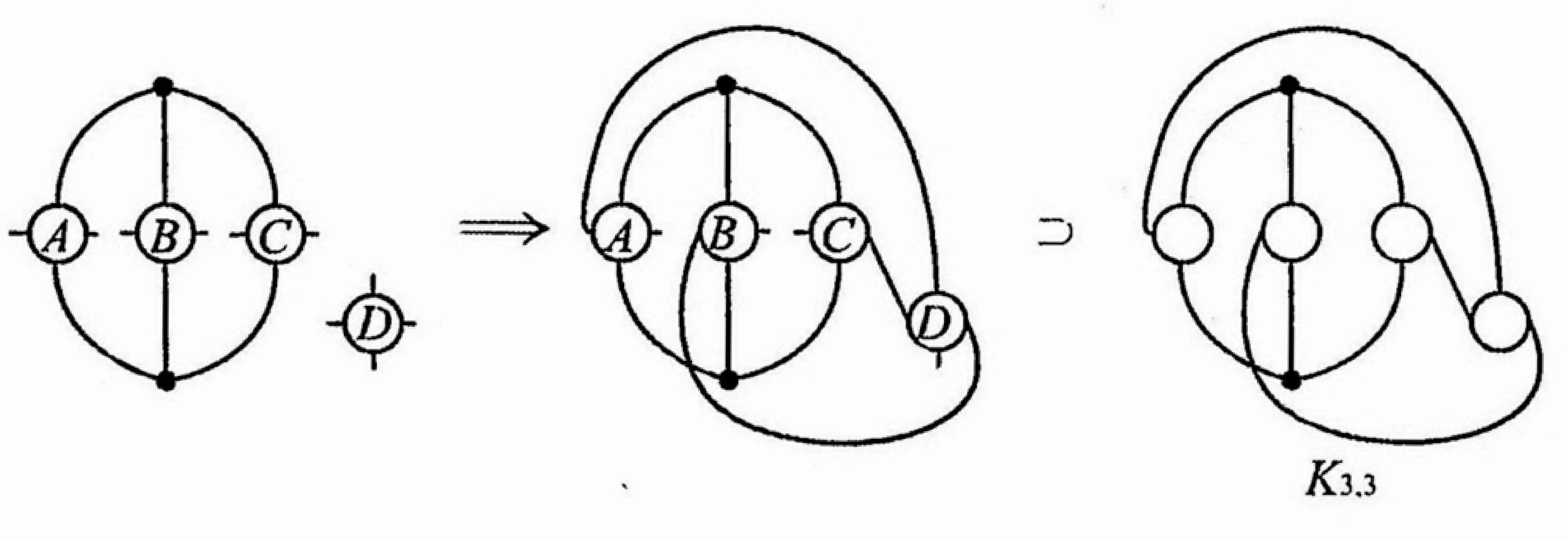


FIG. 3.13. $t = 4$ (a).

Pattern (b). See Fig. 3.14. Each of the vertices C and D has three remaining hands, and so we consider how the hands of D connect. By the condition (P2), $D \sim A$, B, C. Then $C \sim A$, B, and we obtain 4^1_*.

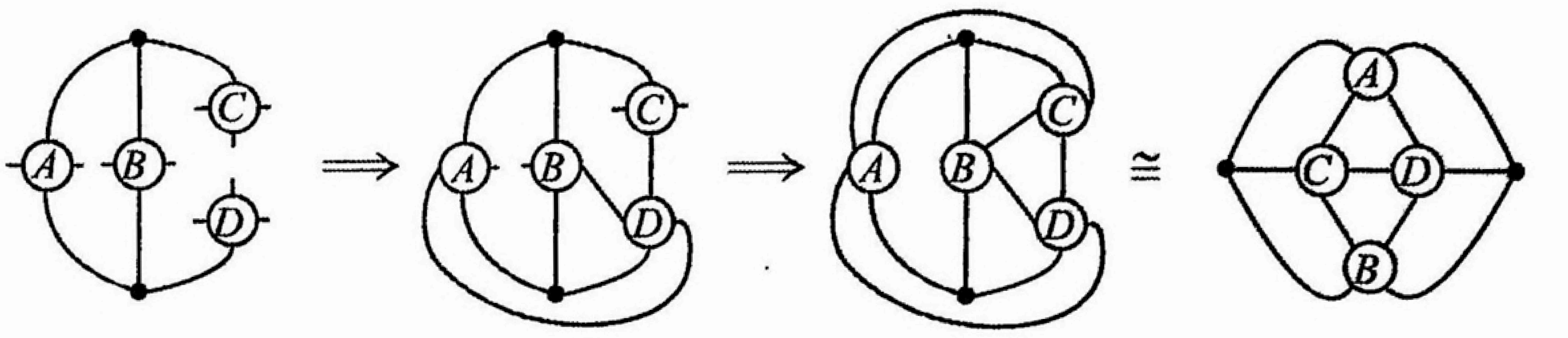


FIG. 3.14. $t = 4$ (b).

2.3. $t = 5$. We consider the patterns (a), (b), and (c).
Pattern (a). By an argument similar to that in the case $t = 4$ (a), this case does not give a planar graph.
Pattern (b). See Fig. 3.15. We consider how the hands of E connect. By the condition (P2), $E \sim A$, B, C, D. Then we consider how the hands of D connect. There are three cases.

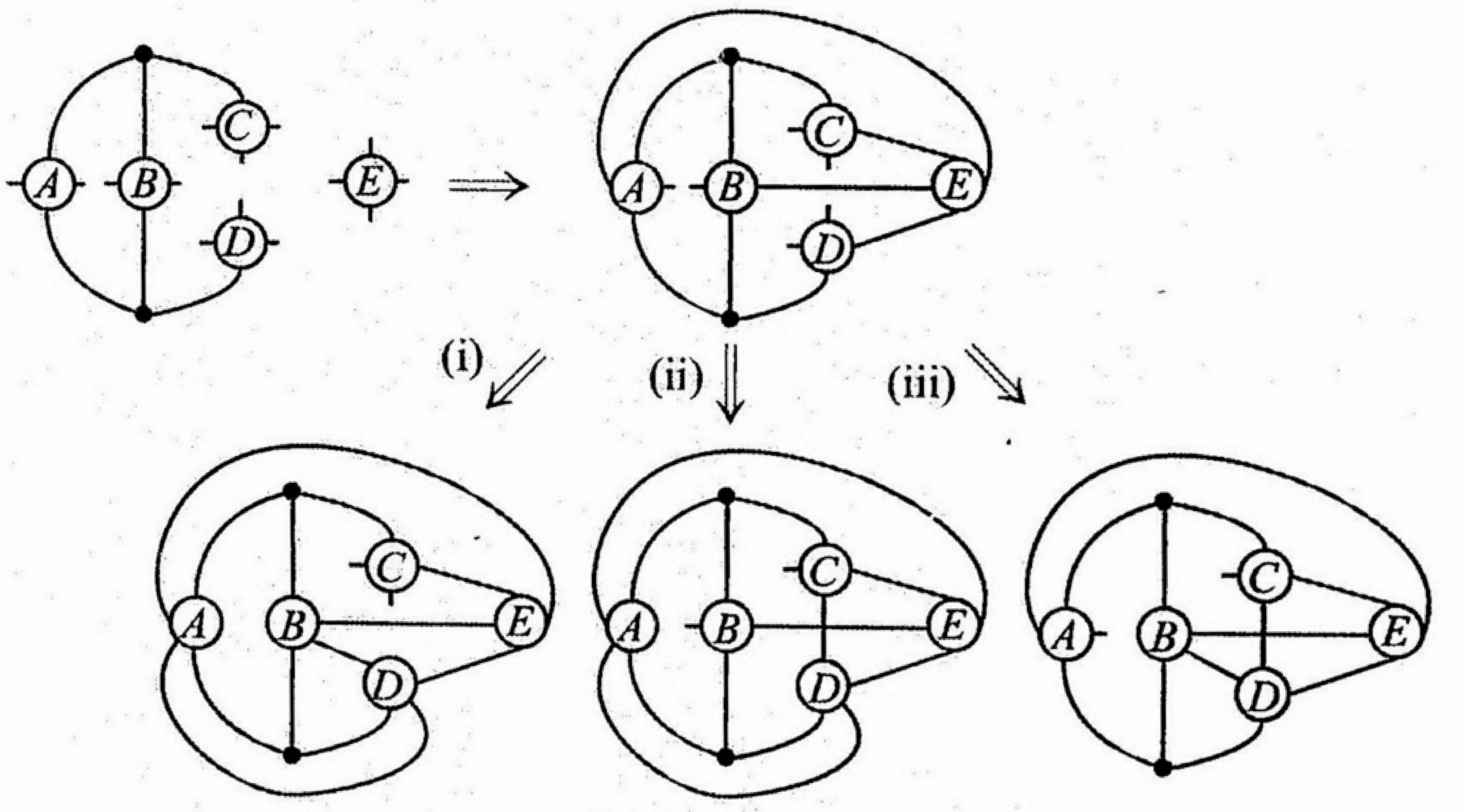


FIG. 3.15. $t = 5$ (b).

(i) $D \sim A$, B. This gives a graph having a loop at C, and so it does not satisfy the condition (P1); see Fig. 3.15.
(ii) $D \sim A$, C. This contains $K_{3,3}$, and so it does not satisfy the condition (P5); see Fig. 3.16.

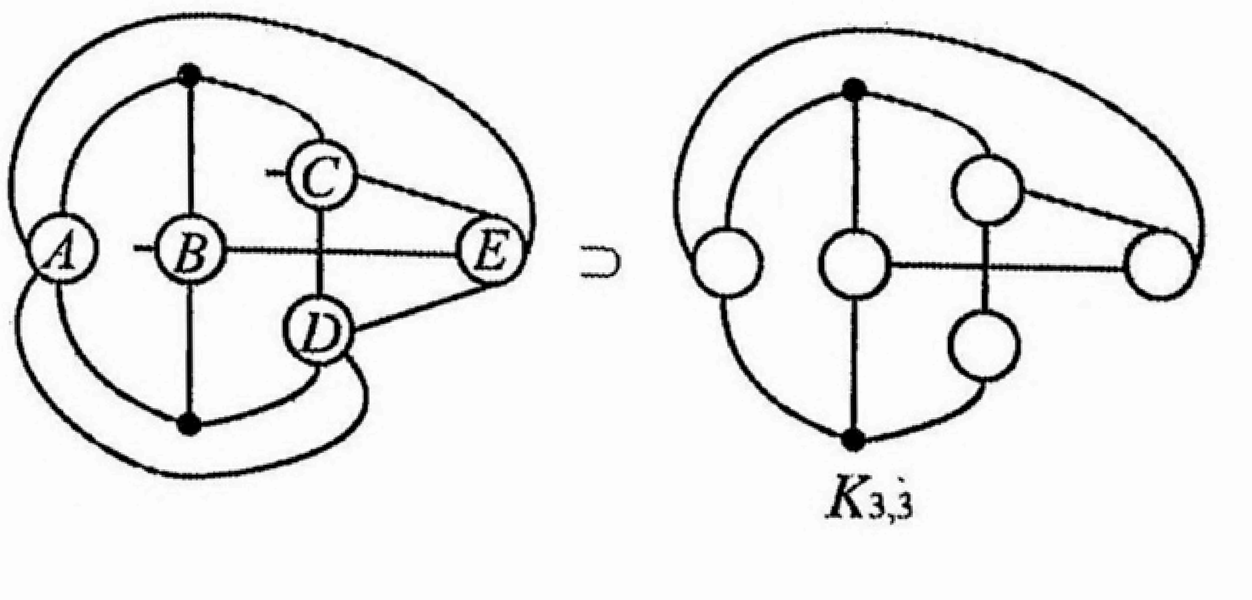


FIG. 3.16. $t = 5$ (b) (ii).

(iii) $D \sim B$, C. This contains $K_{3,3}$, so it does not satisfy the condition (P5); see Fig. 3.17.

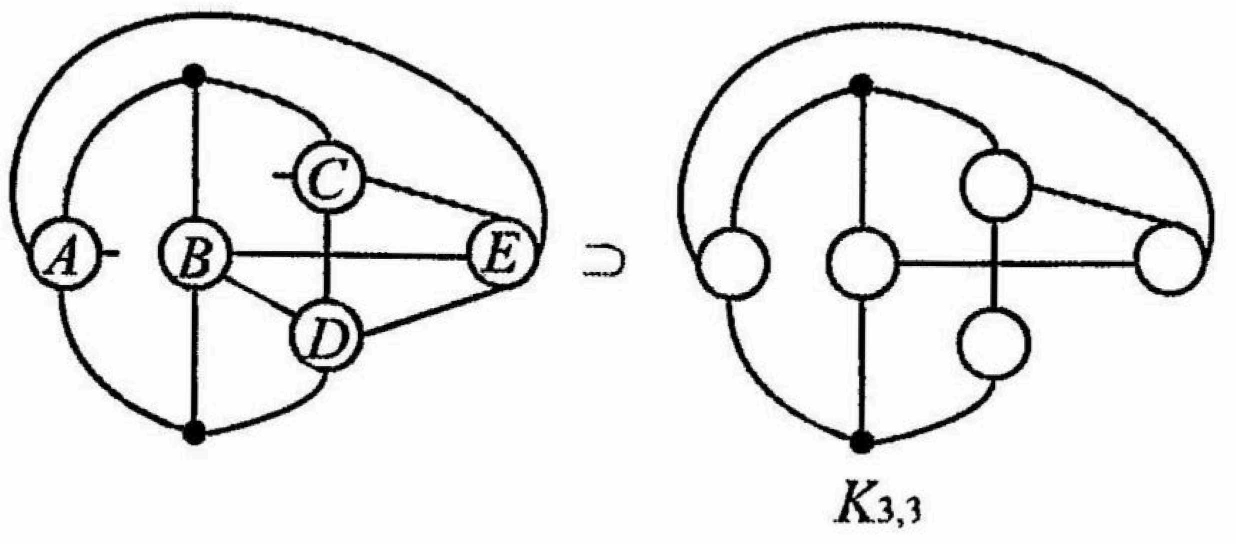


FIG. 3.17. $t = 5$ (b) (iii).

Pattern (c). See Fig. 3.18. Each of the vertices B, C, D and E has three remaining hands, and so we consider how the hands of E connect.

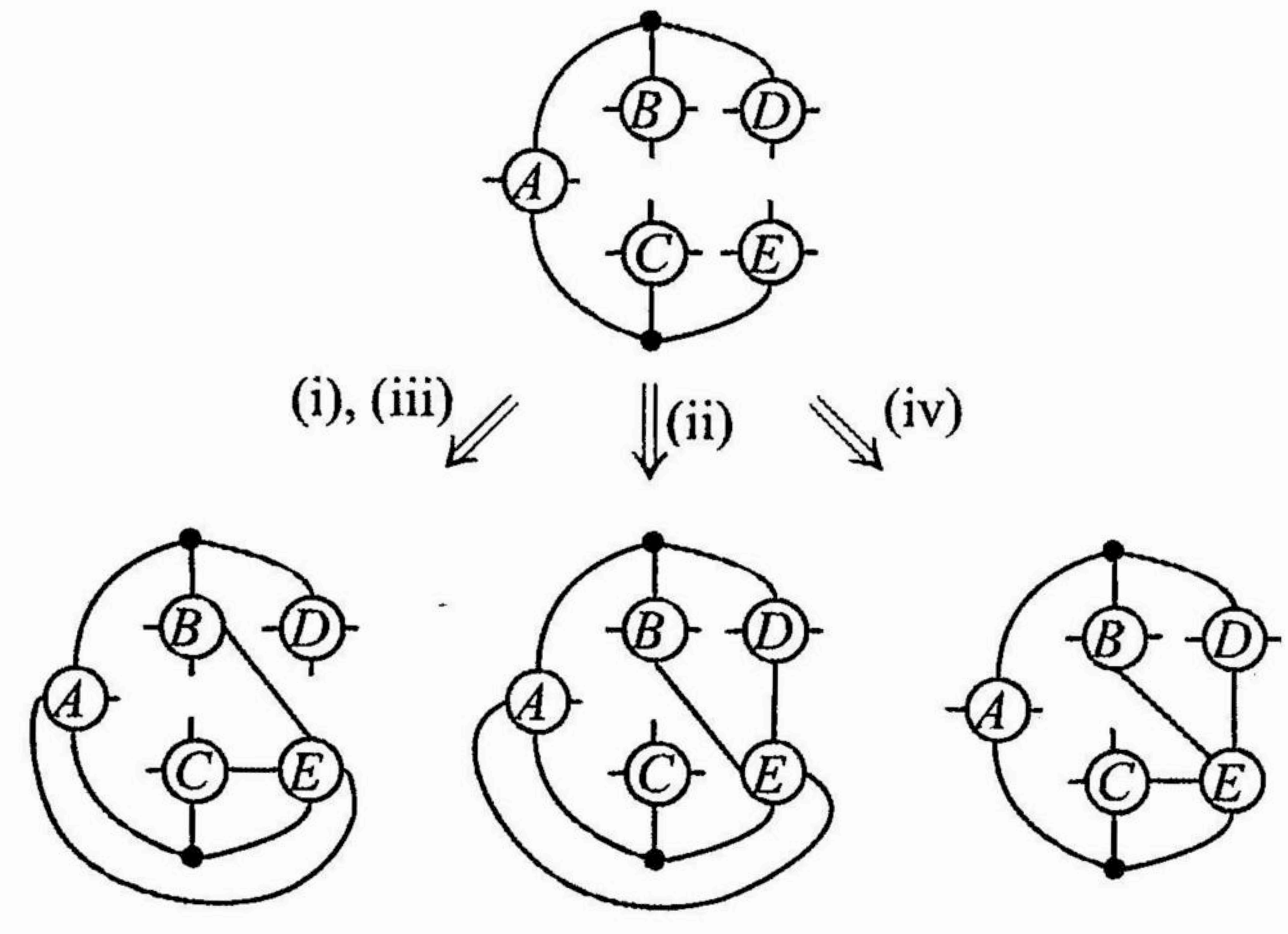


FIG. 3.18. $t = 5$ (c).

(i) $E \sim A, B, C$. Then $D \sim A, B, C$, and we obtain 5^1_*; see Fig. 3.19.

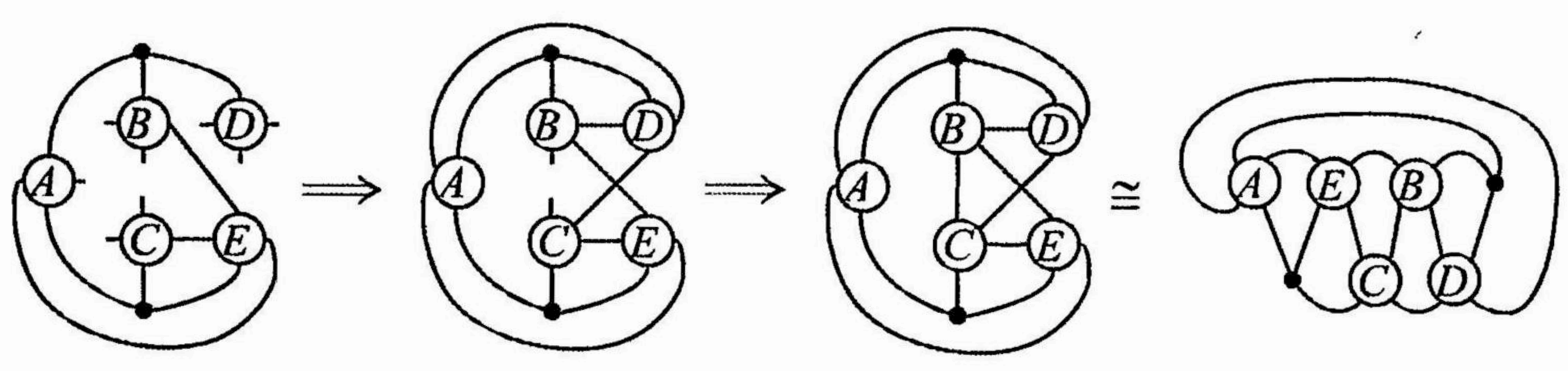


FIG. 3.19. $t = 5$ (c) (i).

(ii) $E \sim A, B, D$. Then $C \sim A, B, D$. This gives a graph containing $K_{3,3}$, and so it does not satisfy the condition (P5); see Fig. 3.20.

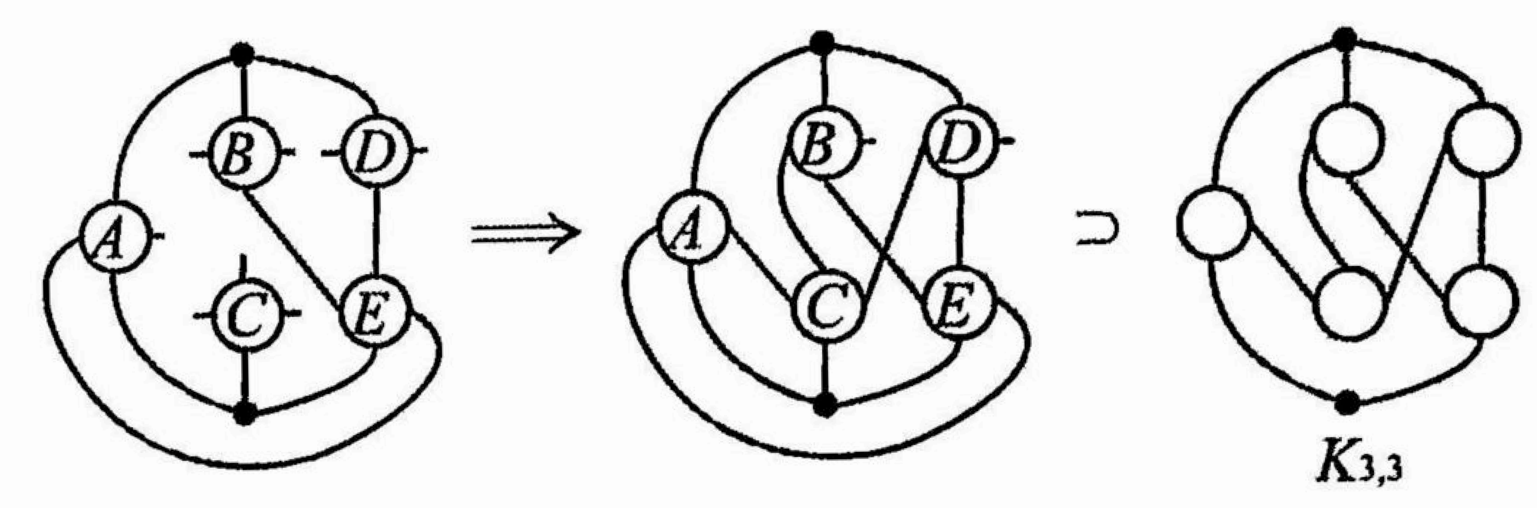


FIG. 3.20. $t = 5$ (c) (ii).

(iii) $E \sim A, C, D$. Since B and D are interchangeable in the first figure in Fig. 3.18, this case is the same as the case (i).
(iv) $E \sim B, C, D$. We consider how the hands of C connect; see Fig. 3.21.

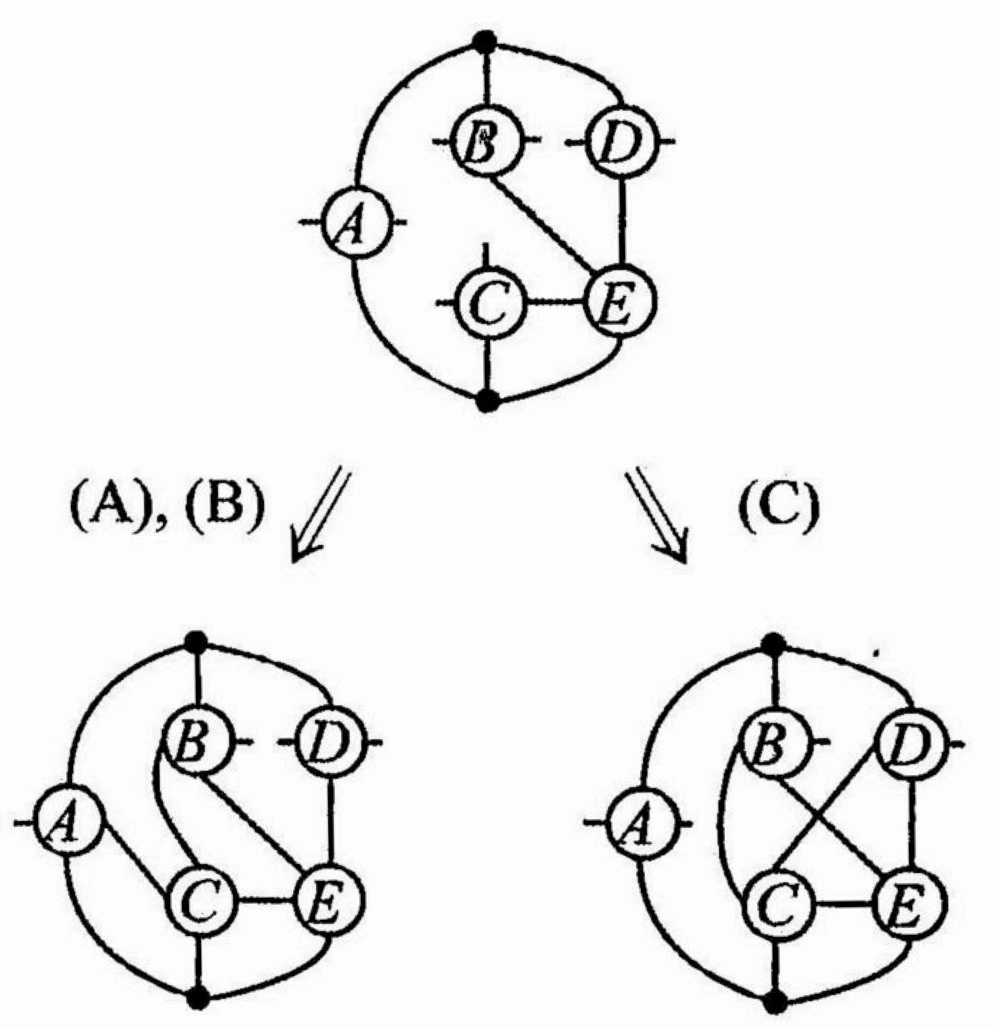


FIG. 3.21. $t = 5$ (c) (iv).

(A) $C \sim A, B$. Then $D \sim A, B$, and we obtain 5^1_*; see Fig. 3.22.

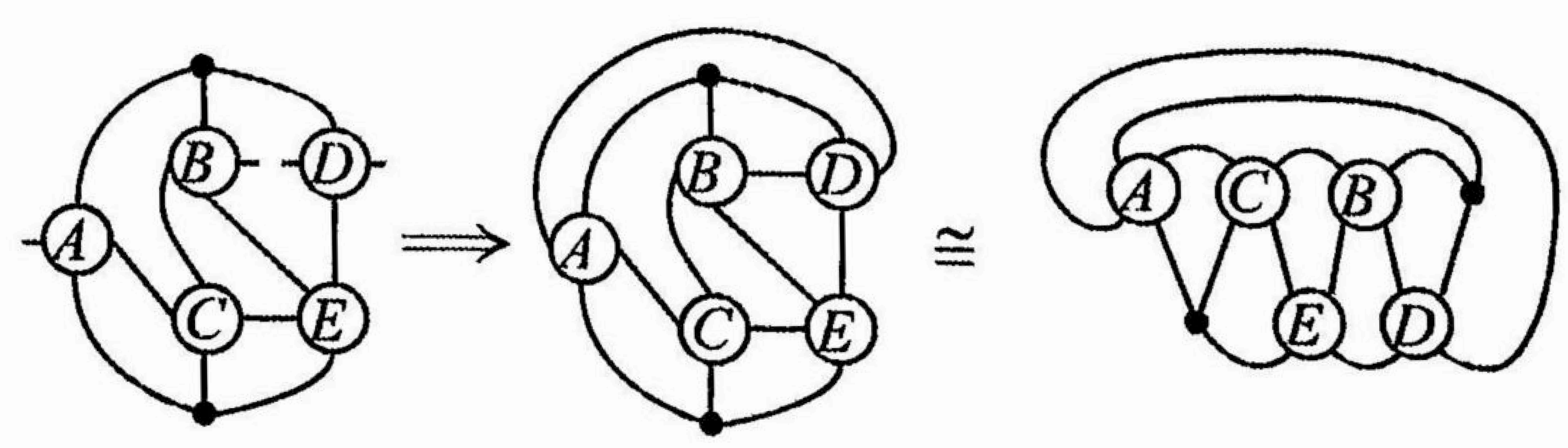


FIG. 3.22. $t = 5$ (c) (iv) (A).

(B) $C \sim A, D$. Since B and D are interchangeable in the first figure in Fig. 3.21, this case is the same as the case (A).
(C) $C \sim B, D$. Then we obtain a graph containing $K_{3,3}$, which does not satisfy the condition (P5); see Fig. 3.23.

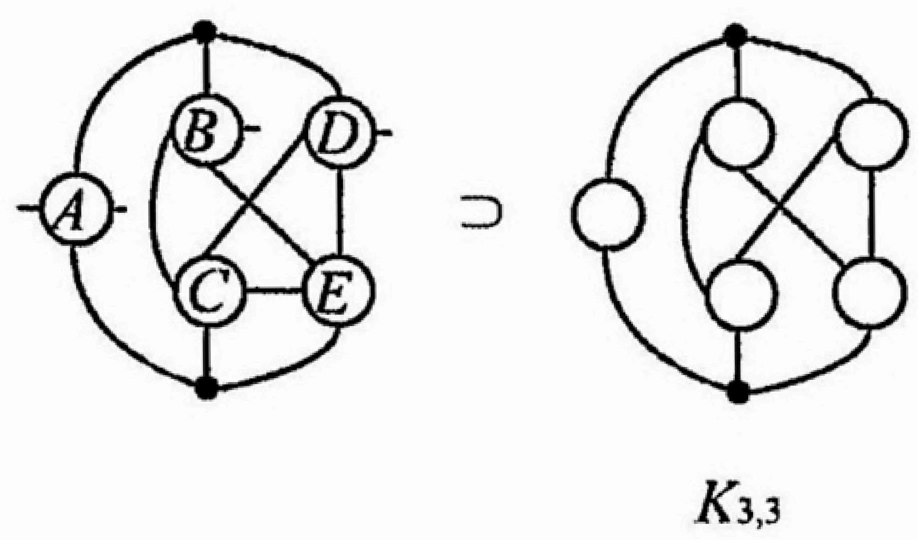


FIG. 3.23. $t = 5$ (c) (iv) (C).

2.4. $t = 6$. We consider all the patterns.
Pattern (a). By an argument similar to that in the case $t = 4$ (a), this case does not give a planar graph.
Pattern (b). See Fig. 3.24. We consider how the hands of F connect. By the condition (P2), there are five cases.

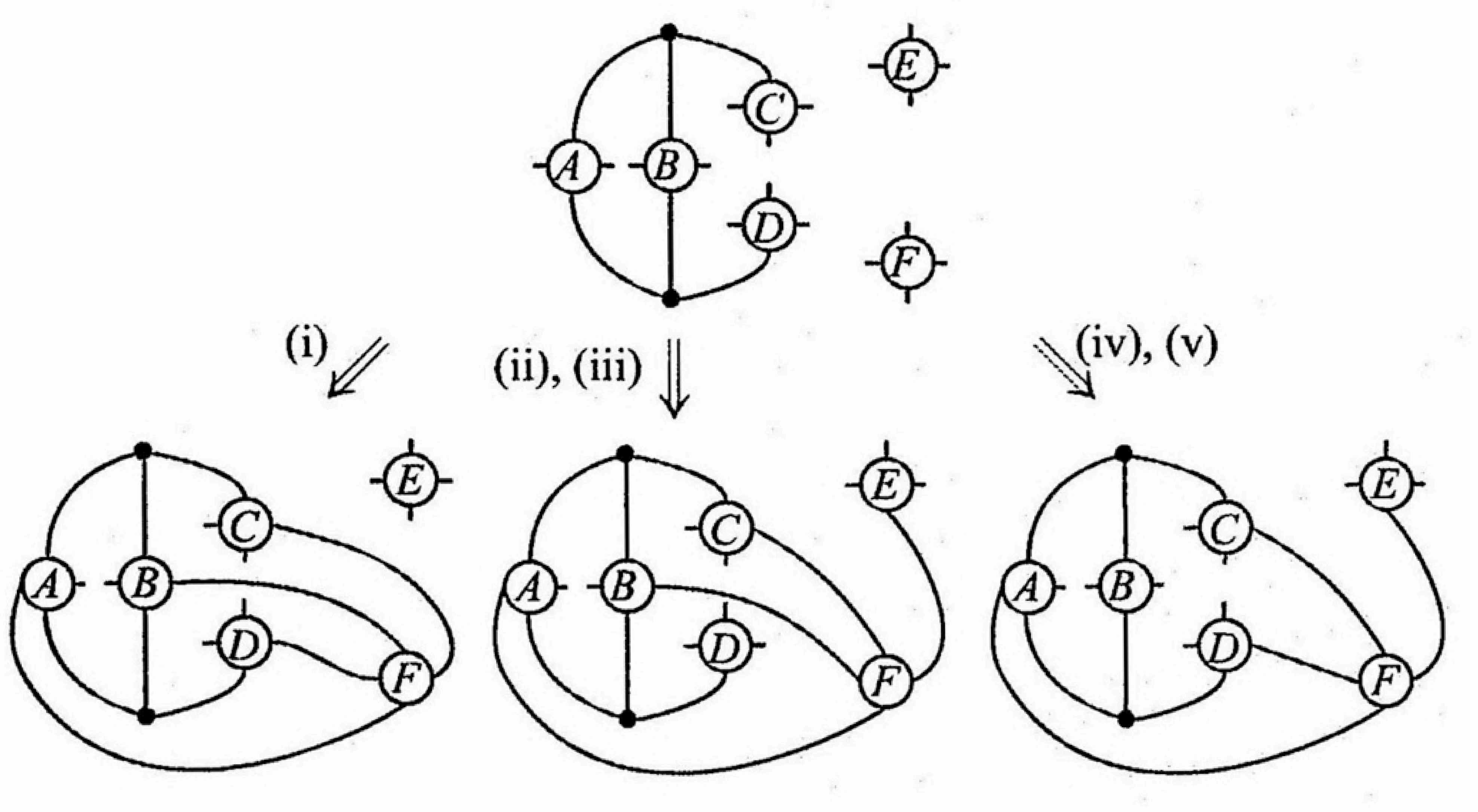


FIG. 3.24. $t = 6$ (b).

(i) $F \sim A, B, C, D$. The vertex E has four remaining hands, and so we consider how the hands of E connect; see Fig. 3.25. By the condition (ii), $E \sim A, B, C, D$. This gives a graph containing $K_{3,3}$, which does not satisfy the condition (P5):

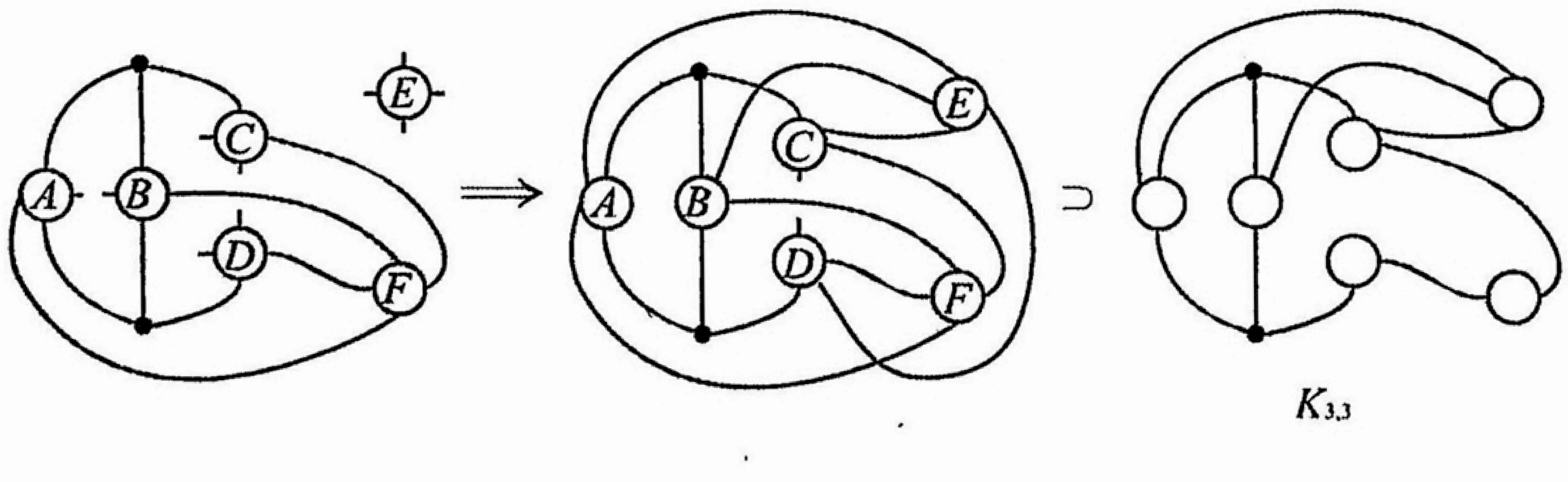


FIG. 3.25. $t = 6$ (b) (i).

(ii) $F \sim A, B, C, E$. The vertex E has three remaining hands, so we consider how the hands of E connect. There are four cases; see Fig. 3.26.

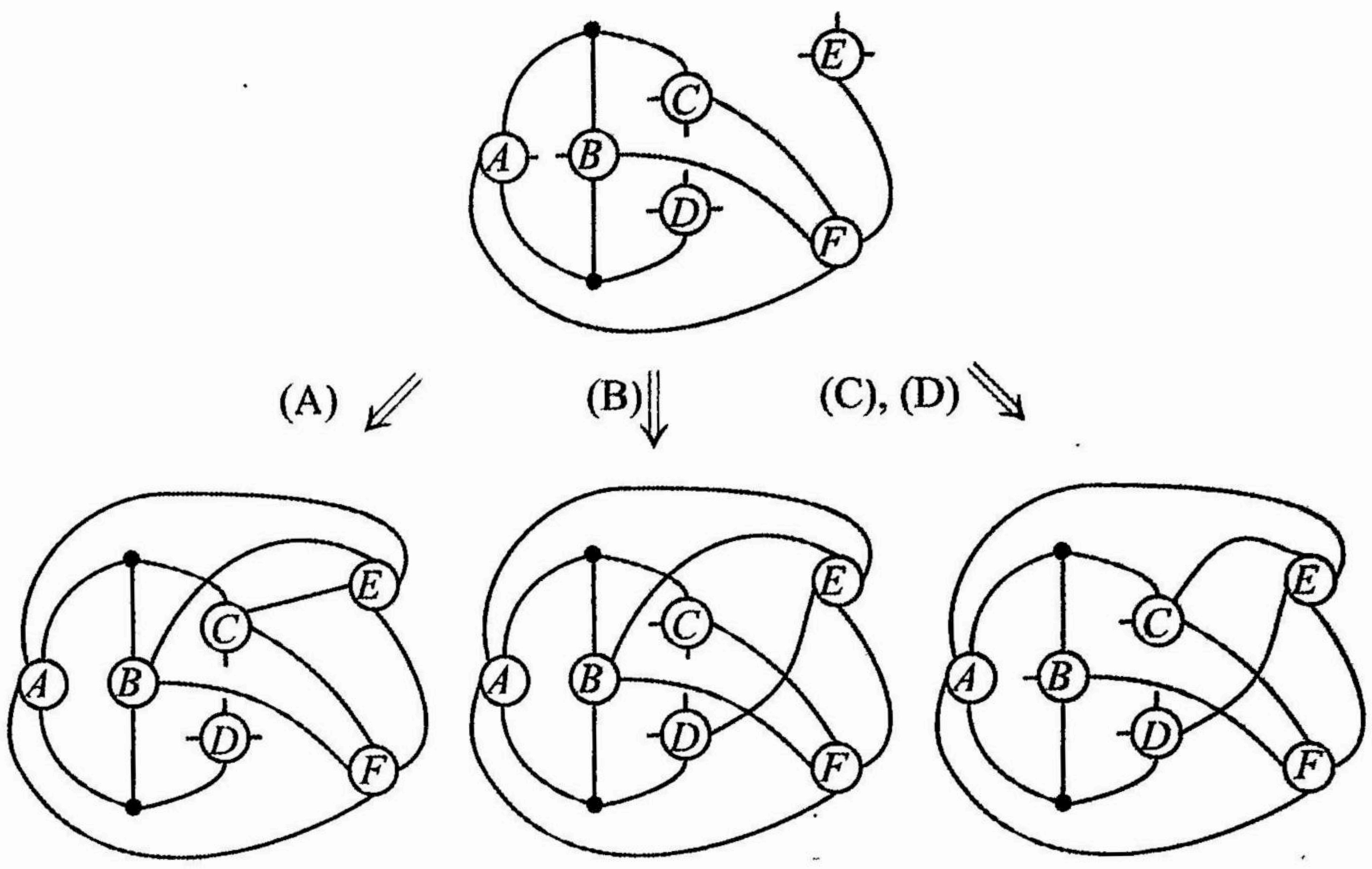


FIG. 3.26. $t = 6$ (b) (ii).

(A) $E \sim A, B, C$. Then $C \sim D$, which gives a graph having a loop at D, and so it does not satisfy the condition (P1); see Fig. 3.27.

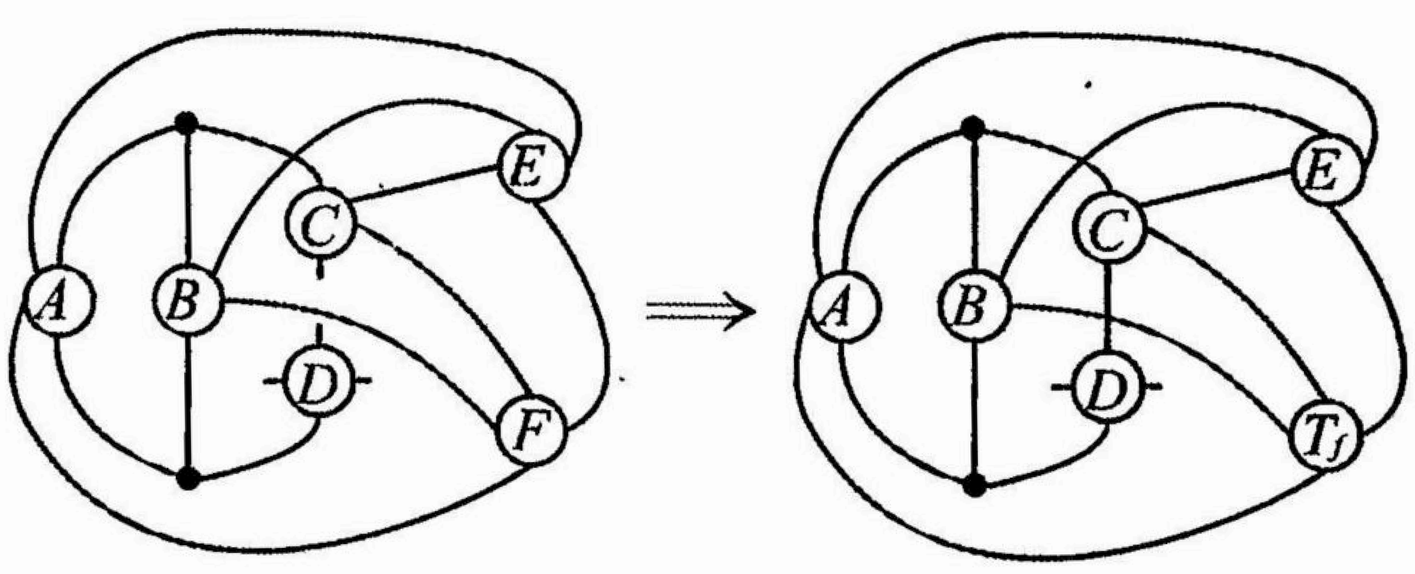


FIG. 3.27. $t = 6$ (b) (ii) (A).

(B) $E \sim A, B, D$. This gives a graph which does not satisfy the conditions (P1) or (P2); see Fig. 3.26.

(C) $E \sim A, C, D$. This gives a graph containing $K_{3,3}$, and so it does not satisfy the condition (P5); see Fig. 3.28.

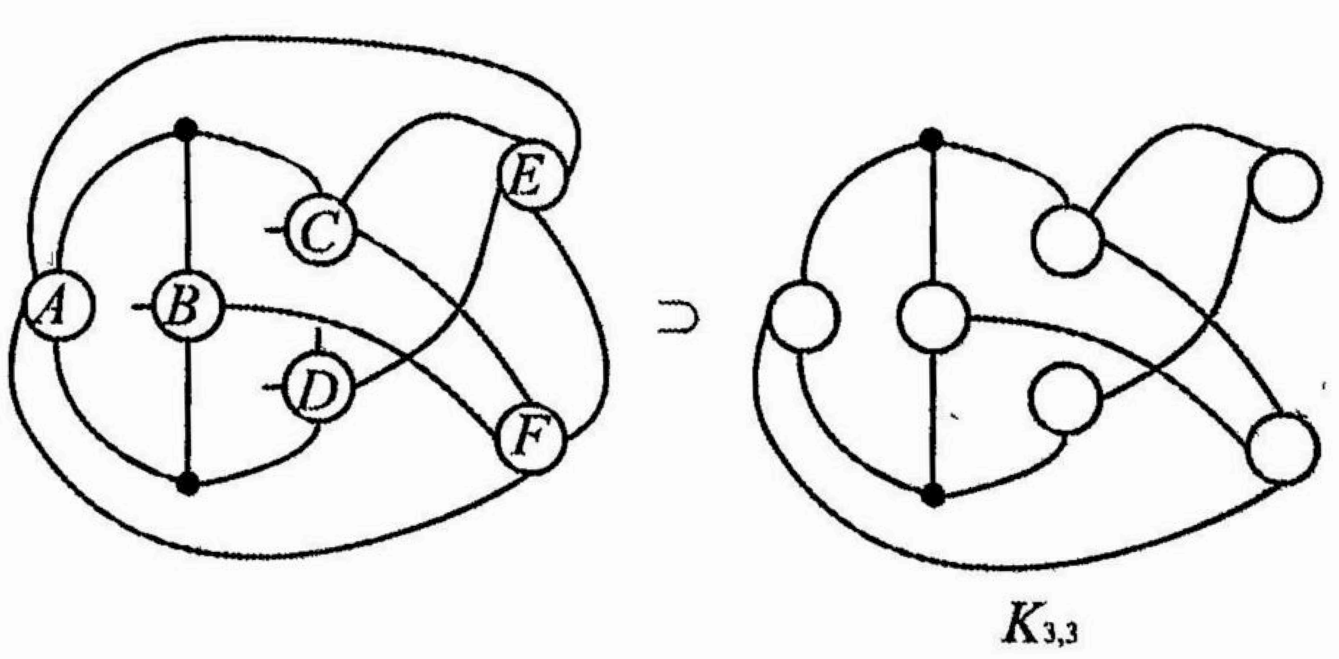


FIG. 3.28. $t = 6$ (b) (ii) (C).

(D) $E \sim B, C, D$. Since A and B are interchangeable in the first figure in Fig. 3.26, this case is the same as the case (C).

(iii) $F \sim A, B, D, E$. Since C and D are interchangeable in the first figure in Fig. 3.24, this case is the same as the case (ii).

(iv) $F \sim A, C, D, E$. The vertex E has three remaining hands, so we consider how the hands of E connect. There are four cases; see Fig. 3.29.

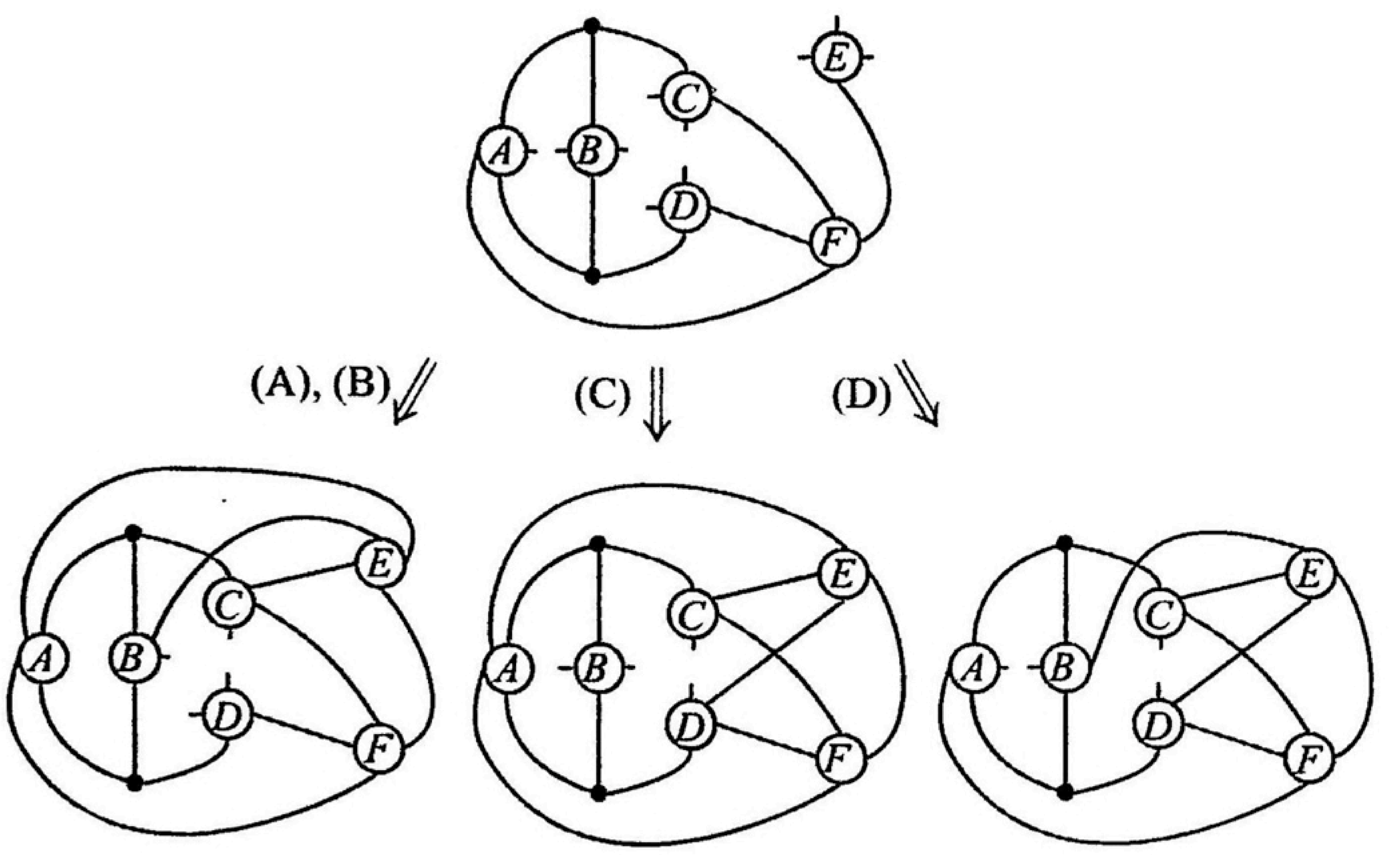


FIG. 3.29. $t = 6$ (b) (iv).

(A) $E \sim A, B, C$. We obtain a graph containing $K_{3,3}$, which does not satisfy the condition (P5); see Fig. 3.30.

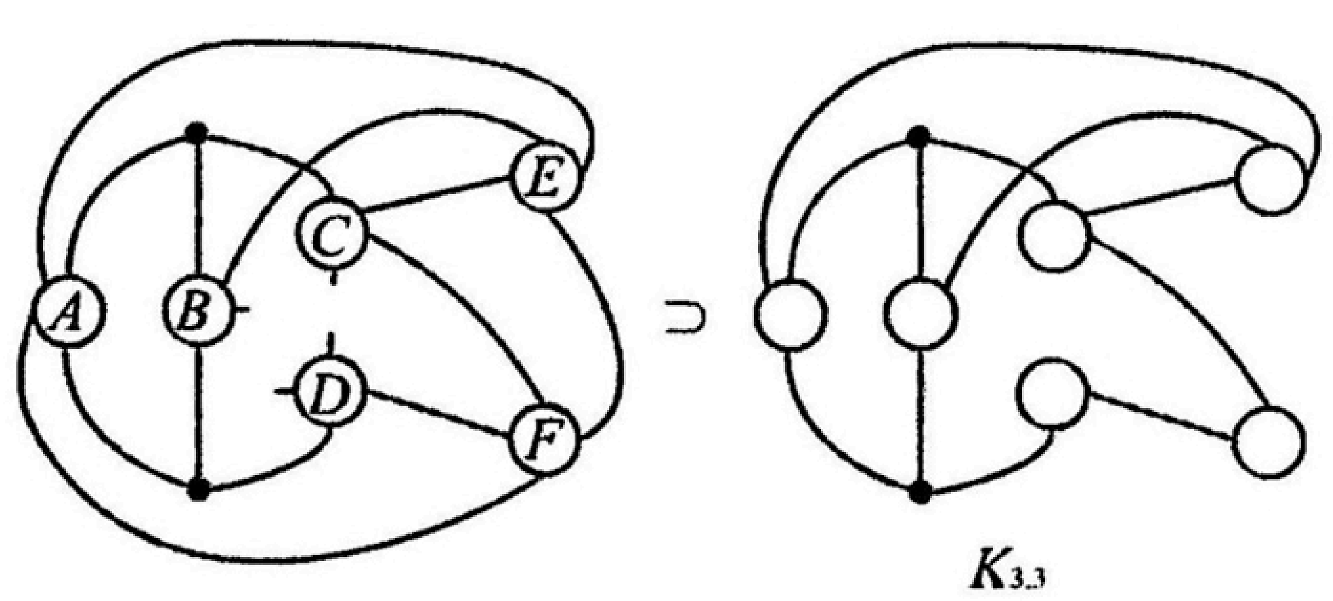


FIG. 3.30. $t = 6$ (b) (iv) (A).

(B) $E \sim A, B, D$. Since C and D are interchangeable in the first figure in Fig. 3.29, this case is the same as the case (A).

(C) $E \sim A, C, D$. Then $B \sim C, D$. We obtain a graph containing K_5, which does not satisfy the condition (P5); see Fig. 3.31.

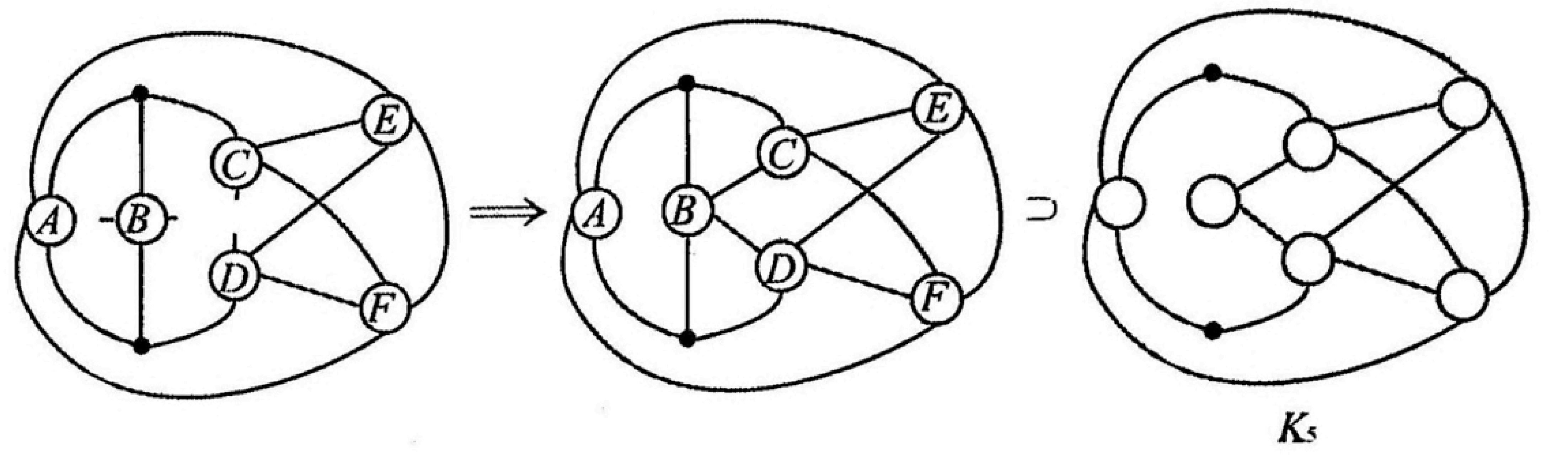


FIG. 3.31. $t = 6$ (b) (iv) (C).

(D) $E \sim B, C, D$. We have three cases; see Fig. 3.32.

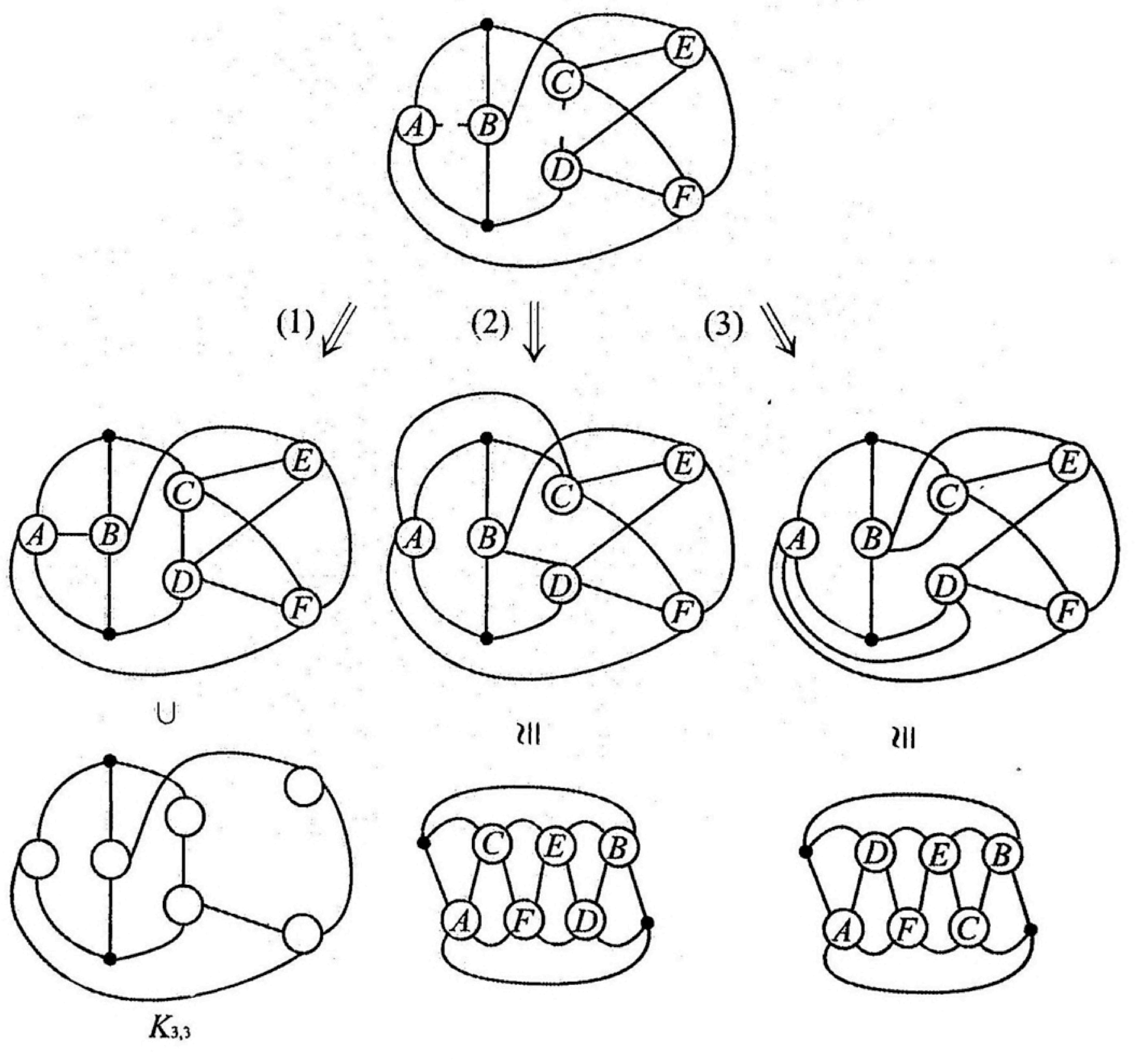


FIG. 3.32. $t = 6$ (b) (iv) (D).

(1) $A \sim B$ and $C \sim D$. We obtain a graph containing $K_{3,3}$, which does not satisfy the condition (P5).

(2) $A \sim C$ and $B \sim D$. We obtain 6^1_*.

(3) $A \sim D$ and $B \sim C$. We obtain 6^1_*.

(v) $F \sim B, C, D, E$. Since A and B are interchangeable in the first figure in Fig. 3.24, this case is the same as the case (iv).

Pattern (C). See Fig. 3.33. The vertex F has four remaining hands, so we consider how the hands of F connect. There are five cases.

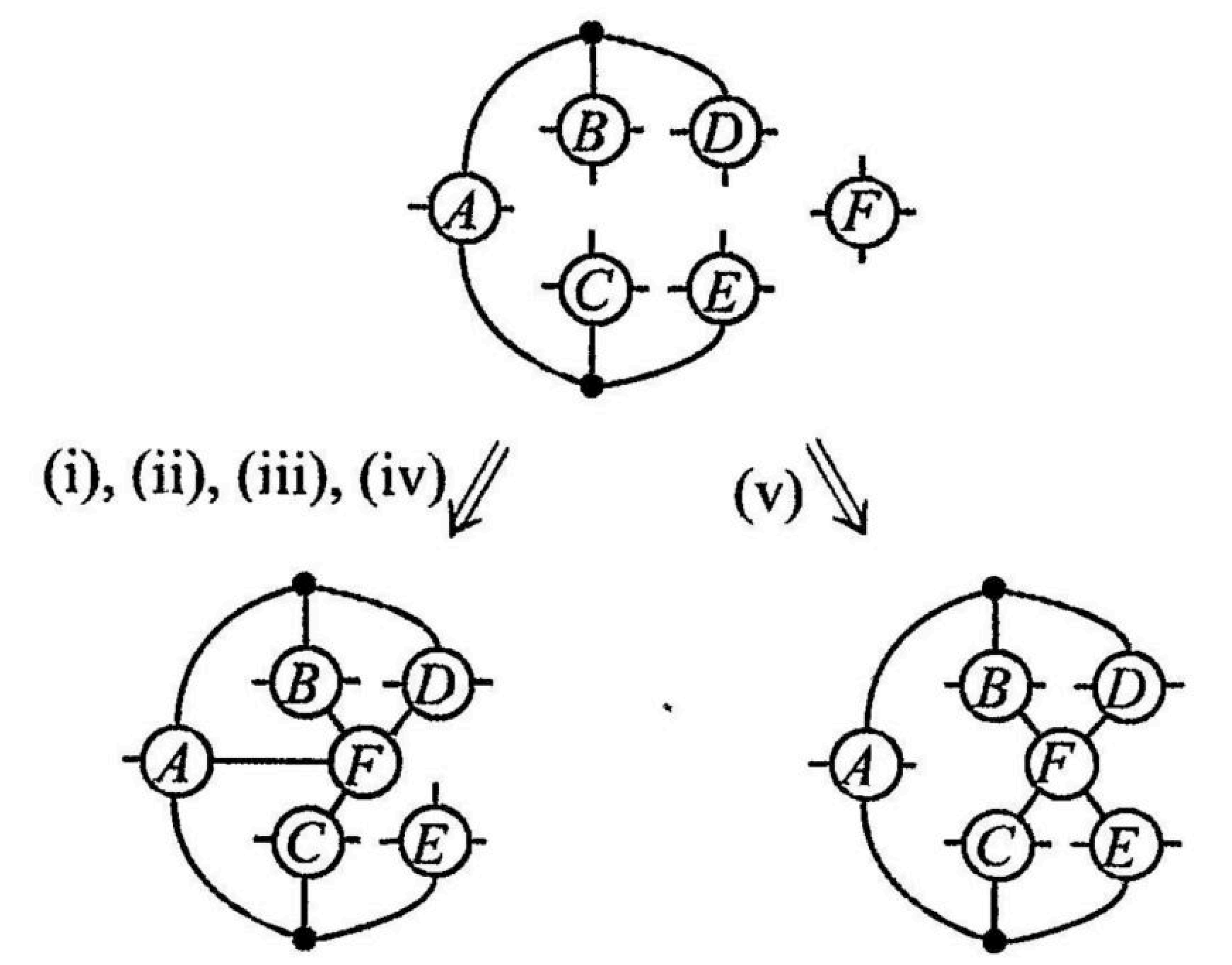


FIG. 3.33. $t = 6$ (c).

(i) $F \sim A, B, C, D$. The vertex E has three remaining hands, so we consider how the hands of E connect. There are four cases; see Fig. 3.34.

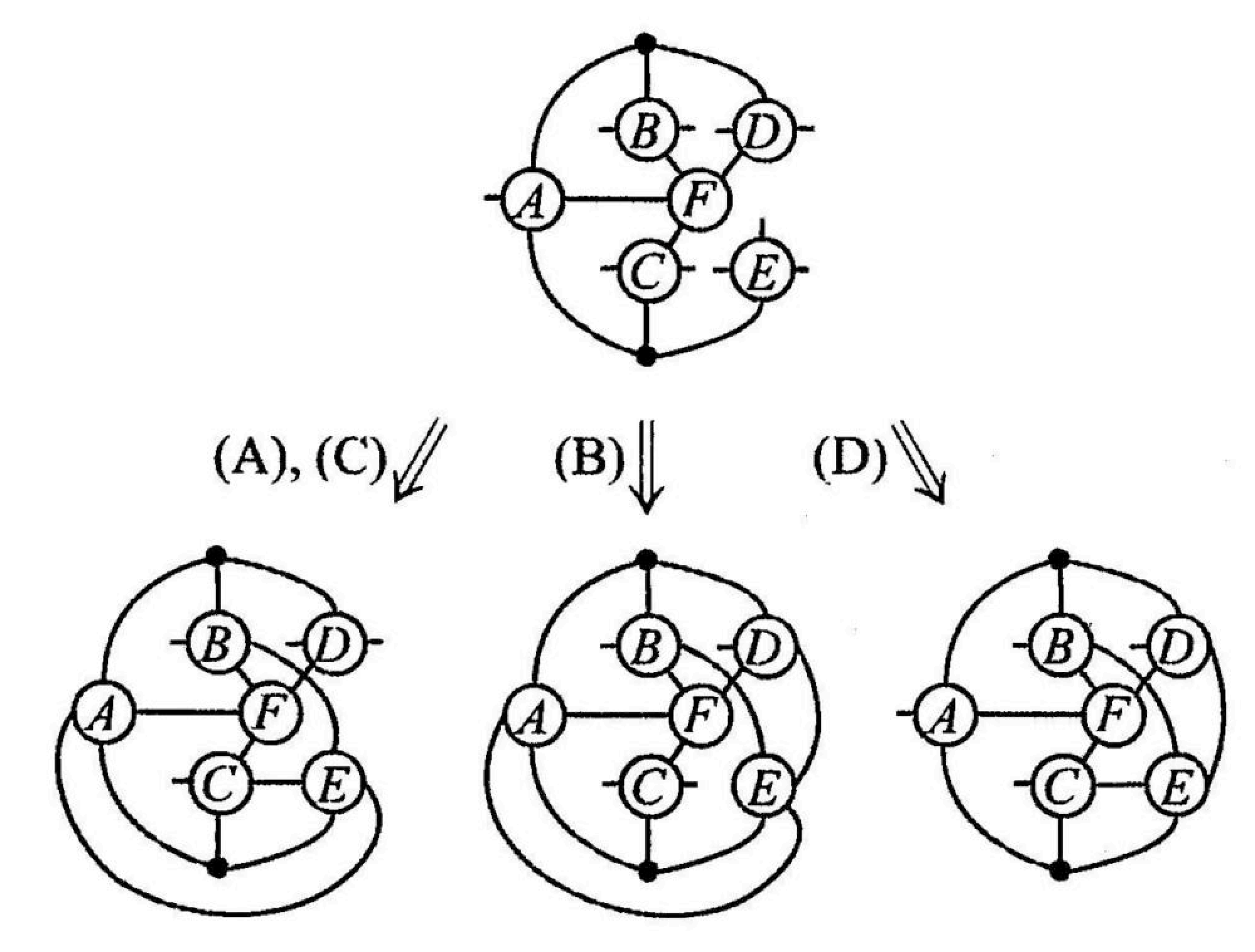


FIG. 3.34. $t = 6$ (c) (i).

(A) $E \sim A, B, C$. Then $D \sim B, C$, and we obtain a graph containing $K_{3,3}$, which does not satisfy the condition (P5); see Fig. 3.35.

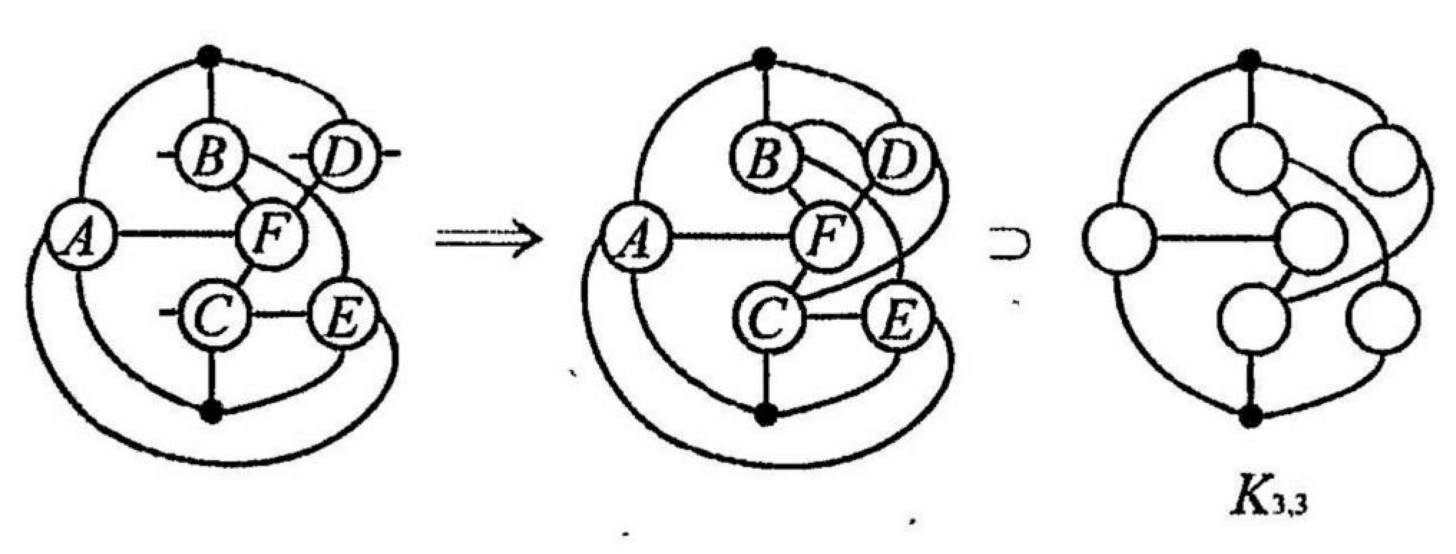


FIG. 3.35. $t = 6$ (c) (i) (A).

(B) $E \sim A, B, D$. Then we obtain a graph containing $K_{3,3}$, which does not satisfy the condition (P5); see Fig. 3.36.

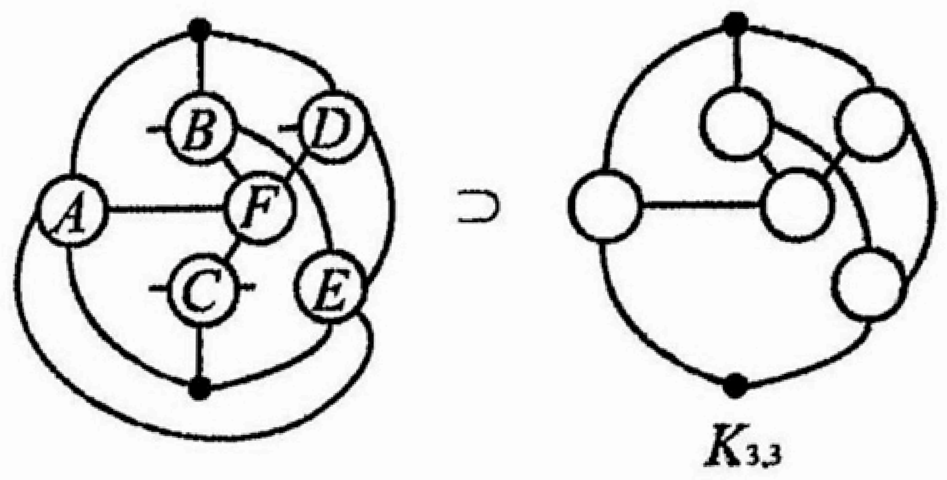


FIG. 3.36. $t = 6$ (c) (i) (B).

(C) $E \sim A, C, D$. Since B and D are interchangeable in the first figure in Fig. 3.34, this case is the same as the case (A).

(D) $E \sim B, C, D$. Then we obtain a graph containing $K_{3,3}$, which does not satisfy the condition (P5); see Fig. 3.37.

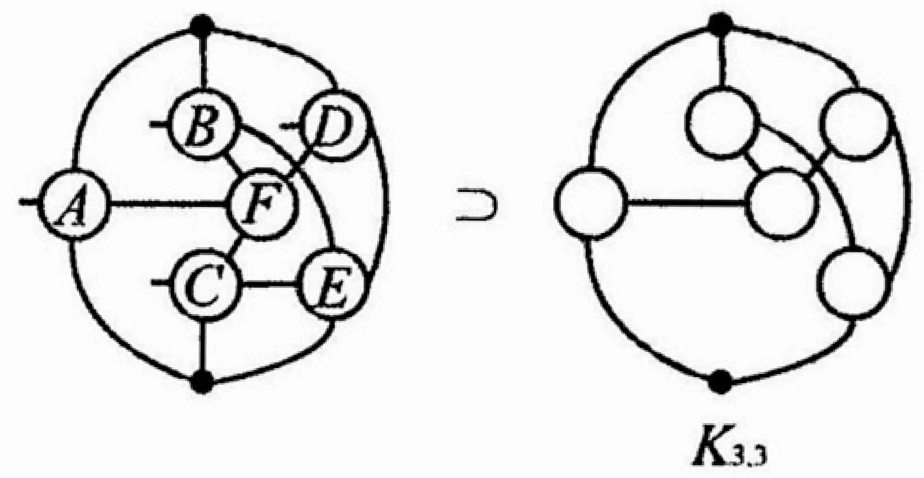


FIG. 3.37. $t = 6$ (c) (i) (D).

(ii) $F \sim A, B, C, E$. Since D and E are interchangeable in the first figure in Fig. 3.33, this case is the same as the case (i).

(iii) $F \sim A, B, D, E$. Since C and E are interchangeable in the first figure in Fig. 3.33, this case is the same as the case (i).

(iv) $F \sim A, C, D, E$. Since B and E are interchangeable in the first figure in Fig. 3.33, this case is the same as the case (i).

(v) $F \sim B, C, D, E$. The vertex E has two remaining hands, so we consider how the hands of E connect. There are six cases; see Fig. 3.38.

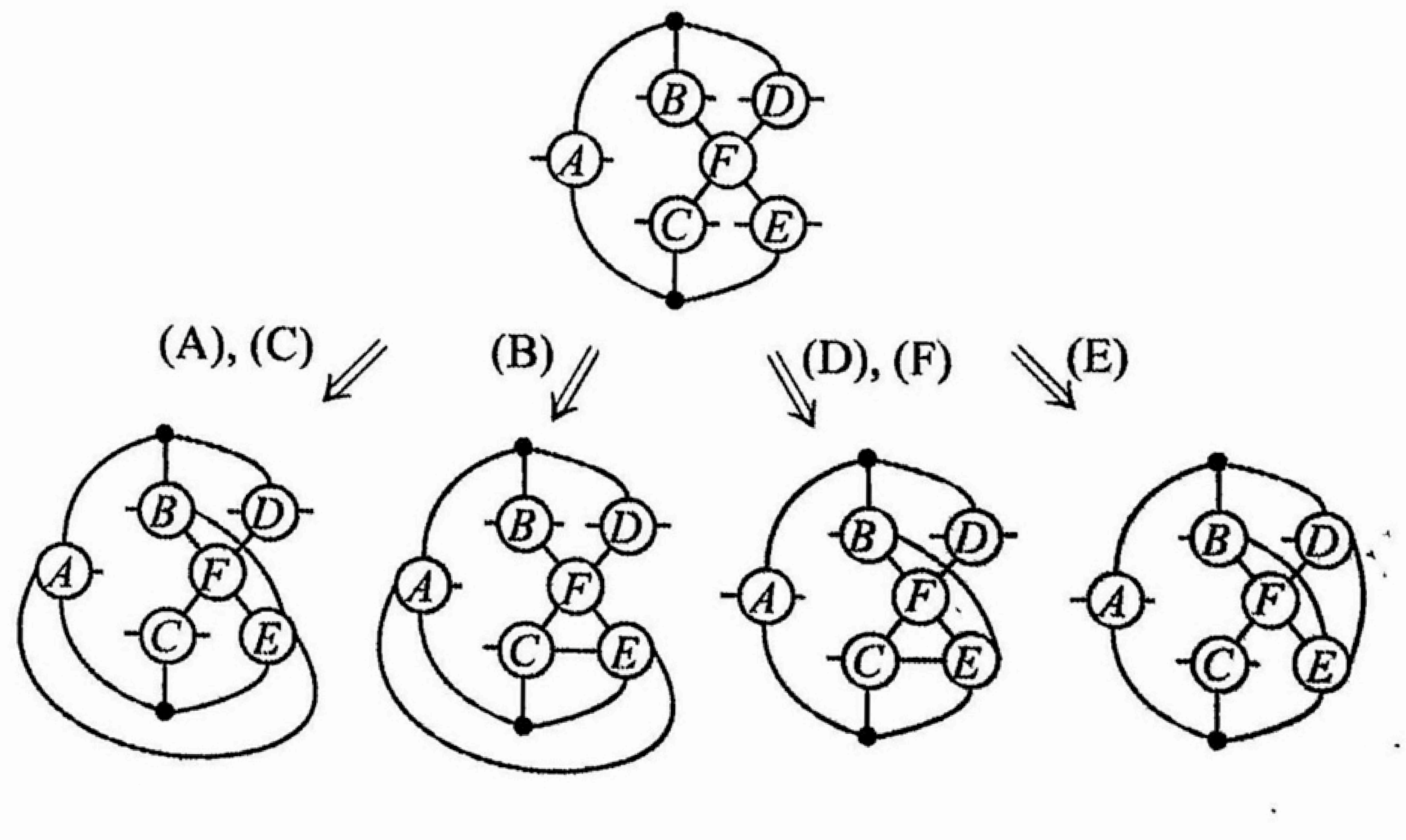


FIG. 3.38. $t = 6$ (c) (v).

(A) $E \sim A, B$. The vertex D has two remaining hands, so we consider how the hands of D connect. There are three cases; see Fig. 3.39.

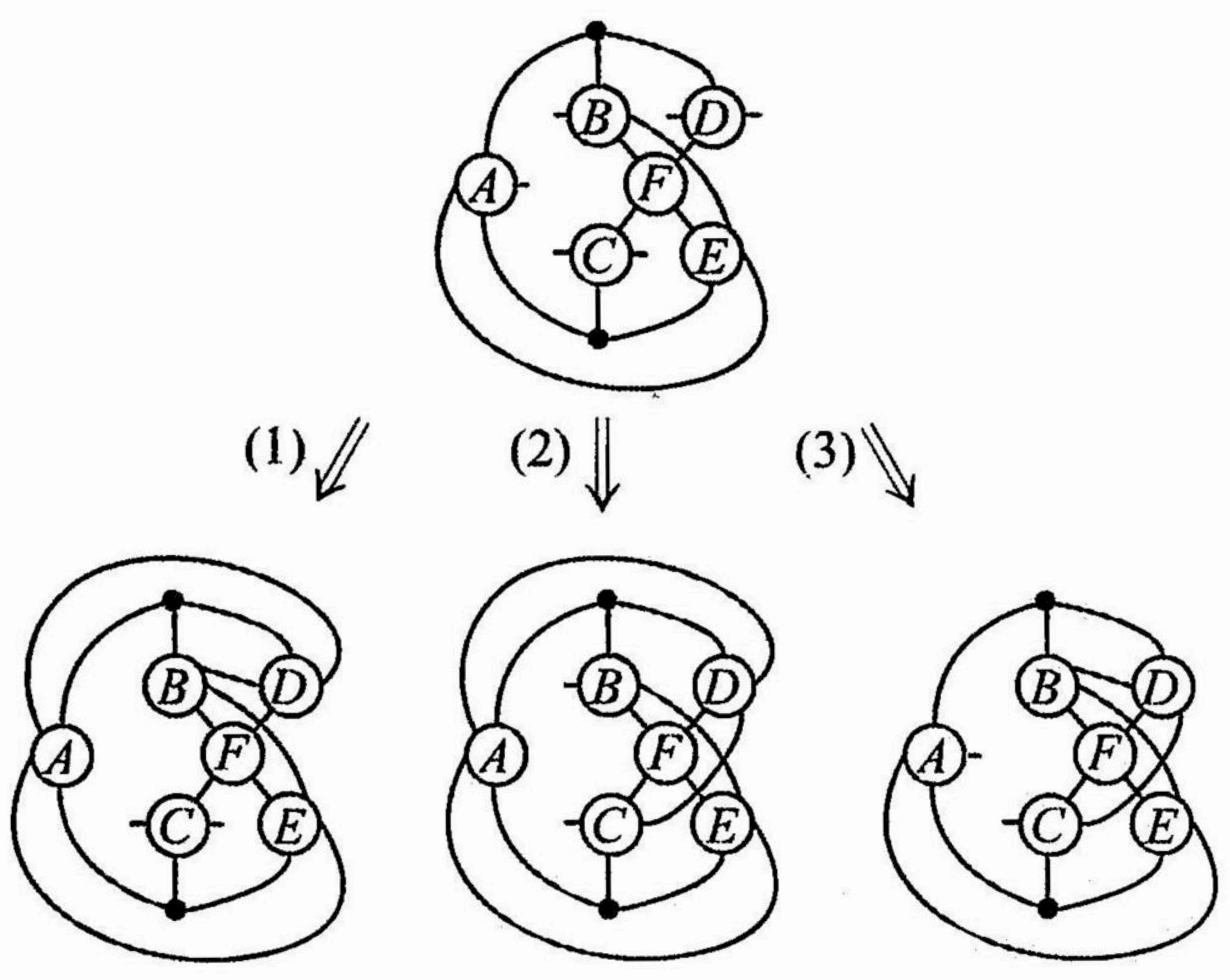


FIG. 3.39. $t = 6$ (c) (v) (A).

(1) $D \sim A$ and $E \sim B$. This gives a graph having a loop at C, and so it does not satisfy the condition (P1); see Fig. 3.39.
(2) $D \sim A$ and $E \sim C$. Then $B \sim C$. This gives a graph containing K_5, and so it does not satisfy the condition (P5); see Fig. 3.40.

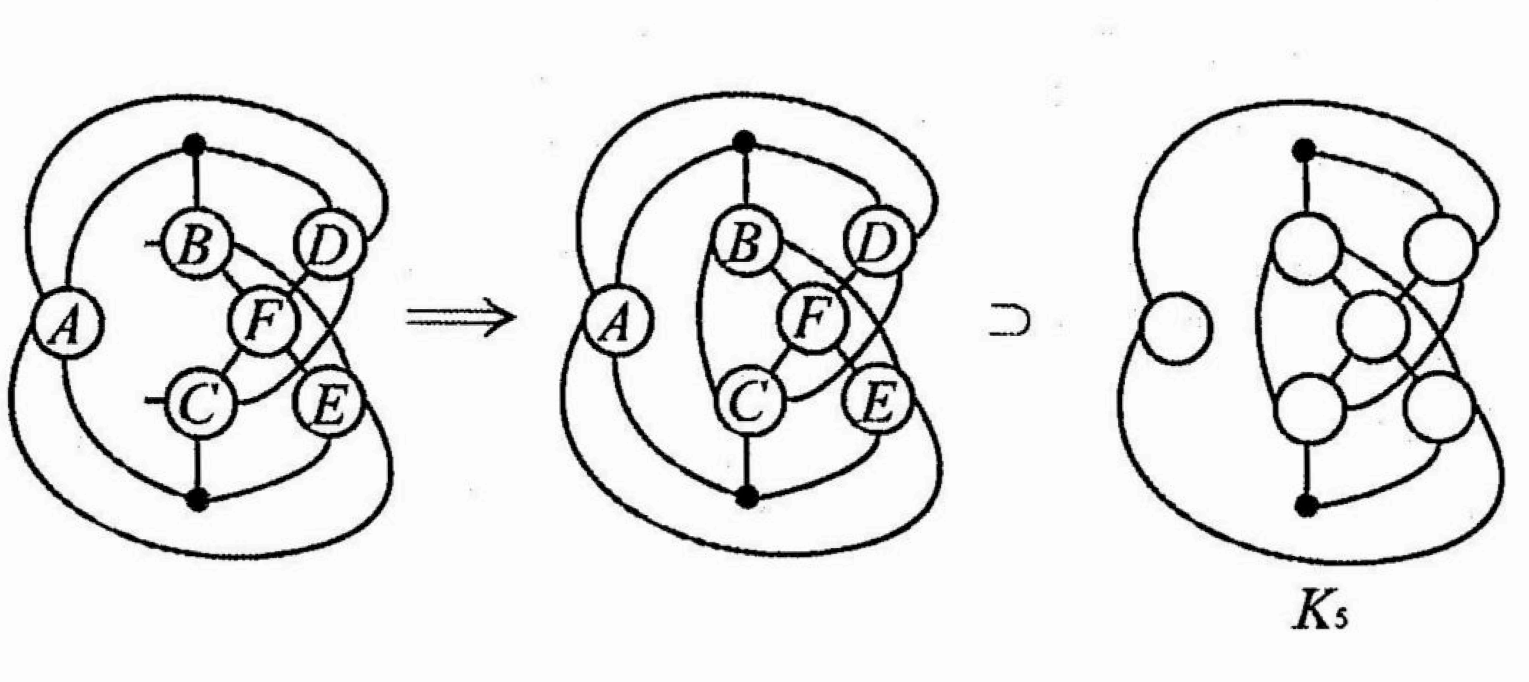


FIG. 3.40. $t = 6$ (c) (v) (A) (2).

(3) $D \sim B$ and $E \sim C$. Then $A \sim C$, and we obtain 6^3_*; see Fig. 3.41.

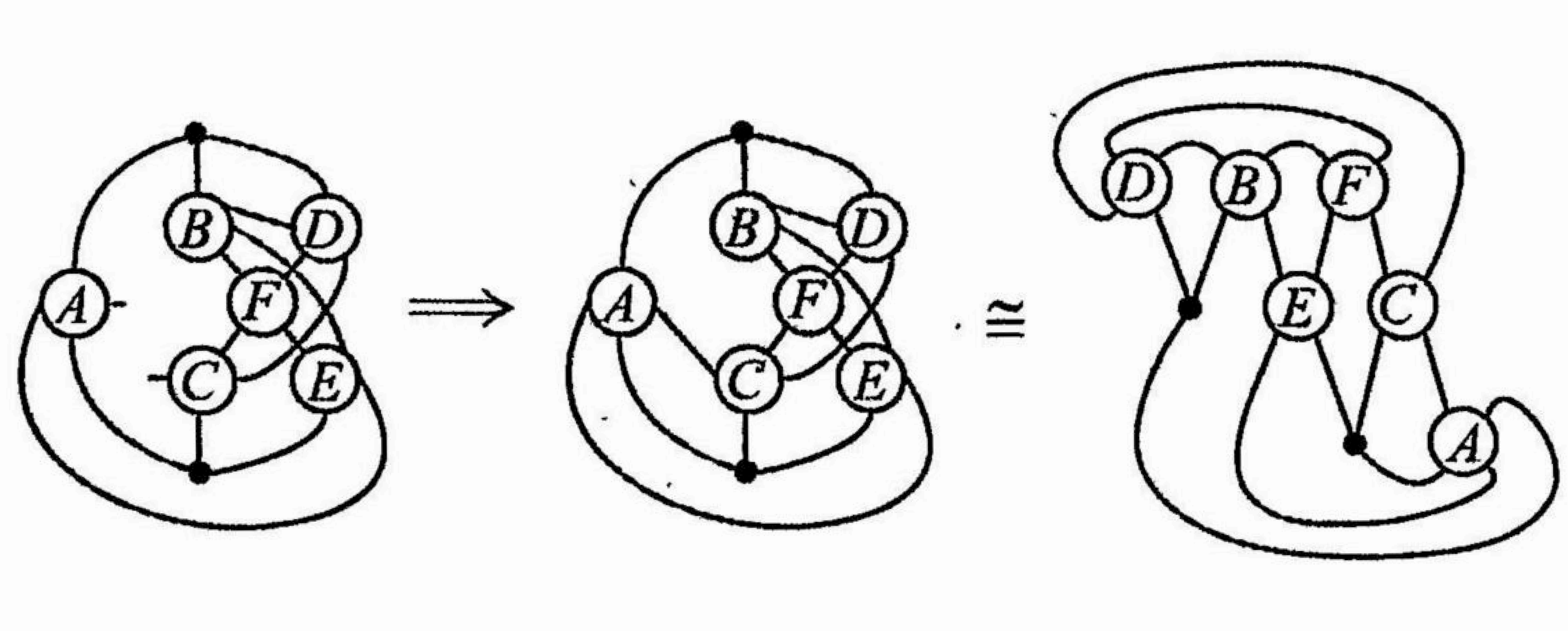


FIG. 3.41. $t = 6$ (c) (v) (A) (3).

(B) $E \sim A, C$. The vertex D has two remaining hands, so we consider how the hands of D connect. There are three cases; see Fig. 3.42.

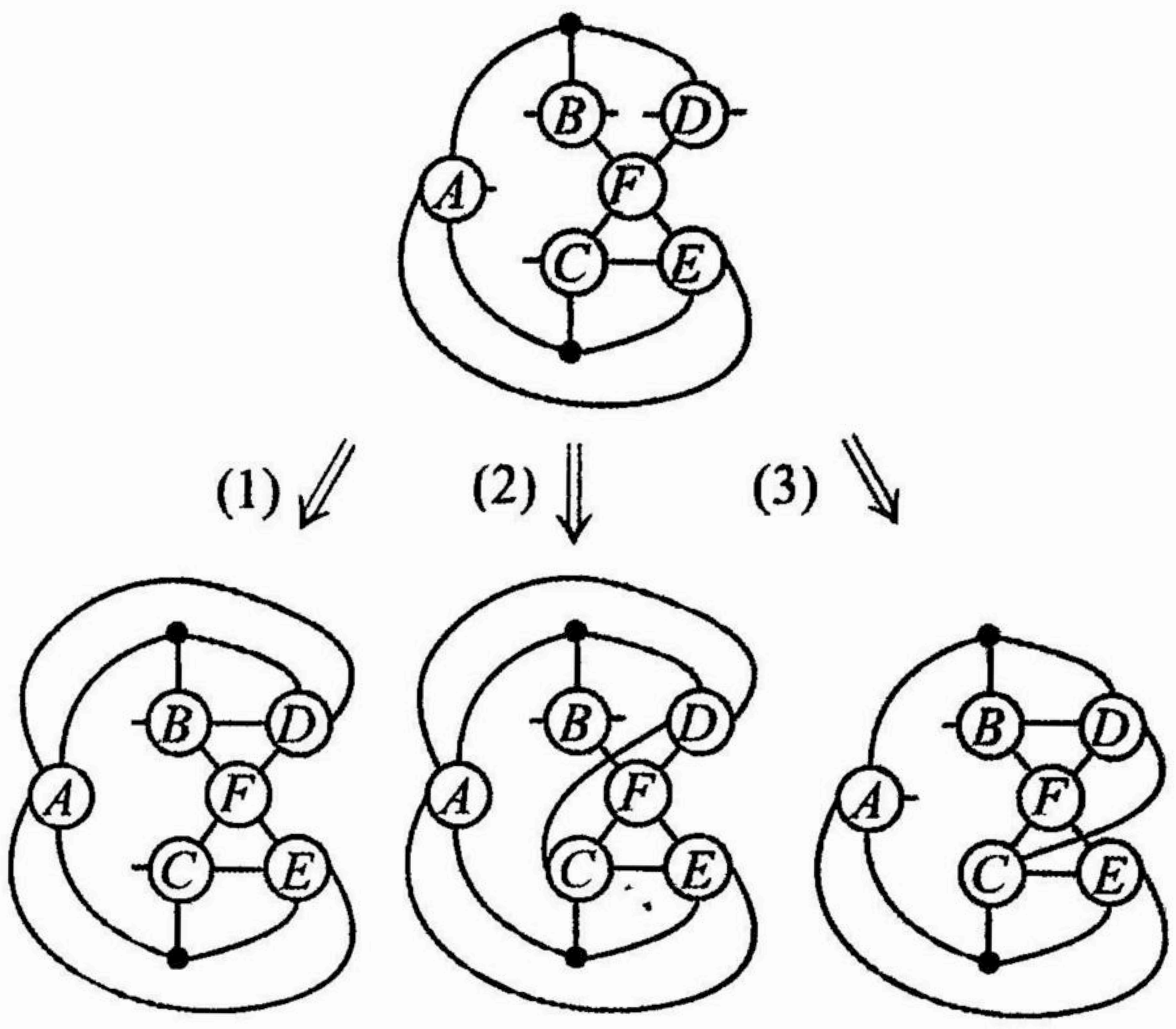


FIG. 3.42. $t = 6$ (c) (v) (B).

(1) $D \sim A, B$. Then $B \sim C$, and we obtain 6^2_*; see Fig. 3.43.

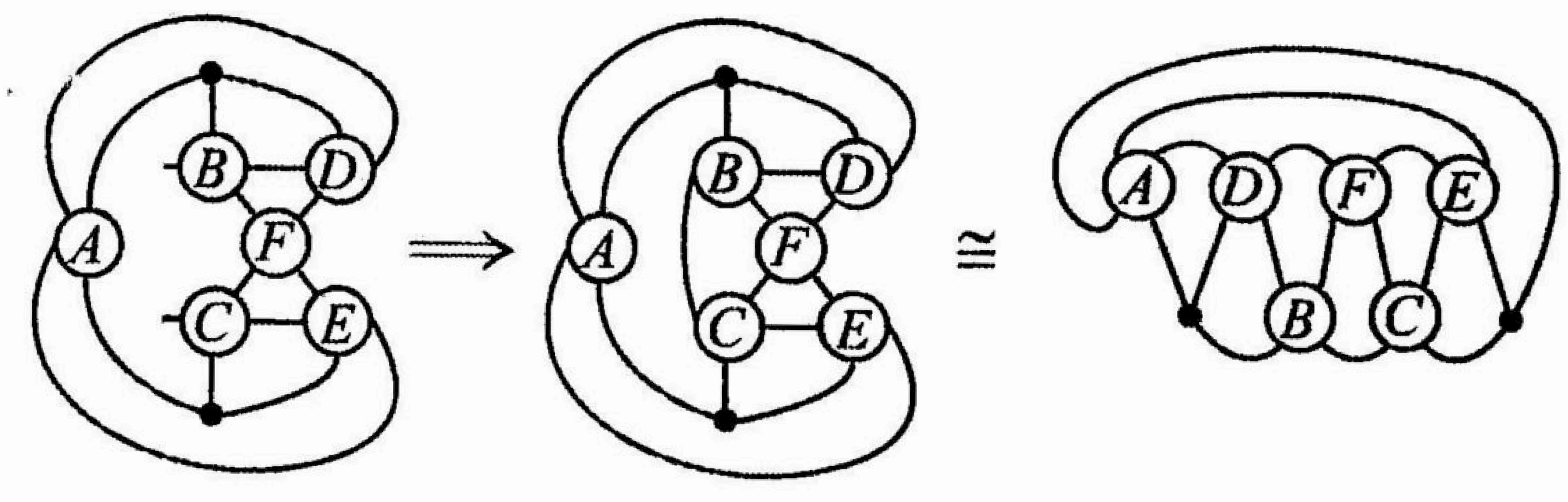


FIG. 3.43. $t = 6$ (c) (v) (B) (1).

(2) $D \sim A, C$. This gives a graph having a loop at B, and so it does not satisfy the condition (P1); see Fig. 3.42.

(3) $D \sim B, C$. Then $A \sim B$, and we obtain 6^2_*; see Fig. 3.44.

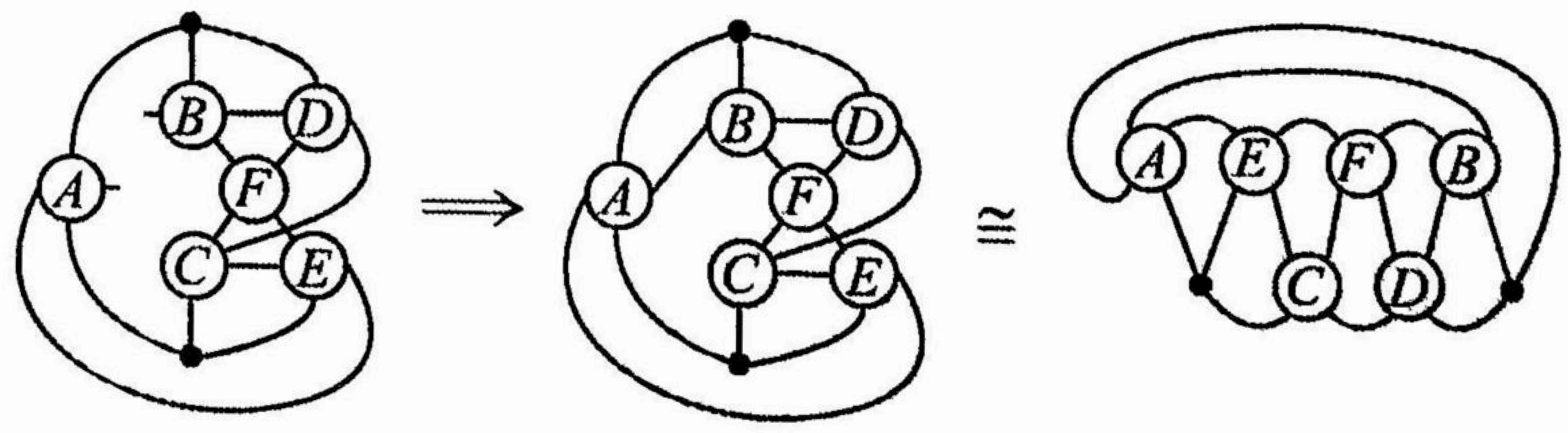


FIG. 3.44. $t = 6$ (c) (v) (B) (3).

(C) $E \sim A, D$. Since B and D are interchangeable in the first figure in Fig. 3.38, this case is the same as the case (A).

(D) $E \sim B, C$. The vertex D has two remaining hands, so we consider how the hands of D connect. There are three cases; see Fig. 3.45.

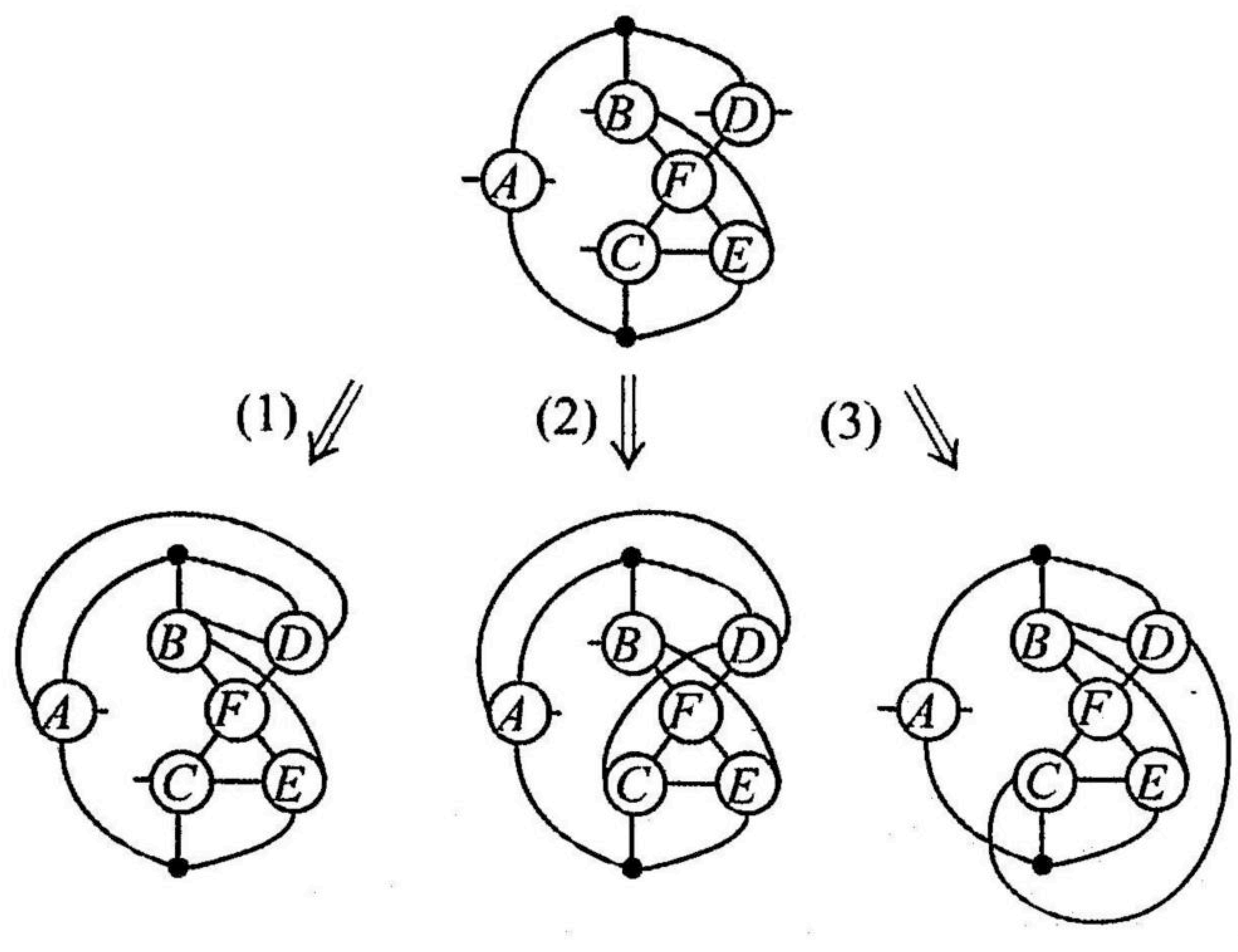


FIG. 3.45. $t = 6$ (c) (v) (D).

(1) $D \sim A, B$. Then $A \sim C$, and we obtain 6^1_*; see Fig. 3.46.

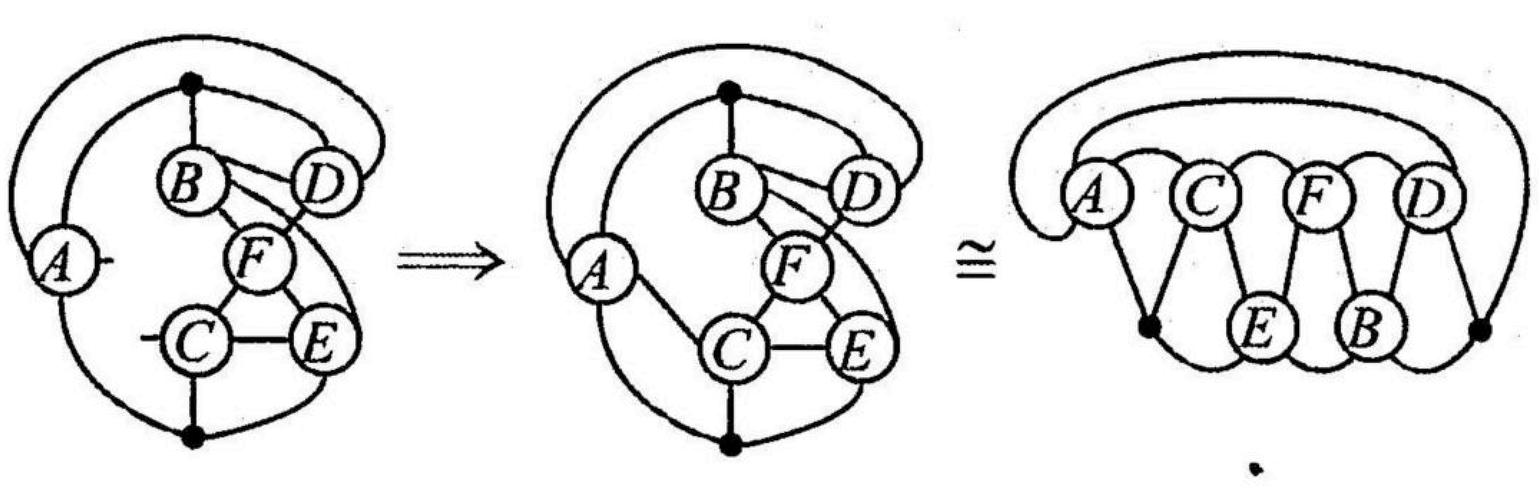


FIG. 3.46. $t = 6$ (c) (v) (D) (1).

(2) $D \sim A, C$. Then $A \sim D$, and we obtain 6^3_*; see Fig. 3.47.

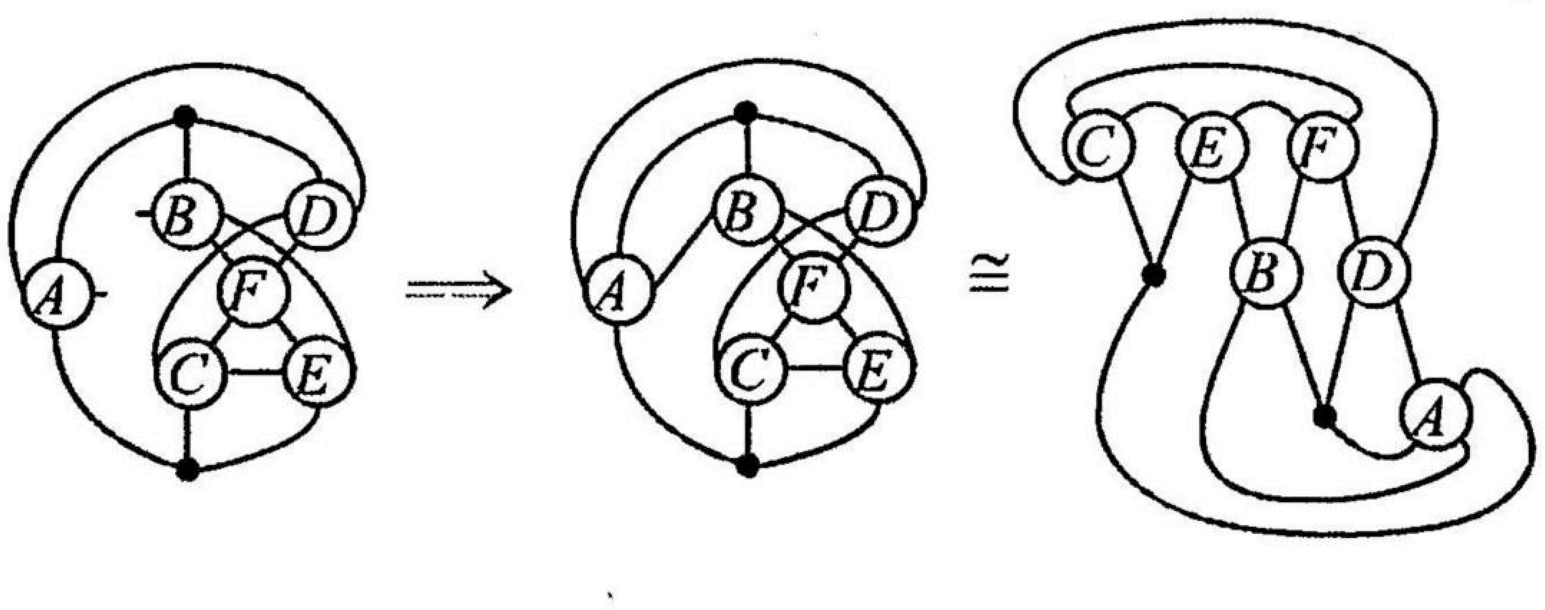


FIG. 3.47. $t = 6$ (c) (v) (D) (2).

(3) $D \sim B, C$. This gives a graph having a loop at A, and so it does not satisfy the condition (P1); see Fig. 3.45.

(E) $E \sim B, D$. We obtain a graph containing $K_{3,3}$, and it does not satisfy the condition (P5); see Fig. 3.48.

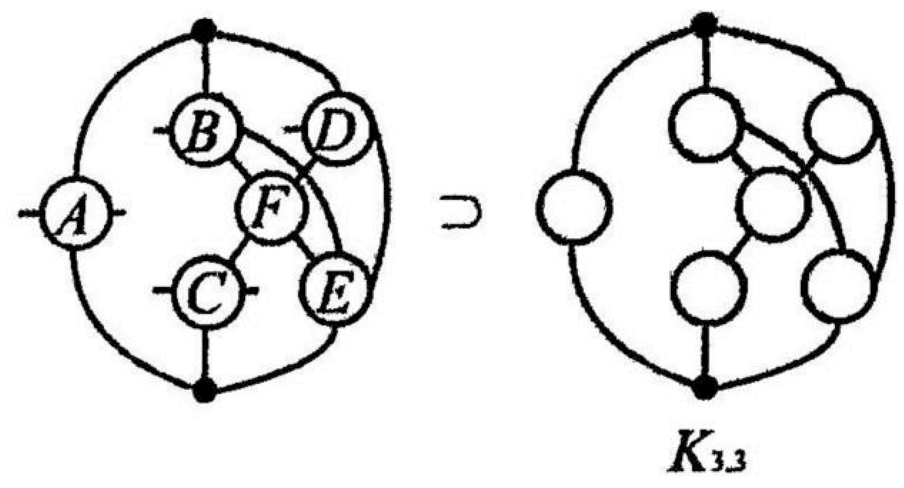


FIG. 3.48. $t = 6$ (c) (v) (E).

(F) $E \sim C, D$. Since B and D are interchangeable in the first figure in Fig. 3.38, this case is the same as the case (D).

Pattern (d). The vertex F has three remaining hands, and so we consider how the hands of F connect. There are ten cases; see Fig. 3.49.

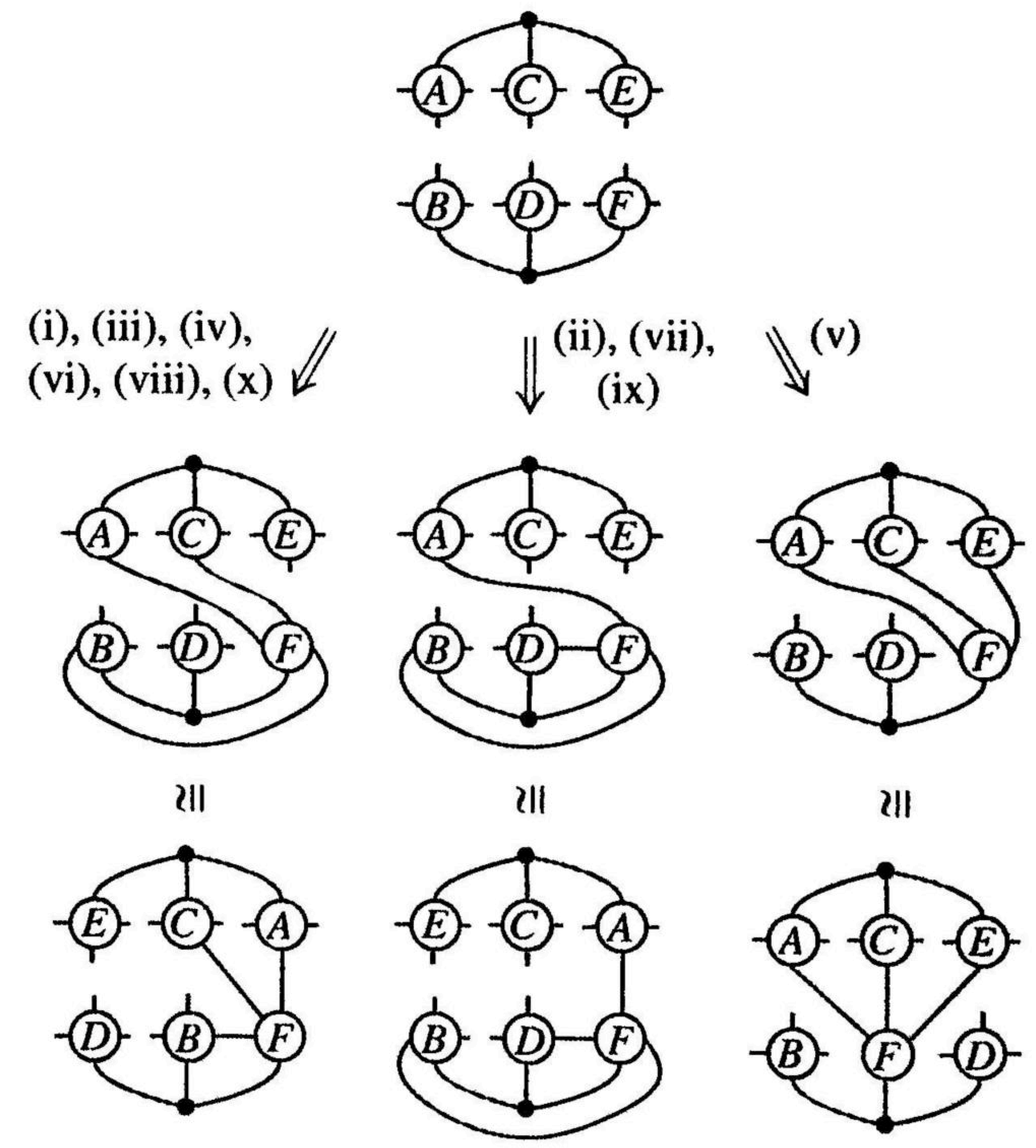


FIG. 3.49. $t = 6$ (d).

(i) $F \sim A, B, C$. The vertex E has three remaining hands, so we consider how the hands of E connect. There are four cases; see Fig. 3.50.

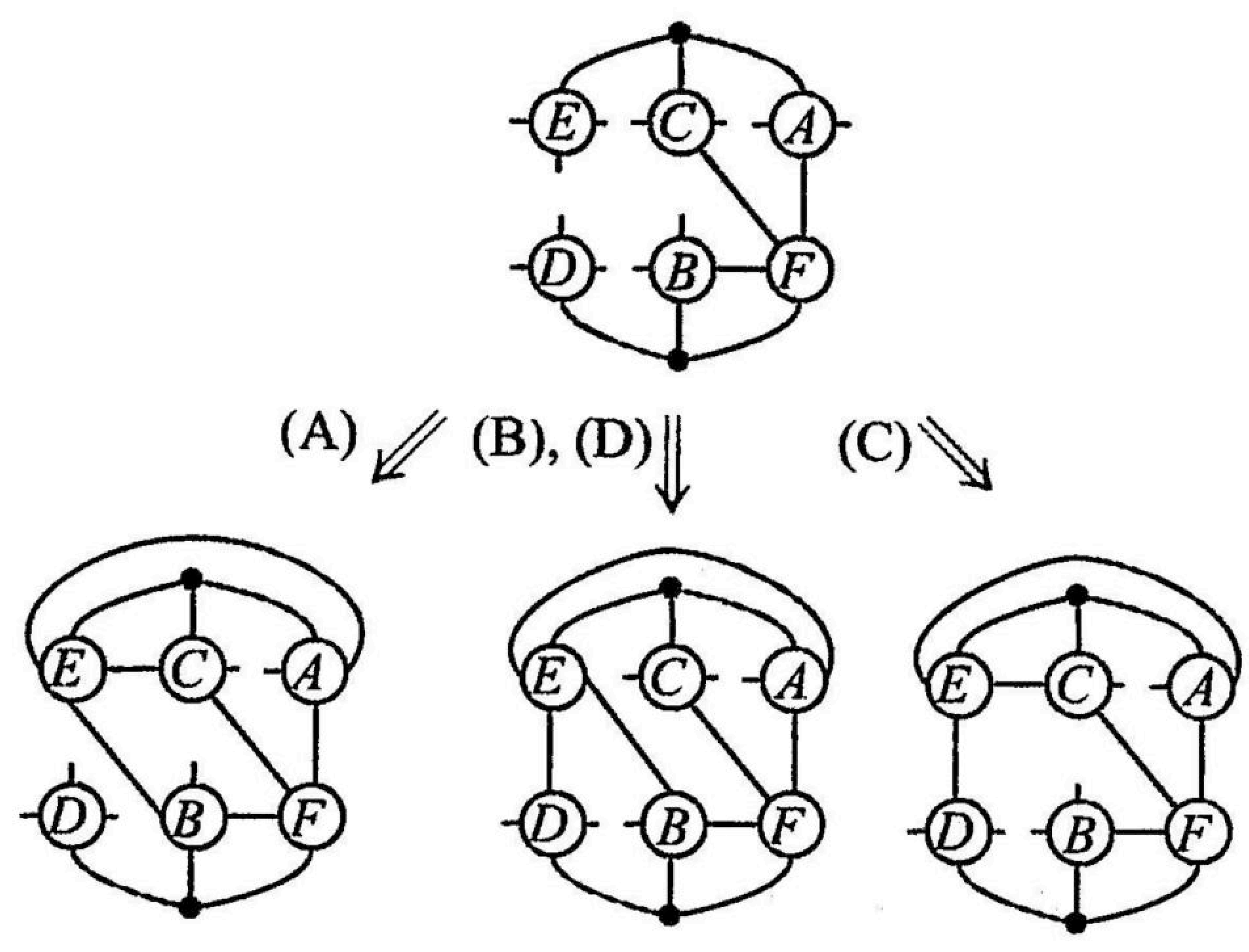


FIG. 3.50. $t = 6$ (d) (i).

(A) $E \sim A, B, C$. Then $D \sim A, B, C$. We obtain a graph containing $K_{3,3}$, and so it does not satisfy the condition (P5); see Fig. 3.51.

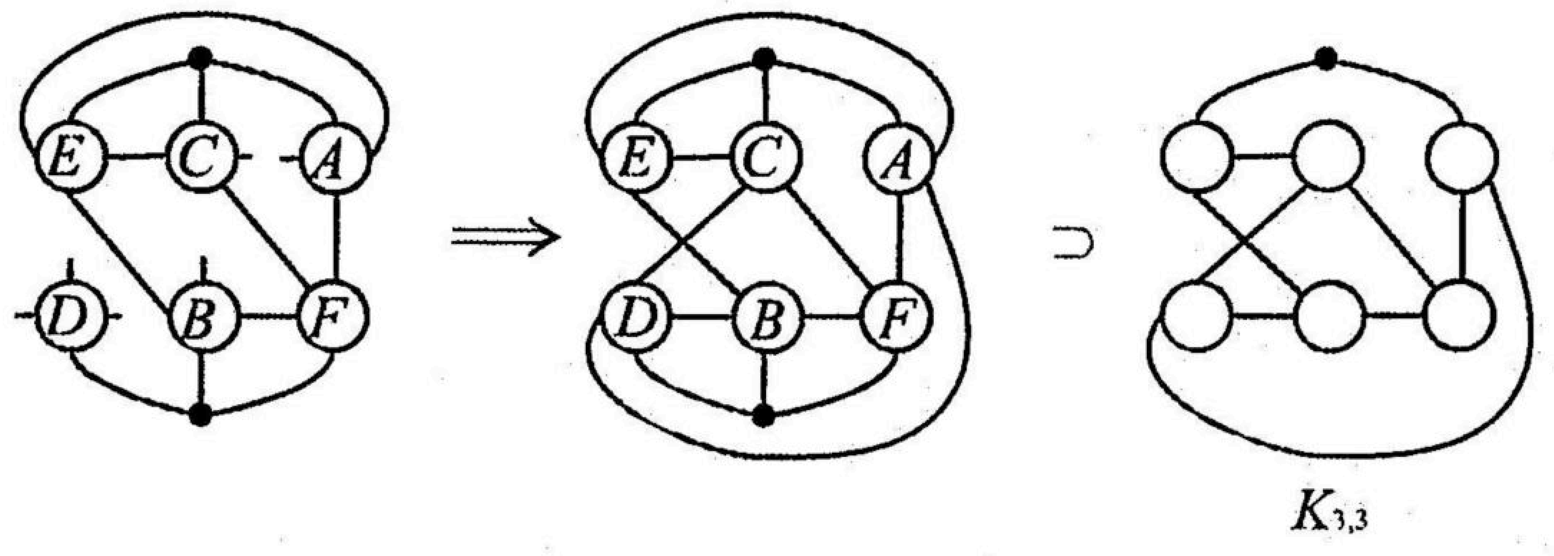


FIG. 3.51. $t = 6$ (d) (i) (A).

(B) $E \sim A, B, D$. The vertex D has two remaining hands, so we consider how the hands of D connect. There are three cases; see Fig. 3.52.

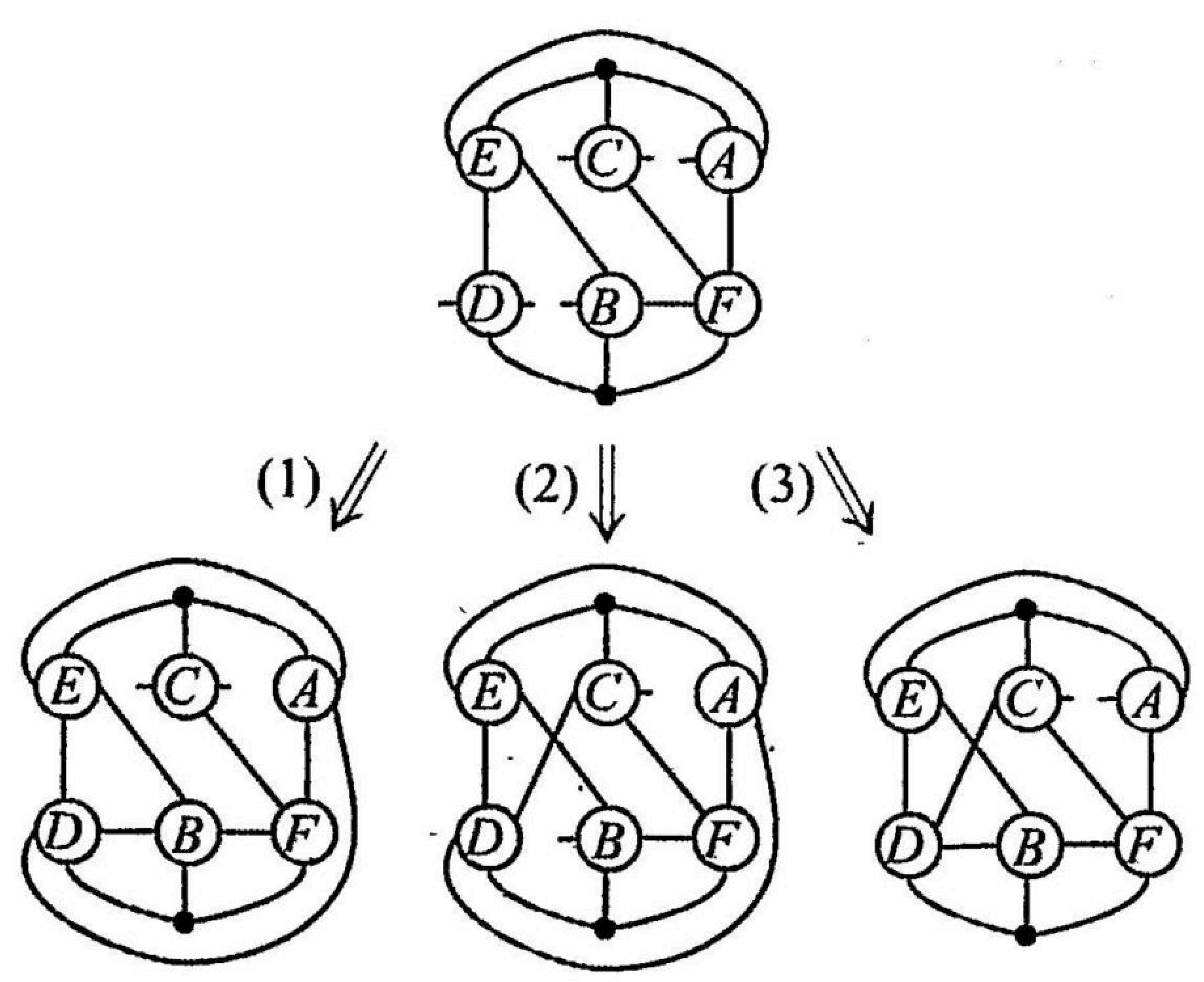


FIG. 3.52. $t = 6$ (d) (i) (B).

(1) $D \sim A, B$. This gives a graph having a loop at C, and so it does not satisfy condition (P1); see Fig. 3.52.
(2) $D \sim A, C$. This gives a graph containing $K_{3,3}$, and so it does not satisfy condition (P5); see Fig. 3.53.

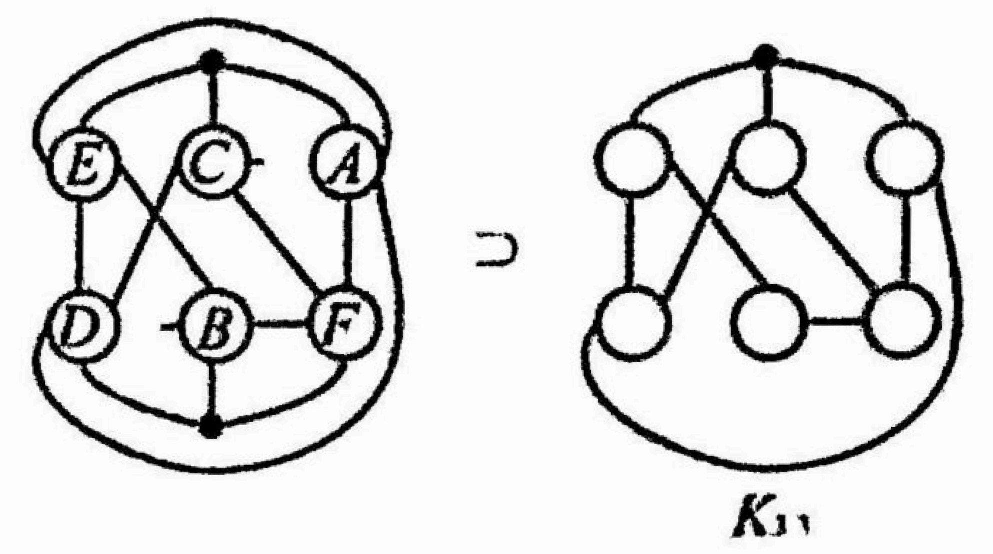


FIG. 3.53. $t = 6$ (d) (i) (B) (2).

(3) $D \sim B, C$. Then $A \sim C$, and we obtain 6^4_*; see Fig. 3.54.

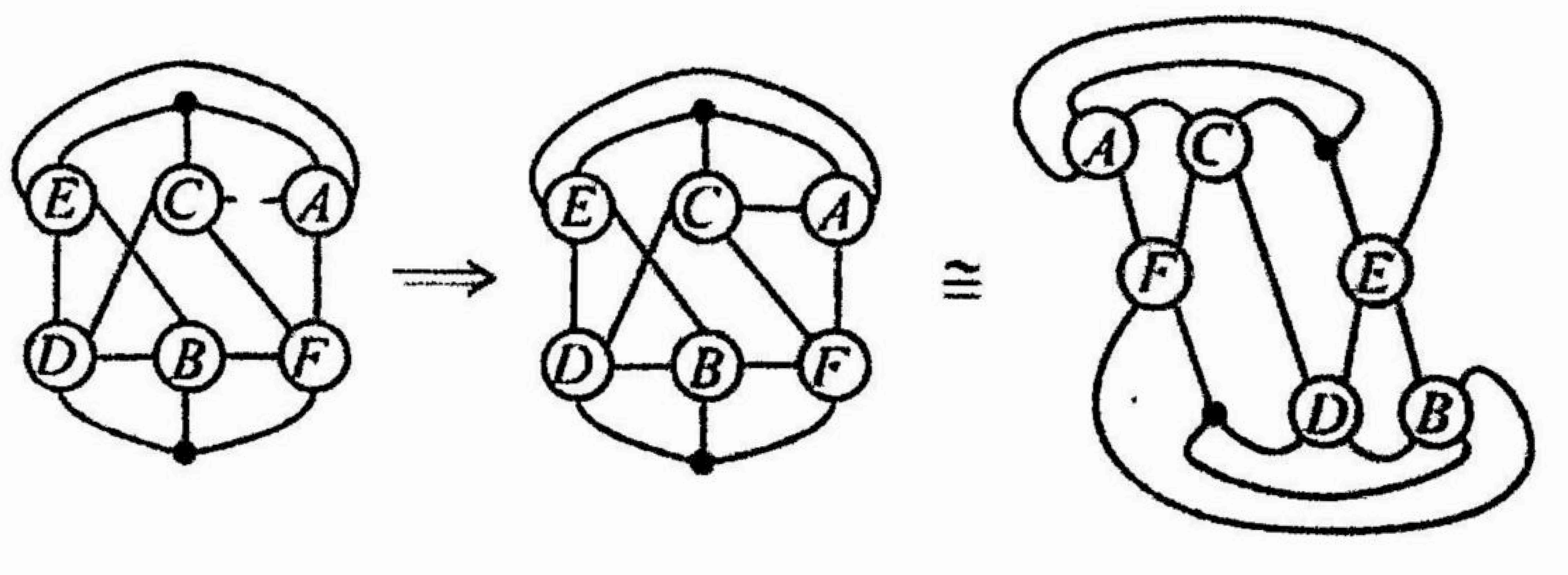


FIG. 3.54. $t = 6$ (d) (i) (B) (3).

(C) $E \sim A, C, D$. The vertex D has two remaining hands, so we consider how the of D connect. There are three cases; see Fig. 3.55.

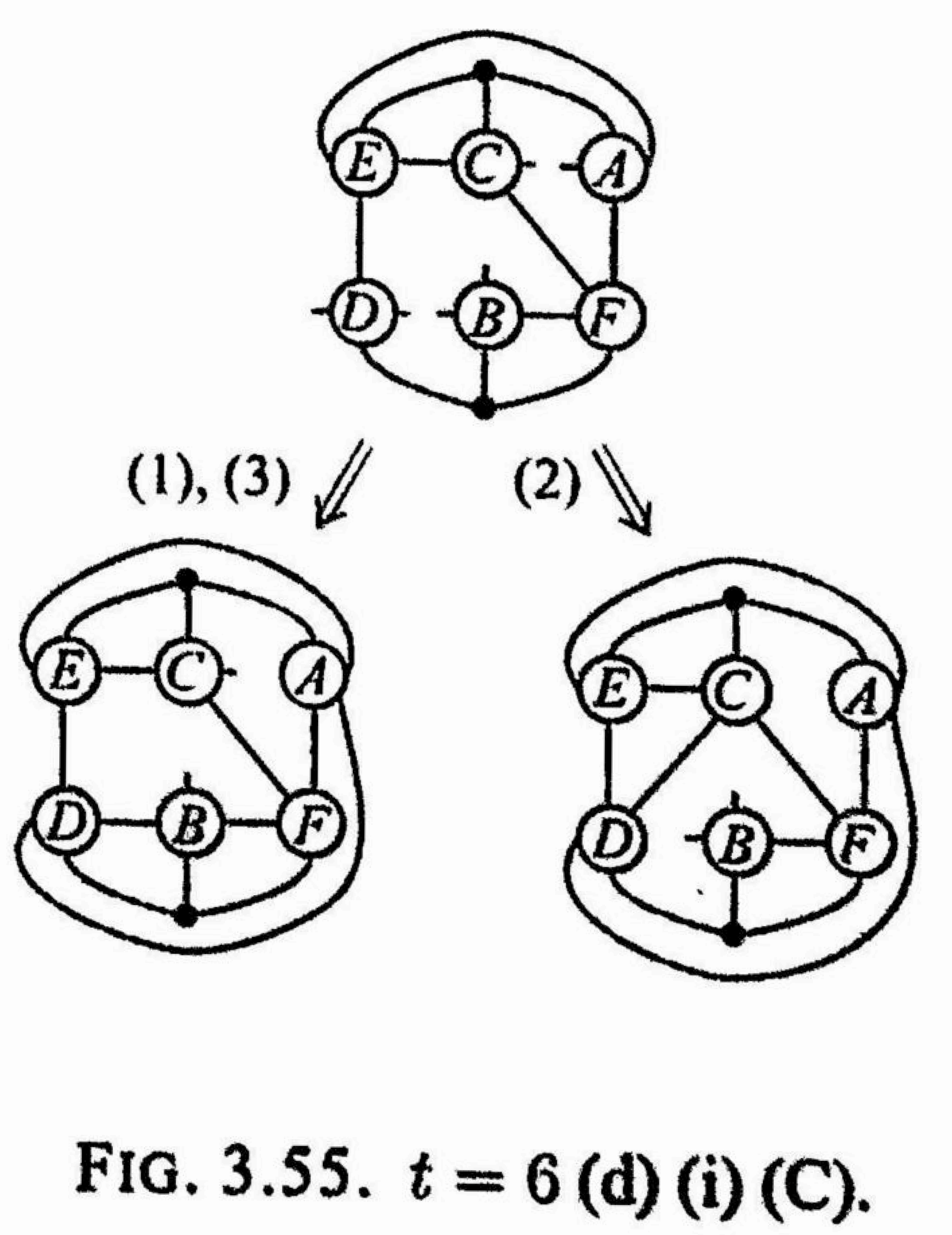


FIG. 3.55. $t = 6$ (d) (i) (C).

(1) $D \sim A, B$. Then $B \sim C$, and we obtain 6^4_*; see Fig. 3.56.

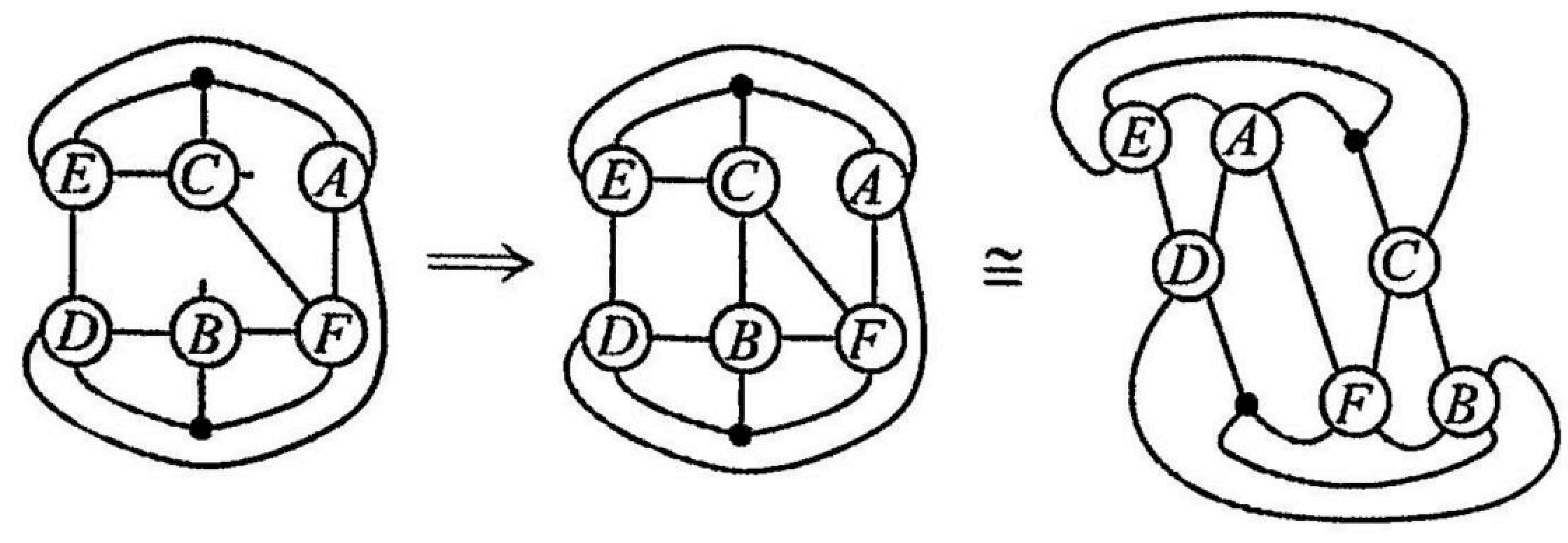


FIG. 3.56. $t = 6$ (d) (i) (C) (1).

(2) $D \sim A, C$. This gives a graph having a loop at B, and so it does not satisfy the condition (P1); see Fig. 3.55.

(3) $D \sim B, C$. Since A and C are interchangeable in the first figure in Fig. 3.55, this case is the same as the case (1).

(D) $E \sim B, C, D$. Since A and C are interchangeable in the first figure in Fig. 3.50, this case is the same as the case (B).

(ii) $F \sim A, B, D$. The vertex E has three remaining hands, so we consider how the hands of E connect. There are four cases; see Fig. 3.57.

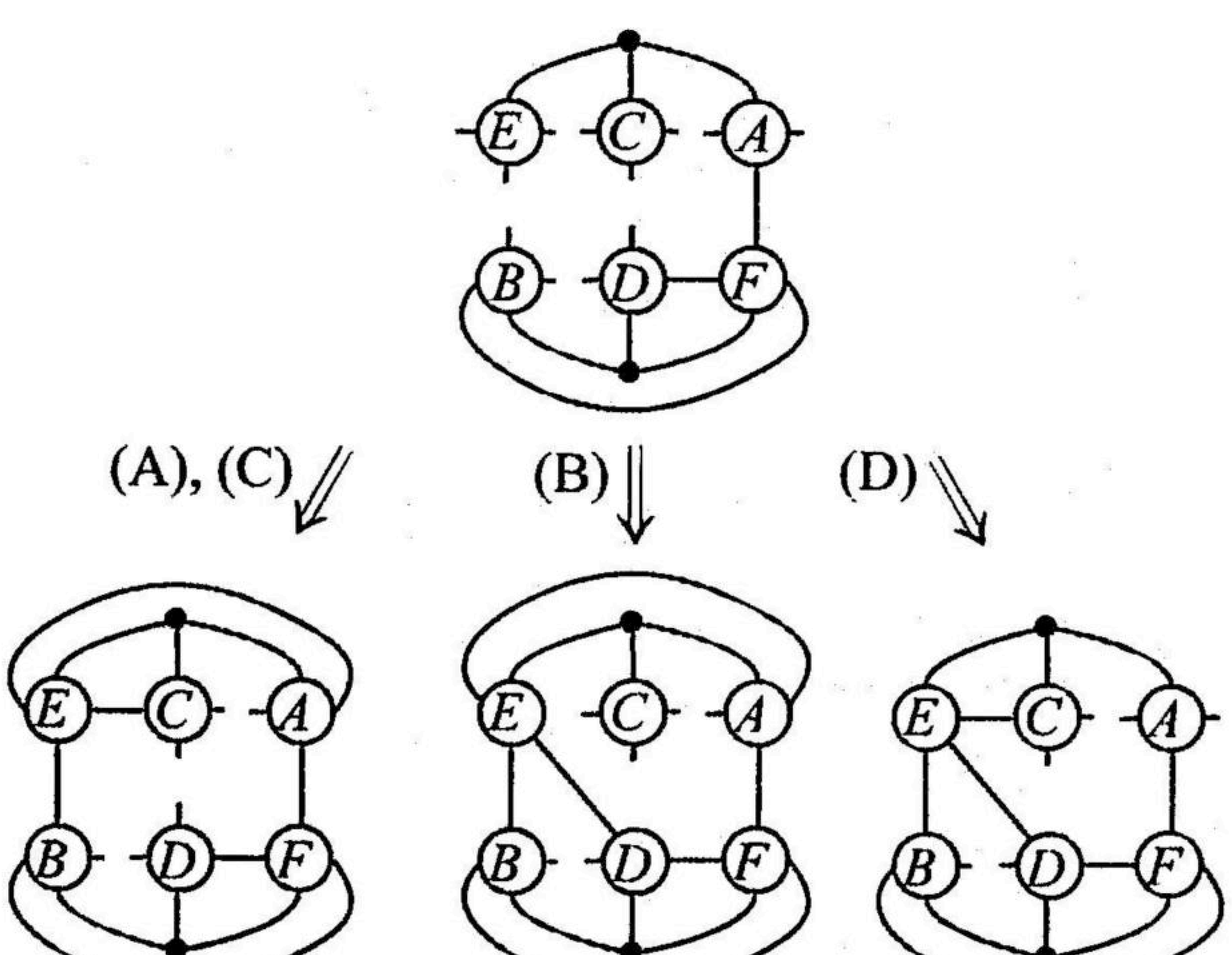


FIG. 3.57. $t = 6$ (d) (ii).

(A) $E \sim A, B, C$. The vertex D has two remaining hands, so we consider how the hands of D connect. There are three cases; see Fig. 3.58.

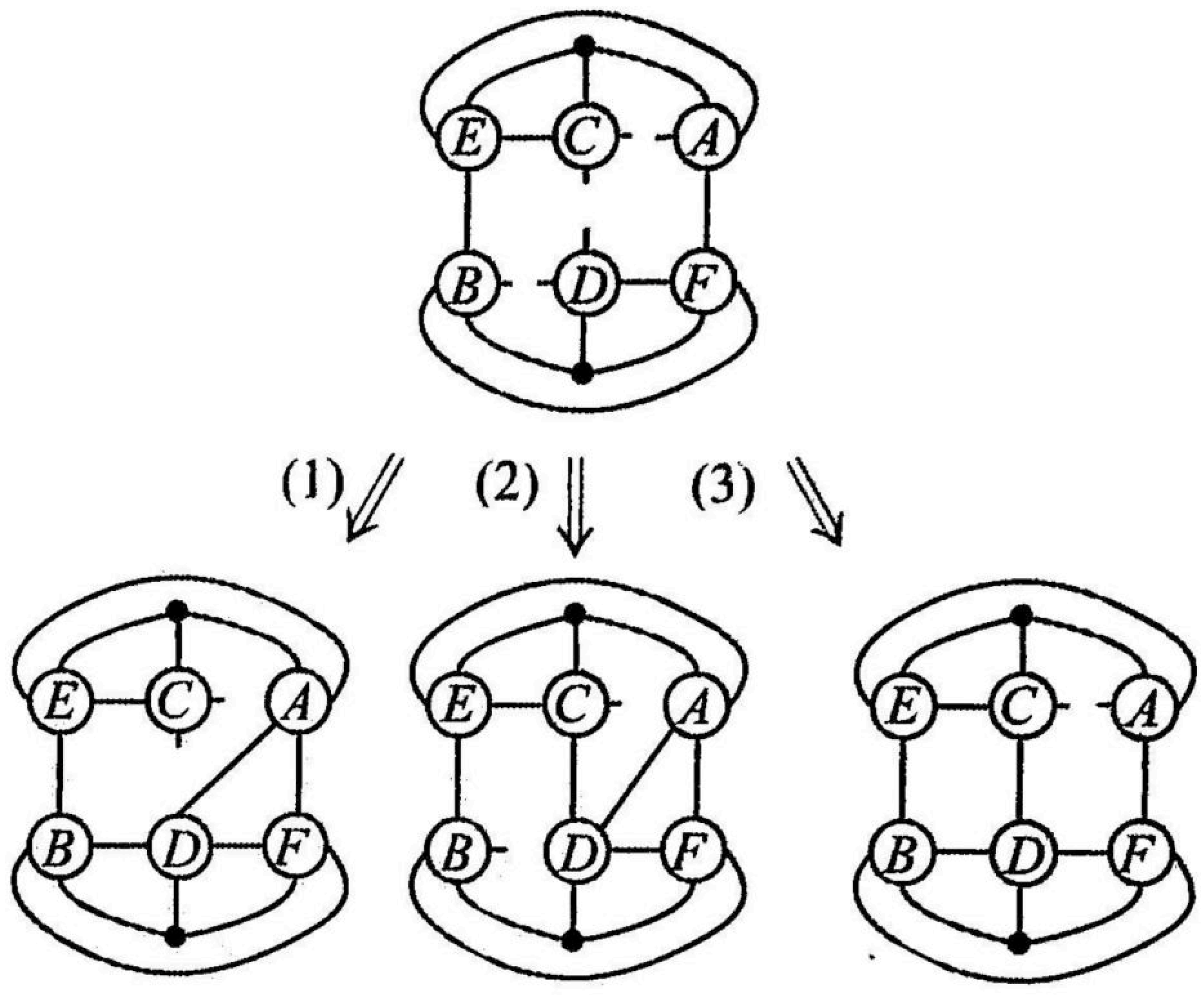


FIG. 3.58. $t = 6$ (d) (ii) (A).

(1) $D \sim A, B$. This gives a graph having a loop at C, and so it does not satisfy the condition (P1); see Fig. 3.58.
(2) $D \sim A, C$. Then $B \sim C$, and we obtain 6^4_*; see Fig. 3.59.

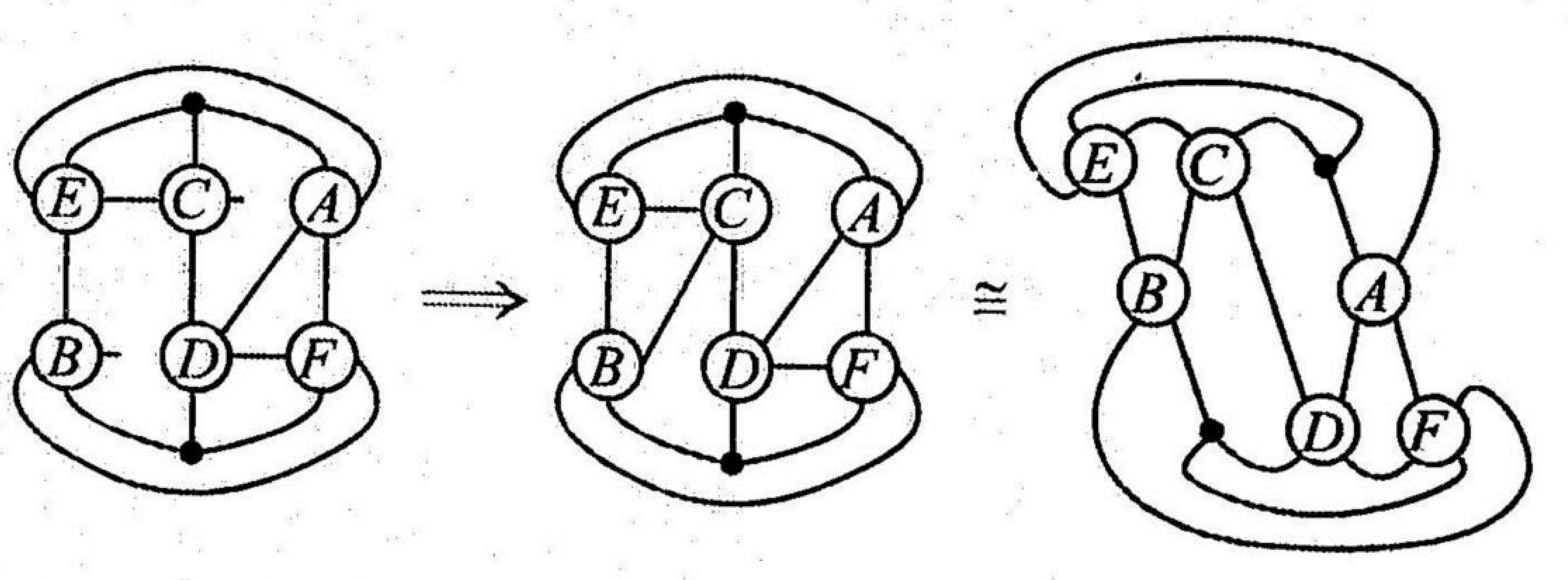


FIG. 3.59. $t = 6$ (d) (ii) (A) (2).

(3) $D \sim B, C$. Then $A \sim C$. This gives a nonprime θ-polyhedron; see Fig. 3.60.

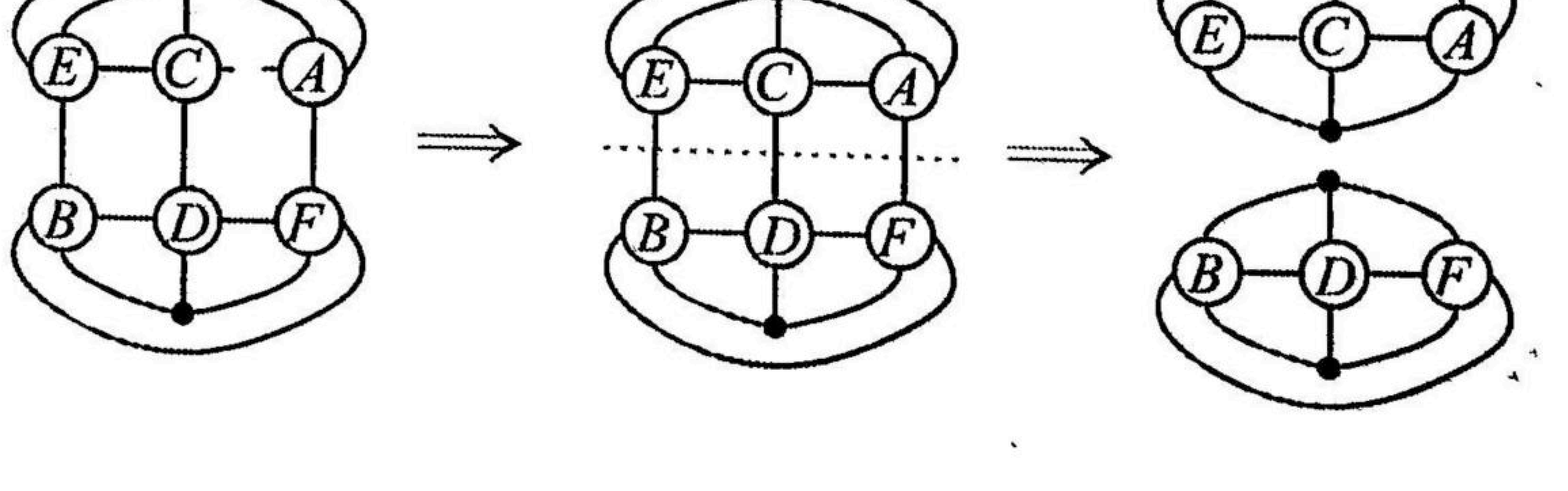


FIG. 3.60. $t = 6$ (d) (ii) (A) (3).

(B) $E \sim A, B, D$. Then $C \sim A, B, D$. This gives a graph containing $K_{3,3}$, and so it does not satisfy the condition (P5); see Fig. 3.61.

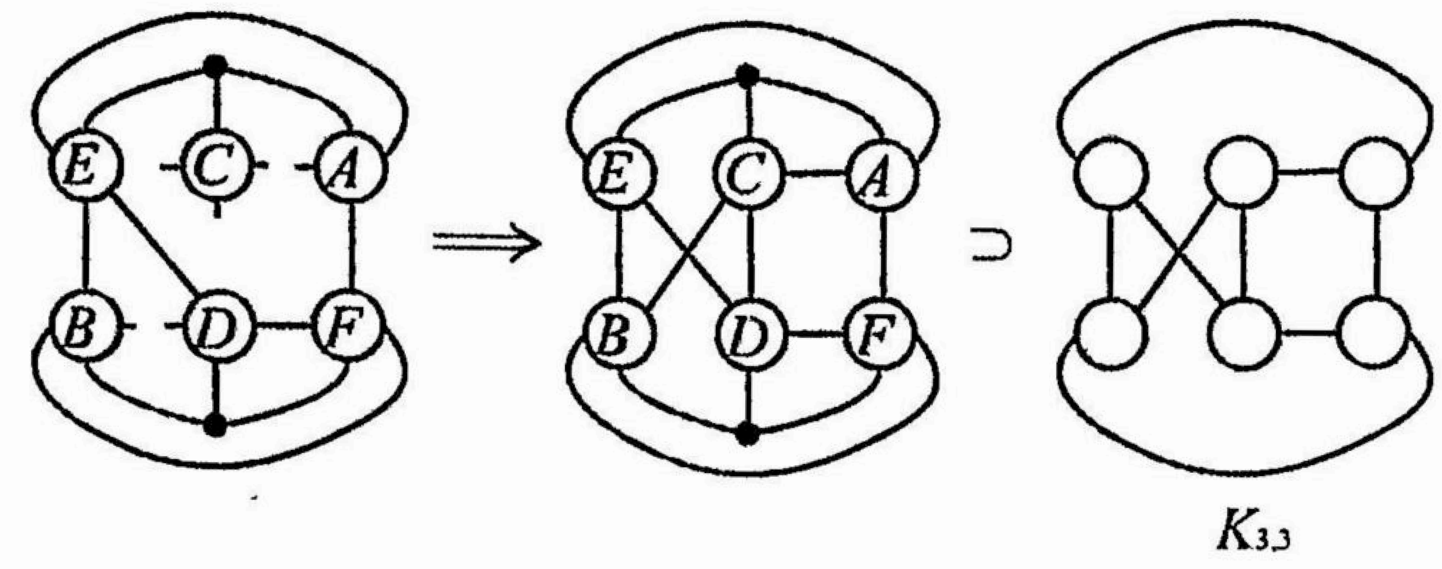


FIG. 3.61. $t = 6$ (d) (ii) (B).

(C) $E \sim A, C, D$. Since B and D are interchangeable in the first figure in Fig. 3.57, this case is the same as the case (A).
(D) $E \sim B, C, D$. The vertex C has two remaining hands, so we consider how the hands of C connect. There are three cases; see Fig. 3.62.

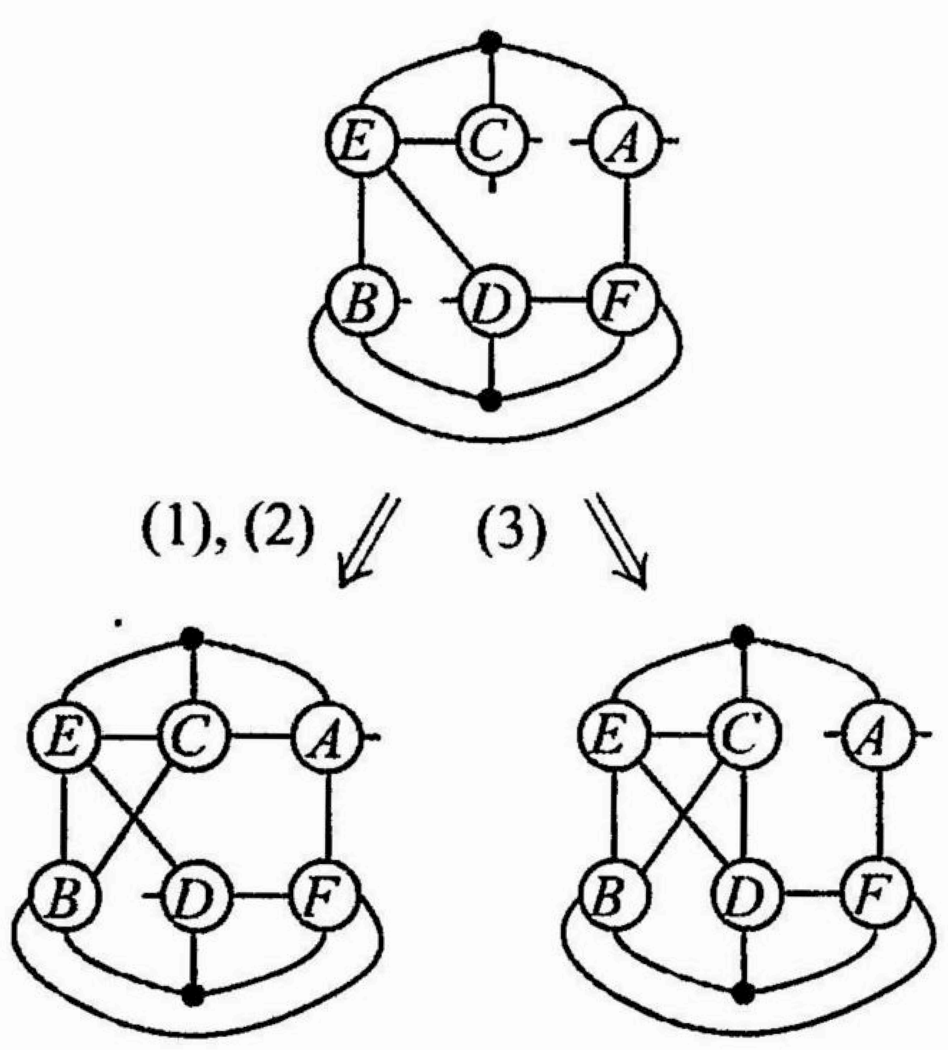


FIG. 3.62. $t = 6$ (d) (ii) (D).

(1) $C \sim A, B$. Then $A \sim D$, and we obtain 6^4_*; see Fig. 3.63.

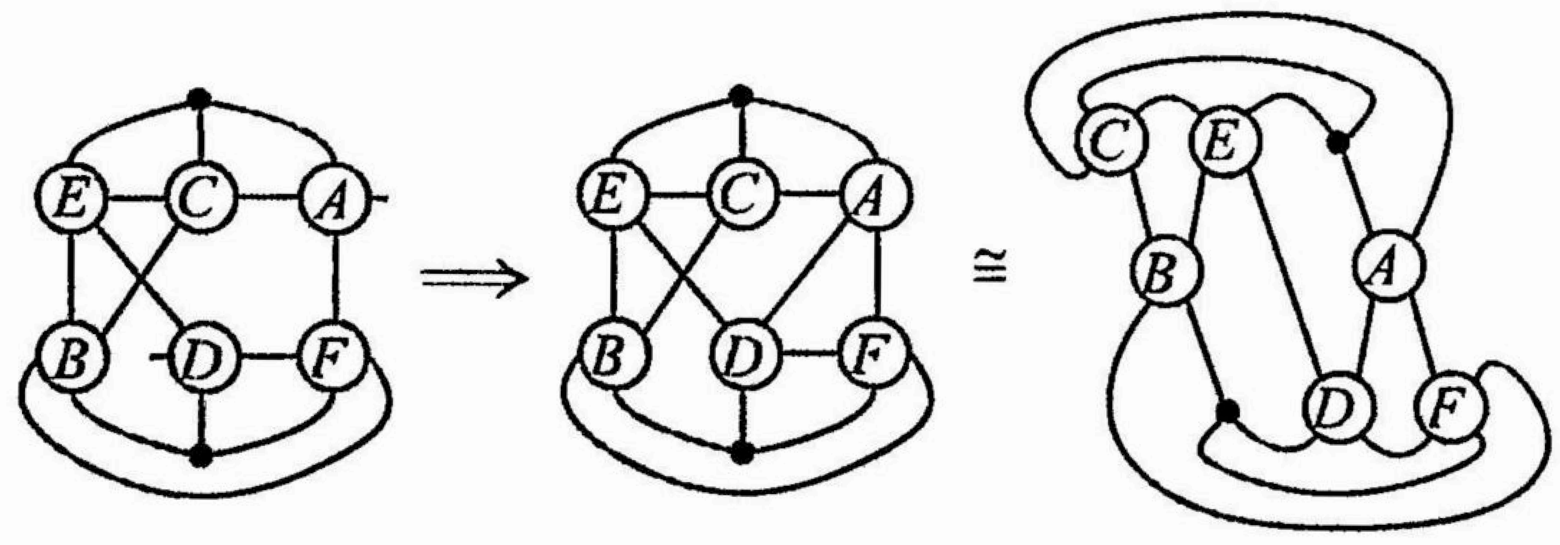


FIG. 3.63. $t = 6$ (d) (ii) (D) (1).

(2) $C \sim A, D$. Since B and D are interchangeable in the first figure in Fig. 3.62, this case is the same as the case (1).
(3) $C \sim B, D$. This gives a graph having a loop at A, and so it does not satisfy the condition (P1); see Fig. 3.62.

(iii) $F \sim A, B, E$. Since C and E are interchangeable in the first figure in Fig. 3.49, this case is the same as the case (i).
(iv) $F \sim A, C, D$. Since B and D are interchangeable in the first figure in Fig. 3.49, this case is the same as the case (i).
(v) $F \sim A, C, E$. The vertex D has three remaining hands, so we consider how the hands of E connect. There are four cases; see Fig. 3.64.

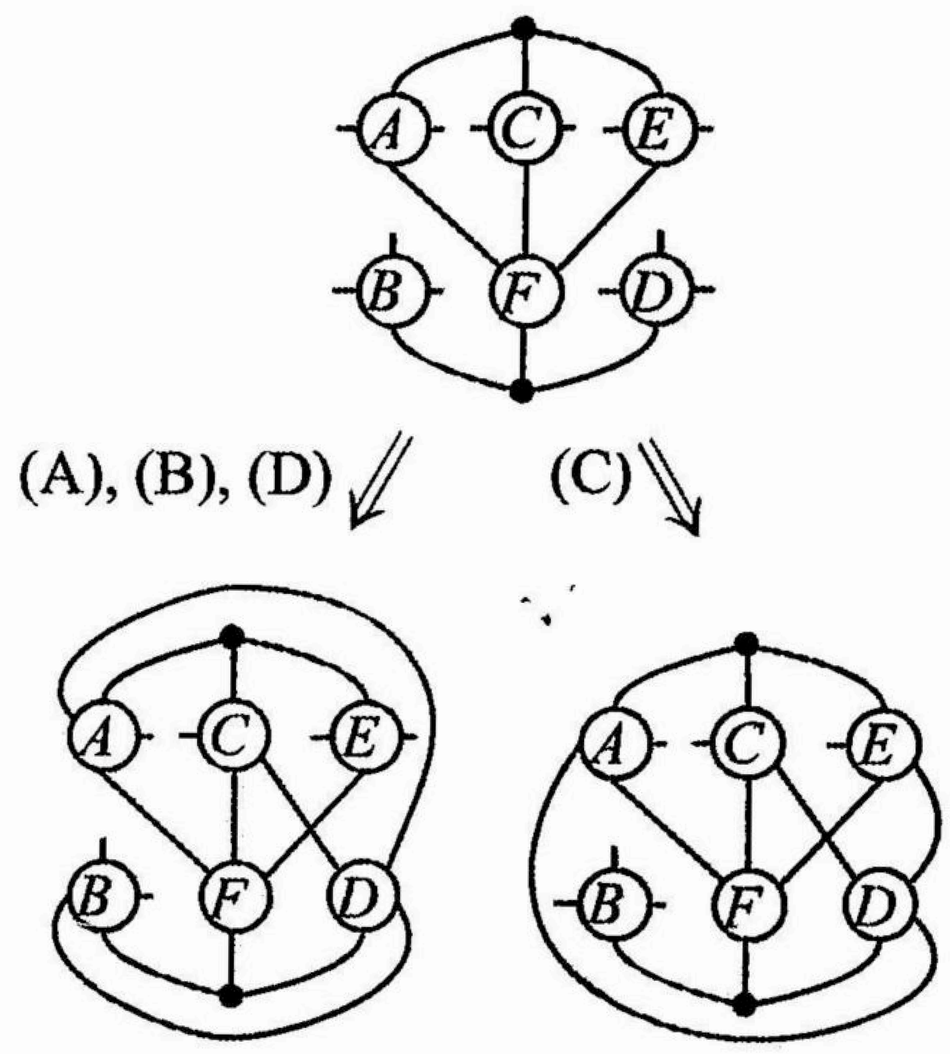


FIG. 3.64. $t = 6$ (d) (v).

(A) $D \sim A, B, C$. The vertex B has two remaining hands, so we consider how the hands of D connect. There are three cases; see Fig. 3.65.

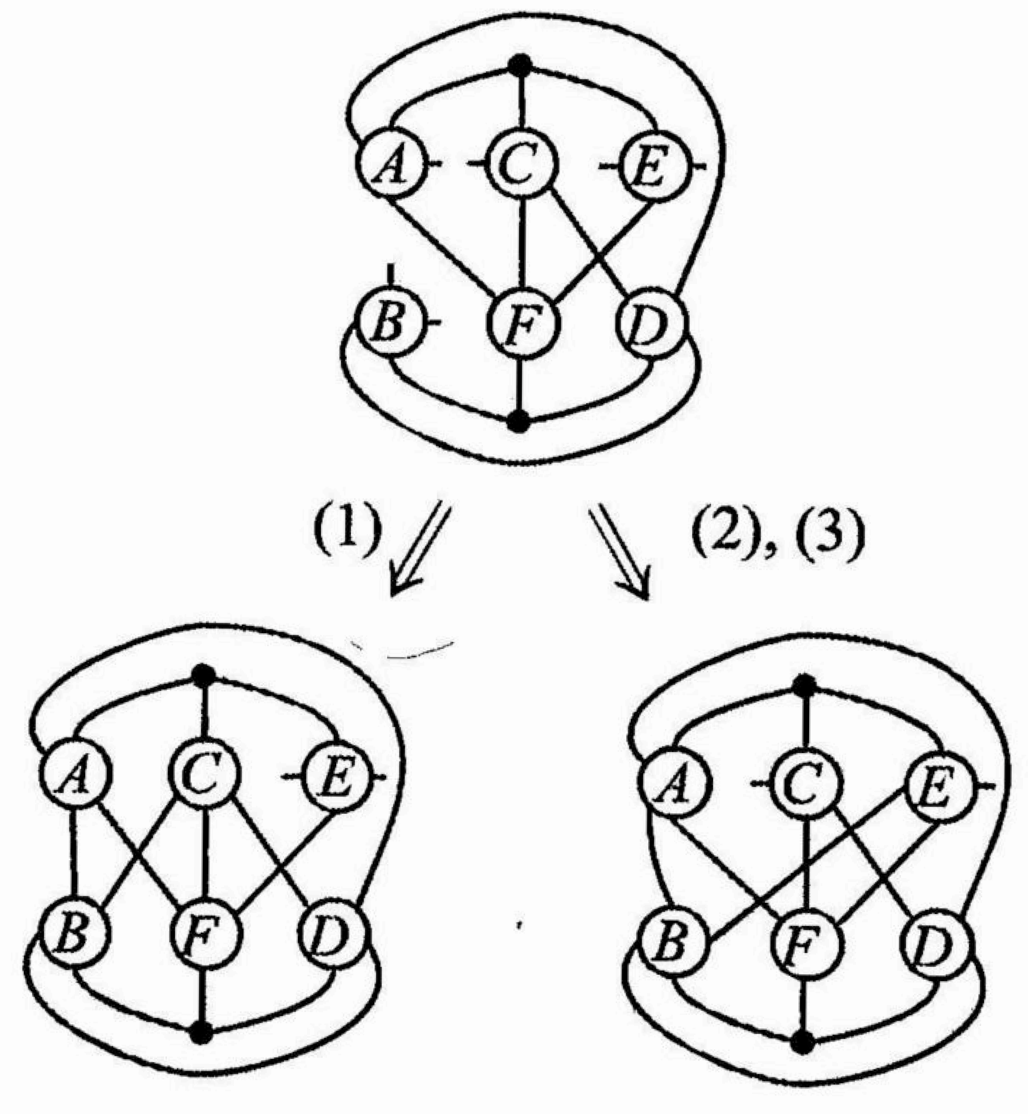


FIG. 3.65. $t = 6$ (d) (v) (A).

(1) $B \sim A, C$. This gives a graph having a loop at E, and so it does not satisfy the condition (P1); see Fig. 3.65.
(2) $B \sim A, E$. We obtain a graph containing $K_{3,3}$, and so it does not satisfy the condition (P5); see Fig. 3.66.

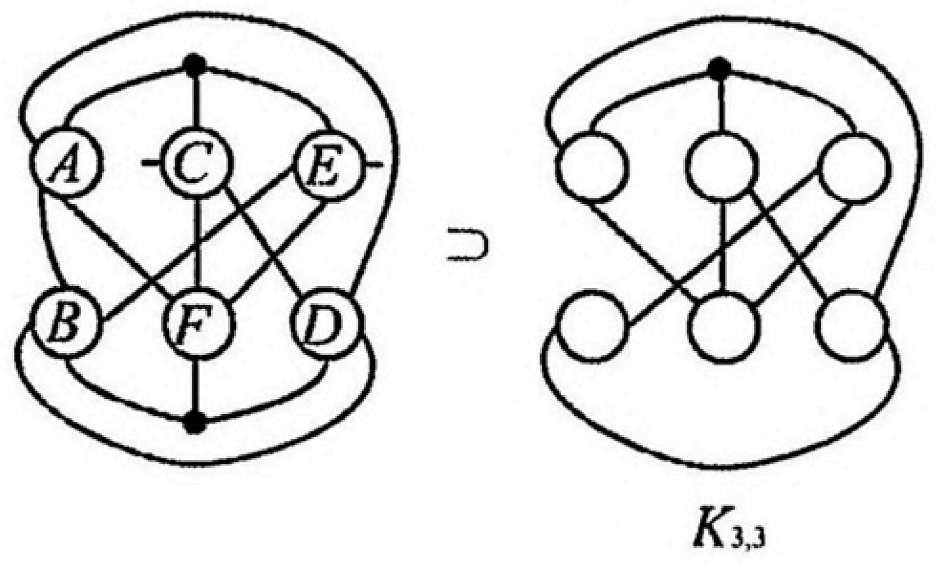


FIG. 3.66. $t = 6$ (d) (v) (A) (2).

(3) $B \sim C, E$. Since A and C are interchangeable in the first figure in Fig. 3.65, this case is the same as the case (2).

(B) $D \sim A, B, E$. Since C and E are interchangeable in the first figure in Fig. 3.64, this case is the same as the case (A).

(C) $D \sim A, C, E$. We obtain a graph containing $K_{3,3}$, and so it does not satisfy the condition (P5); see Fig. 3.67.

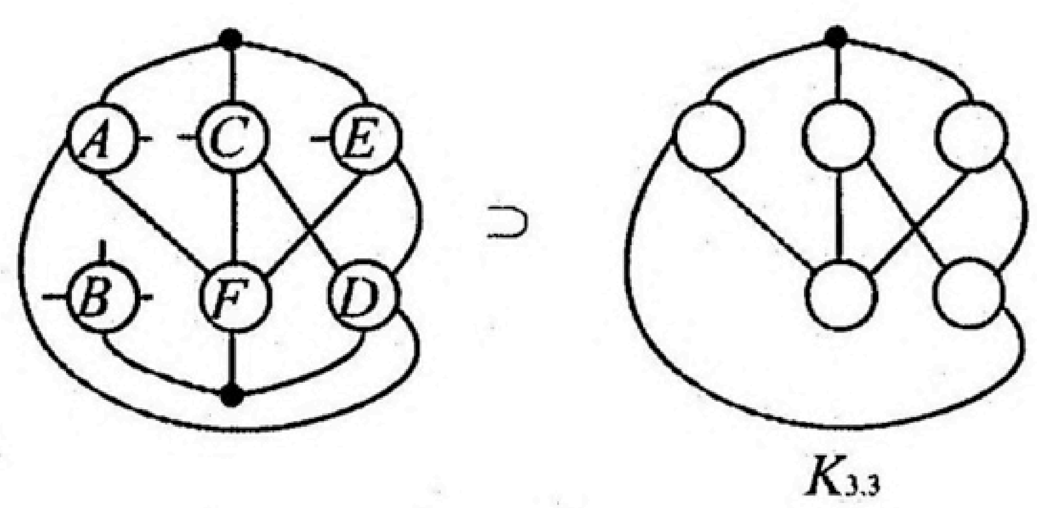


FIG. 3.67. $t = 6$ (d) (v) (C).

(D) $D \sim B, C, E$. Since A and E are interchangeable in the first figure in Fig. 3.64, this case is the same as the case (A).

(vi) $F \sim A, D, E$. Since B and D, C and E are interchangeable in the first figure in Fig. 3.49, this case is the same as the case (i).

(vii) $F \sim B, C, D$. Since A and C are interchangeable in the first figure in Fig. 3.49, this case is the same as the case (ii).

(viii) $F \sim B, C, E$. Since A and E are interchangeable in the first figure in Fig. 3.49, this case is the same as the case (i).

(ix) $F \sim B, D, E$. Since A and E are interchangeable in the first figure in Fig. 3.49, this case is the same as the case (ii).

(x) $F \sim C, D, E$. Since A and E, B and D are interchangeable in the first figure in Fig. 3.49, this case is the same as the case (ii).

2.5. $t = 7$. We consider all the patterns.

Pattern (a). By an argument similar to that in the case $t = 4$ (a), this case does not give a planar graph.

Pattern (b). See Fig. 3.68. We consider how the hands of G connect. By the condition (P2), there are fifteen cases.

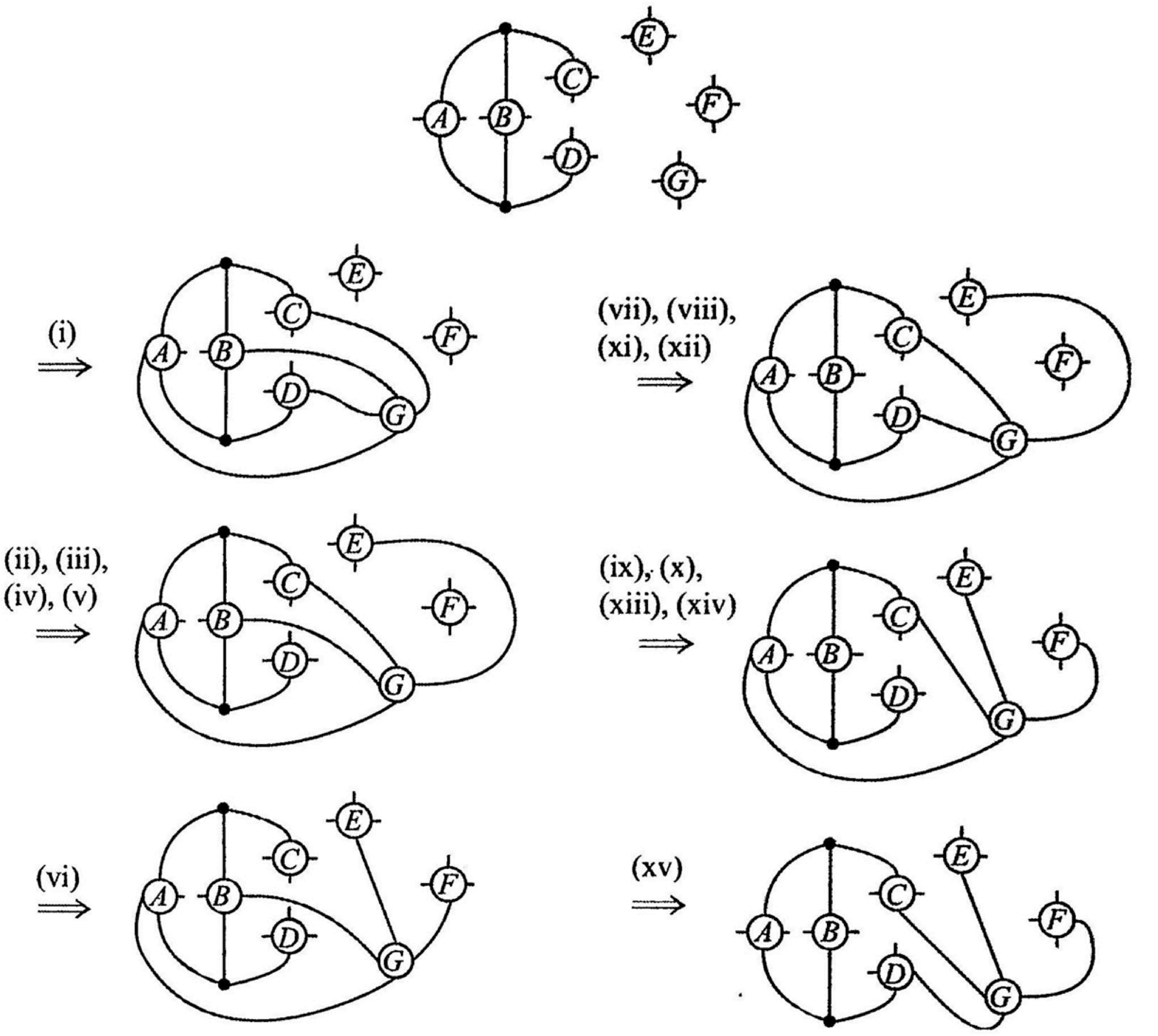


FIG. 3.68. $t = 7$ (b).

(i) $G \sim A, B, C, D$. The vertex F has four remaining hands, so we consider how the hands of F connect. There are five cases; see Fig. 3.69.

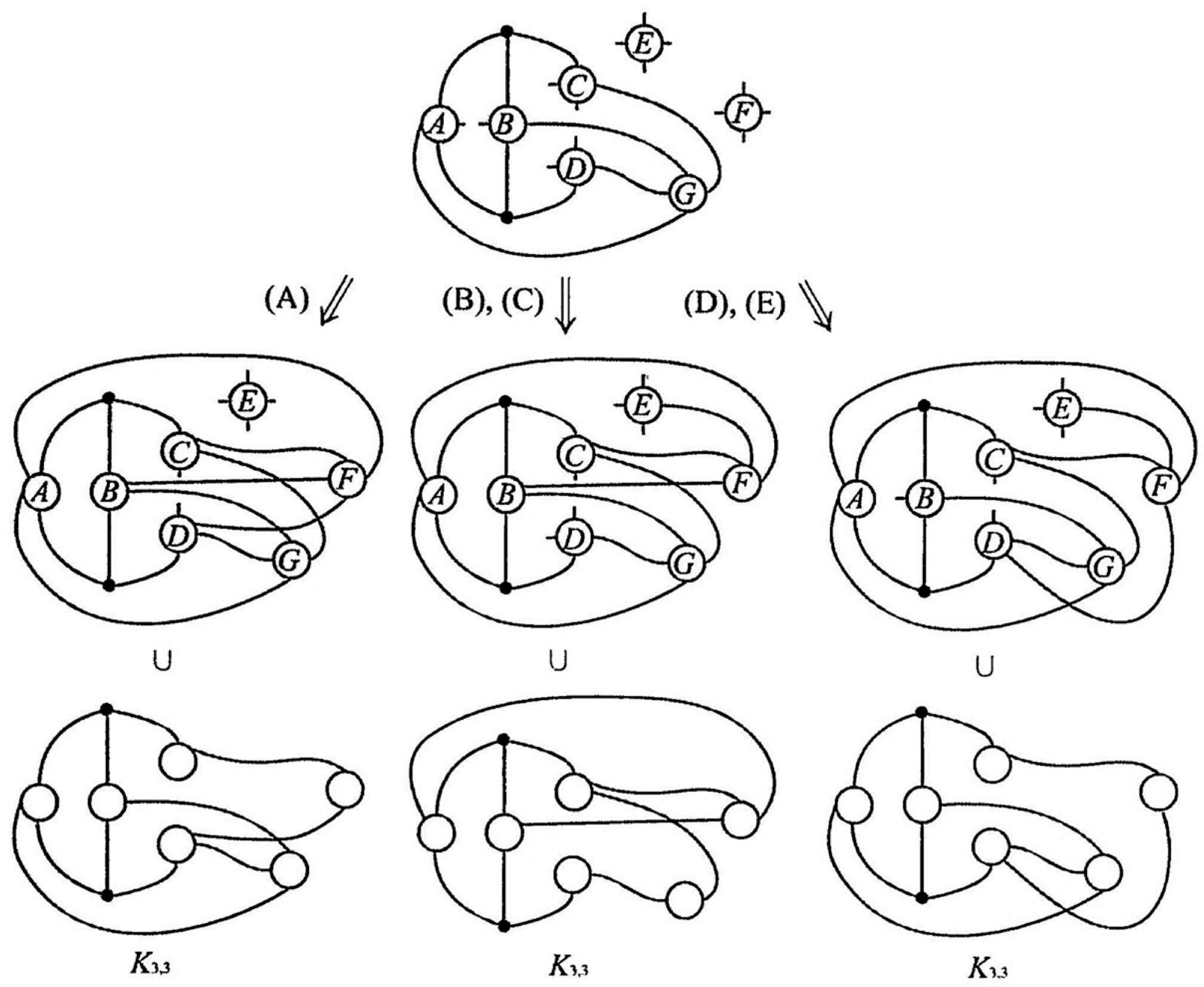


FIG. 3.69. $t = 7$ (b) (i).

(A) $F \sim A, B, C, D$. This gives a graph containing $K_{3,3}$, and so it does not satisfy the condition (P5).
(B) $F \sim A, B, C, E$. This gives a graph containing $K_{3,3}$, and so it does not satisfy the condition (P5).
(C) $F \sim A, B, D, E$. Since C and D are interchangeable in the first figure in Fig. 3.69, this case is the same as the case (B).
(D) $F \sim A, C, D, E$. This gives a graph containing $K_{3,3}$, and so it does not satisfy the condition (P5).
(E) $F \sim B, C, D, E$. Since A and B are interchangeable in the first figure in Fig. 3.69, this case is the same as the case (D).

(ii) $G \sim A, B, C, E$. The vertex F has four remaining hands, so we consider how the hands of F connect. There are five cases; see Fig. 3.70.

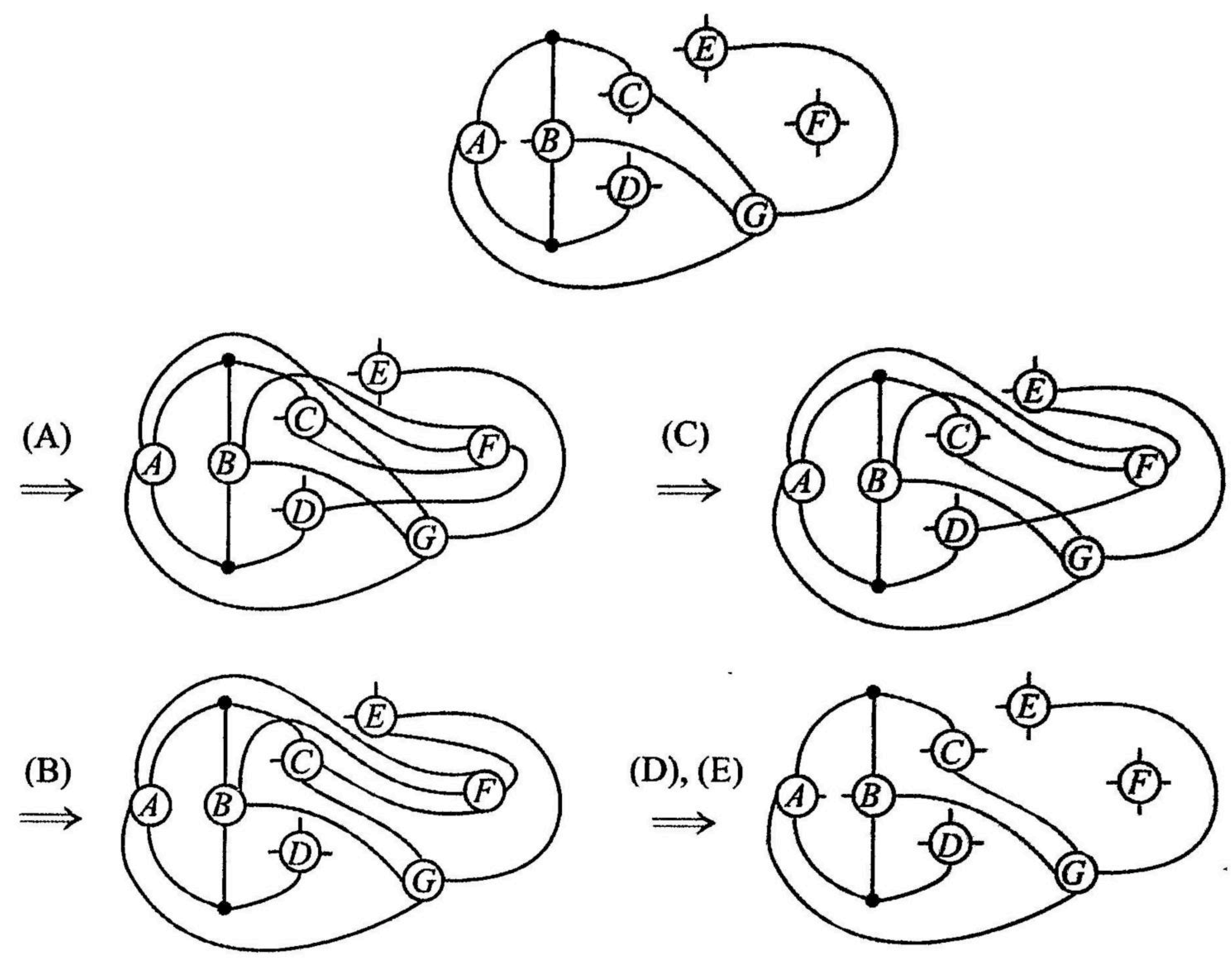


FIG. 3.70. $t = 7$ (b) (ii).

(A) $F \sim A, B, C, D$. This gives a graph containing $K_{3,3}$, and so it does not satisfy the condition (P5); see Fig. 3.71.

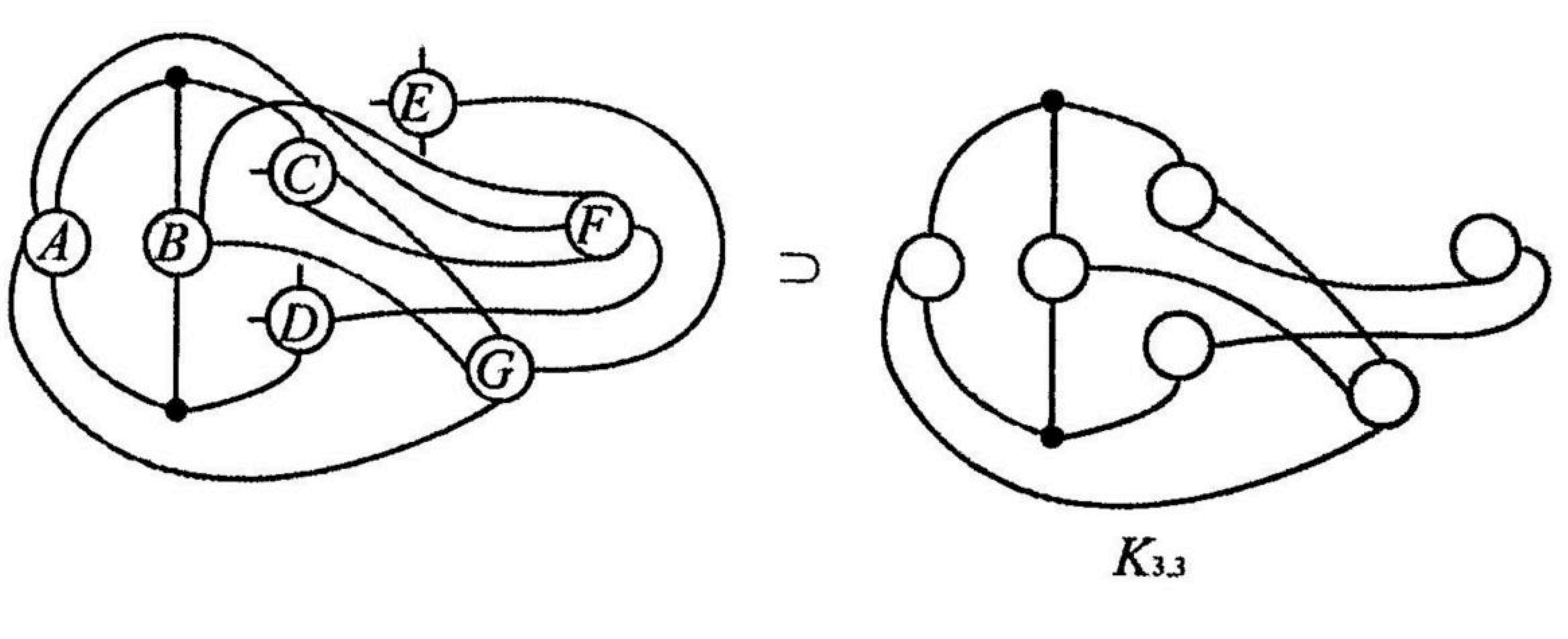


FIG. 3.71. $t = 7$ (b) (ii) (A).

(B) $F \sim A, B, C, E$. This gives a graph containing $K_{3,3}$, and so it does not satisfy the condition (P5); see Fig. 3.72.

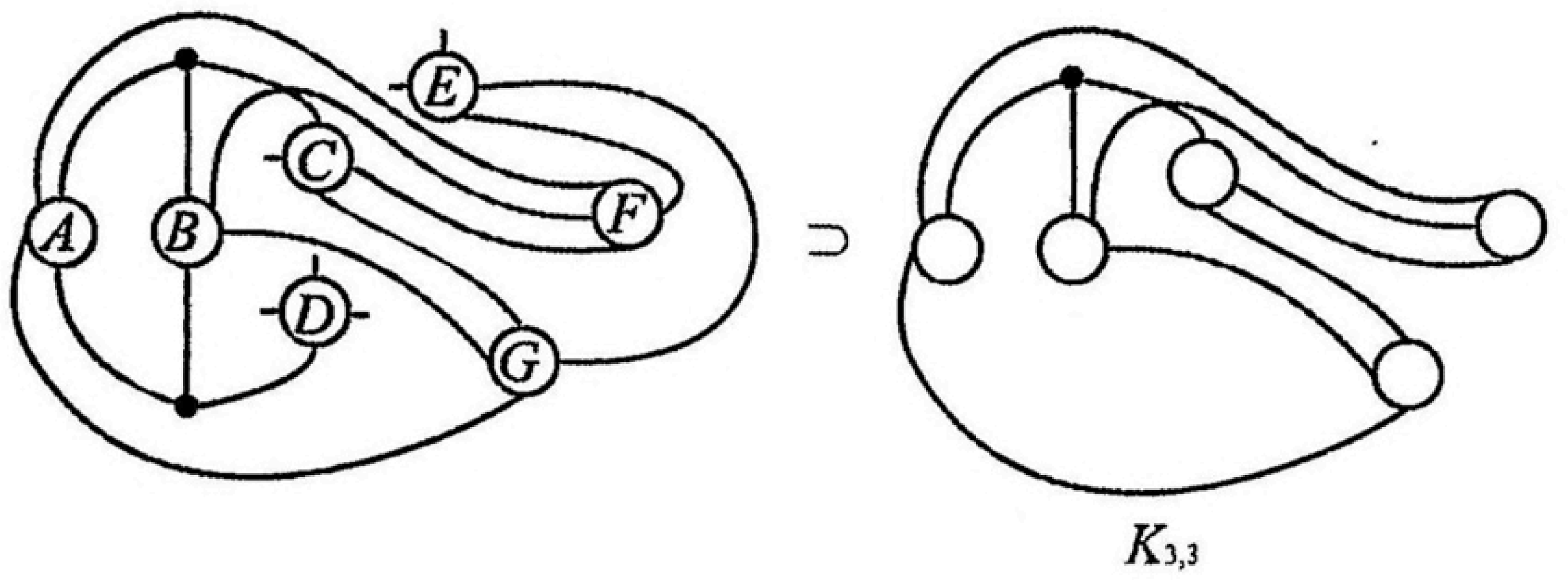


FIG. 3.72. $t = 7$ (b) (ii) (B).

(C) $F \sim A, B, D, E$. Then $C \sim D, E$, $D \sim E$. We obtain a graph containing $K_{3,3}$, which does not satisfy the condition (P5); see Fig. 3.73.

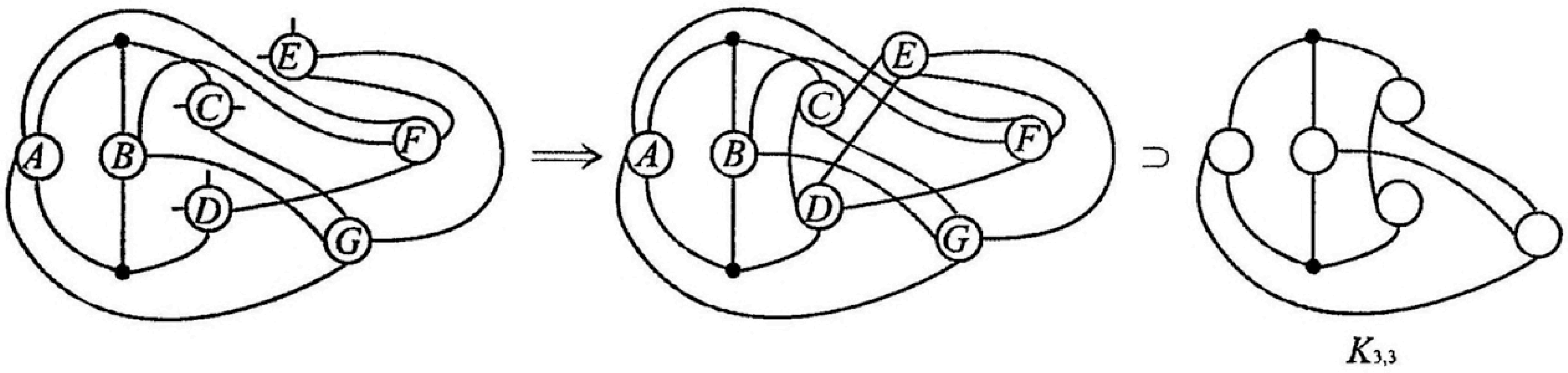


FIG. 3.73. $t = 7$ (b) (ii) (C).

(D) $F \sim A, C, D, E$. This gives a graph containing $K_{3,3}$, and so it does not satisfy the condition (P5); see Fig. 3.74.

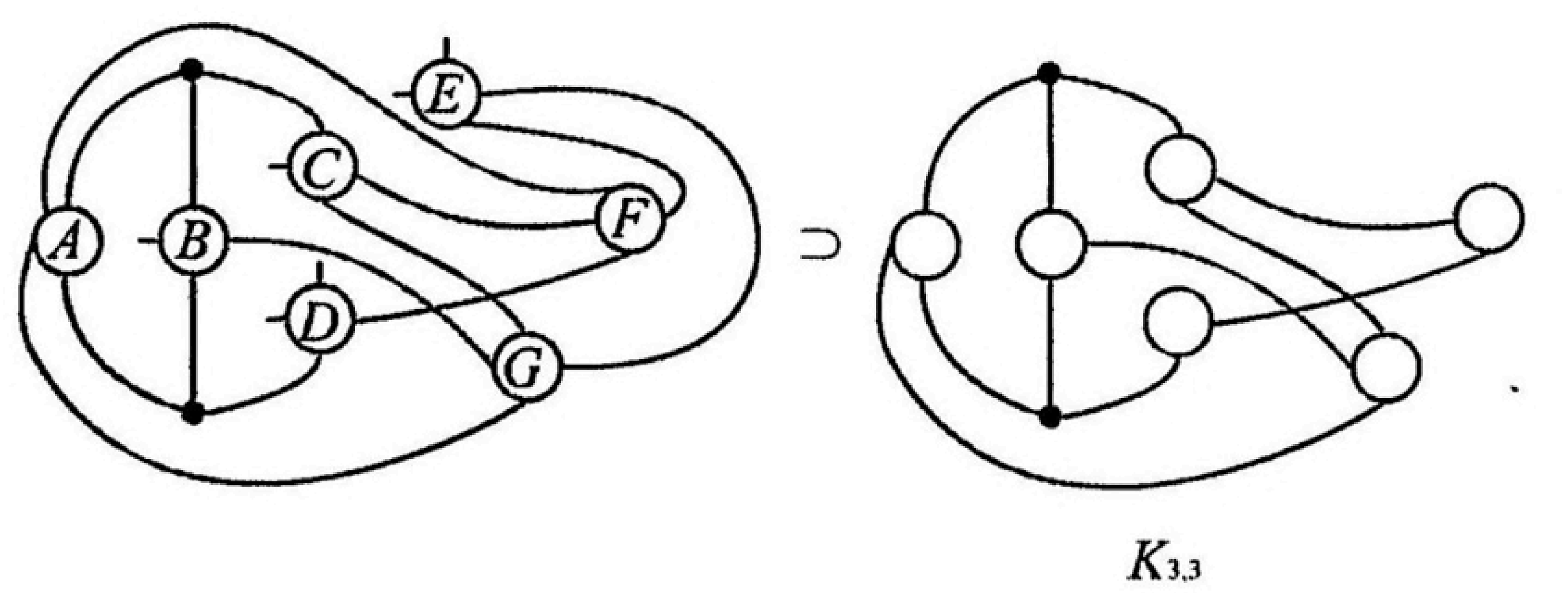


FIG. 3.74. $t = 7$ (b) (ii) (D).

(E) $F \sim B, C, D, E$. Since A and B are interchangeable in the first figure in Fig. 3.70, this case is the same as the case (D).

(iii) $G \sim A, B, C, F$. Since E and F are interchangeable in the first figure in Fig. 3.70, this case is the same as the case (ii).

(iv) $G \sim A, B, D, E$. Since C and D are interchangeable in the first figure in Fig. 3.70, this case is the same as the case (ii).

(v) $G \sim A, B, D, F$. Since C and D, E and F are interchangeable in the first figure in Fig. 3.70, this case is the same as the case (ii).

(vi) $G \sim A, B, E, F$. The vertex F has three remaining hands, so we consider how the hands of F connect. There are ten cases; see Fig. 3.75.

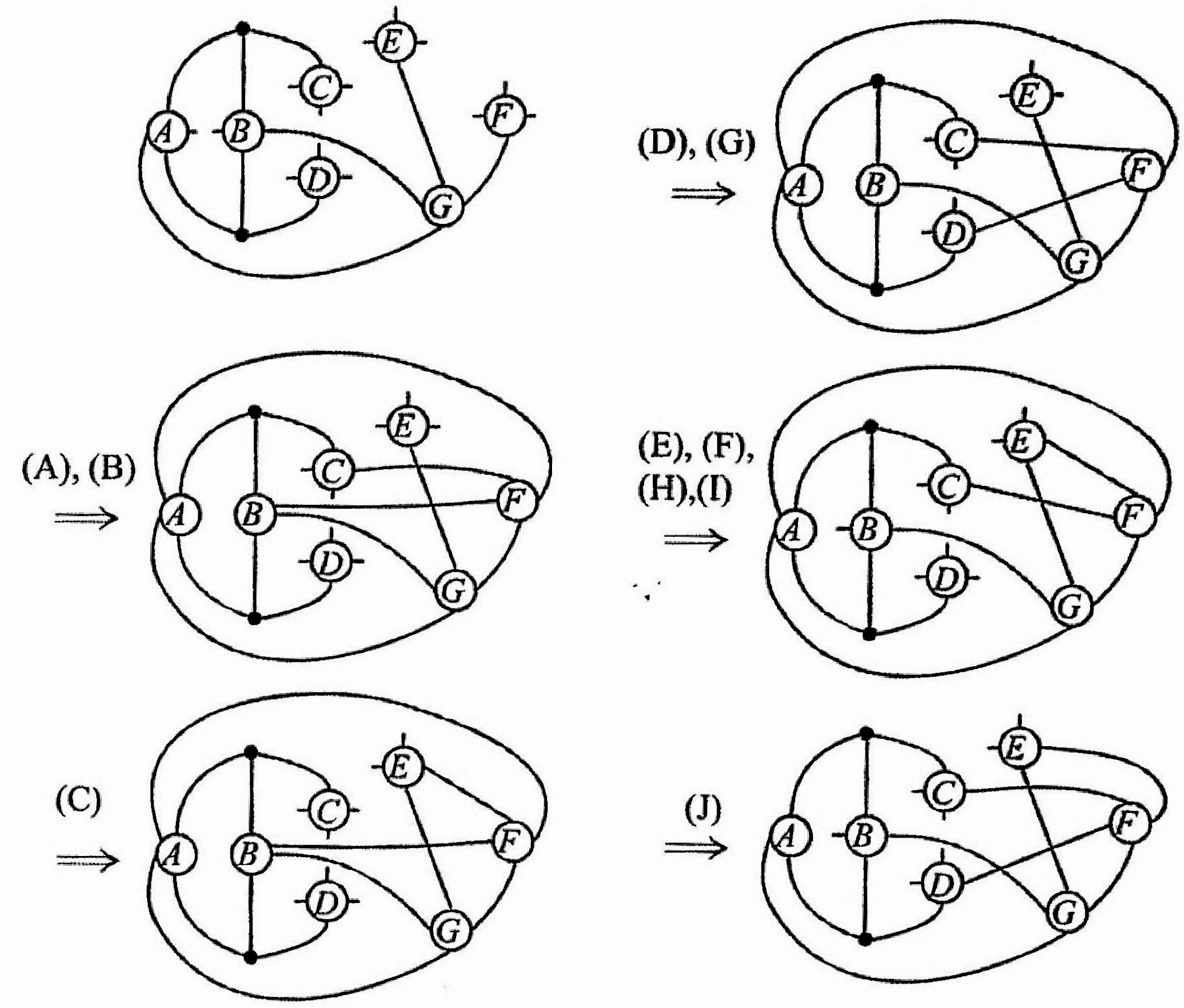


FIG. 3.75. $t = 7$ (b) (vi).

(A) $F \sim A, B, C$. Then $C \sim D, E, D \sim E$. This gives a graph which does not satisfy the condition (P2); see Fig. 3.76.

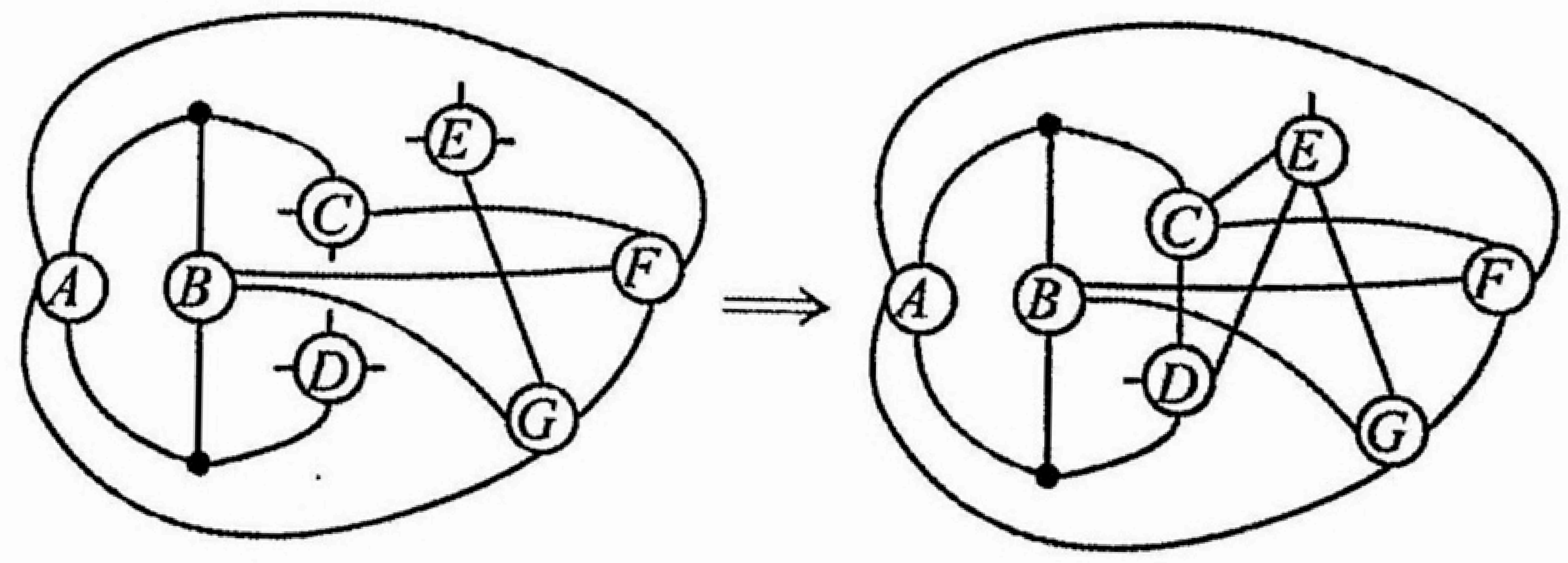


FIG. 3.76. $t = 7$ (b) (vi) (A).

(B) $F \sim A, B, D$. Since C and D are interchangeable in the first figure in Fig. 3.75, this case is the same as the case (A).

(C) $F \sim A, B, E$. Then $C \sim D, E, D \sim E$. This gives a graph which does not satisfy the condition (P2); see Fig. 3.77.

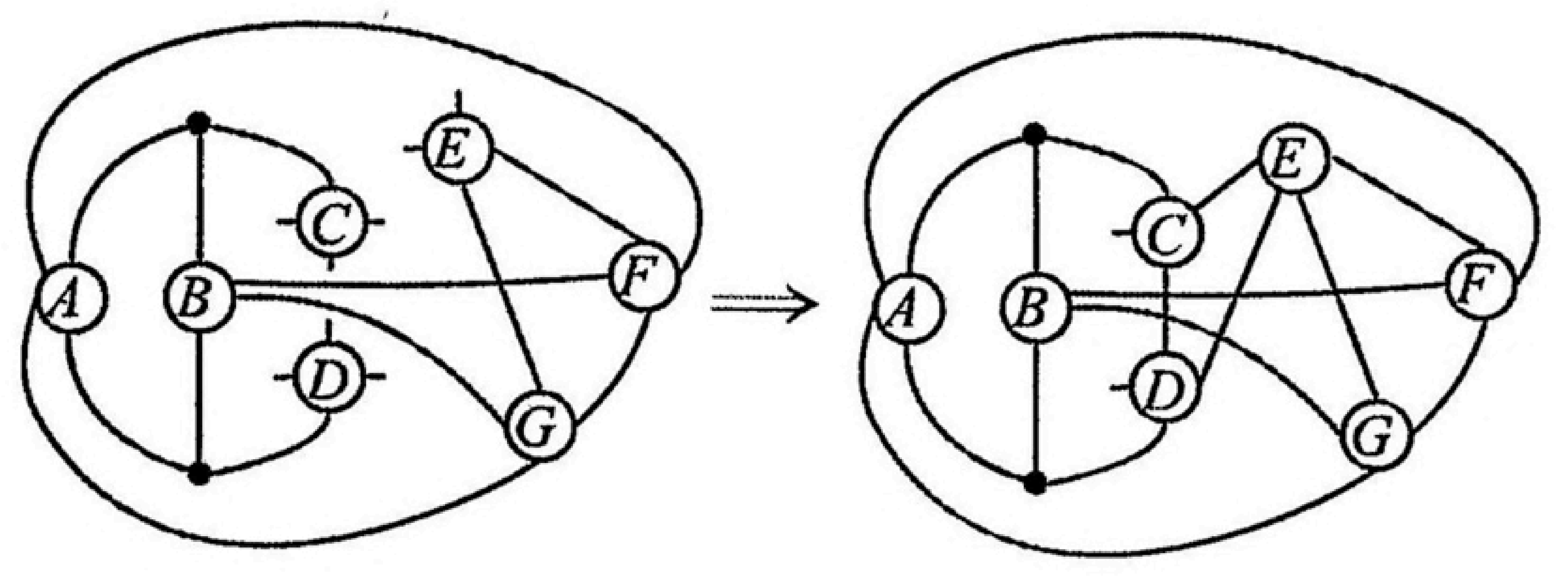


FIG. 3.77. $t = 7$ (b) (vi) (C).

(D) $F \sim A, C, D$. This gives a graph containing $K_{3,3}$, and so it does not satisfy the condition (P5); see Fig. 3.78.

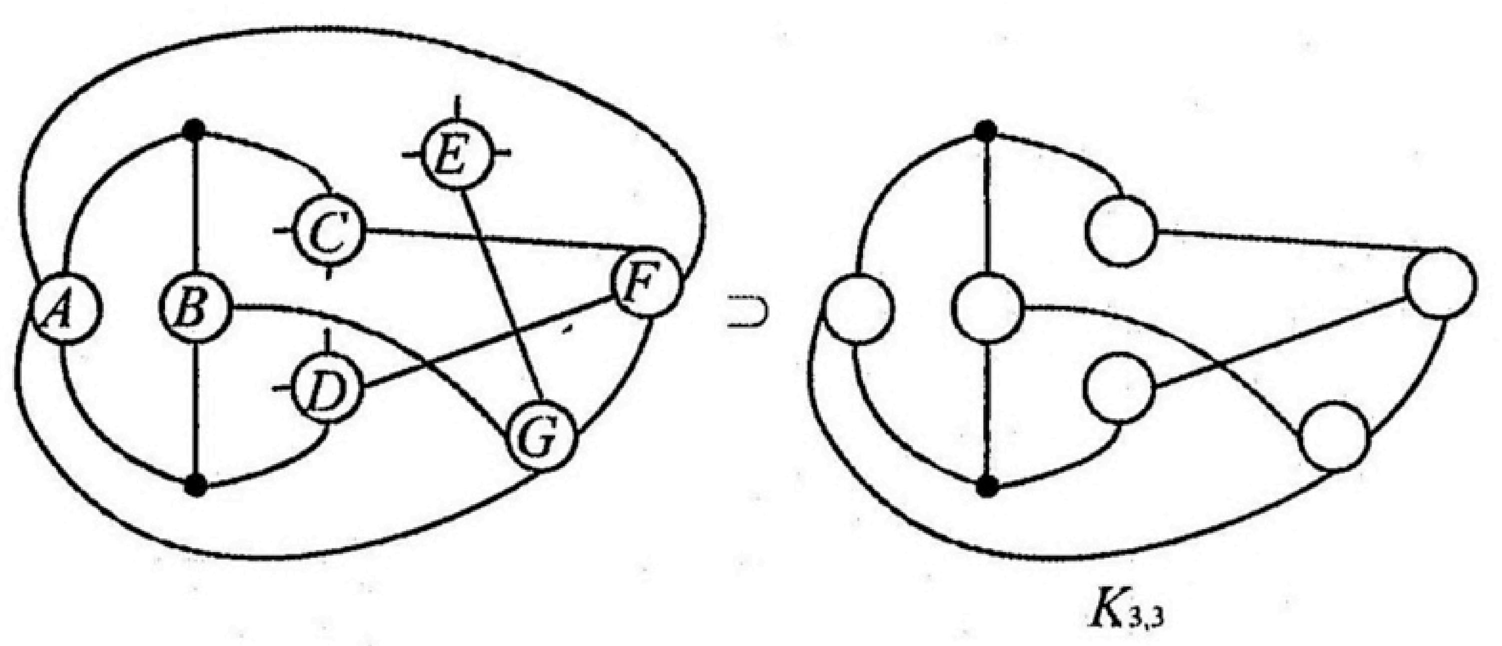


FIG. 3.78. $t = 7$ (b) (vi) (D).

(E) $F \sim A, C, E$. Then $D \sim B, C, E$. This gives a graph containing $K_{3,3}$, and so it does not satisfy the condition (P5); see Fig. 3.79.

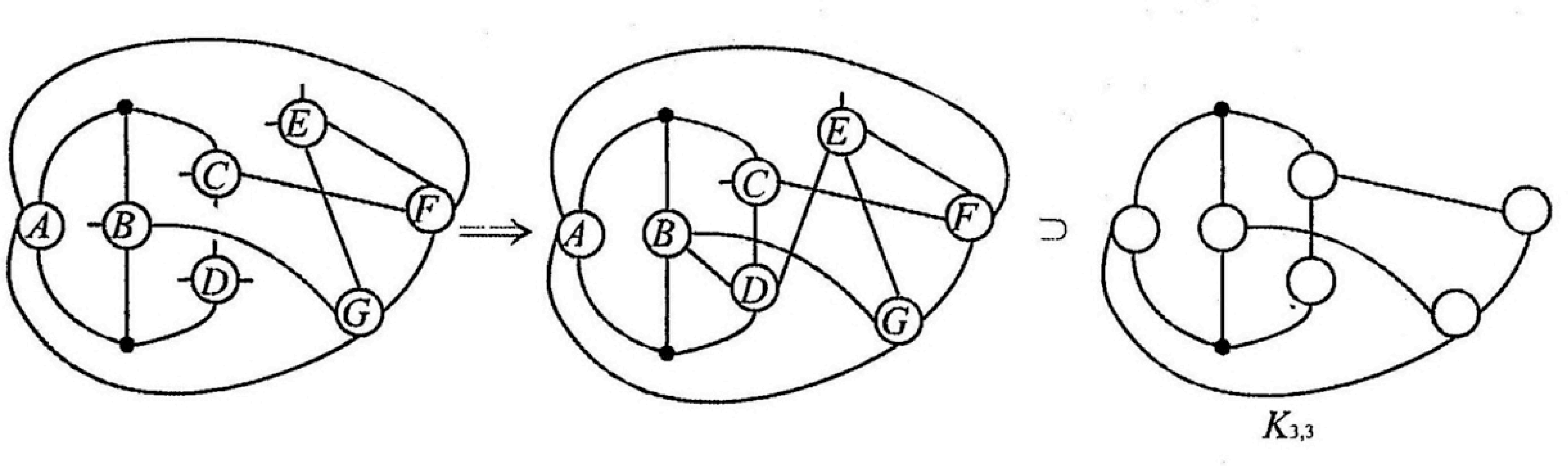


FIG. 3.79. $t = 7$ (b) (vi) (E).

(F) $F \sim A, D, E$. Since C and D are interchangeable in the first figure in Fig. 3.75, this case is the same as the case (E).
(G) $F \sim B, C, D$. Since A and B are interchangeable in the first figure in Fig. 3.75, this case is the same as the case (D).
(H) $F \sim B, C, E$. Since A and B are interchangeable in the first figure in Fig. 3.75, this case is the same as the case (E).
(I) $F \sim B, D, E$. Since A and B, C and D are interchangeable in the first figure in Fig. 3.75, this case is the same as the case (E).

(J) $F \sim C, D, E$. This gives a graph containing $K_{3,3}$, and so it does not satisfy the condition (P5); see Fig. 3.80.

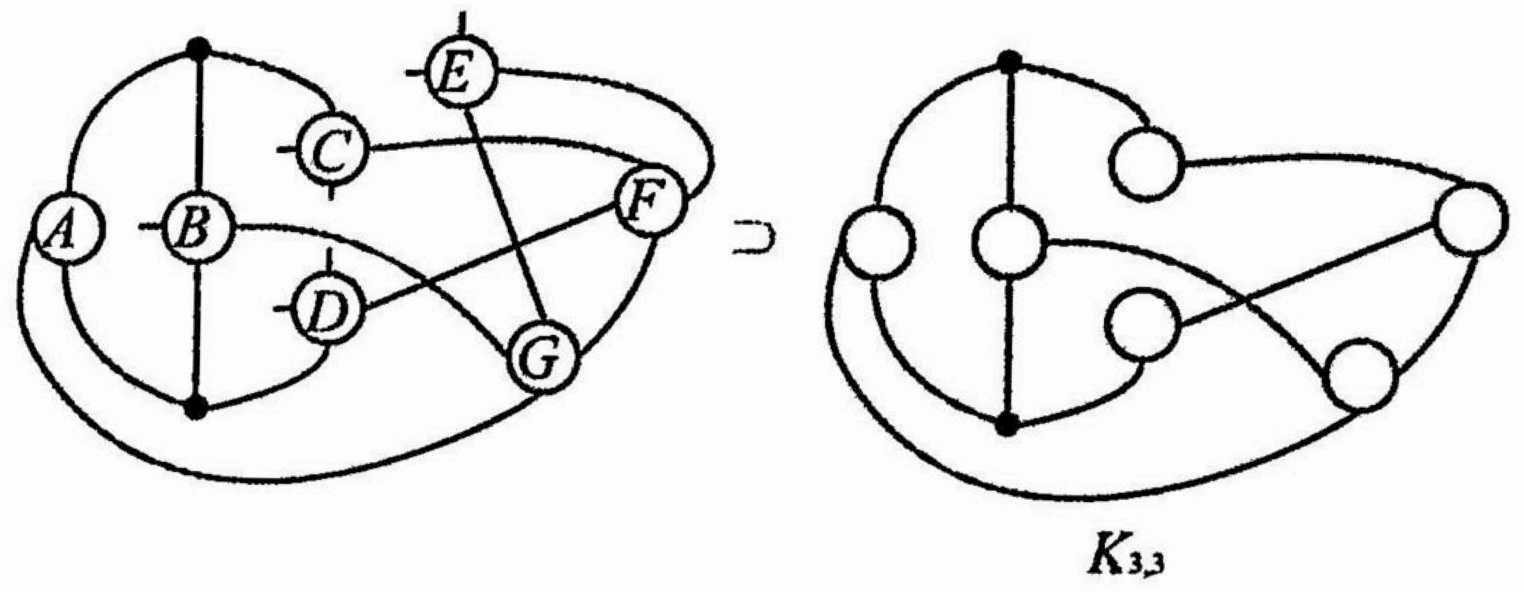


FIG. 3.80. $t = 7$ (b) (vi) (J).

(vii) $G \sim A, C, D, E$. The vertex F has three remaining hands, so we consider how the hands of F connect. There are five cases; see Fig. 3.81.

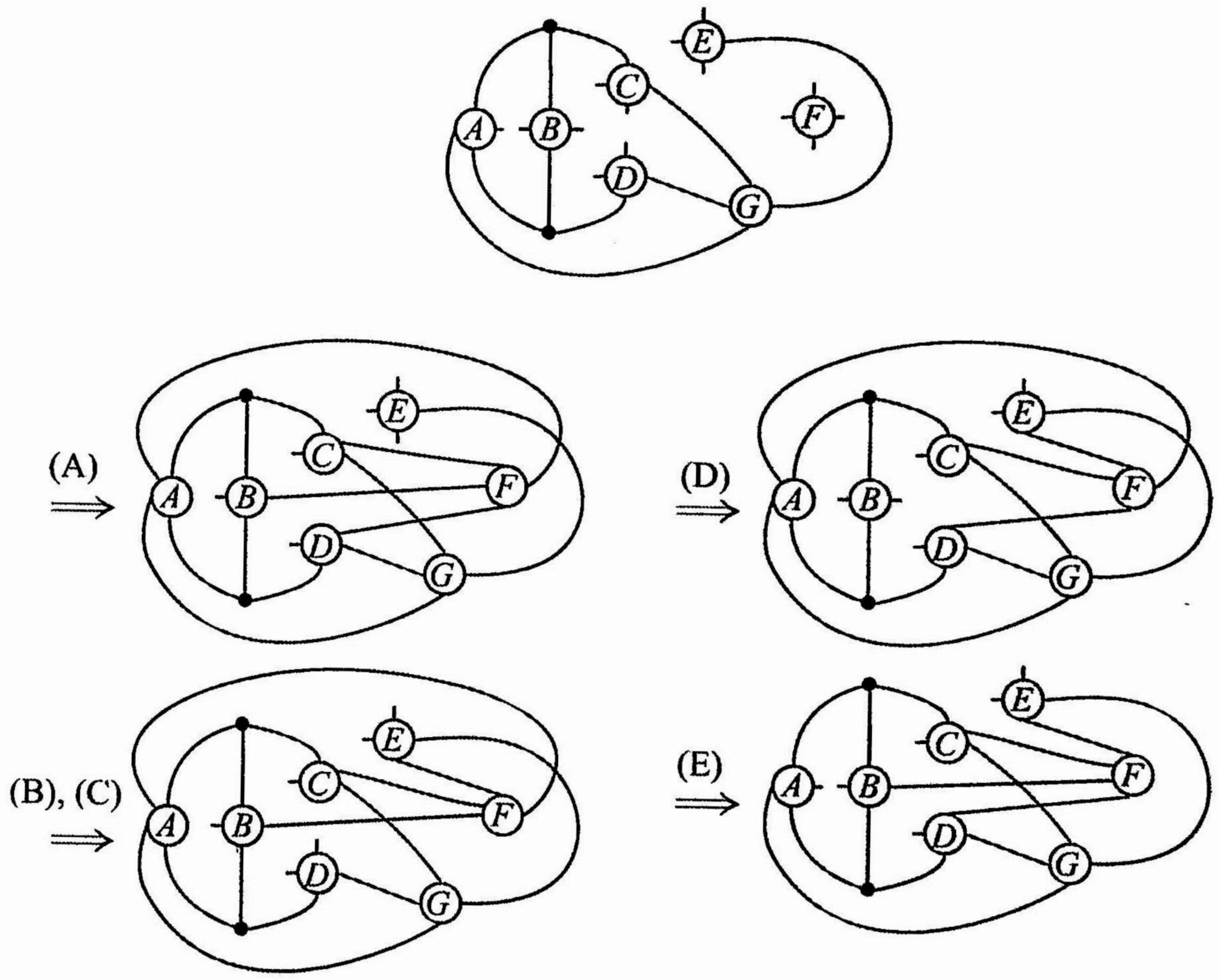


FIG. 3.81. $t = 7$ (b) (vii).

(A) $F \sim A, B, C, D$. This gives a graph containing $K_{3,3}$, and so it does not satisfy the condition (P5); see Fig. 3.82.

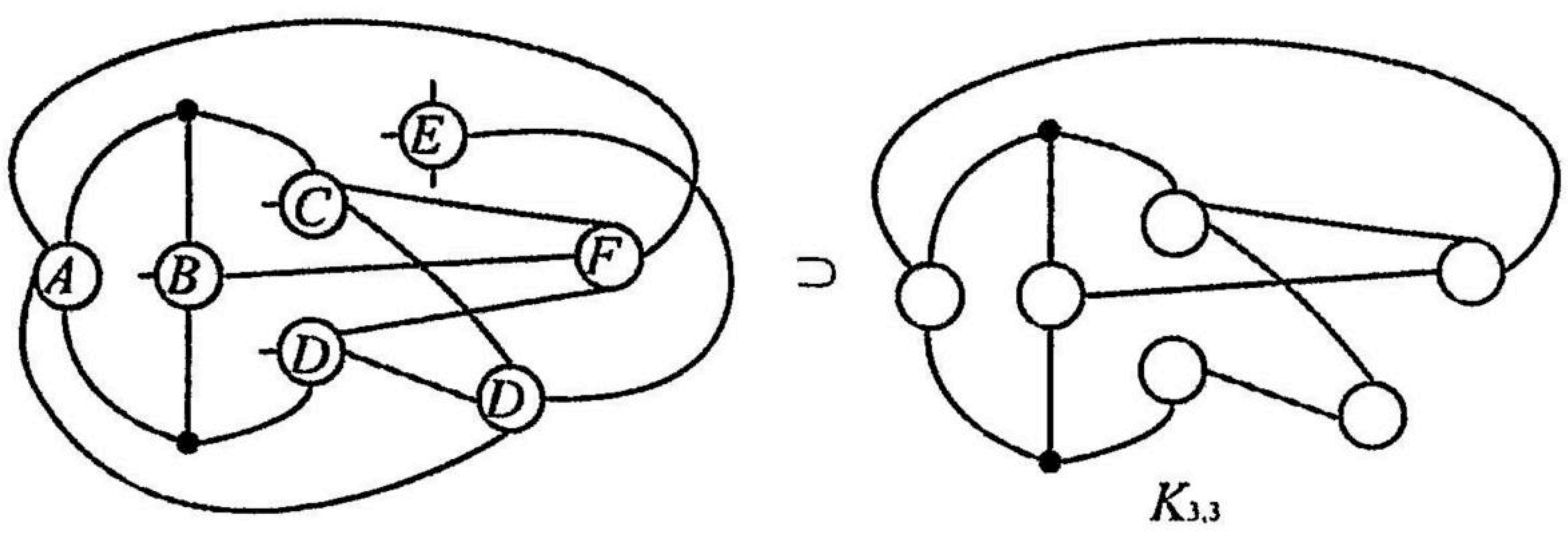


FIG. 3.82. $t = 7$ (b) (vii) (A).

(B) $F \sim A, B, C, E$. This gives a graph containing $K_{3,3}$, and so it does not satisfy the condition (P5); see Fig. 3.83.

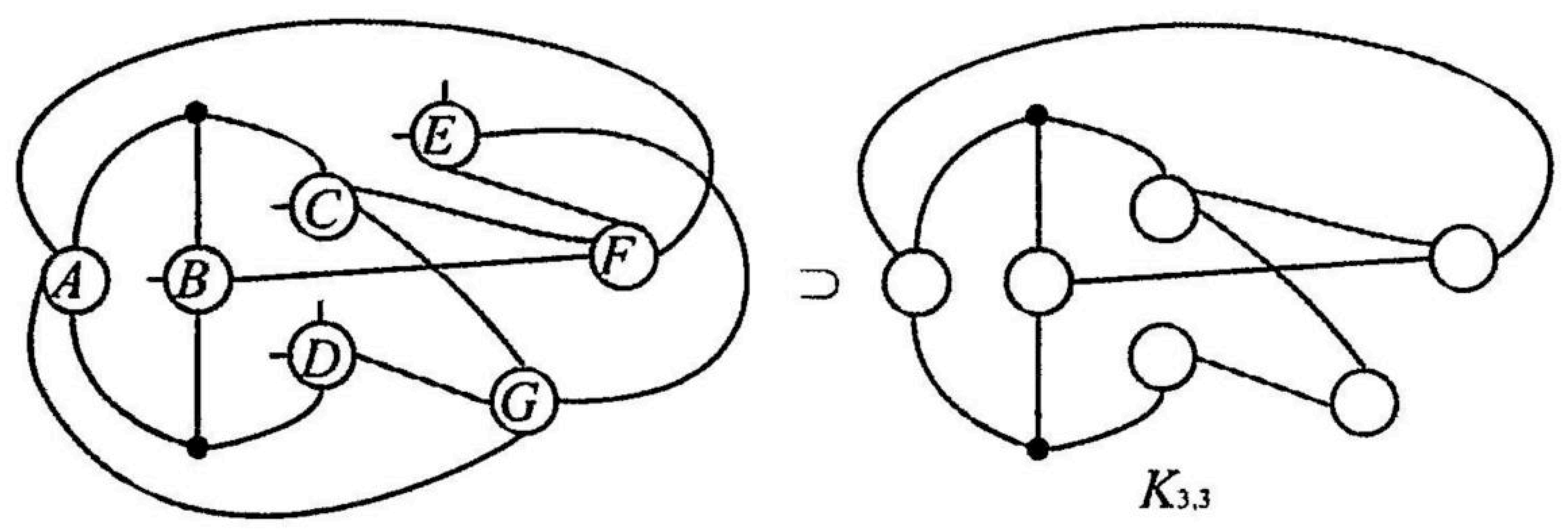


FIG. 3.83. $t = 7$ (b) (vii) (B).

(C) $F \sim A, B, D, E$. Since C and D are interchangeable in the first figure in Fig. 3.81, this case is the same as the case (B).

(D) $F \sim A, C, D, E$. This gives a graph containing $K_{3,3}$, and so it does not satisfy the condition (P5); see Fig. 3.84.

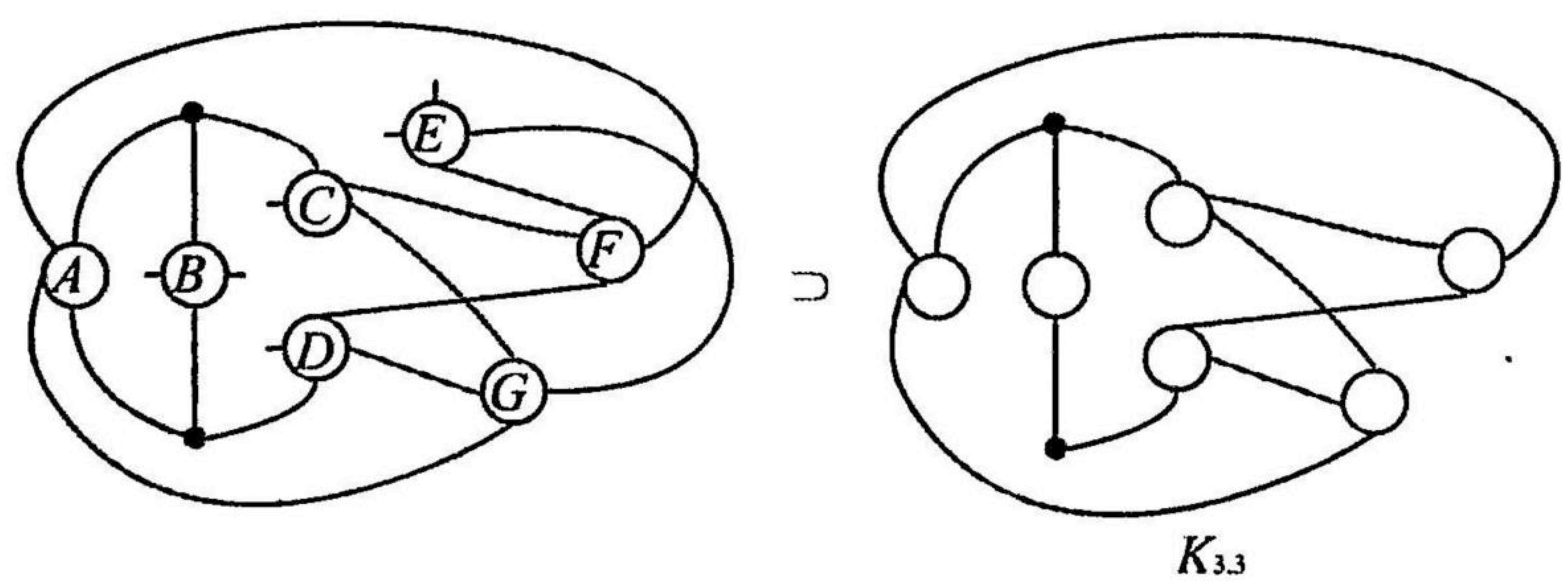


FIG. 3.84. $t = 7$ (b) (vii) (D).

(E) $F \sim B, C, D, E$. The vertex E has two remaining hands, so we consider how the hands of E connect. There are six cases; see Fig. 3.85.

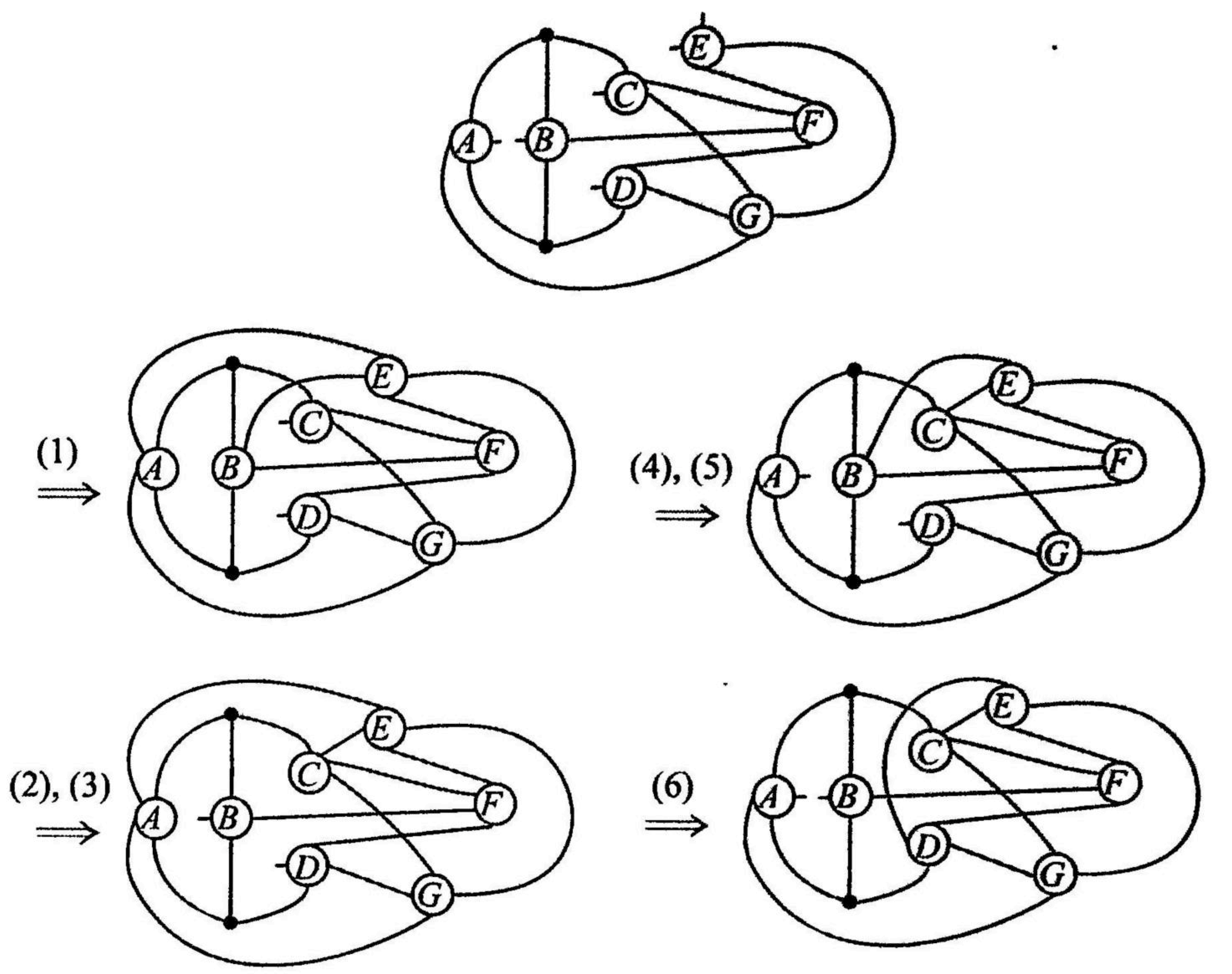


FIG. 3.85. $t = 7$ (b) (vii) (E).

(1) $E \sim A, B$. This gives a graph containing $K_{3,3}$, and so it does not satisfy the condition (P5); see Fig. 3.86.

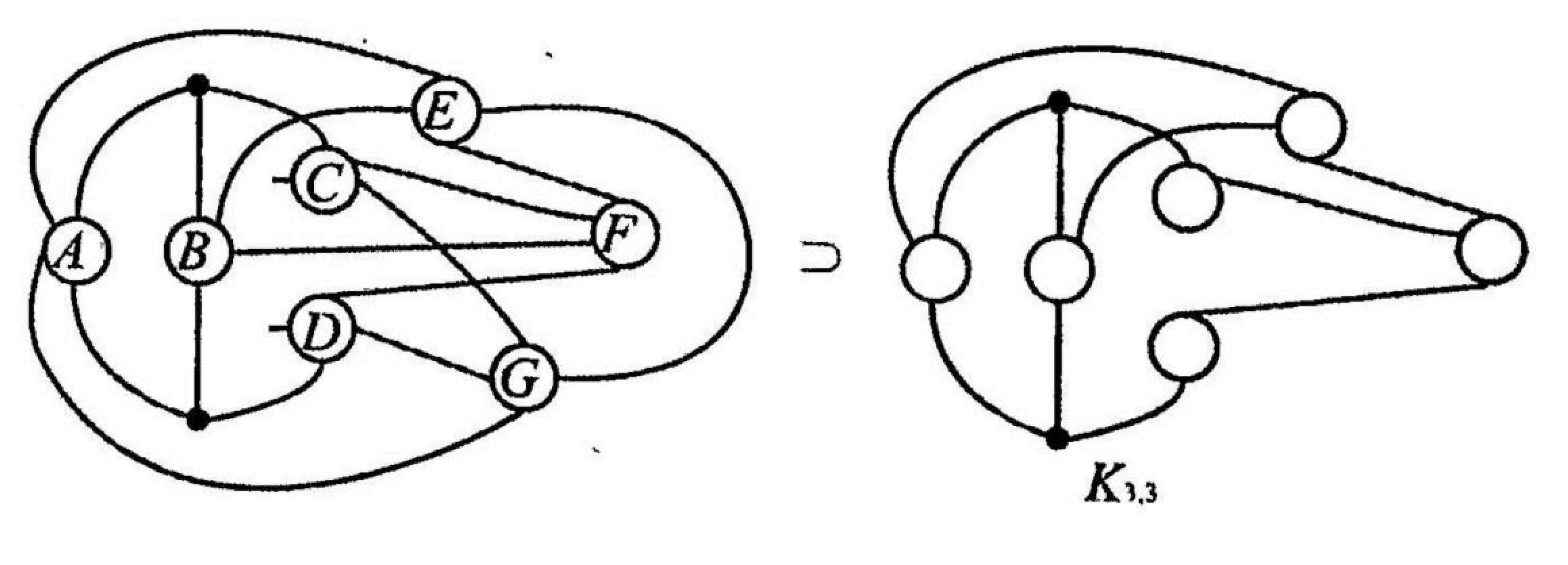


FIG. 3.86. $t = 7$ (b) (vii) (E) (1).

(2) $E \sim A, C$. This gives a graph containing $K_{3,3}$, and so it does not satisfy the condition (P5); see Fig. 3.87.

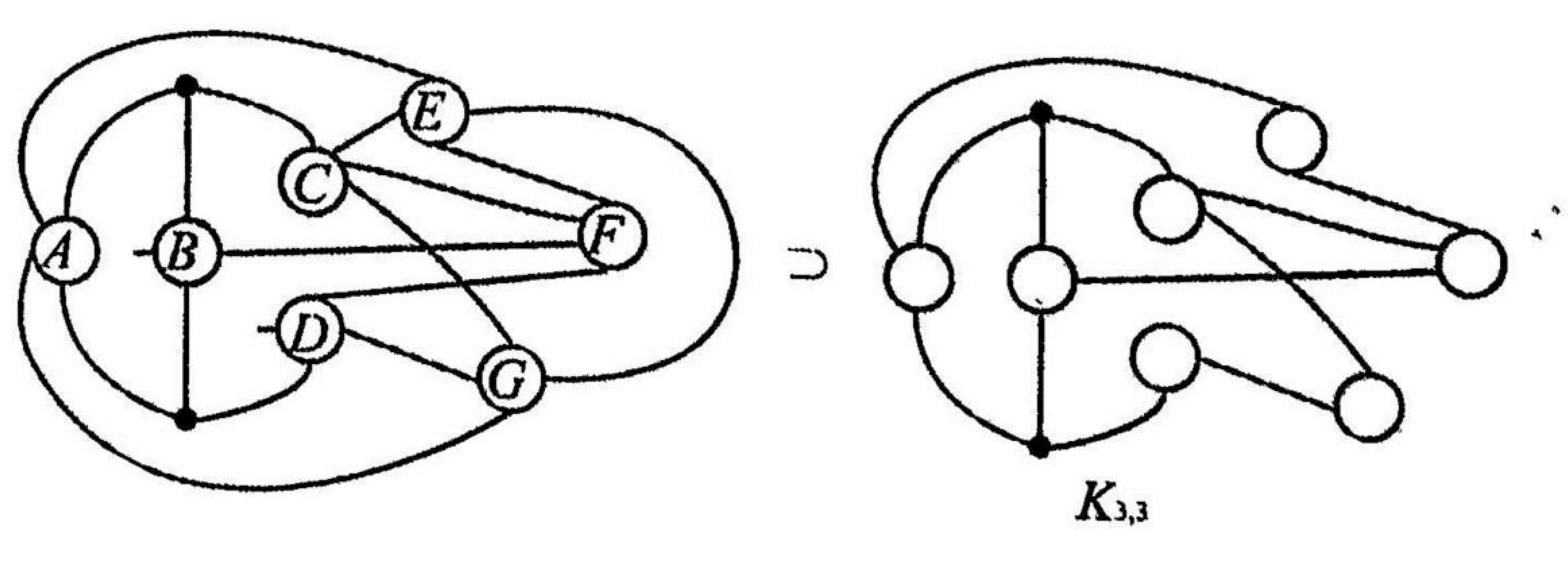


FIG. 3.87. $t = 7$ (b) (vii) (E) (2).

(3) $E \sim A, D$. Since C and D are interchangeable in the first figure in Fig. 3.85, this case is the same as the case (2).

(4) $E \sim B, C$. This gives a graph containing $K_{3,3}$, and so it does not satisfy the condition (P5); see Fig. 3.88.

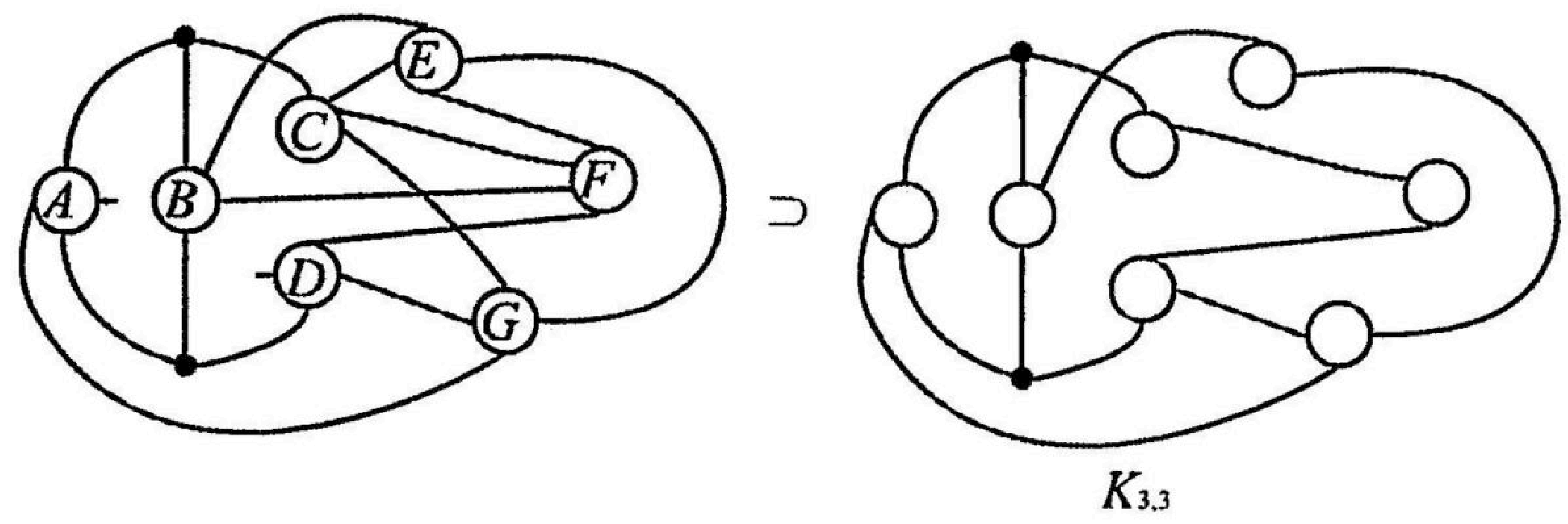


FIG. 3.88. $t = 7$ (b) (vii) (E) (4).

(5) $E \sim B, D$. Since C and D are interchangeable in the first figure in Fig. 3.85, this case is the same as the case (4).

(6) $E \sim C, D$. Then $A \sim B$, and we obtain 7^1_*; see Fig. 3.89.

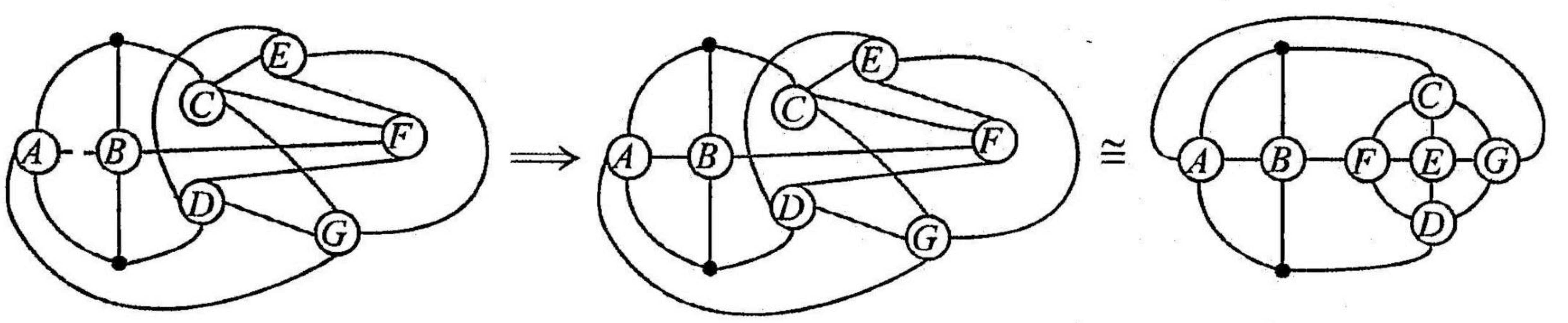


FIG. 3.89. $t = 7$ (b) (vii) (E) (6).

(viii) $G \sim A, C, D, F$. Since E and F are interchangeable in the first figure in Fig. 3.68, this case is the same as the case (vii).

(ix) $G \sim A, C, E, F$. The vertex F has three remaining hands, so we consider how the hands of F connect. There are ten cases; see Fig. 3.90.

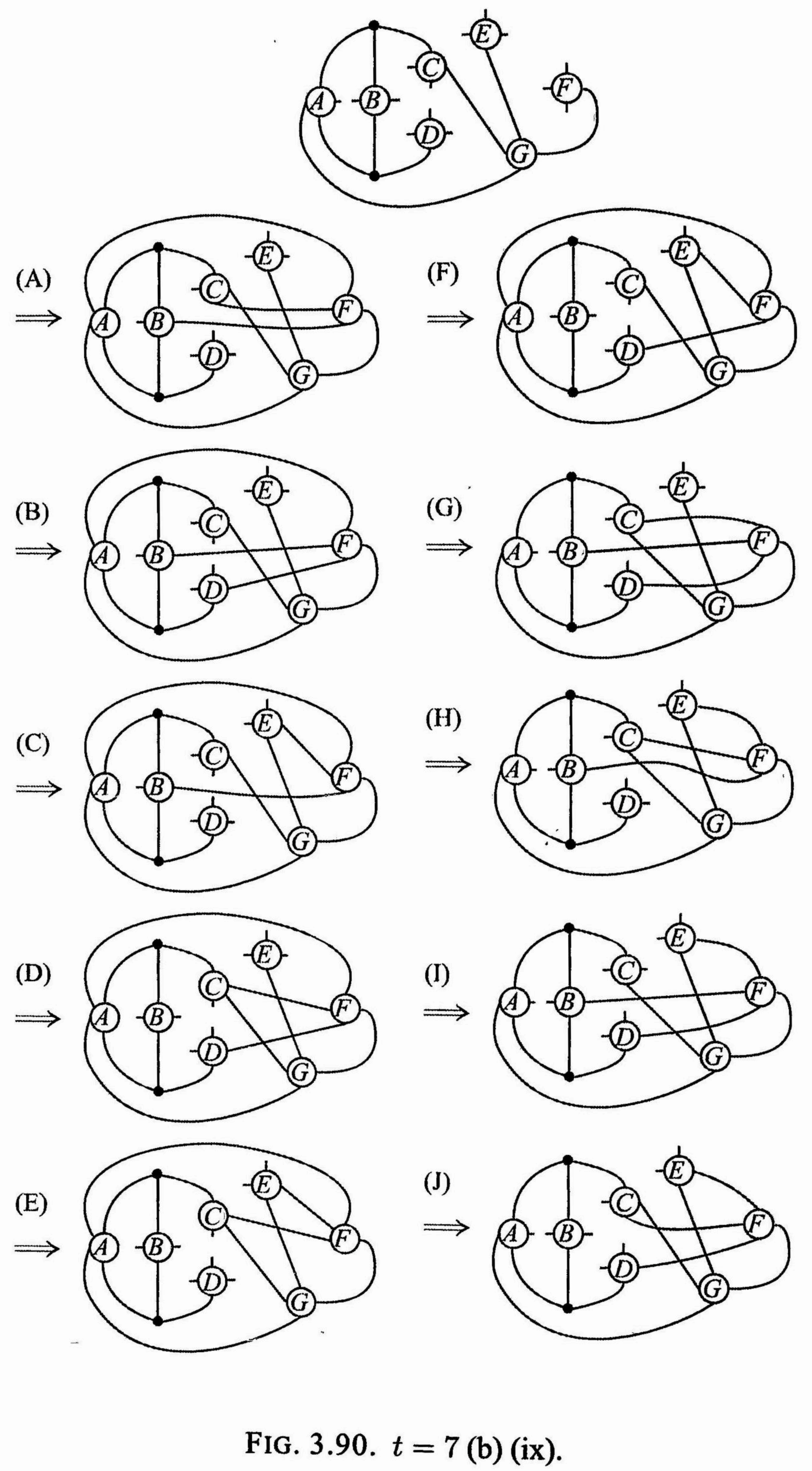


FIG. 3.90. $t = 7$ (b) (ix).

(A) $F \sim A, B, C$. Then $E \sim B, C, D$. This gives a graph having a loop at D, and so it does not satisfy the condition (P1); see Fig. 3.91.

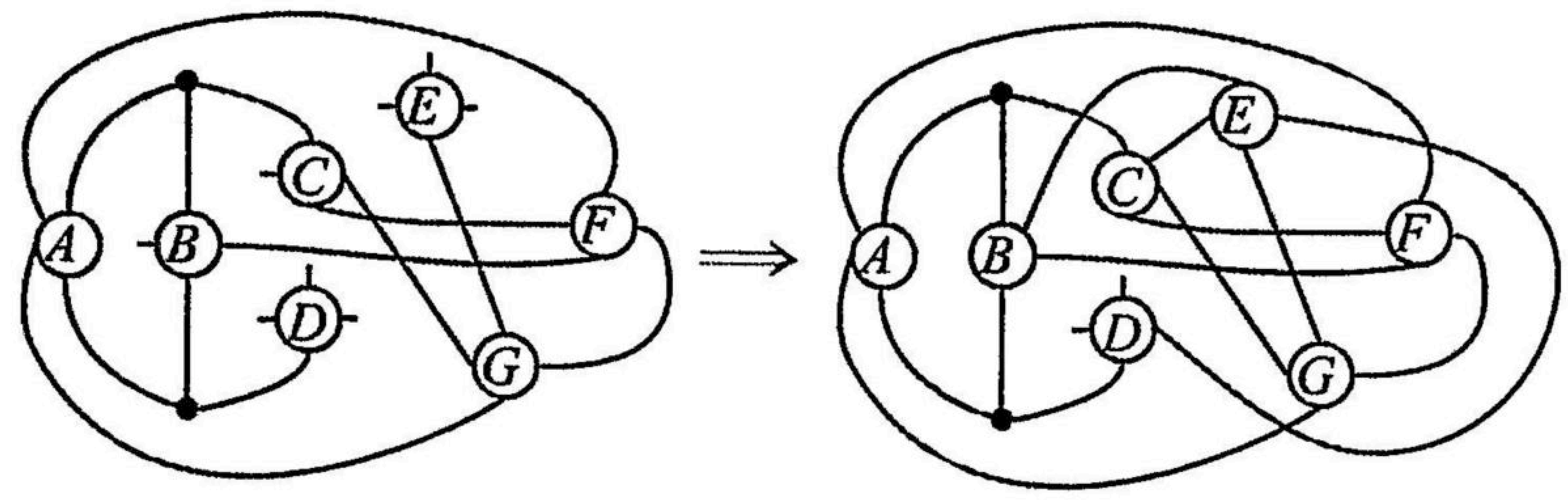


FIG. 3.91. $t = 7$ (b) (ix) (A).

(B) $F \sim A, B, D$. Then $E \sim B, C, D$. This gives a graph containing $K_{3,3}$, and so it does not satisfy the condition (P5); see Fig. 3.92.

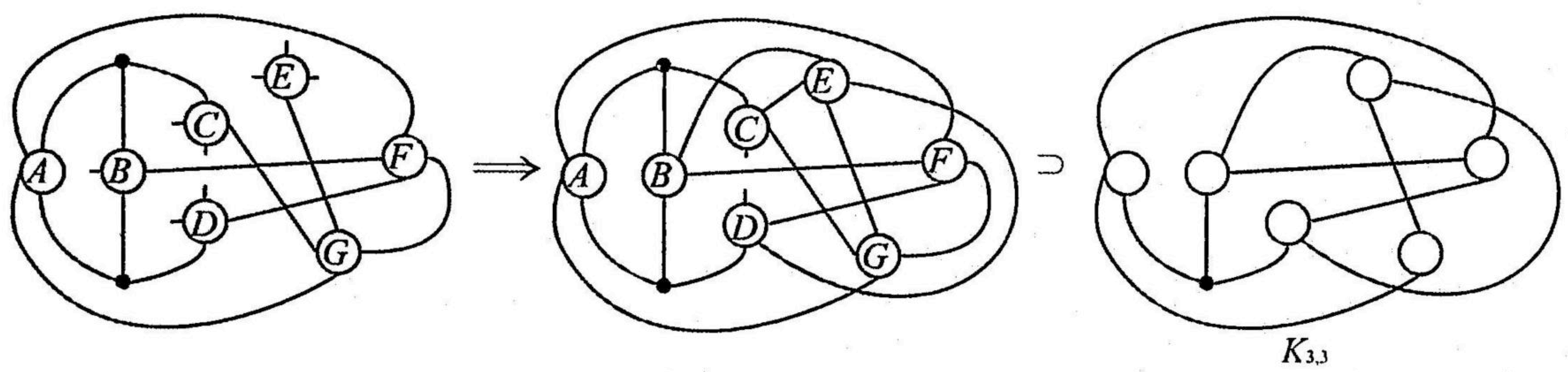


FIG. 3.92. $t = 7$ (b) (ix) (B).

(C) $F \sim A, B, E$. Then $D \sim B, C, E$. This gives a graph containing $K_{3,3}$, and so it does not satisfy the condition (P5); see Fig. 3.93.

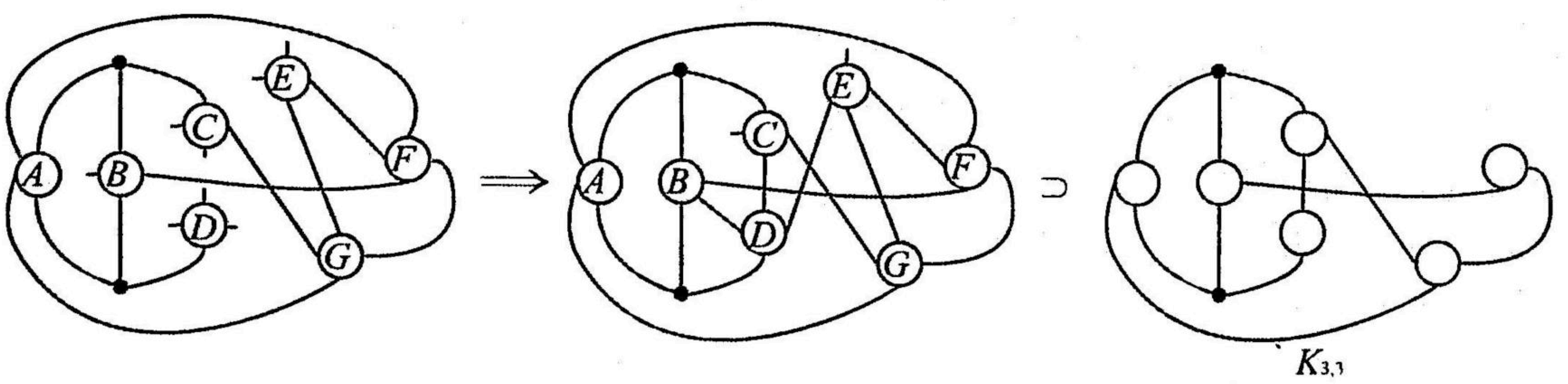


FIG. 3.93. $t = 7$ (b) (ix) (C).

(D) $F \sim A, C, D$. Then $E \sim B, C, D$. This gives a graph containing $K_{3,3}$, and so it does not satisfy the condition (P5); see Fig. 3.94.

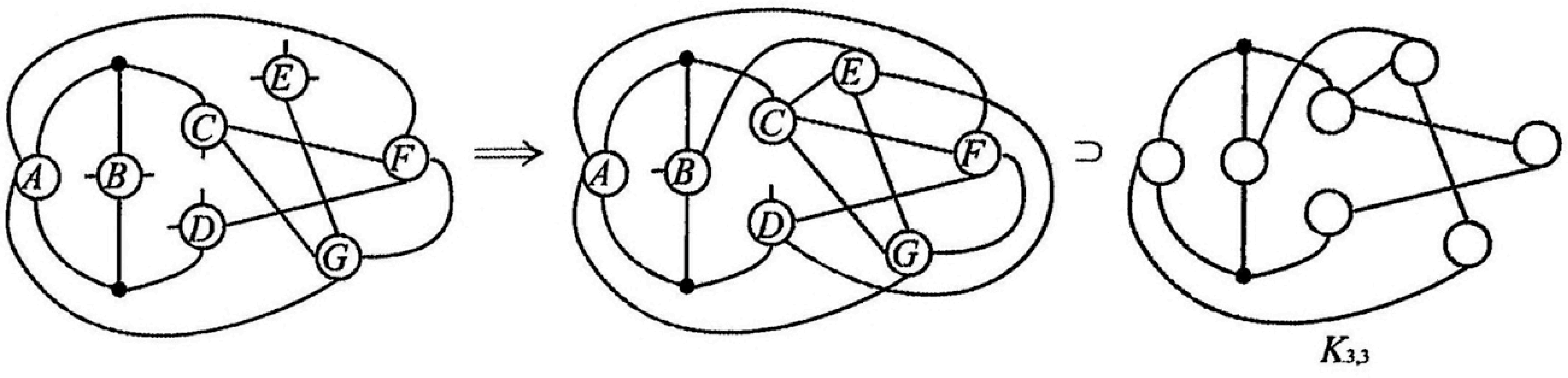


FIG. 3.94. $t = 7$ (b) (ix) (D).

(E) $F \sim A, C, E$. Then $D \sim B, C, E$. This gives a graph containing $K_{3,3}$, and so it does not satisfy the condition (P5); see Fig. 3.95.

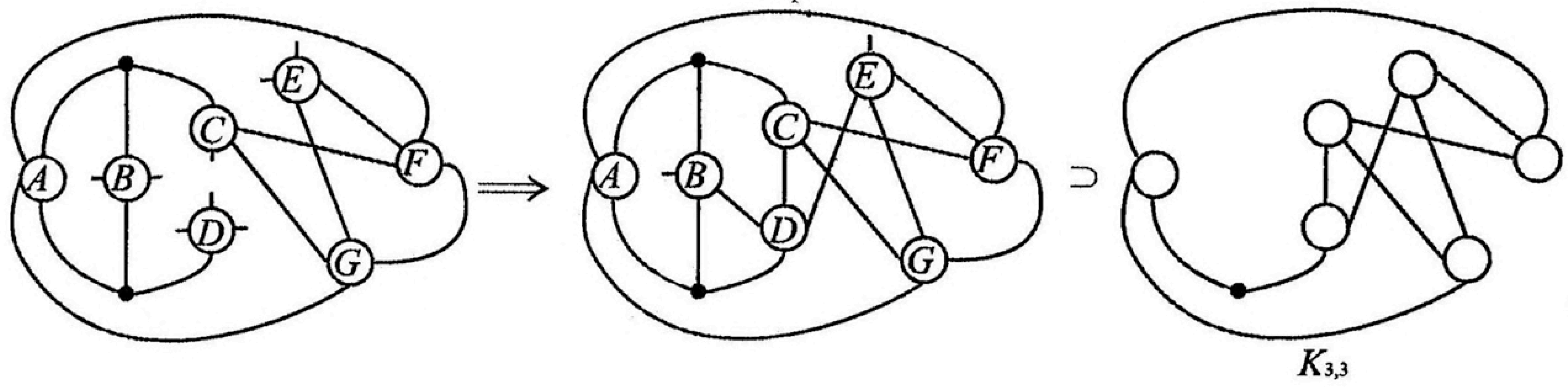


FIG. 3.95. $t = 7$ (b) (ix) (E).

(F) $F \sim A, D, E$. The vertex E has two remaining hands, so we consider how the hands of E connect. There are three cases; see Fig. 3.96.

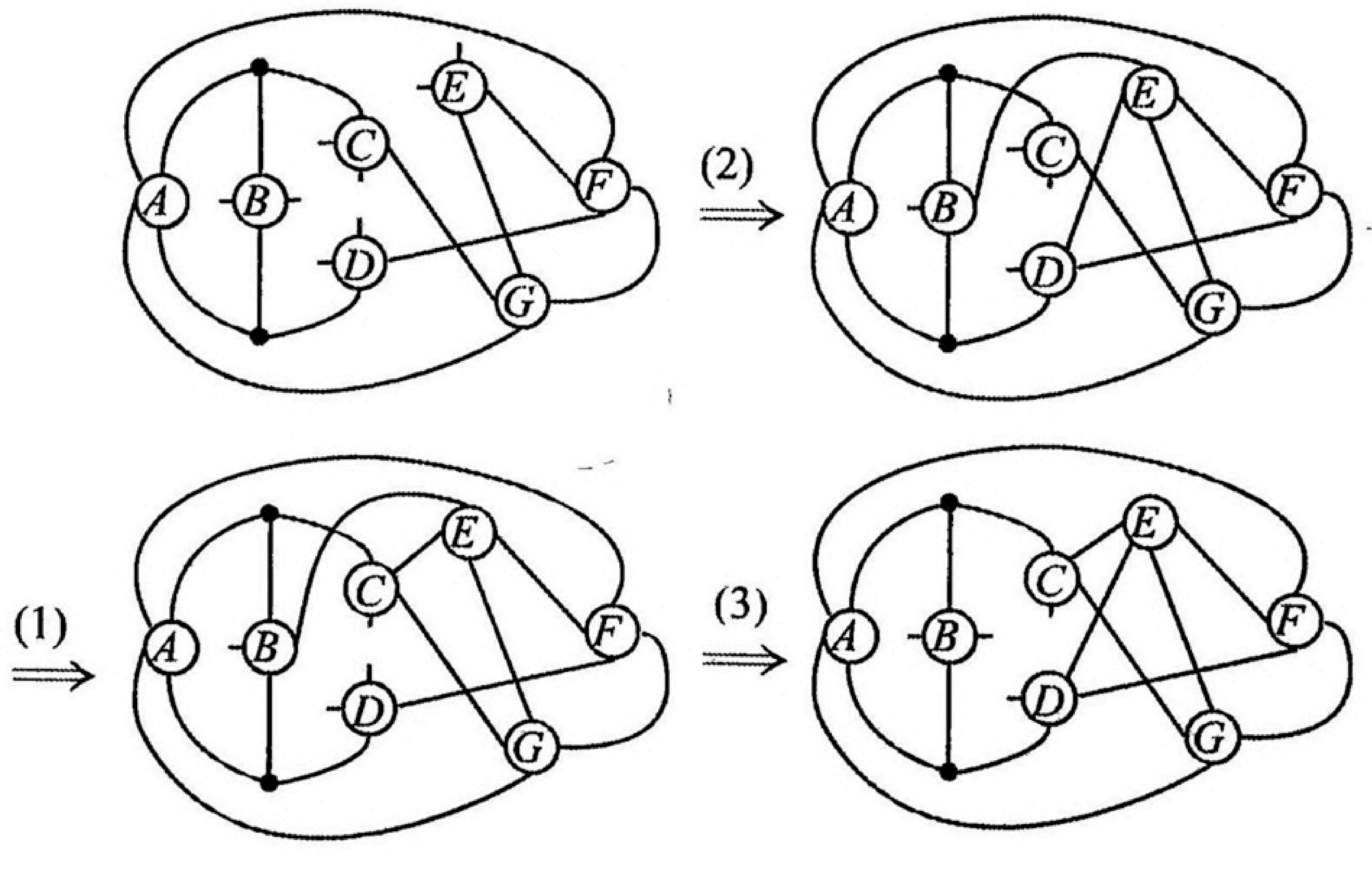


FIG. 3.96. $t = 7$ (b) (ix) (F).

(1) $E \sim B, C$. Then $D \sim B, C$. This gives a graph containing $K_{3,3}$, and so it does not satisfy the condition (P5); see Fig. 3.97.

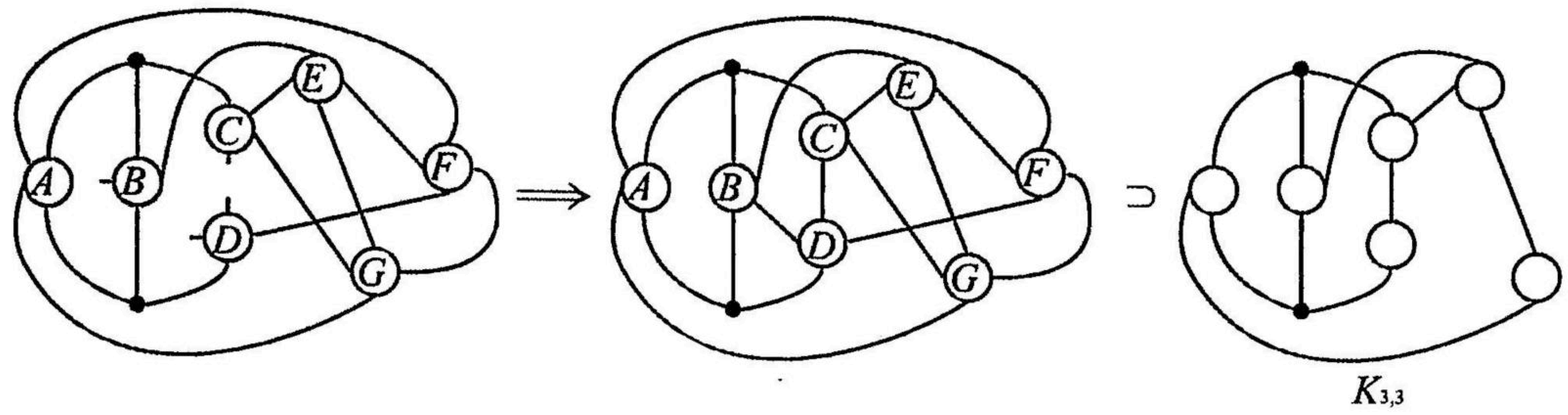


FIG. 3.97. $t = 7$ (b) (ix) (F) (1).

(2) $E \sim B, D$. Then $C \sim B, D$. This gives a graph containing $K_{3,3}$, and so it does not satisfy the condition (P5); see Fig. 3.98.

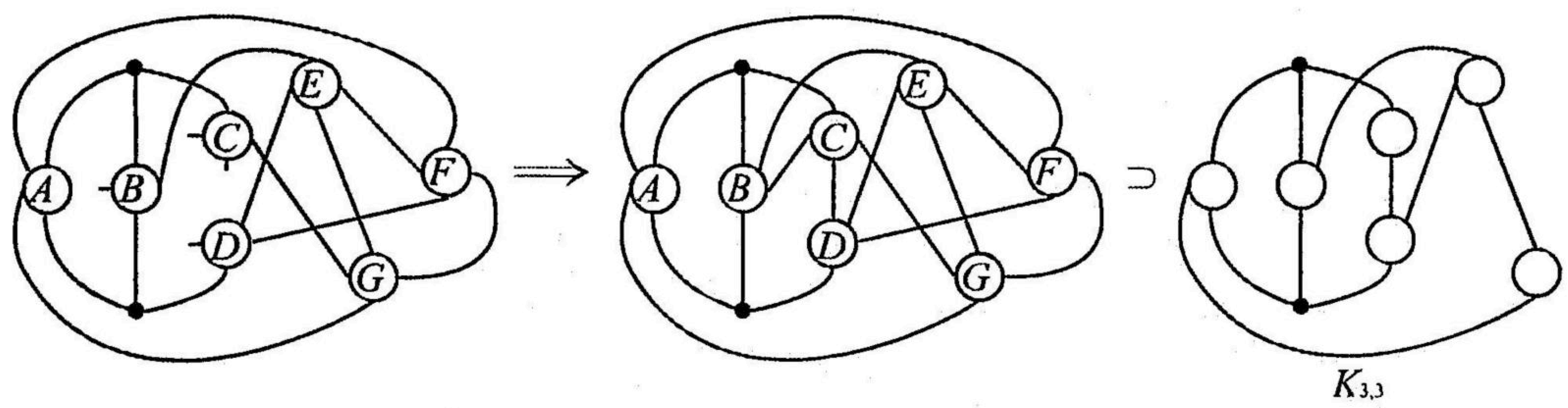


FIG. 3.98. $t = 7$ (b) (ix) (F) (2).

(3) $E \sim C, D$. Then $B \sim C, D$, and we obtain 7^2_*; see Fig. 3.99.

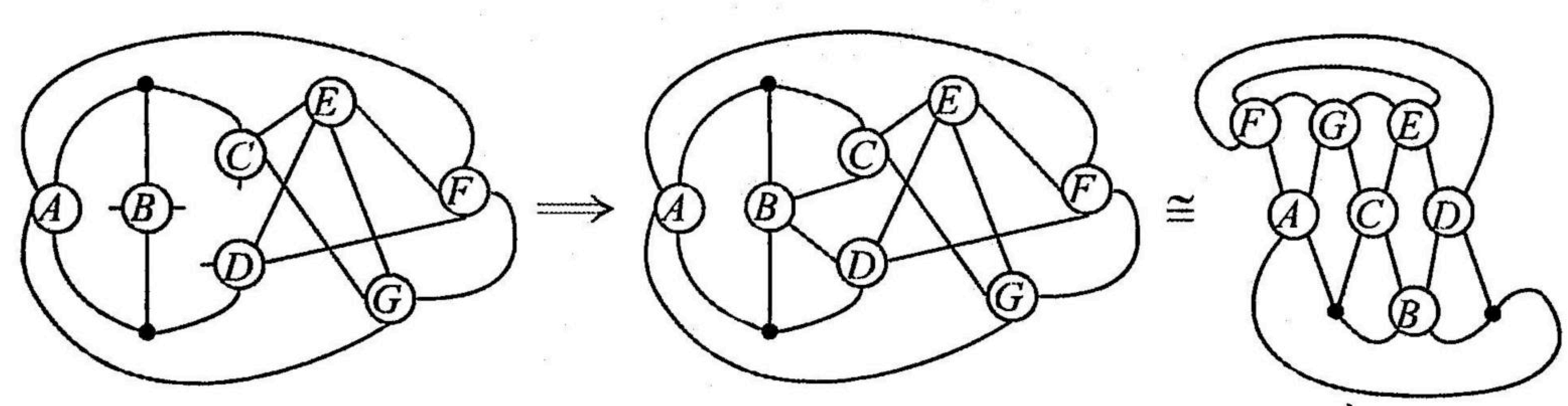


FIG. 3.99. $t = 7$ (b) (ix) (F) (3).

(G) $F \sim B, C, D$. The vertex E has three remaining hands, so we consider how the hands of E connect. There are four cases; see Fig. 3.100.

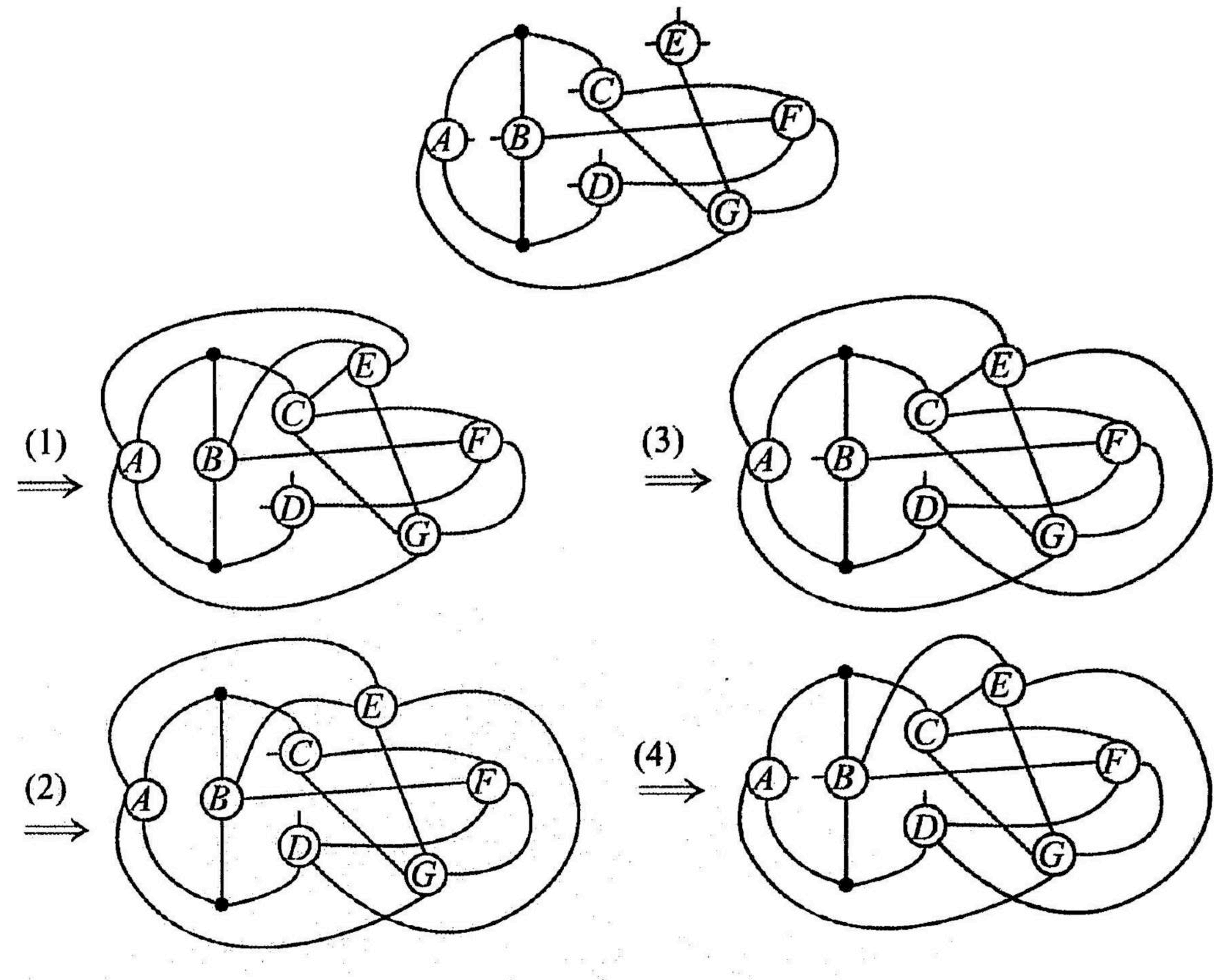


FIG. 3.100. $t = 7$ (b) (ix) (G).

(1) $E \sim A, B, C$. This gives a graph having a loop at D, and so it does not satisfy the condition (P1); see Fig. 3.100.
(2) $E \sim A, B, D$. This gives a graph containing $K_{3,3}$, and so it does not satisfy the condition (P5); see Fig. 3.101.

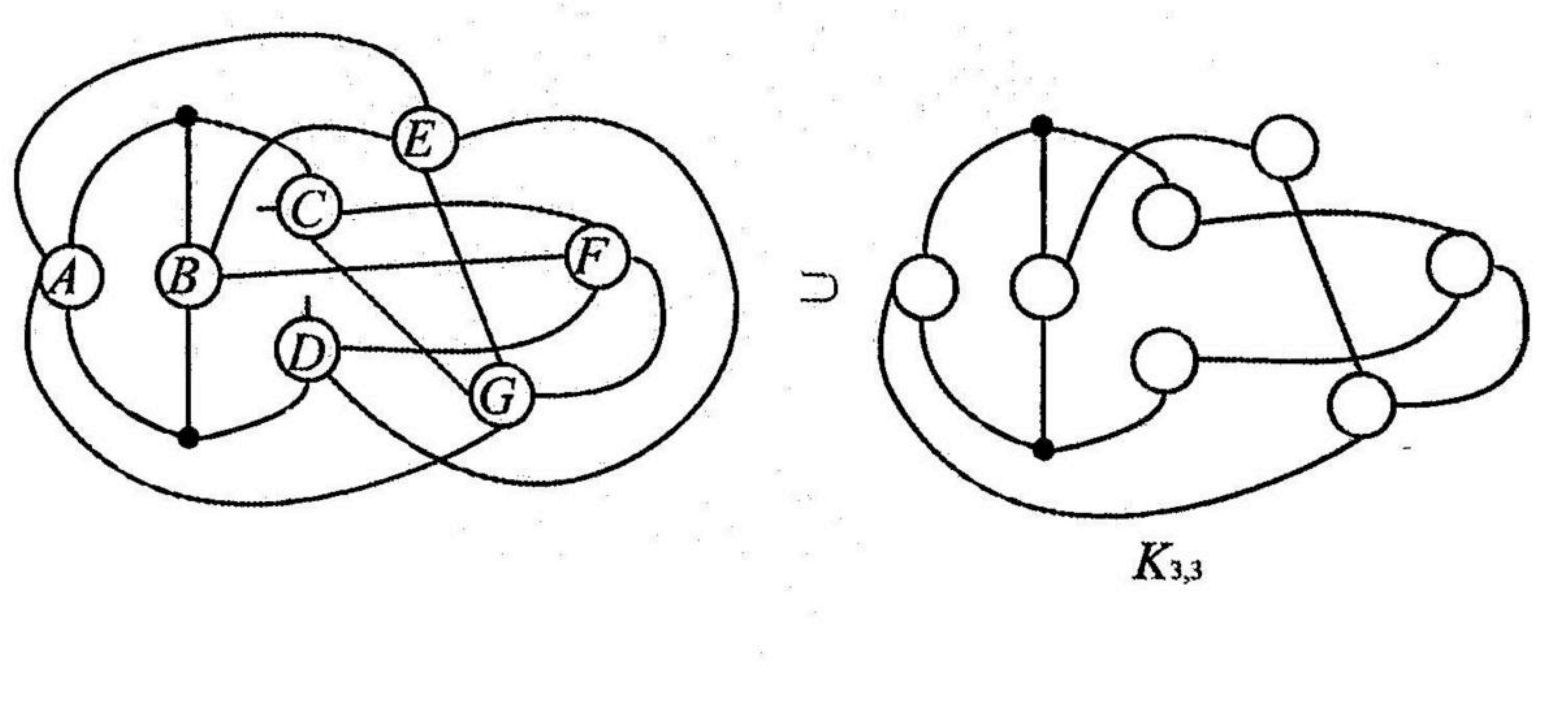


FIG. 3.101. $t = 7$ (b) (ix) (G) (2).

(3) $E \sim A, C, D$. This gives a graph containing $K_{3,3}$, and so it does not satisfy the condition (P5); see Fig. 3.102.

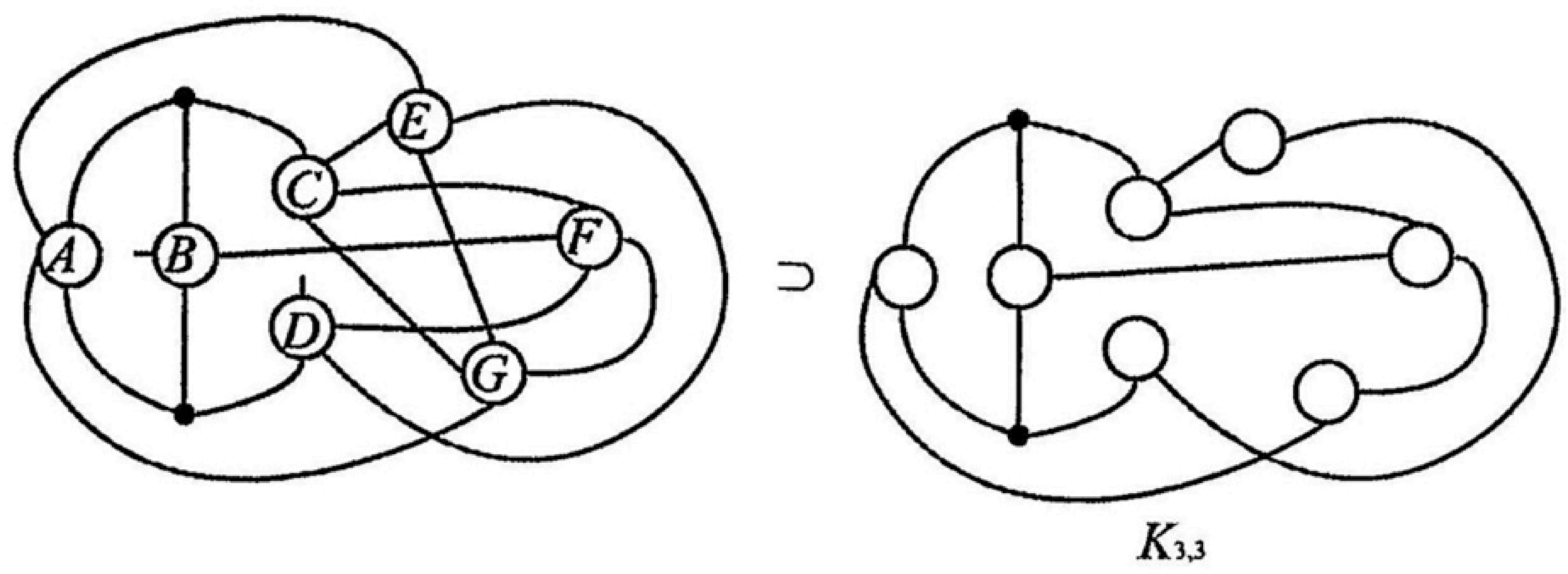


FIG. 3.102. $t = 7$ (b) (ix) (G) (3).

(4) $E \sim B, C, D$. This gives a graph containing $K_{3,3}$, and so it does not satisfy the condition (P5); see Fig. 3.103.

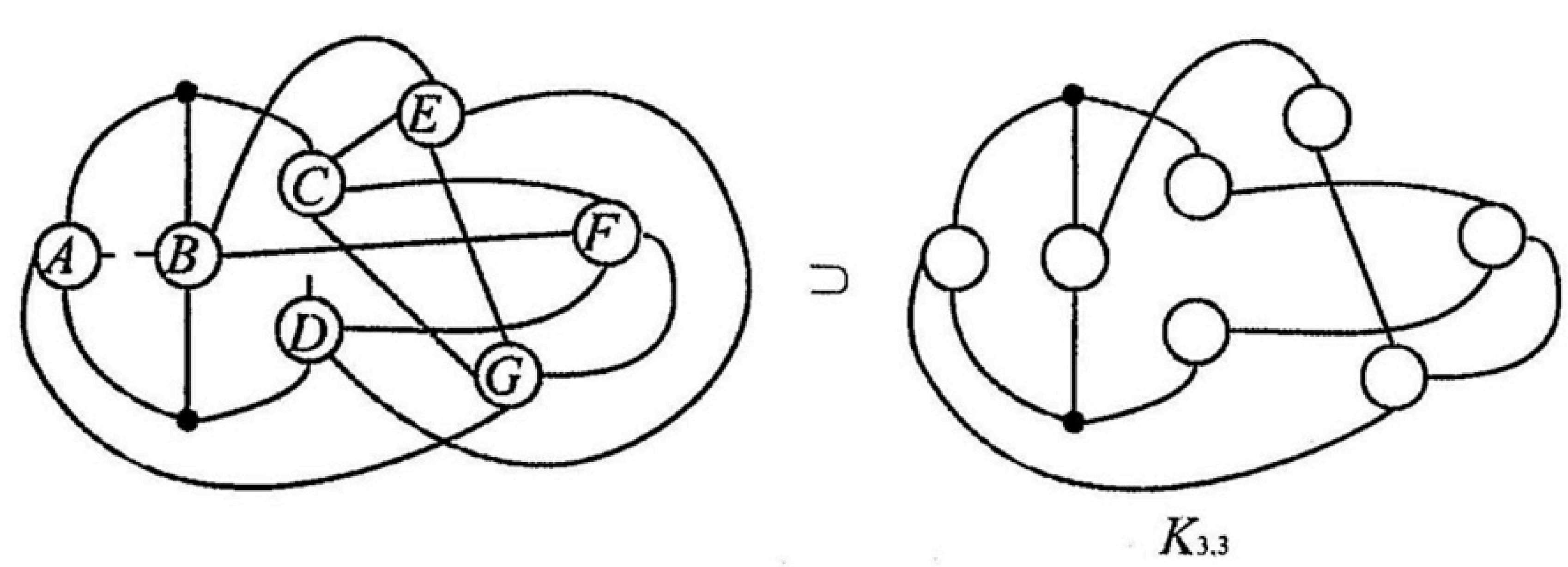


FIG. 3.103. $t = 7$ (b) (ix) (G) (4).

(H) $F \sim B, C, E$. The vertex D has three remaining hands, so we consider how the hands of D connect. There are four cases; see Fig. 3.104.

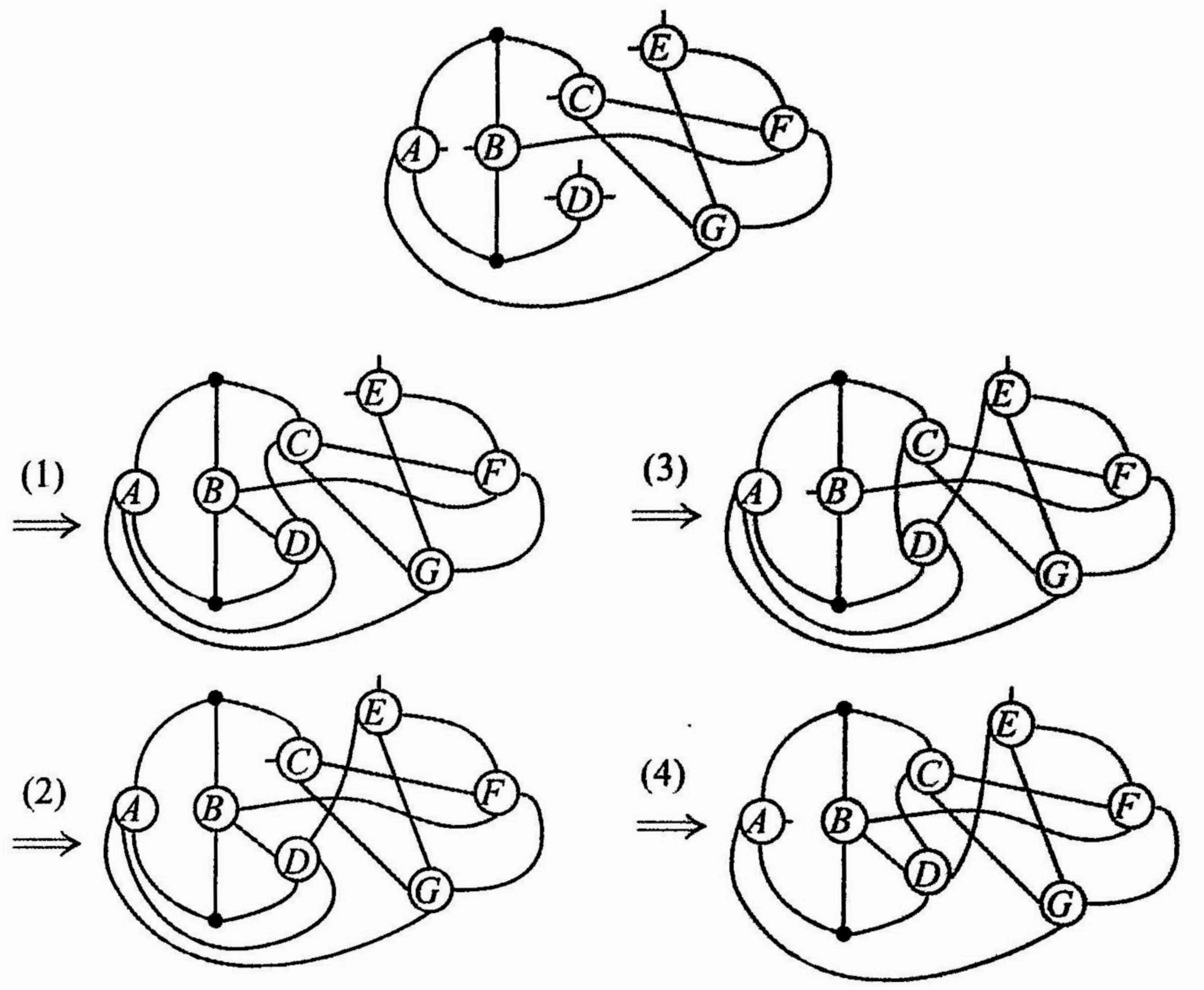


FIG. 3.104. $t = 7$ (b) (ix) (H).

(1) $D \sim A, B, C$. This gives a graph having a loop at E, and so it does not satis the condition (P1); see Fig. 3.104.
(2) $D \sim A, B, E$. Then $C \sim E$. This gives a graph containing $K_{3,3}$, and so it do not satisfy the condition (P5); see Fig. 3.105.

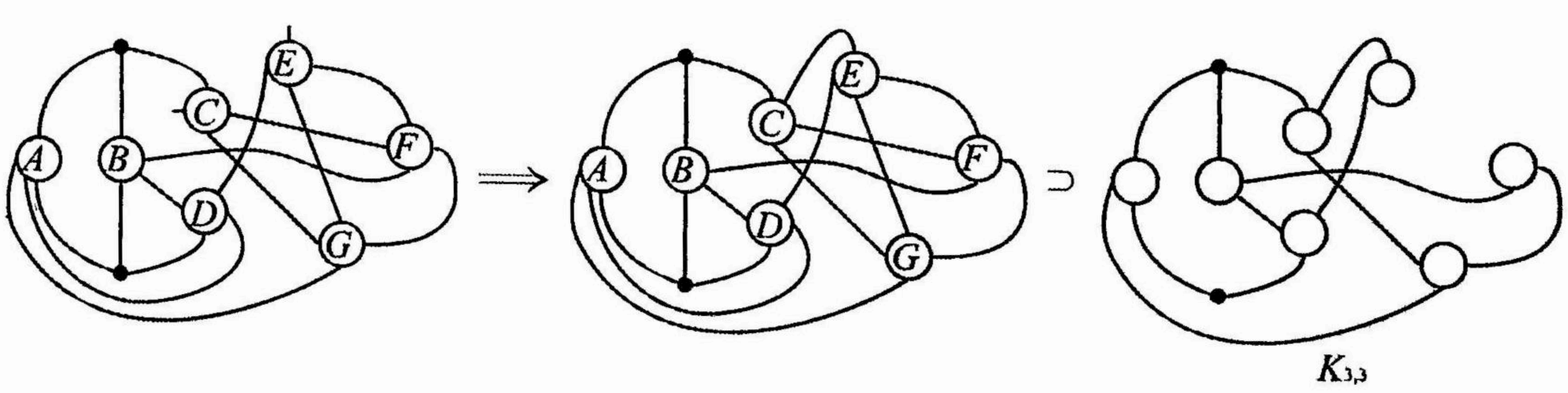


FIG. 3.105. $t = 7$ (b) (ix) (H) (2).

(3) $D \sim A, C, E$. This gives a graph containing $K_{3,3}$, and so it does not satisfy the condition (P5); see Fig. 3.106.

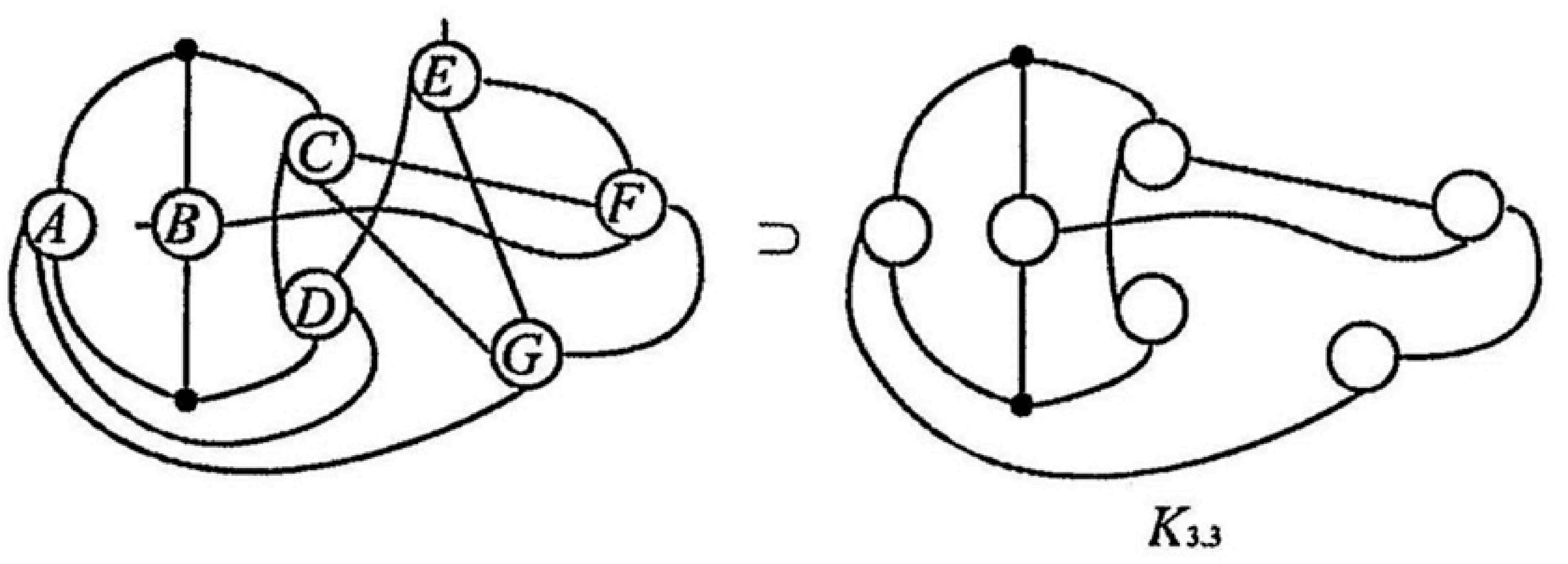


FIG. 3.106. $t = 7$ (b) (ix) (H) (3).

(4) $D \sim B, C, E$. This gives a graph containing $K_{3,3}$, and so it does not satisfy the condition (P5); see Fig. 3.107.

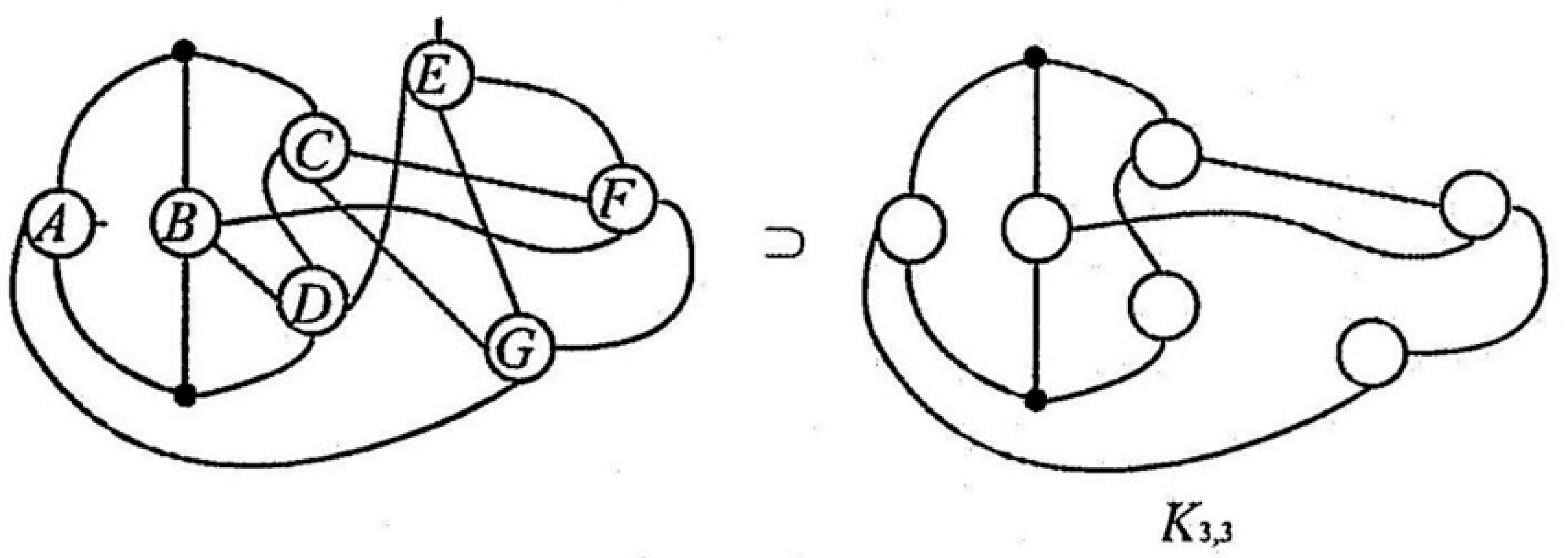


FIG. 3.107. $t = 7$ (b) (ix) (H) (4).

(I) $F \sim B, D, E$. The vertex E has two remaining hands, so we consider how the hands of E connect. There are six cases; see Fig. 3.108.

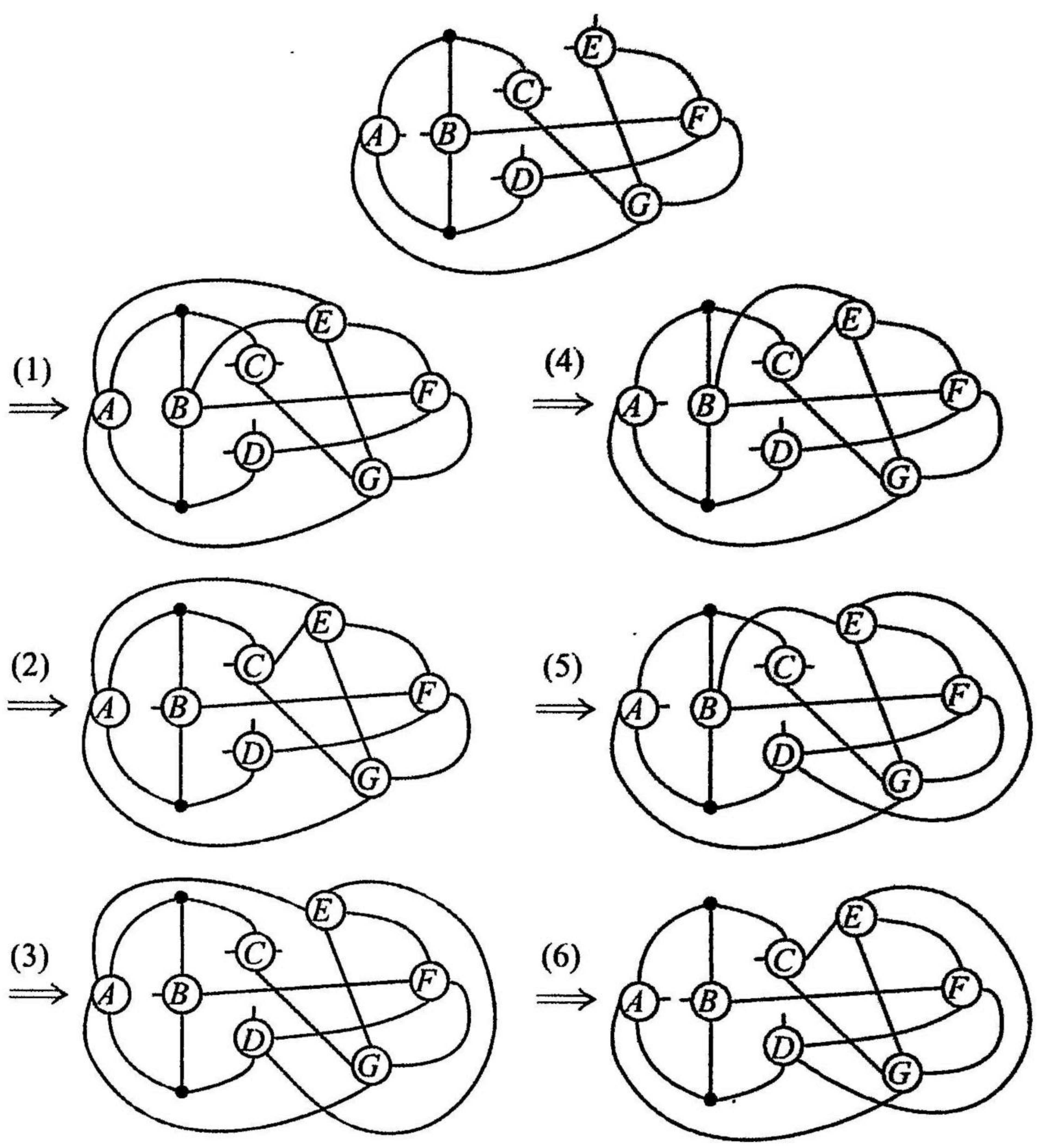


FIG. 3.108. $t = 7$ (b) (ix) (I).

(1) $E \sim A, B$. This gives a graph which does not satisfy the conditions (P1) or (P2); see Fig. 3.108.
(2) $E \sim A, C$. This gives a graph containing $K_{3,3}$, and so it does not satisfy the condition (P5); see Fig. 3.109.

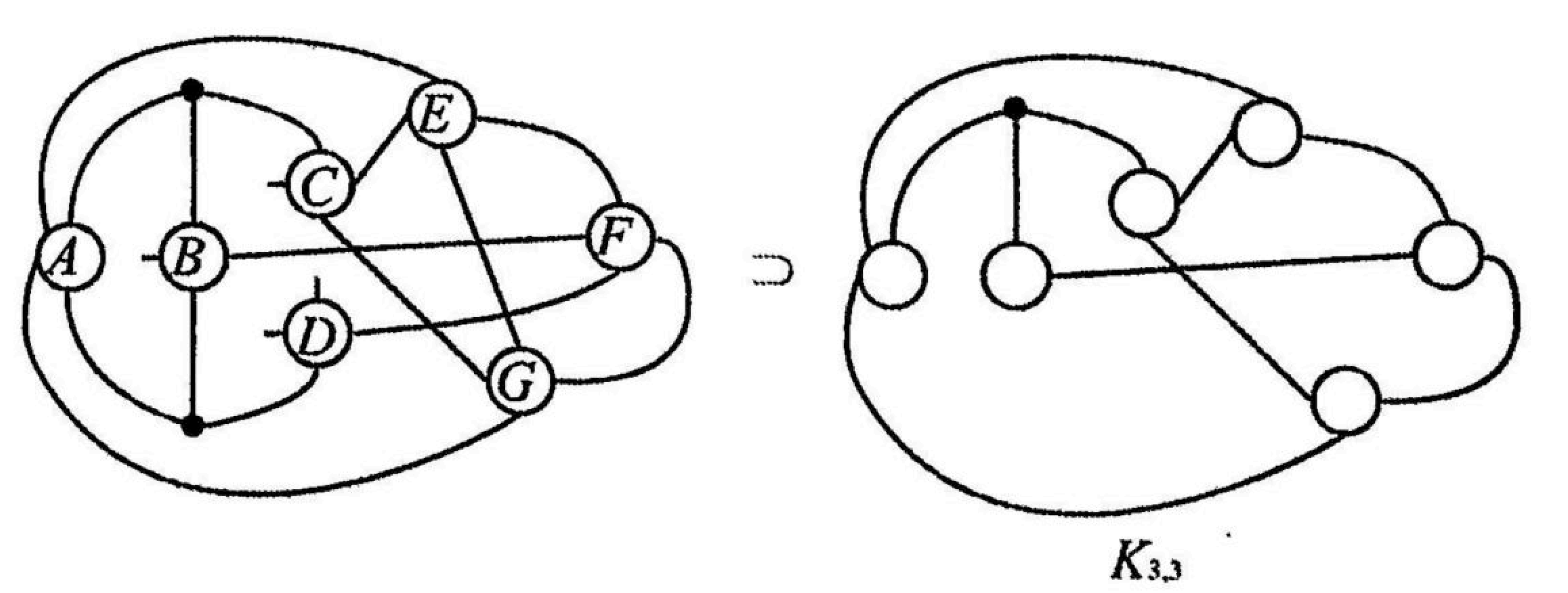


FIG. 3.109. $t = 7$ (b) (ix) (I) (2).

(3) $E \sim A, D$. Then $B \sim C, C \sim D$. This gives a graph containing $K_{3,3}$, and so it does not satisfy the condition (P5); see Fig. 3.110.

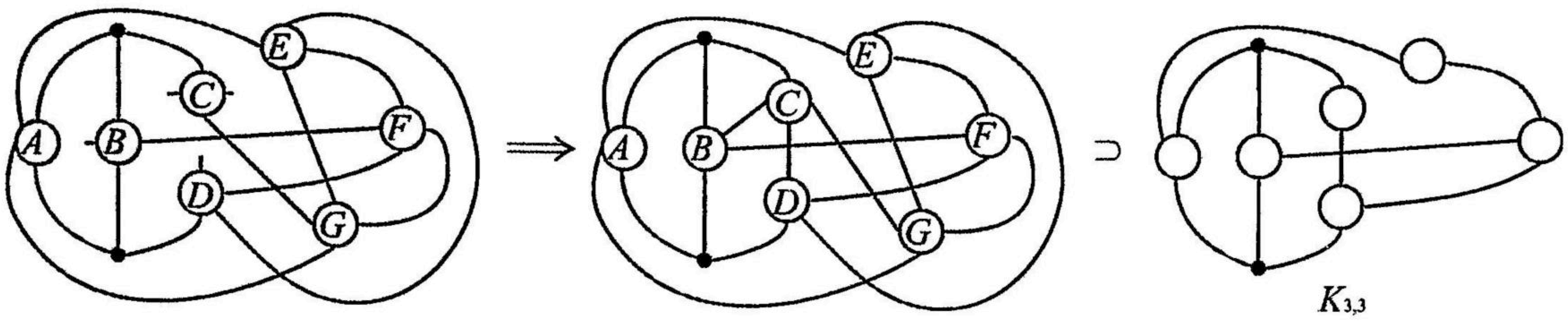


FIG. 3.110. $t = 7$ (b) (ix) (I) (3).

(4) $E \sim B, C$. Then $A \sim D, C \sim D$. This gives a graph containing $K_{3,3}$, and so it does not satisfy the condition (P5); see Fig. 3.111.

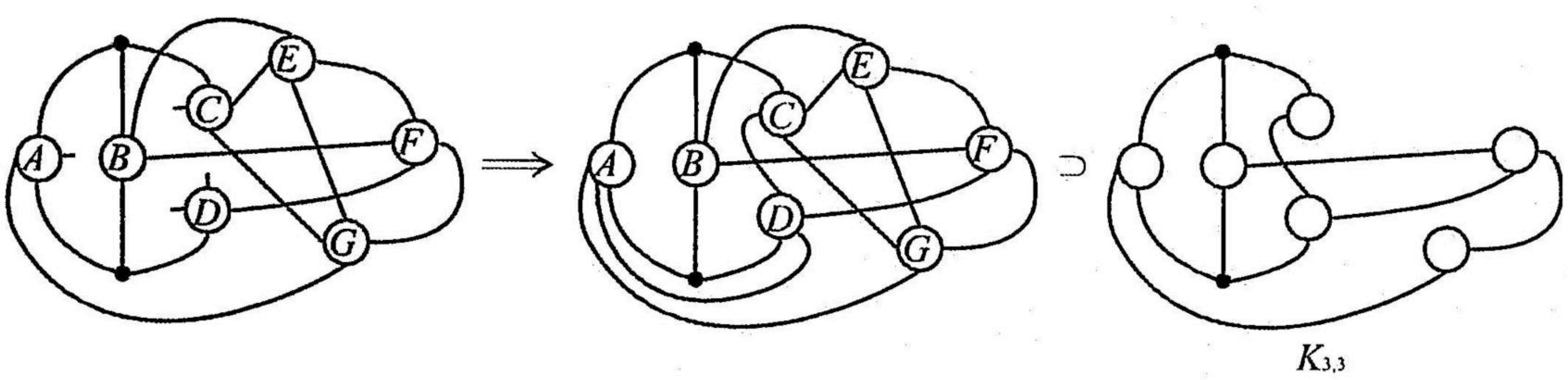


FIG. 3.111. $t = 7$ (b) (ix) (I) (4).

(5) $E \sim B, D$. This gives a graph containing $K_{3,3}$, and so it does not satisfy the condition (P5); see Fig. 3.112.

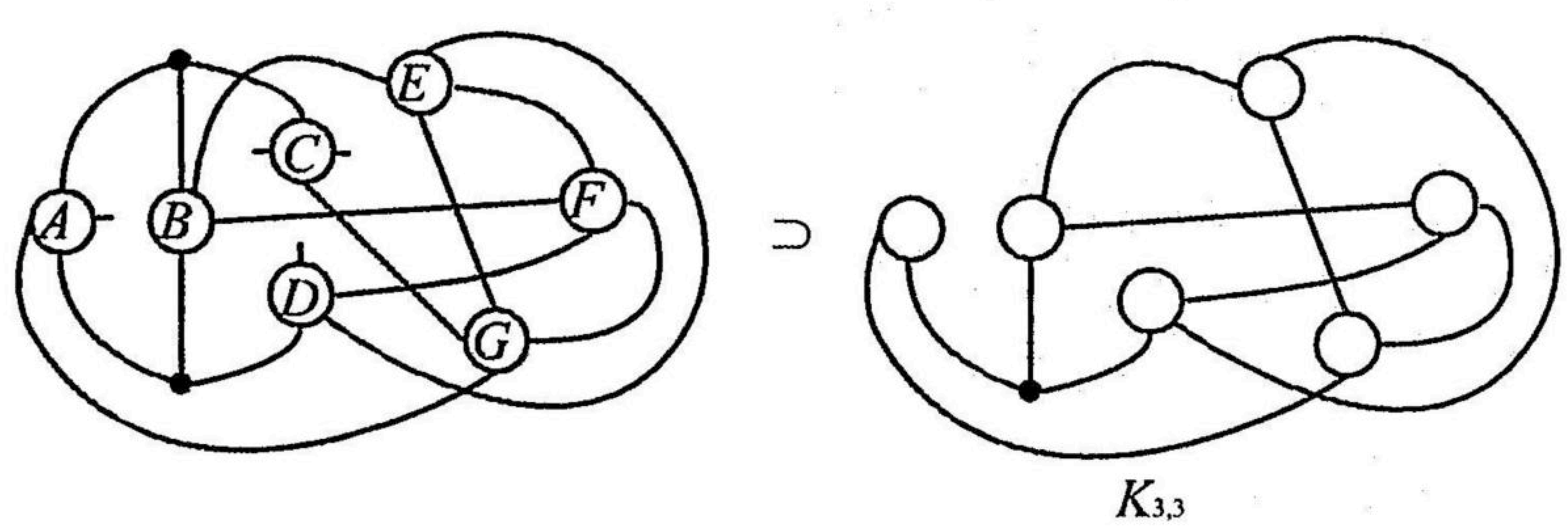


FIG. 3.112. $t = 7$ (b) (ix) (I) (5).

(6) $E \sim C, D$. This gives a graph containing $K_{3,3}$, and so it does not satisfy the condition (P5); see Fig. 3.113.

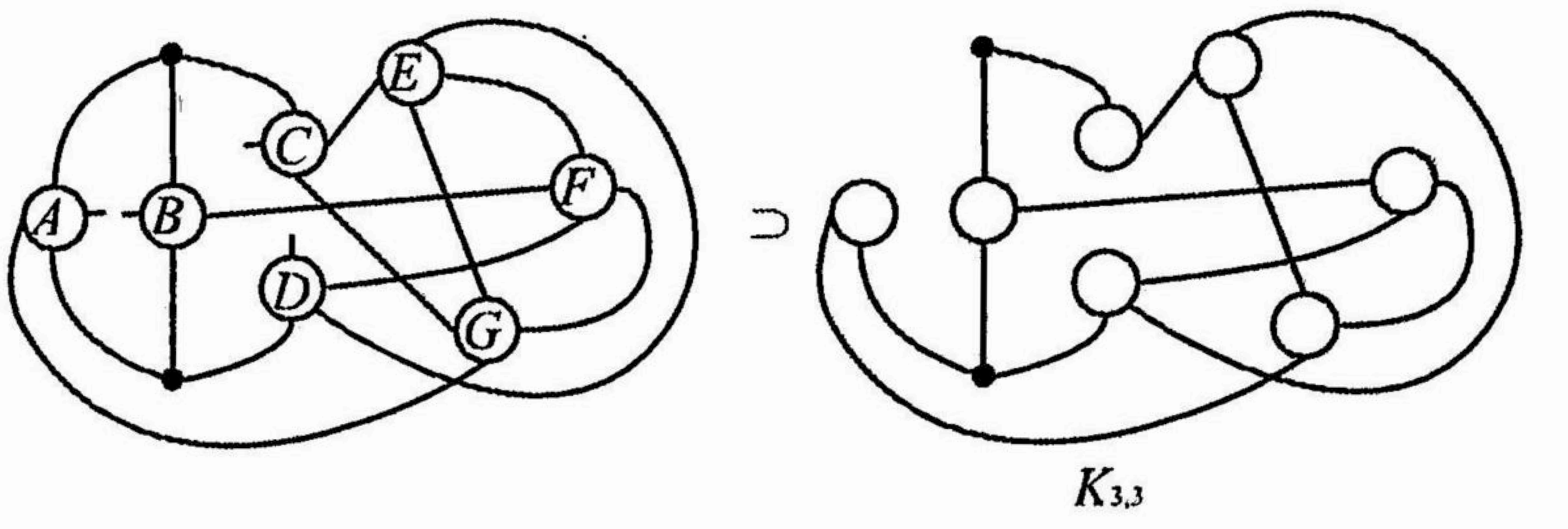


FIG. 3.113. $t = 7$ (b) (ix) (I) (6).

(J) $F \sim C, D, E$. The vertex E has two remaining hands, so we consider how the hand of E connect. There are six cases; see Fig. 3.114.

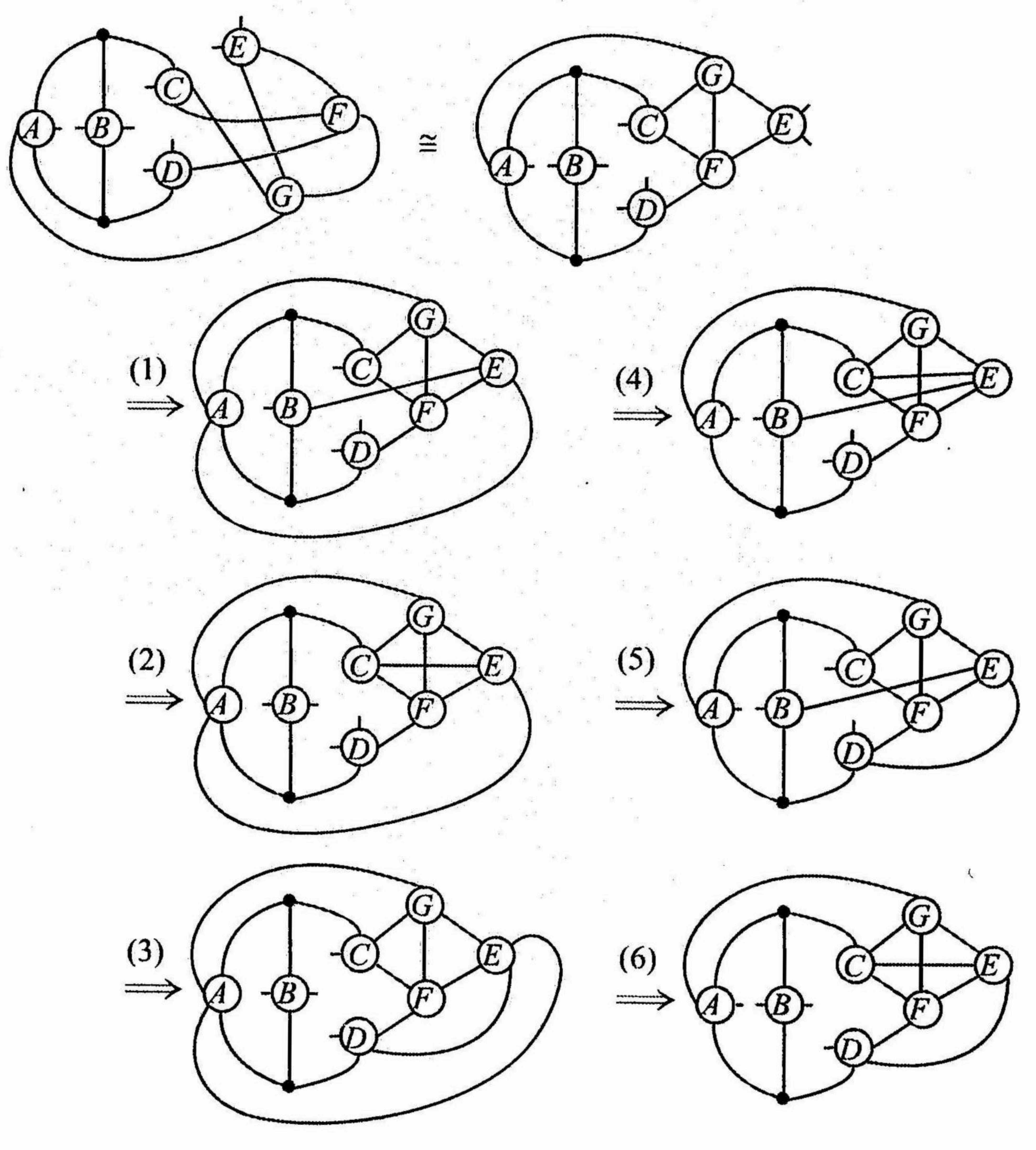


FIG. 3.114. $t = 7$ (b) (ix) (J).

(1) $E \sim A, B$. This gives a graph containing $K_{3,3}$, and so it does not satisfy the condition (P5); see Fig. 3.115.

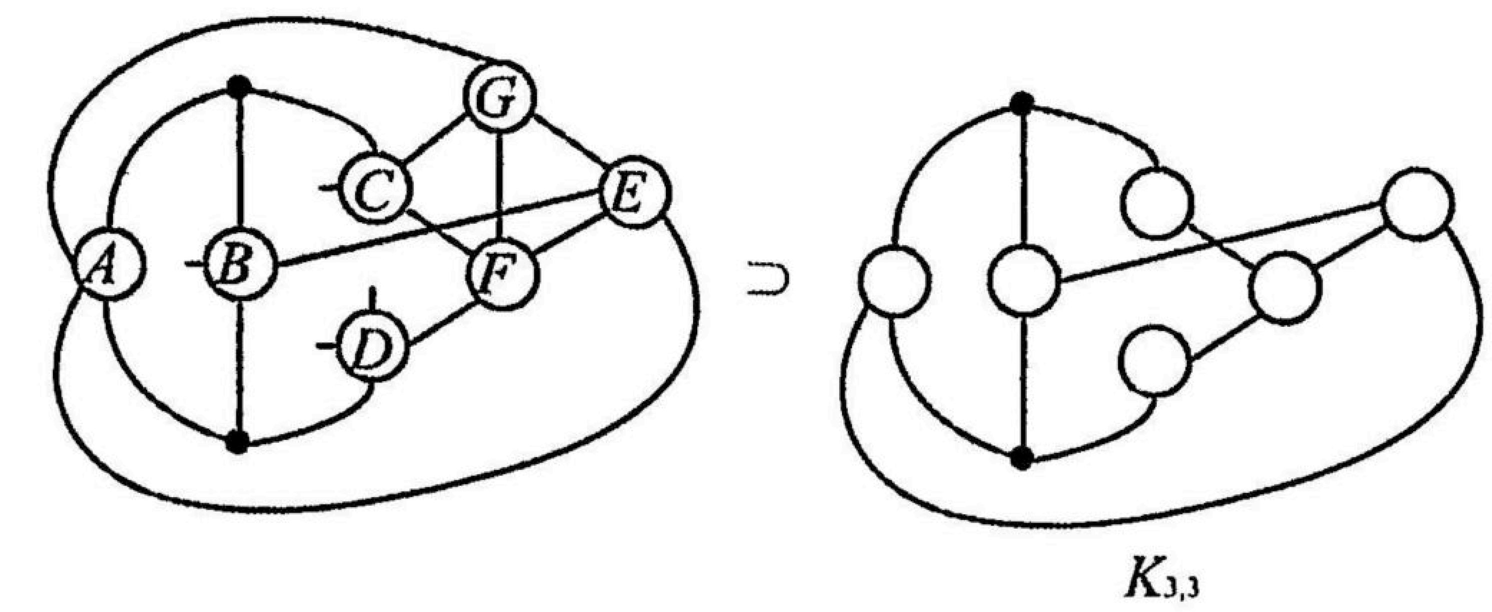


FIG. 3.115. $t = 7$ (b) (ix) (J) (1).

(2) $E \sim A, C$. This gives a graph which does not satisfy the conditions (P1) or (P2); see Fig. 3.114.
(3) $E \sim A, D$. Then $B \sim C, D$, and we obtain 7^2_*; see Fig. 3.116.

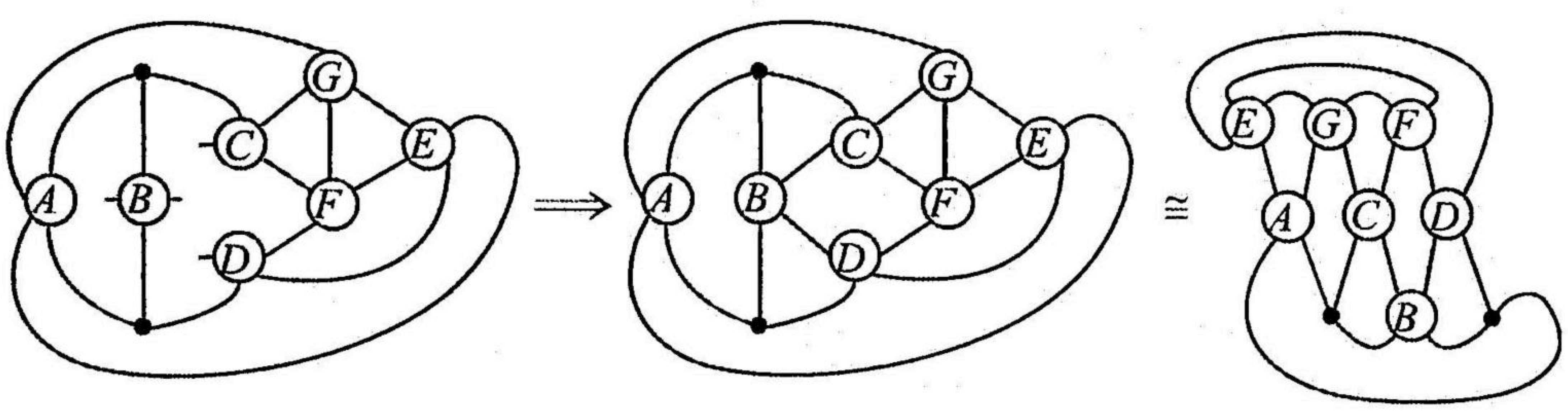


FIG. 3.116. $t = 7$ (b) (ix) (J) (3).

(4) $E \sim B, C$. This gives a graph containing $K_{3,3}$, and so it does not satisfy the condition (P5); see Fig. 3.117.

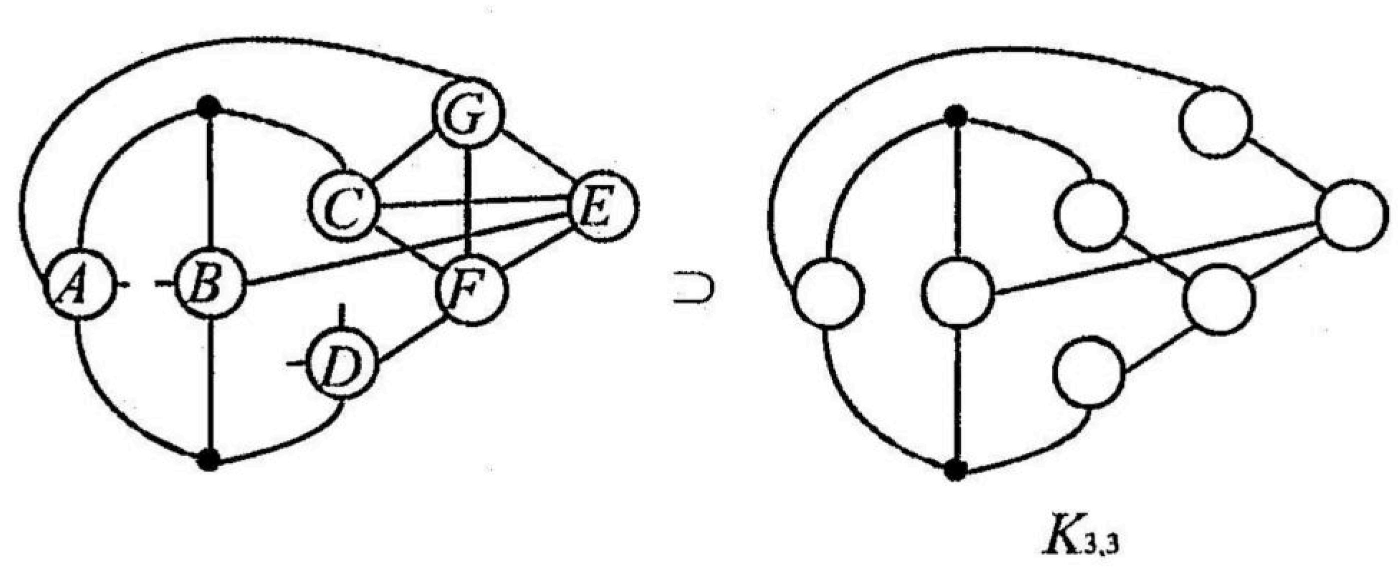


FIG. 3.117. $t = 7$ (b) (ix) (J) (4).

(5) $E \sim B, D$. This gives a graph containing $K_{3,3}$, and so it does not satisfy the condition (P5); see Fig. 3.118.

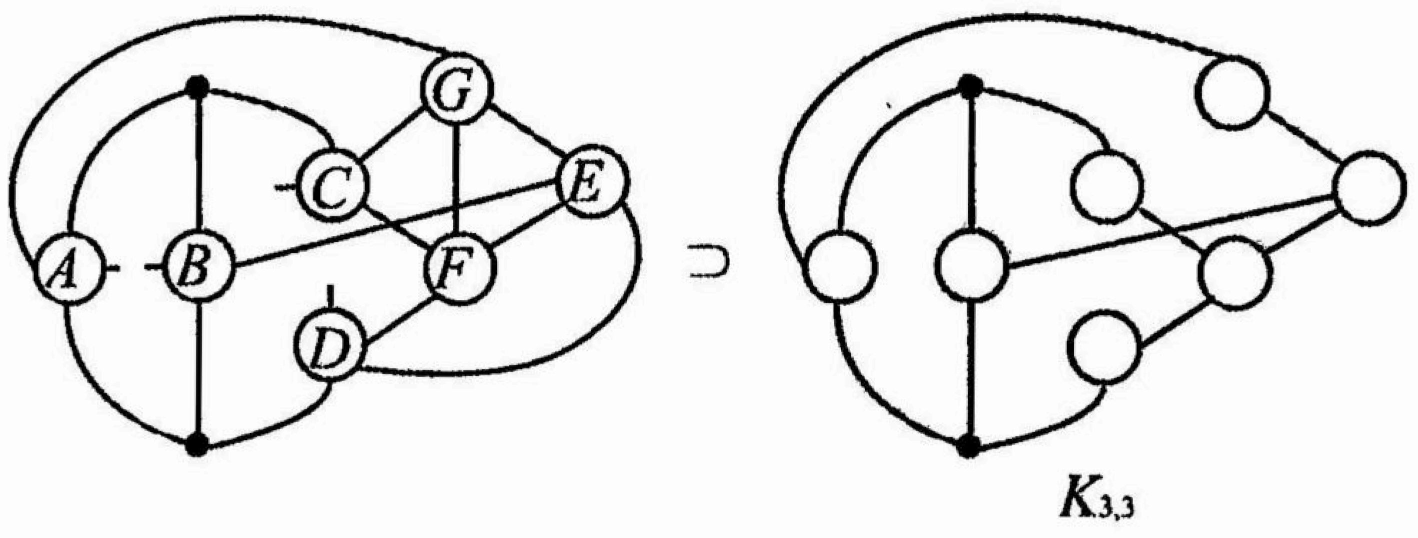


FIG. 3.118. $t = 7$ (b) (ix) (J) (5).

(6) $E \sim C, D$. This gives a graph containing $K_{3,3}$, and so it does not satisfy the condition (P5); see Fig. 3.119.

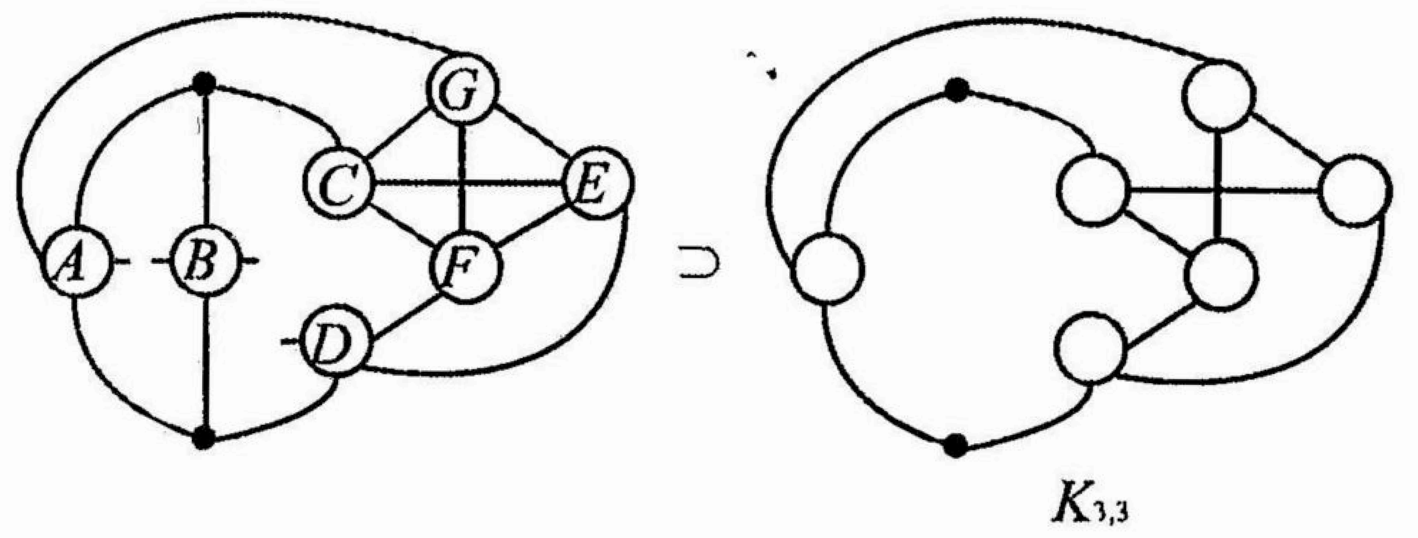


FIG. 3.119. $t = 7$ (b) (ix) (J) (6).

(x) $G \sim A, D, E, F$. Since C and D are interchangeable in the first figure in Fig. 3.68, this case is the same as the case (ix).

(xi) $G \sim B, C, D, E$. Since A and B are interchangeable in the first figure in Fig. 3.68, this case is the same as the case (vii).

(xii) $G \sim B, C, D, F$. Since A and B, E and F are interchangeable in the first figure in Fig. 3.68, this case is the same as the case (vii).

(xiii) $G \sim B, C, E, F$. Since A and B are interchangeable in the first figure in Fig. 3.68, this case is the same as the case (ix).

(xiv) $G \sim B, D, E, F$. Since A and B, C and D are interchangeable in the first figure in Fig. 3.68, this case is the same as the case (ix).

(xv) $G \sim C, D, E, F$. The vertex F has three remaining hands, so we consider how the hands of F connect. There are ten cases; see Fig. 3.120.

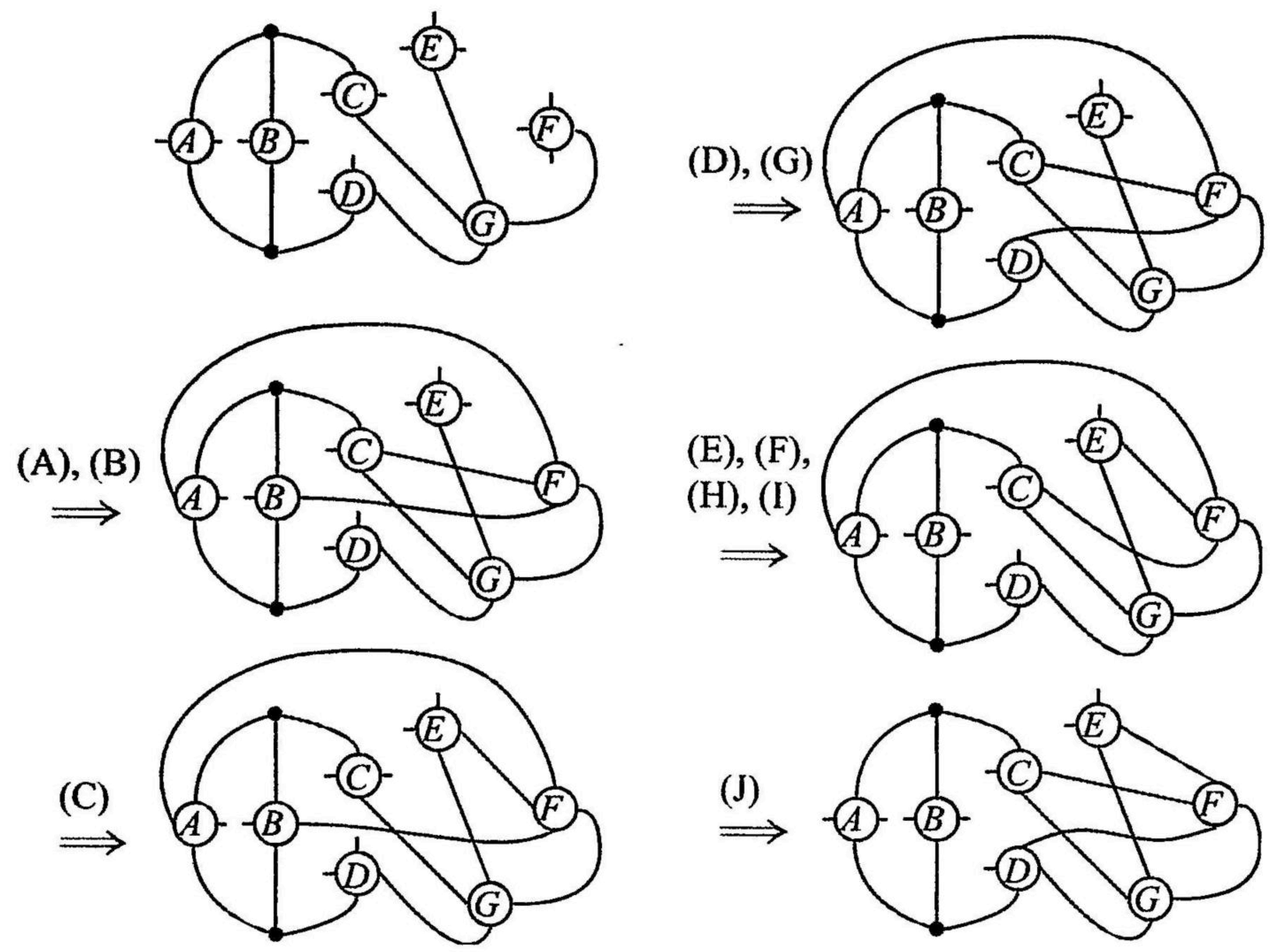


FIG. 3.120. $t = 7$ (b) (xv).

(A) $F \sim A, B, C$. This gives a graph containing $K_{3,3}$, and so it does not satisfy the condition (P5); see Fig. 3.121.

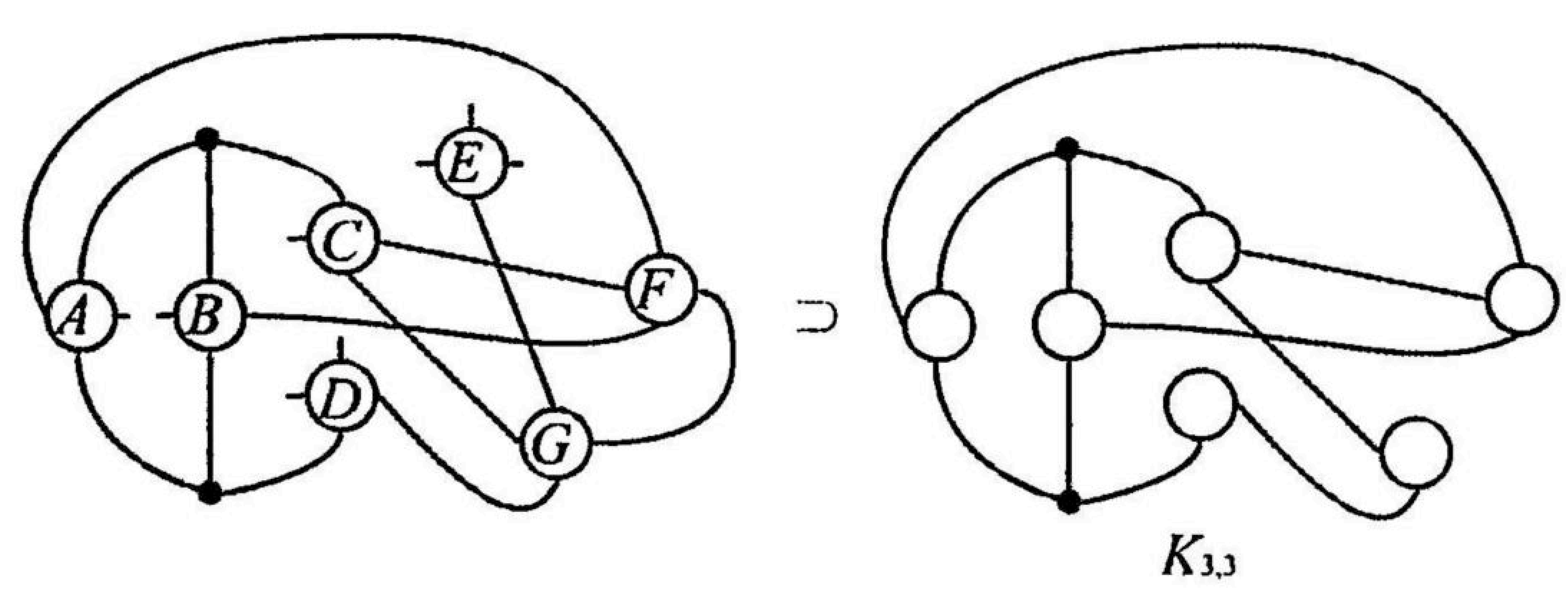


FIG. 3.121. $t = 7$ (b) (xv) (A).

(B) $F \sim A, B, D$. Since C and D are interchangeable in the first figure in Fig. 3.120, this case is the same as the case (A).

(C) $F \sim A, B, E$. This gives a graph containing $K_{3,3}$, and so it does not satisfy the condition (P5); see Fig. 3.122.

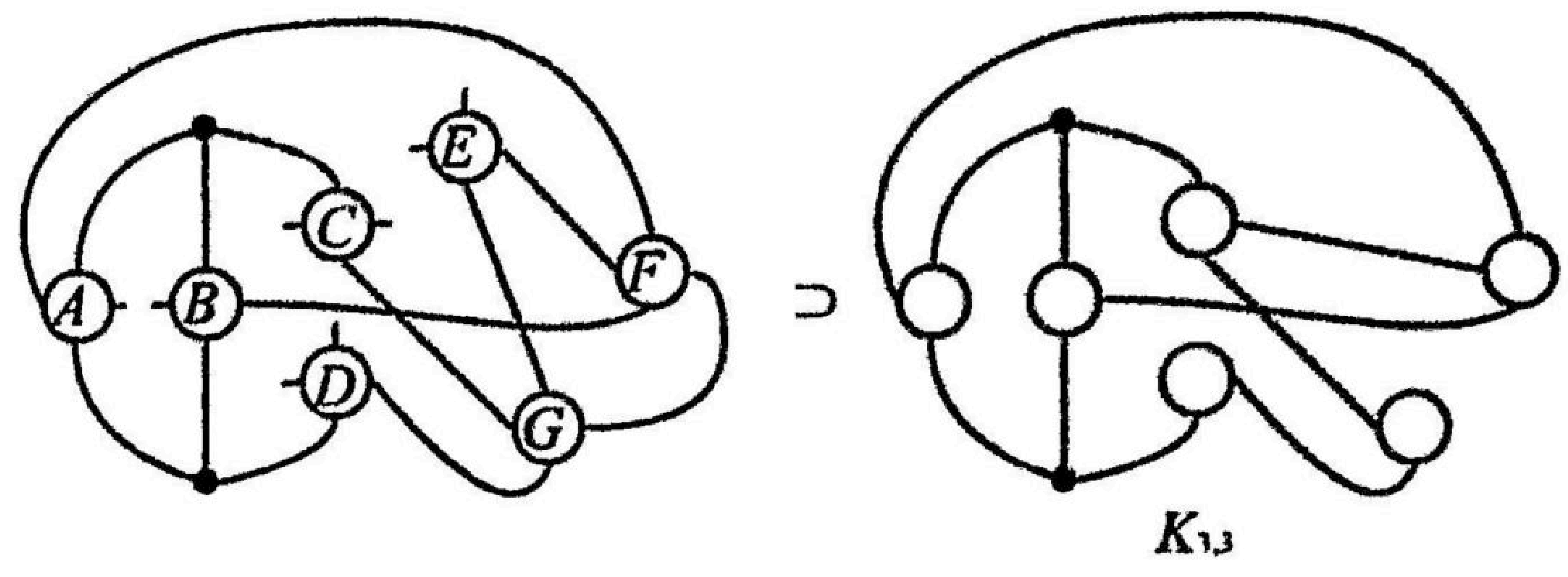


FIG. 3.122. $t = 7$ (b) (xv) (C).

(D) $F \sim A, C, D$. The vertex E has three remaining hands, so we consider how the of E connect. There are four cases; see Fig. 3.123.

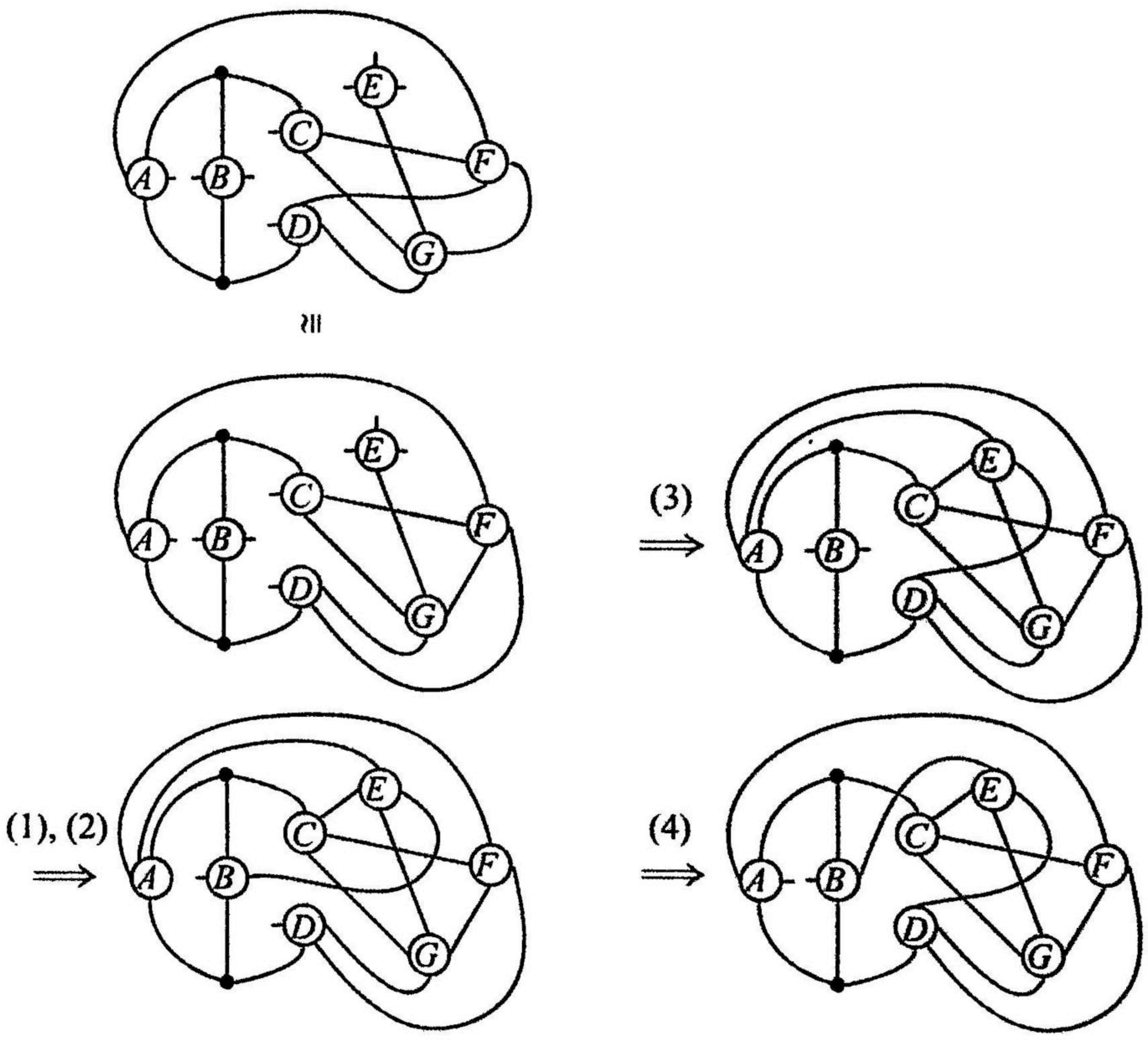


FIG. 3.123. $t = 7$ (b) (xv) (D).

(1) $E \sim A, B, C$. This gives a graph containing $K_{3,3}$, and so it does not satisfy the condition (P5); see Fig. 3.124.

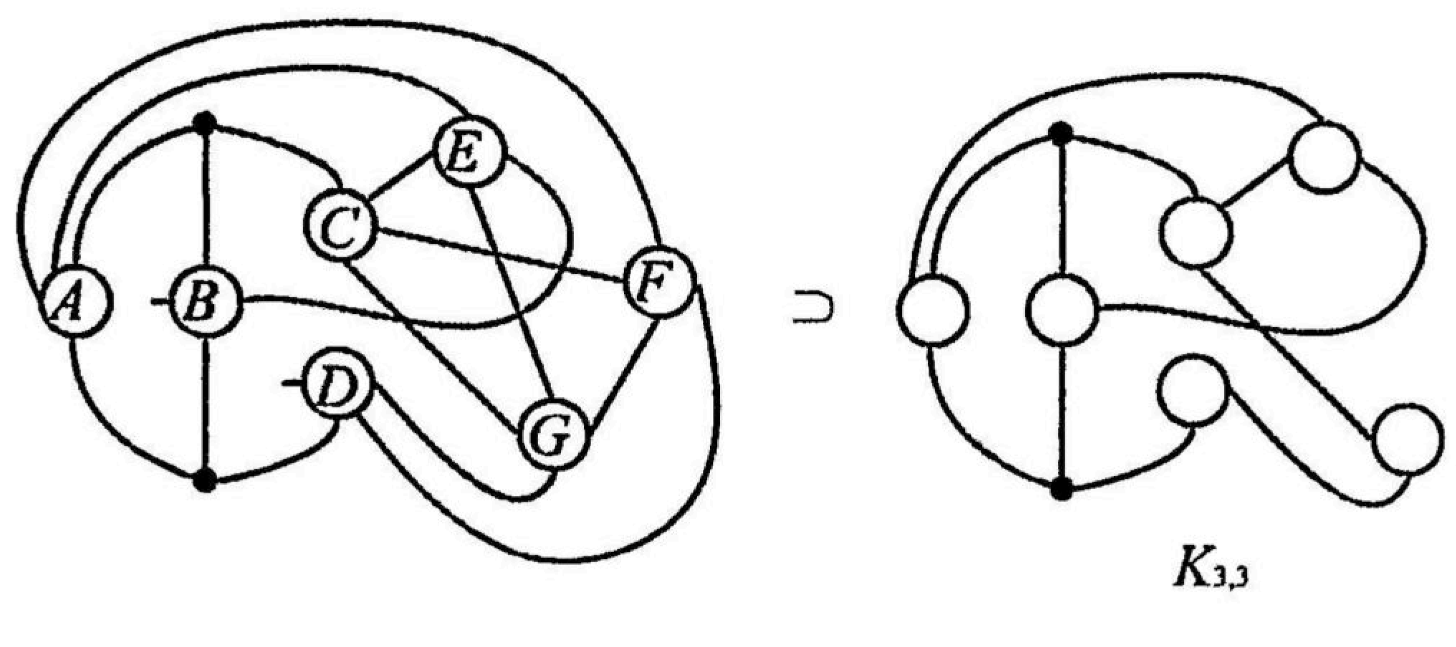


FIG. 3.124. $t = 7$ (b) (xv) (D) (1).

(2) $E \sim A, B, D$. Since C and D are interchangeable in the first figure in Fig. 3.123, this case is the same as the case (1).
(3) $E \sim A, C, D$. This gives a graph having a loop at B, and so it does not satisfy the condition (P1); see Fig. 3.123.
(4) $E \sim B, C, D$. Then $A \sim B$, and we obtain 7^1_*; see Fig. 3.125.

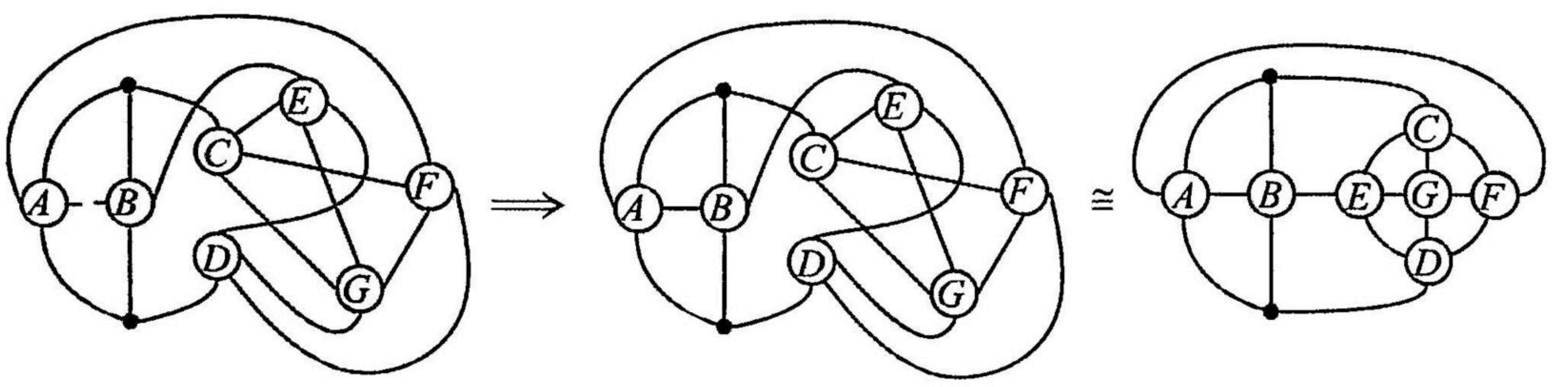


FIG. 3.125. $t = 7$ (b) (xv) (D) (4).

(E) $F \sim A, C, E$. The vertex E has two remaining hands, so we consider how the hands of E connect. There are six cases; see Fig. 3.126.

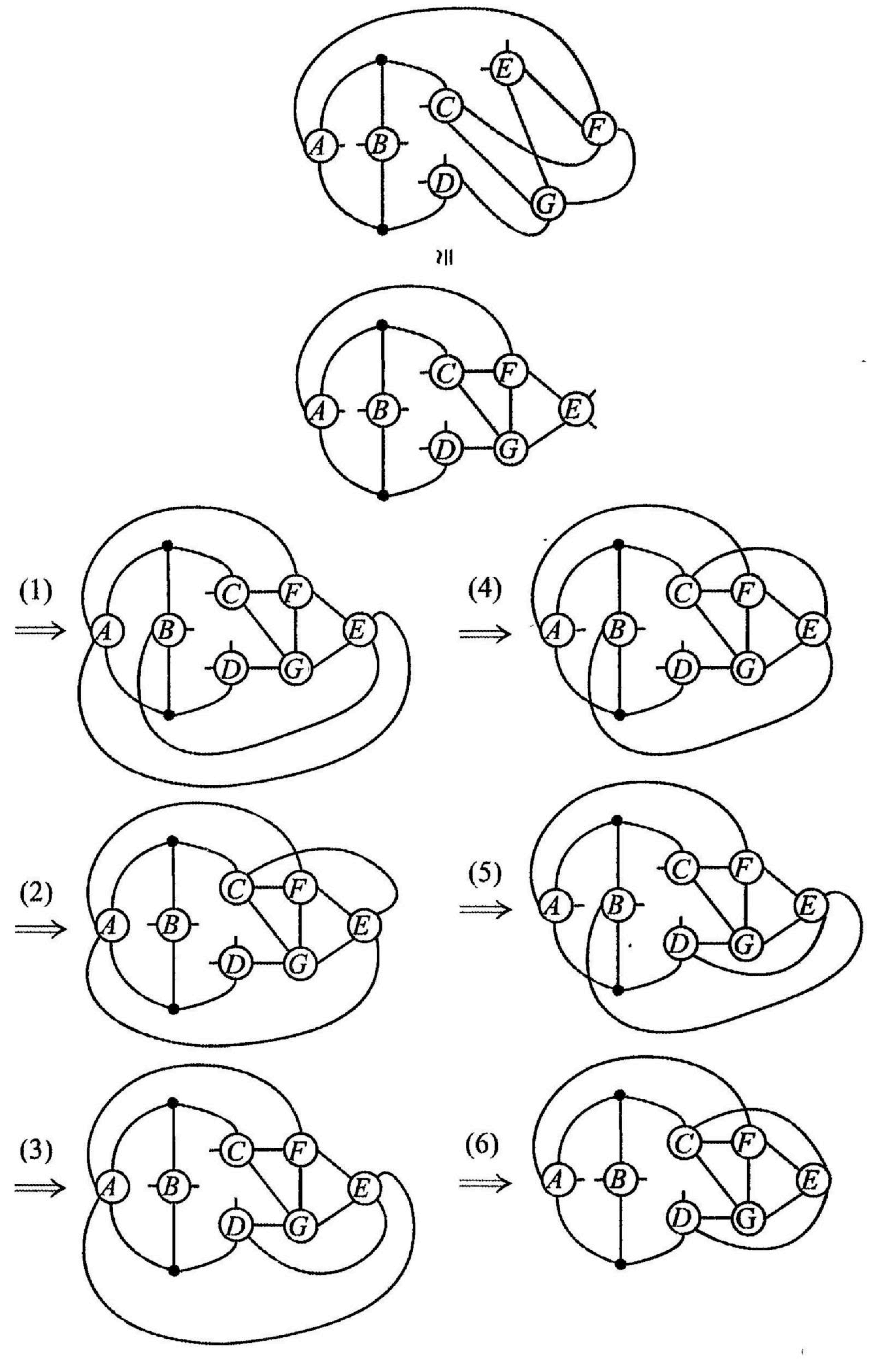


FIG. 3.126. $t = 7$ (b) (xv) (E).

(1) $E \sim A, B$. This gives a graph containing $K_{3,3}$, and so it does not satisfy the condition (P5); see Fig. 3.127.

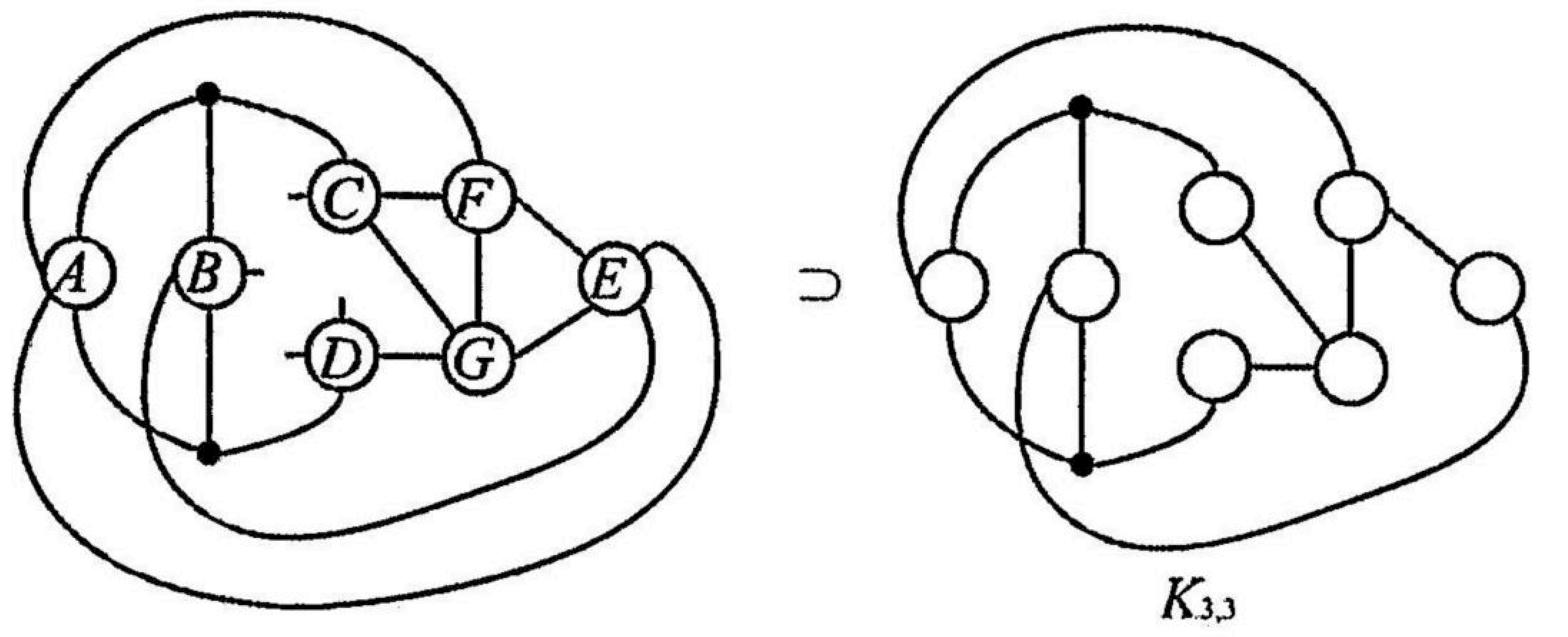


FIG. 3.127. $t = 7$ (b) (xv) (E) (1).

(2) $E \sim A, C$. This gives a graph which does not satisfy the conditions (P1) or (P2); see Fig. 3.126.
(3) $E \sim A, D$. Then $B \sim C, D$, and we obtain 7^2_* ; see Fig. 3.128.

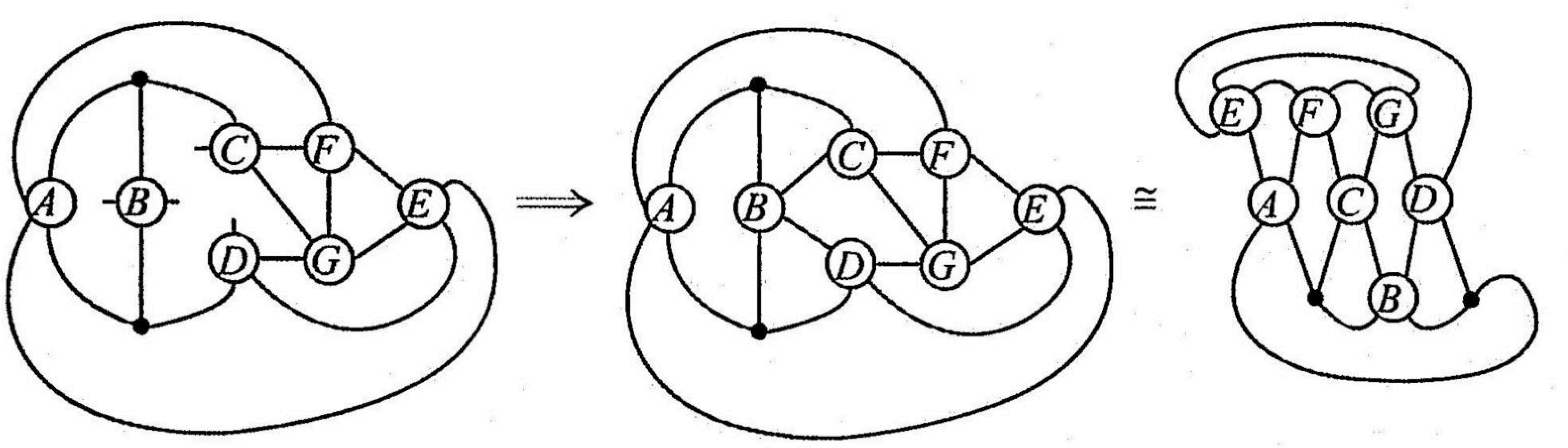


FIG. 3.128. $t = 7$ (b) (xv) (E) (3).

(4) $E \sim B, C$. This gives a graph containing $K_{3,3}$, and so it does not satisfy the condition (P5); see Fig. 3.129.

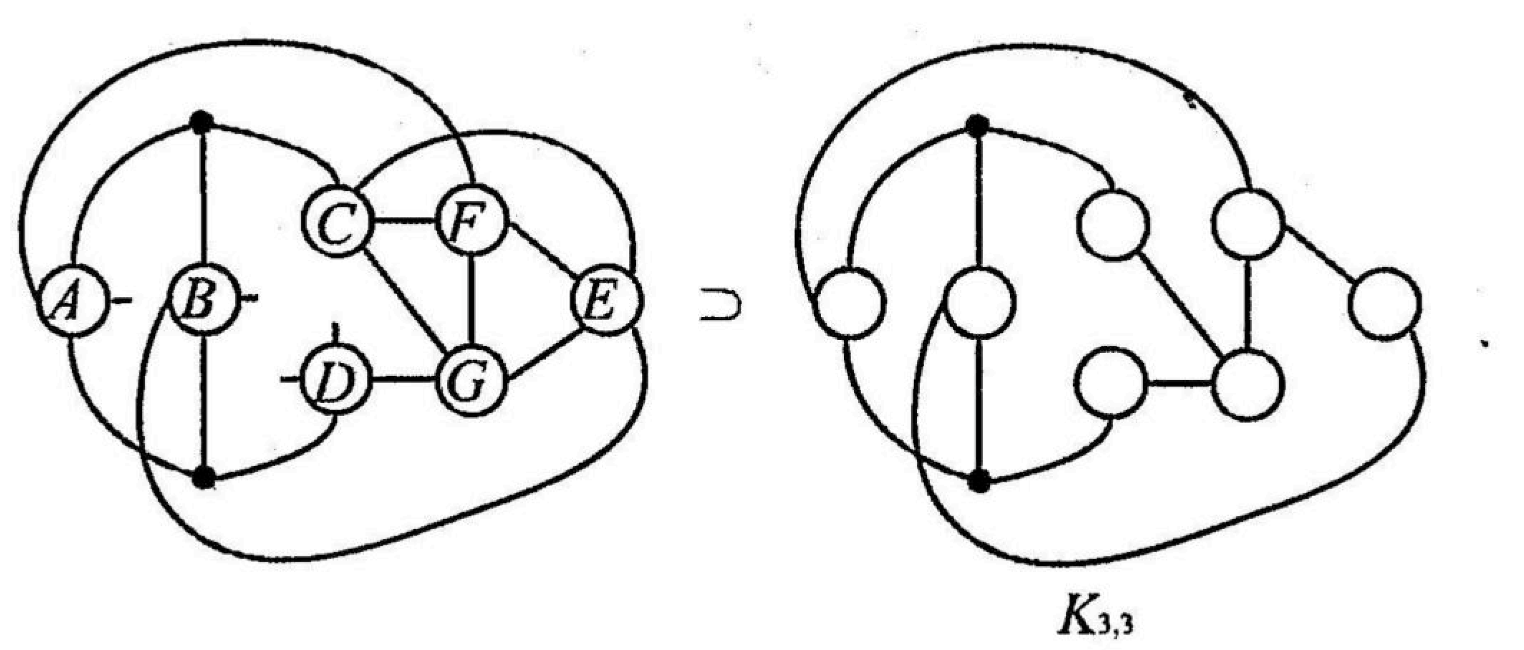


FIG. 3.129. $t = 7$ (b) (xv) (E) (4).

(5) $E \sim B, D$. This gives a graph containing $K_{3,3}$, and so it does not satisfy the condition (P5); see Fig. 3.130.

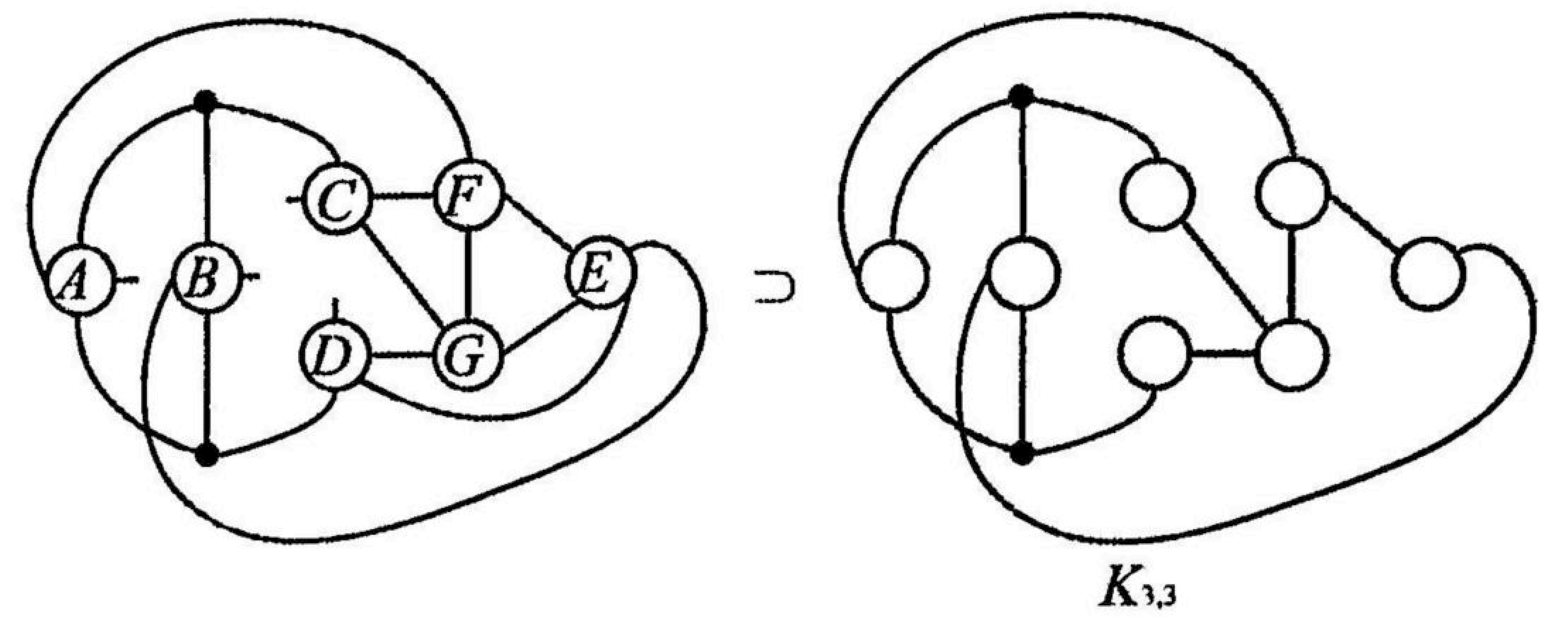


FIG. 3.130. $t = 7$ (b) (xv) (E) (5).

(6) $E \sim C, D$. This gives a graph containing $K_{3,3}$, and so it does not satisfy the condition (P5); see Fig. 3.131.

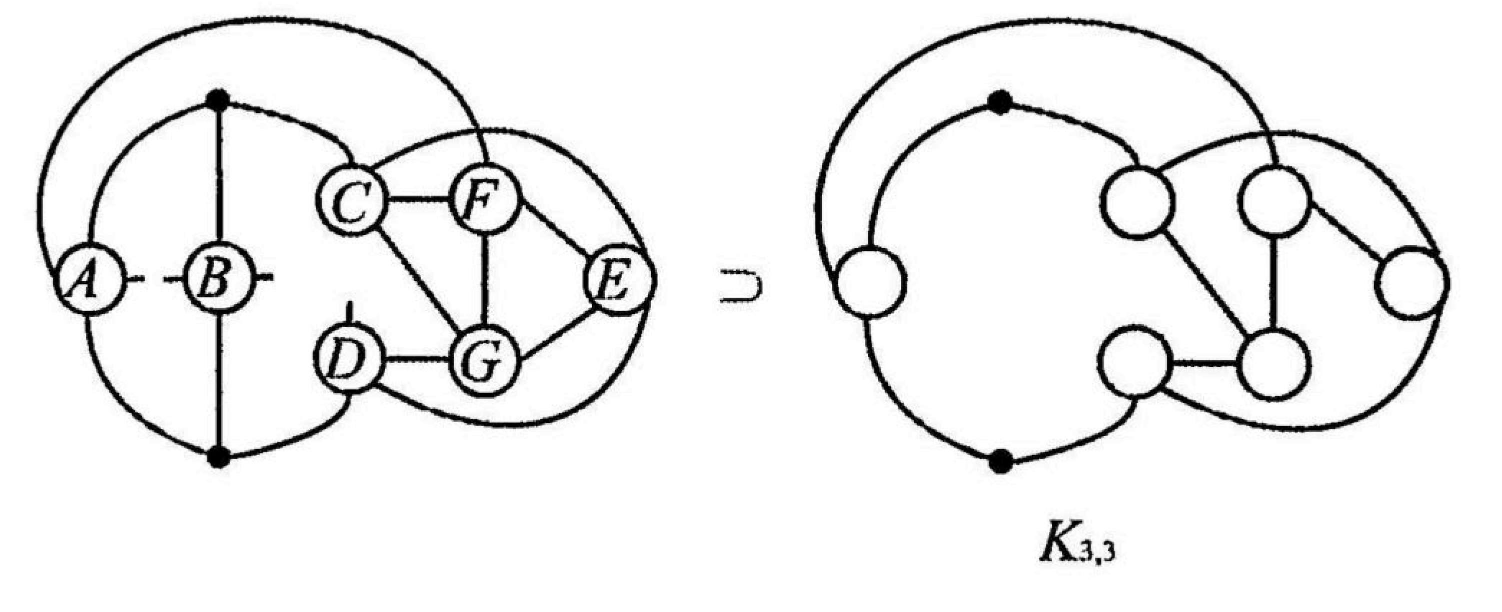


FIG. 3.131. $t = 7$ (b) (xv) (E) (6).

(F) $F \sim A, D, E$. Since C and D are interchangeable in the first figure in Fig. 3.120, this case is the same as the case (E).
(G) $F \sim B, C, D$. Since A and B are interchangeable in the first figure in Fig. 3.120, this case is the same as the case (D).
(H) $F \sim B, C, E$. Since A and B are interchangeable in the first figure in Fig. 3.120, this case is the same as the case (E).
(I) $F \sim B, D, E$. Since A and B, C and D are interchangeable in the first figure in Fig. 3.120, this case is the same as the case (E).
(J) $F \sim C, D, E$. The vertex E has two remaining hands, so we consider how the hands of E connect. There are six cases; see Fig. 3.132.

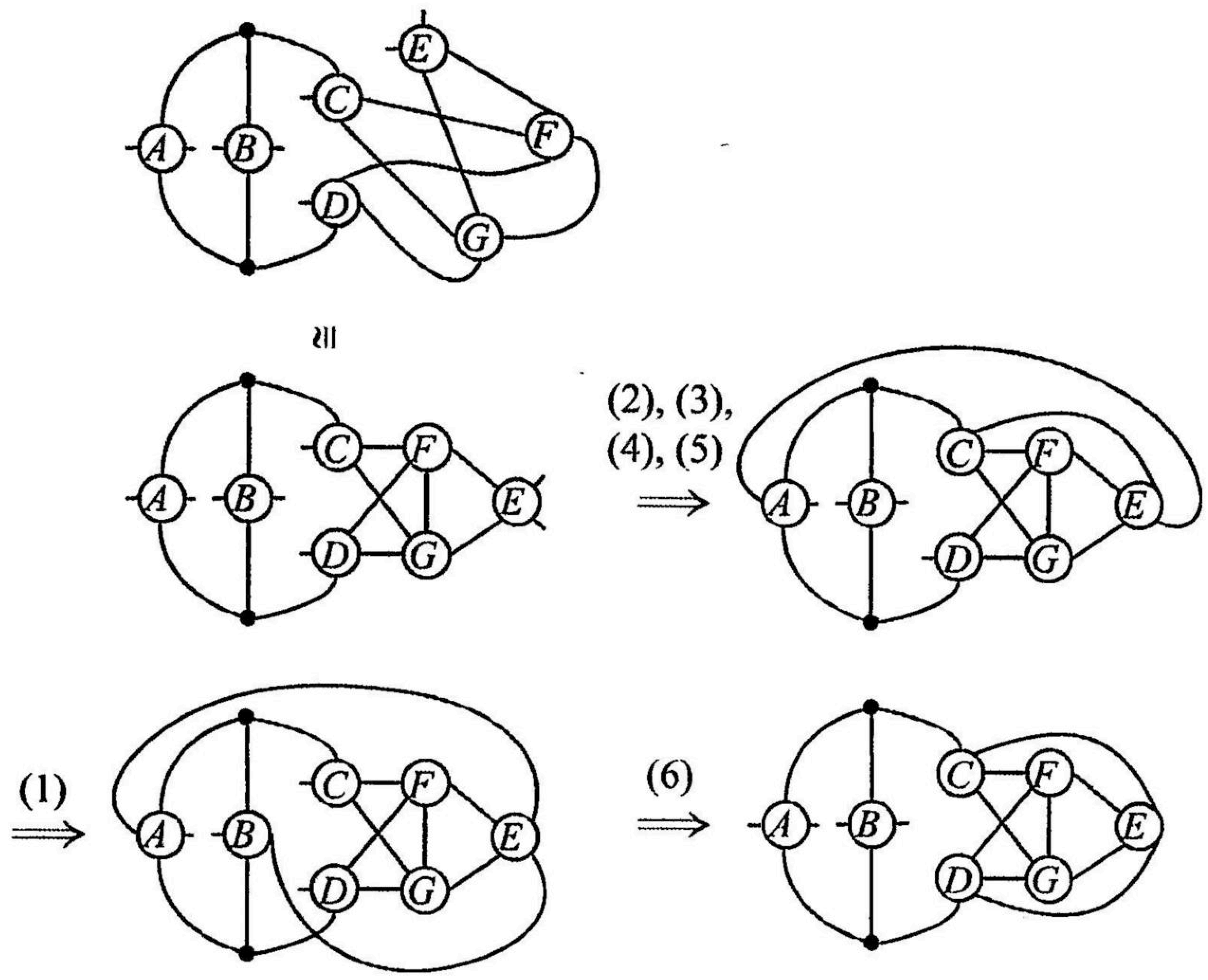


FIG. 3.132. $t = 7$ (b) (xv) (J).

(1) $E \sim A$, B. This gives a graph containing $K_{3,3}$, and so it does not satisfy the condition (P5); see Fig. 3.133.

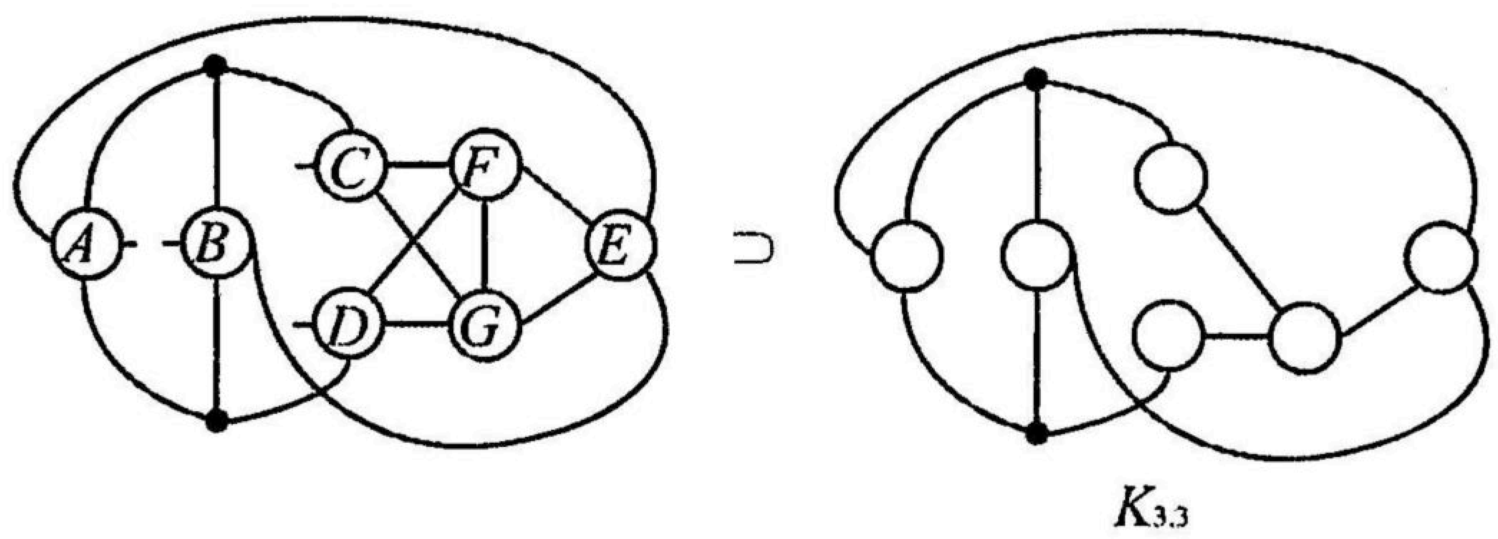


FIG. 3.133. $t = 7$ (b) (xv) (J) (1).

(2) $E \sim A$, C. This gives a graph containing $K_{3,3}$, and so it does not satisfy the condition (P5); see Fig. 3.134.

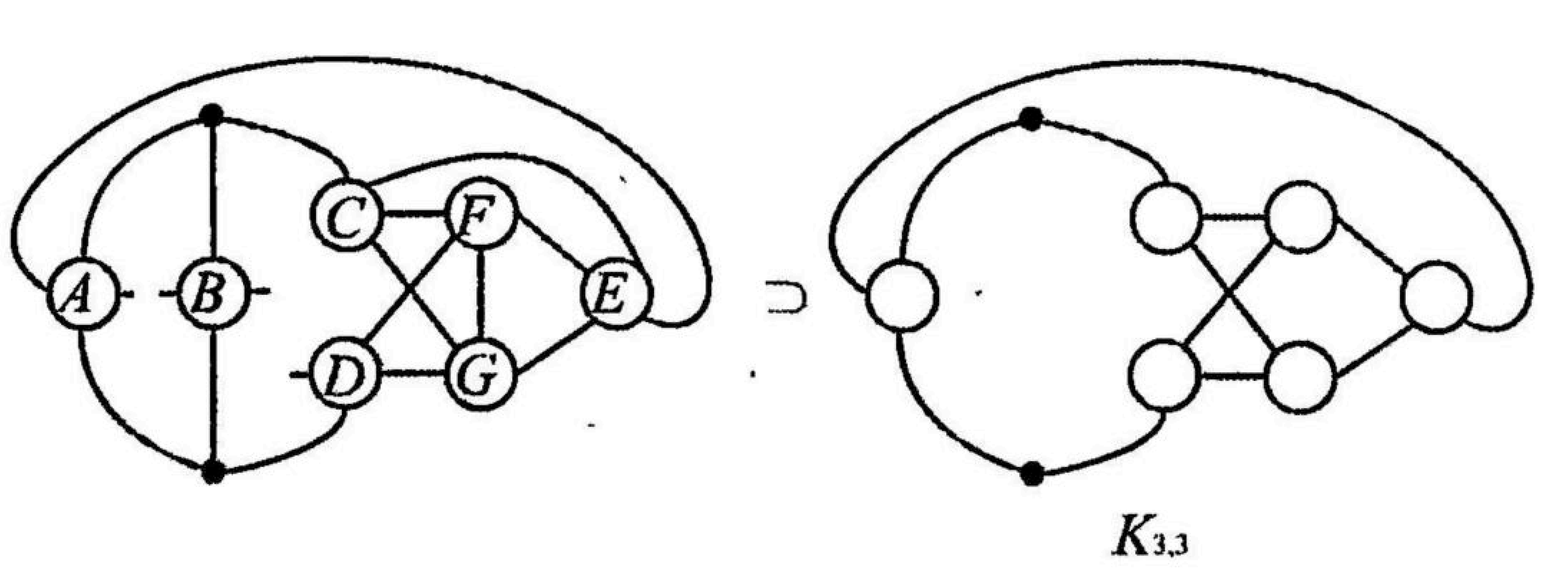


FIG. 3.134. $t = 7$ (b) (xv) (J) (2).

(3) $E \sim A, D$. Since C and D are interchangeable in the first figure in Fig. 3.1 this case is the same as the case (2).
(4) $E \sim B, C$. Since A and B are interchangeable in the first figure in Fig. 3.132, this case is the same as the case (2).
(5) $E \sim B, D$. Since A and B, C and D are interchangeable in the first figure in Fig. 3.132, this case is the same as the case (2).
(6) $E \sim C, D$. This gives a graph which does not satisfy the conditions (P1) or (P2); see Fig. 3.132.

Pattern (c). The vertex G has four remaining hands, so we consider how the hands of G connect. There are fifteen cases; see Fig. 3.135.

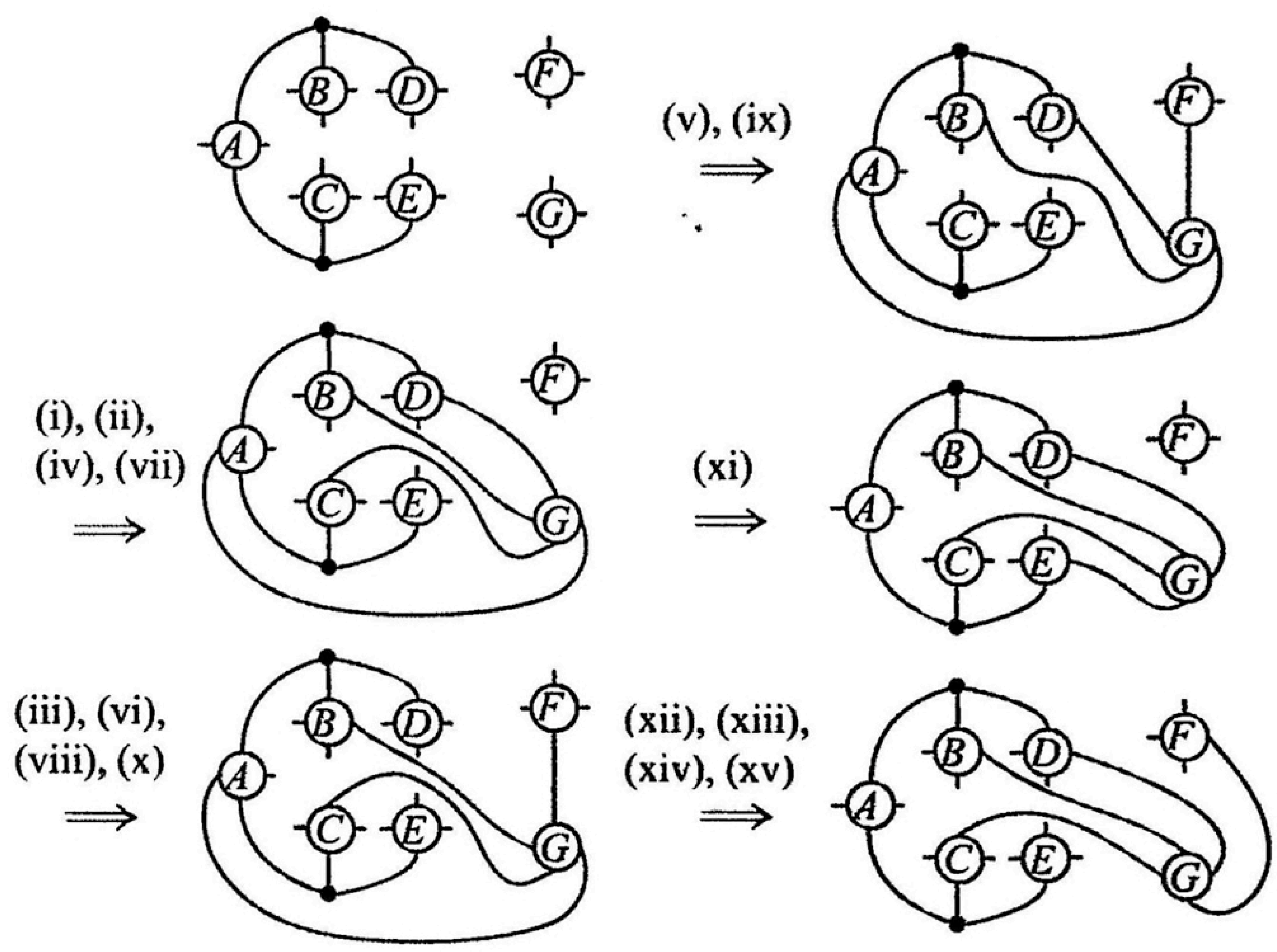


FIG. 3.135. $t = 7$ (c).

(i) $G \sim A, B, C, D$. The vertex F has four remaining hands, so we consider how the hands of F connect. There are five cases; see Fig. 3.136.

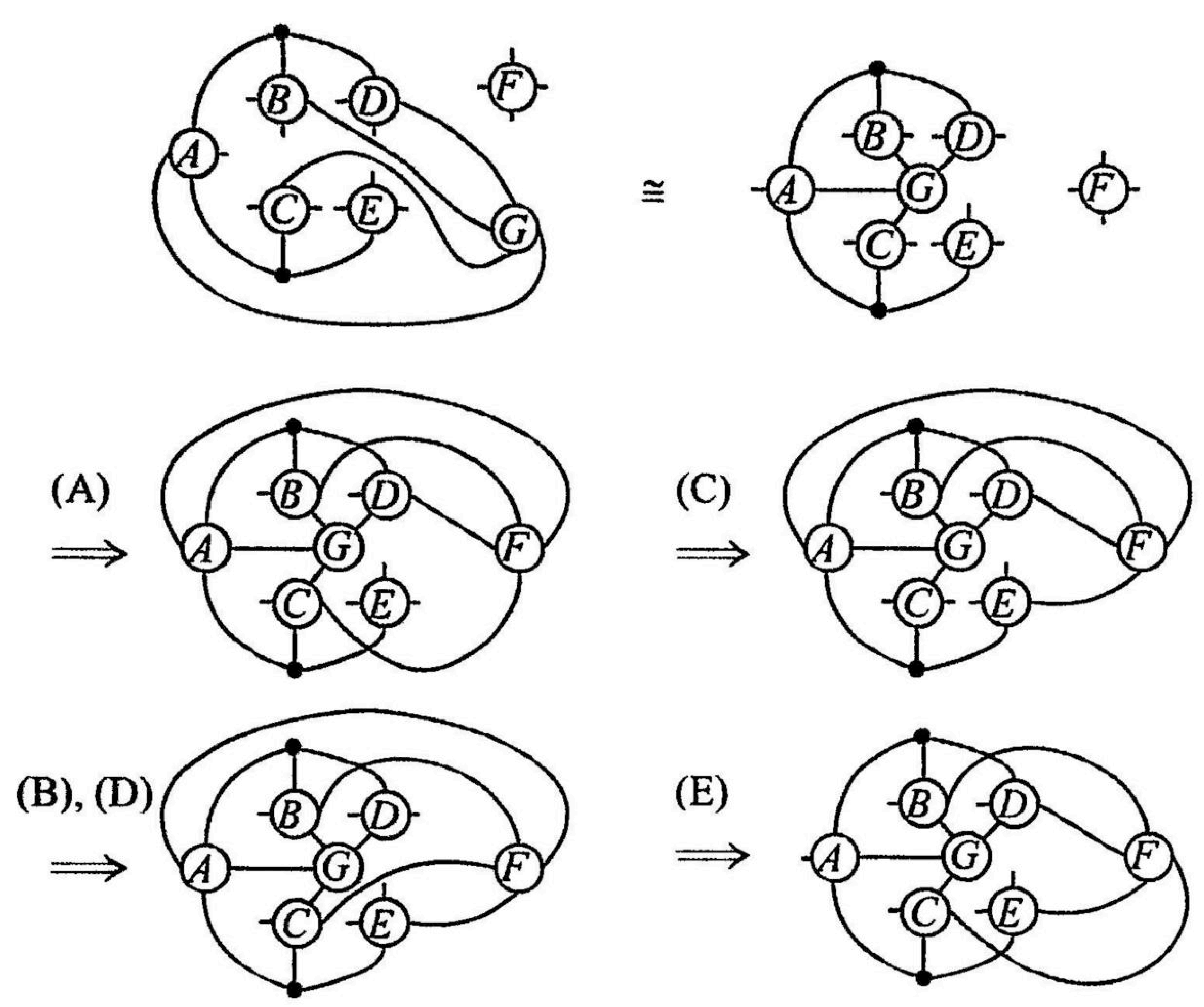


FIG. 3.136. $t = 7$ (c) (i).

(A) $F \sim A, B, C, D$. This gives a graph containing $K_{3,3}$, and so it does not satisfy the condition (P5); see Fig. 3.137.

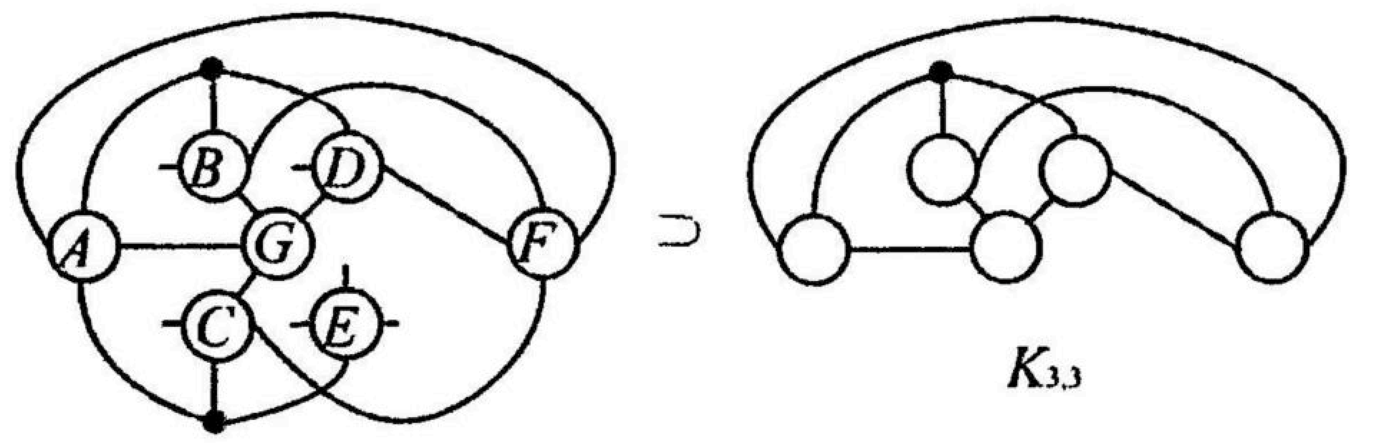


FIG. 3.137. $t = 7$ (c) (i) (A).

(B) $F \sim A, B, C, E$. The vertex E has two remaining hands, so we consider how the hands of E connect. There are three cases; see Fig. 3.138.

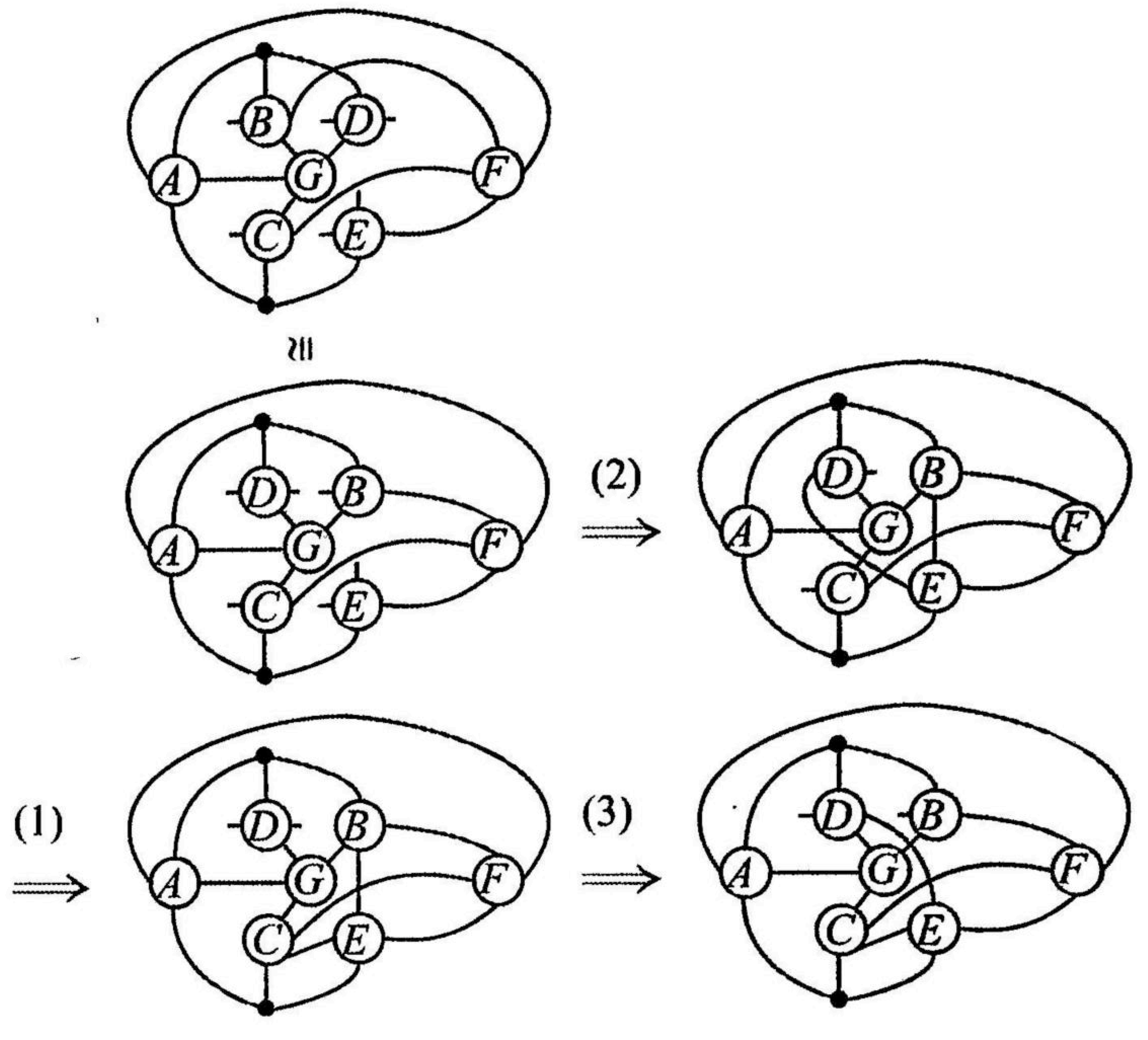


FIG. 3.138. $t = 7$ (c) (i) (B).

(1) $E \sim B, C$. This gives a graph having a loop at D, and so it does not satisfy the condition (P1); see Fig. 3.138.
(2) $E \sim B, D$. This gives a graph containing $K_{3,3}$, and so it does not satisfy the condition (P5); see Fig. 3.139.

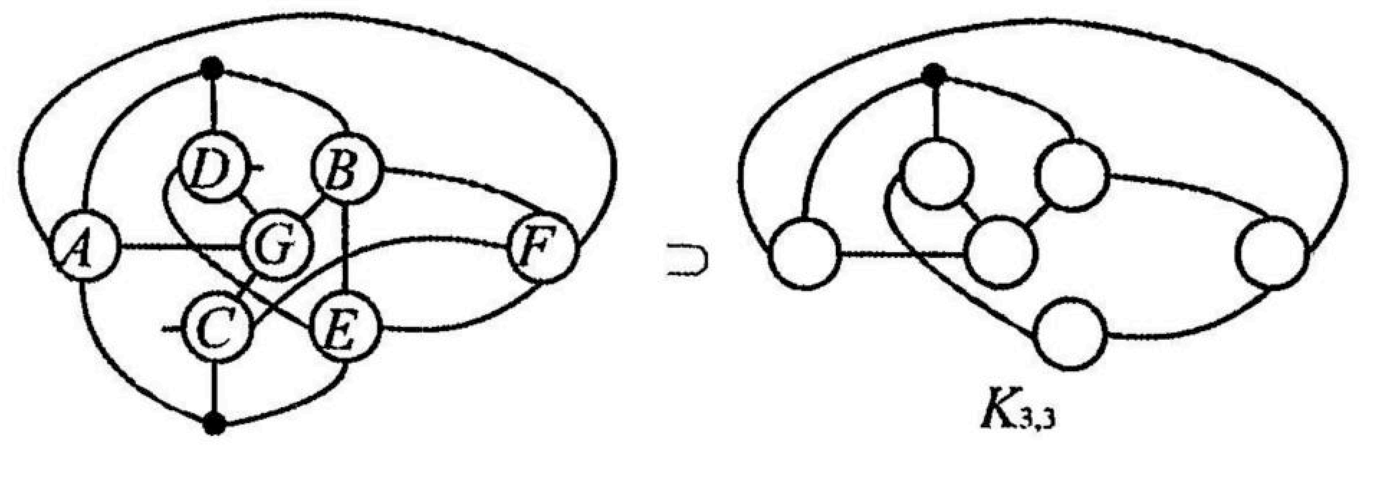


FIG. 3.139. $t = 7$ (c) (i) (B) (2).

(3) $E \sim C$ and $E \sim D$. This gives a graph containing $K_{3,3}$, and so it does not satisfy the condition (P5); see Fig. 3.140.

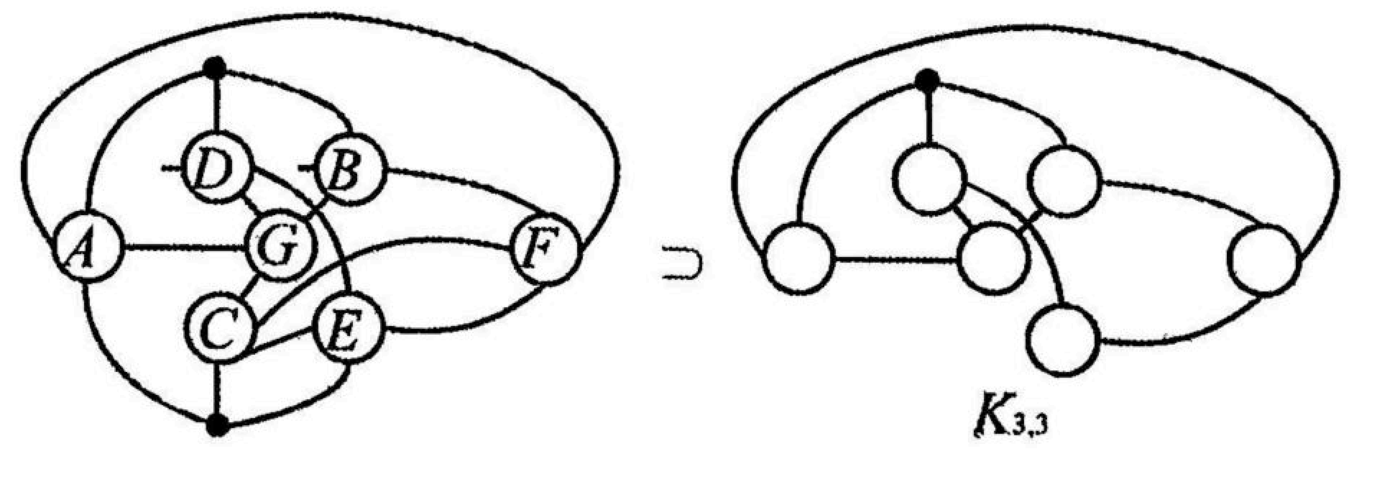


FIG. 3.140. $t = 7$ (c) (i) (B) (3).

(C) $F \sim A, B, D, E$. This gives a graph containing $K_{3,3}$, and so it does not satisfy the condition (P5); see Fig. 3.141.

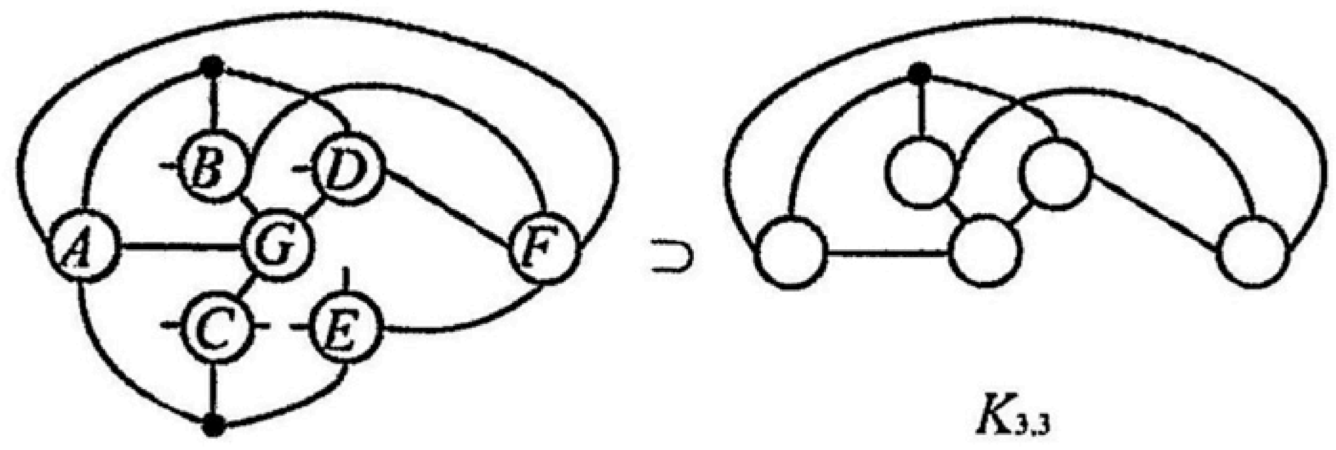


FIG. 3.141. $t = 7$ (c) (i) (C).

(D) $F \sim A, C, D, E$. Since B and D are interchangeable in the first figure in Fig. 3.136, this case is the same as the case (B).

(E) $F \sim B, C, D, E$. This gives a graph containing $K_{3,3}$, and so it does not satisfy the condition (P5); see Fig. 3.142.

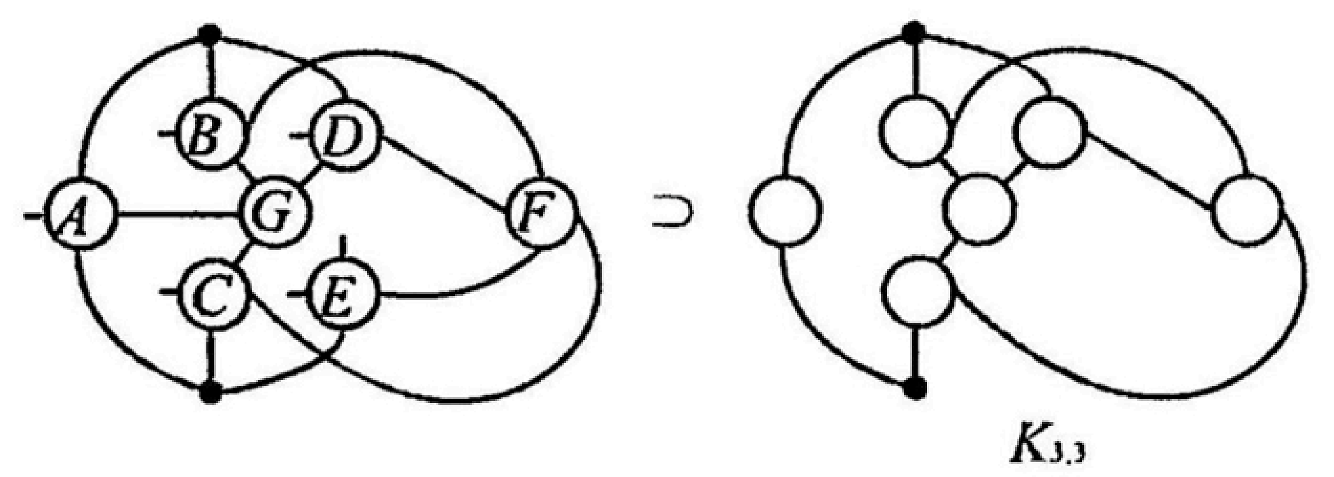


FIG. 3.142. $t = 7$ (c) (i) (E).

(ii) $G \sim A, B, C, E$. Since D and E are interchangeable in the first figure in Fig. 3.135, this case is the same as the case (i).

(iii) $G \sim A, B, C, F$. The vertex F has three remaining hands, so we consider how the hands of F connect. There are ten cases; see Fig. 3.143.

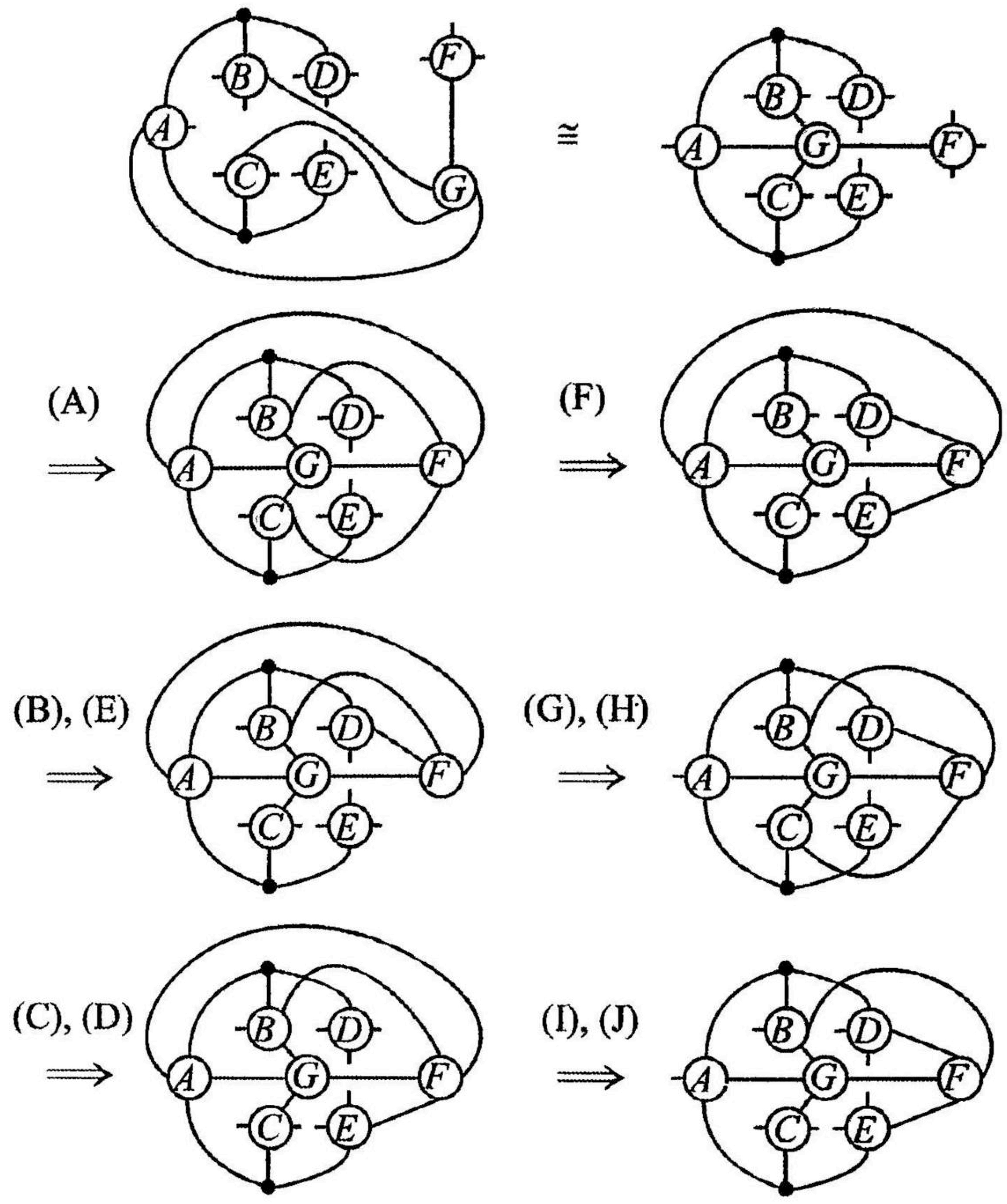


FIG. 3.143. $t = 7$ (c) (iii).

(A) $F \sim A, B, C$. Then $E \sim B, C, D$. This gives a graph having a loop at D, and so it does not satisfy the condition (P1); see Fig. 3.144.

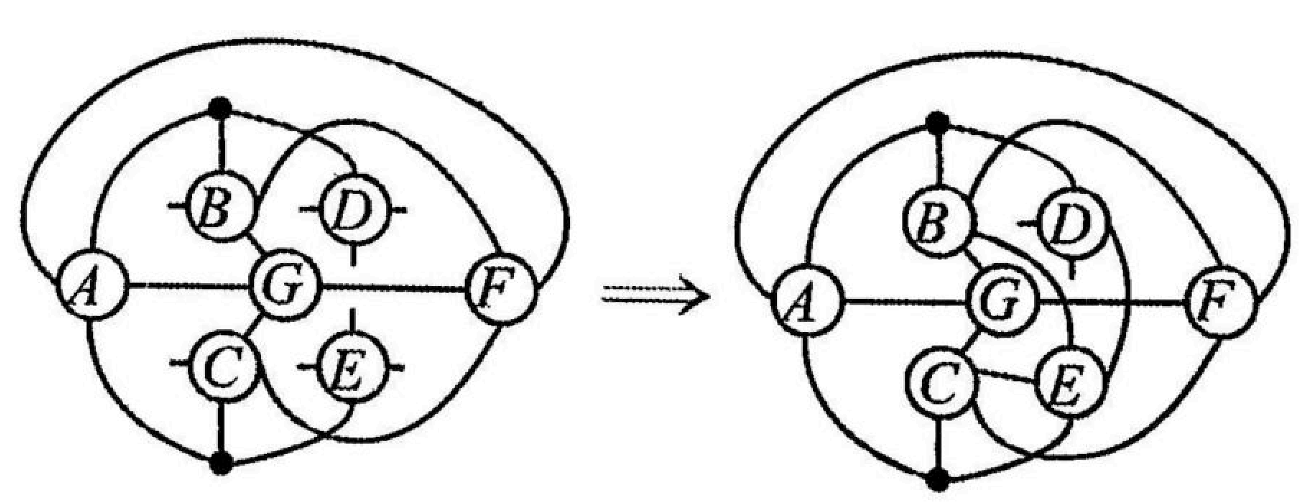


FIG. 3.144. $t = 7$ (c) (iii) (A).

(B) $F \sim A, B, D$. Then $E \sim B, C, D$. This gives a graph containing $K_{3,3}$, and so it does not satisfy the condition (P5); see Fig. 3.145.

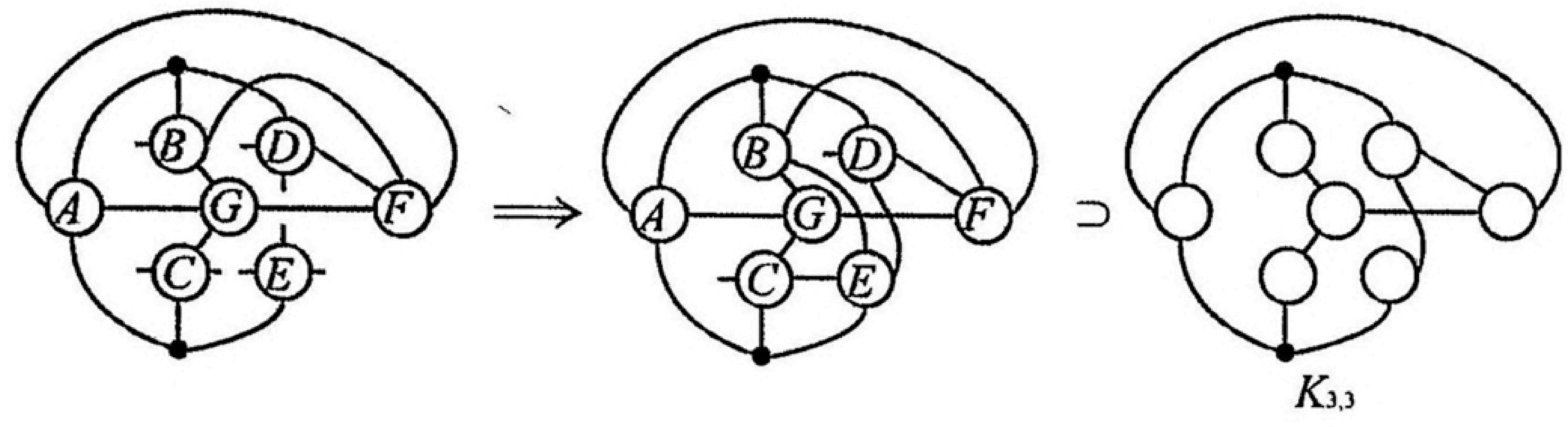


FIG. 3.145. $t = 7$ (c) (iii) (B).

(C) $F \sim A, B, E$. Then $D \sim B, C, E$. This gives a graph containing $K_{3,3}$, and so it does not satisfy the condition (P5); see Fig. 3.146.

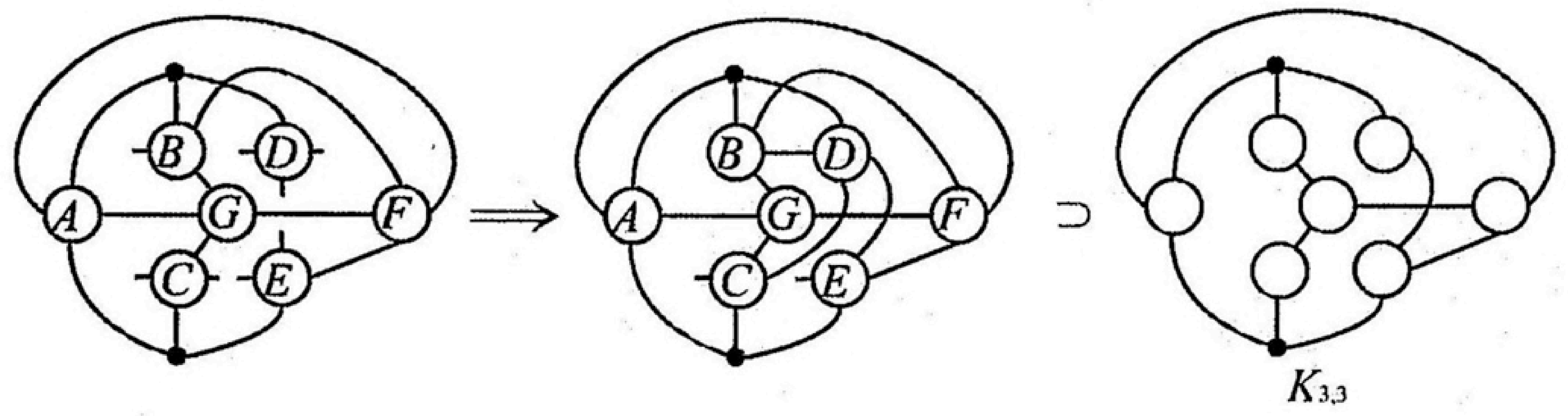


FIG. 3.146. $t = 7$ (c) (iii) (C).

(D) $F \sim A, C, D$. Since B and C, D and E are interchangeable in the first figure in Fig. 3.143, this case is the same as the case (C).
(E) $F \sim A, C, E$. Since B and C, D and E are interchangeable in the first figure in Fig. 3.143, this case is the same as the case (B).
(F) $F \sim A, D, E$. The vertex E has two remaining hands, so we consider how the hands of E connect. There are three cases; see Fig. 3.147.

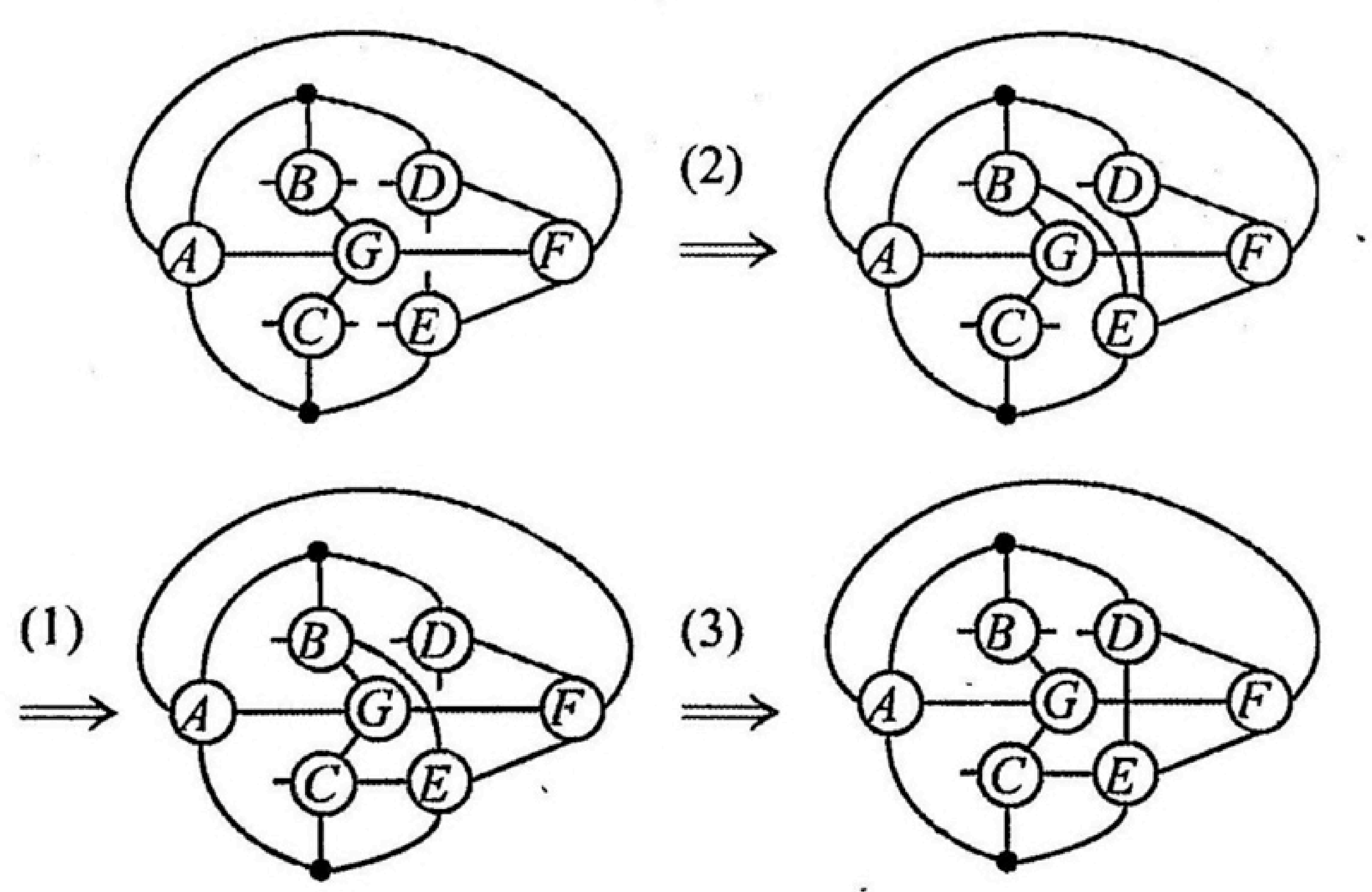


FIG. 3.147. $t = 7$ (c) (iii) (F).

(1) $E \sim B, C$. This gives a graph containing $K_{3,3}$, and so it does not satisfy the condition (P5); see Fig. 3.148.

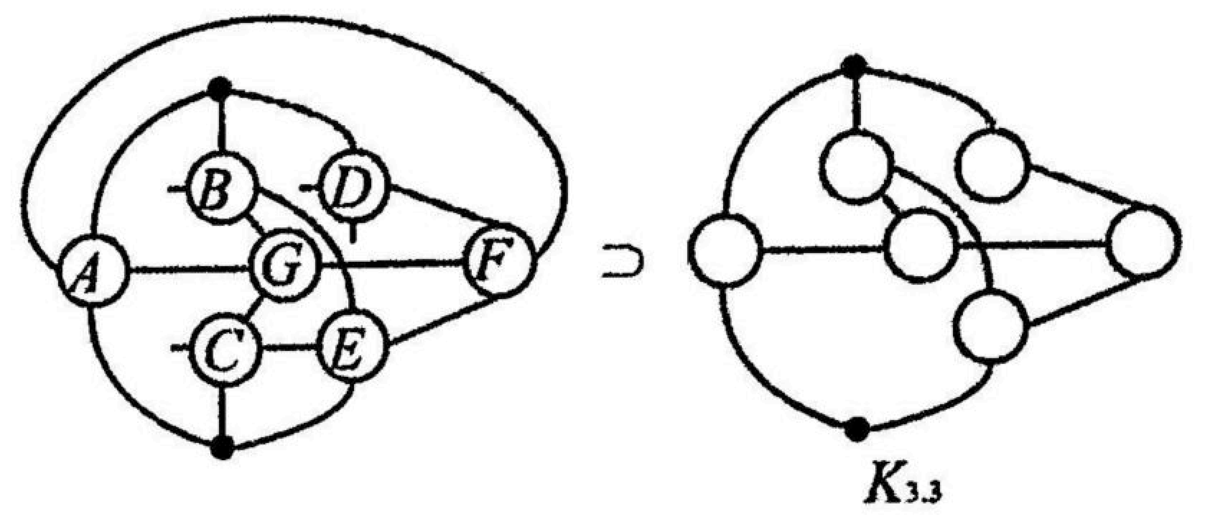


FIG. 3.148. $t = 7$ (c) (iii) (F) (1).

(2) $E \sim B, D$. This gives a graph containing $K_{3,3}$, and so it does not satisfy condition (P5); see Fig. 3.149.

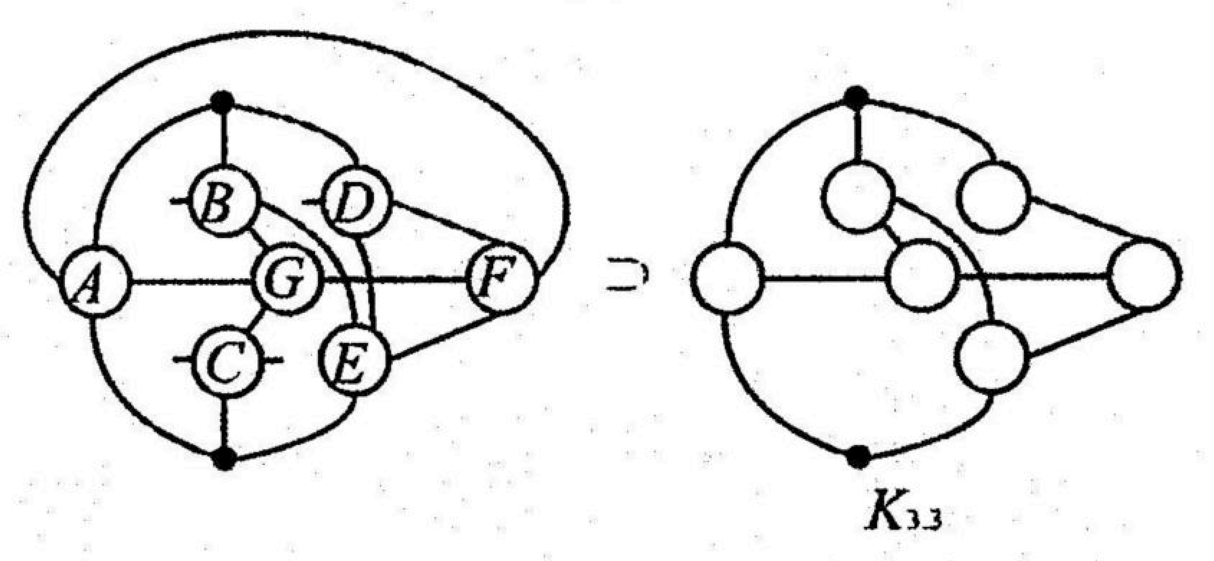


FIG. 3.149. $t = 7$ (c) (iii) (F) (2).

(3) $E \sim C, D$. This gives a graph containing $K_{3,3}$, and so it does not satisfy the condition (P5); see Fig. 3.150.

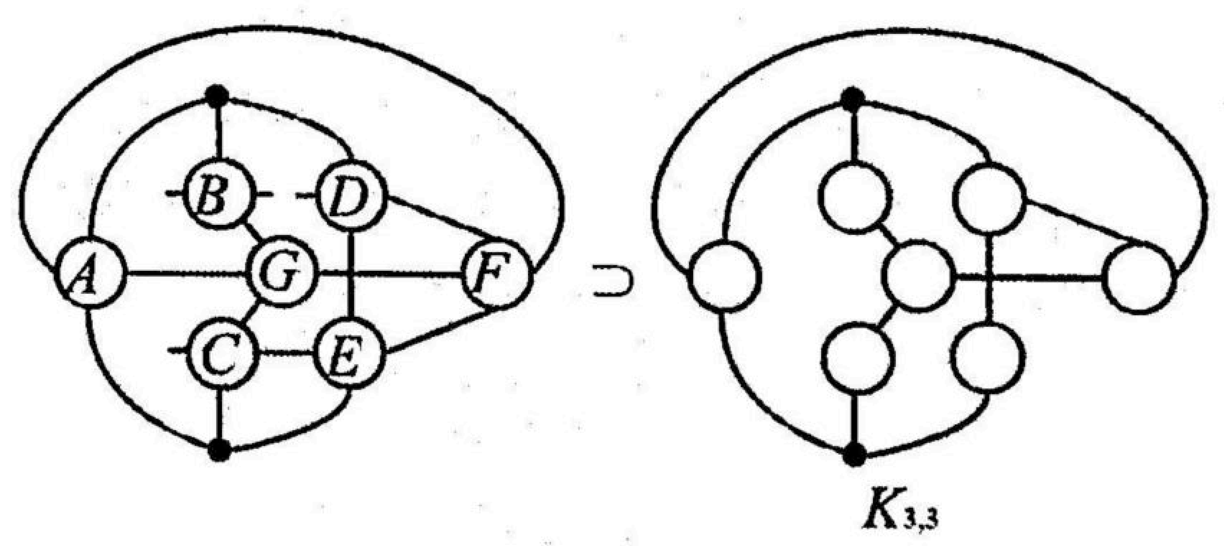


FIG. 3.150. $t = 7$ (c) (iii) (F) (3).

(G) $F \sim B, C, D$. The vertex E has three remaining hands, so we consider how the hands of E connect. There are four cases; see Fig. 3.151.

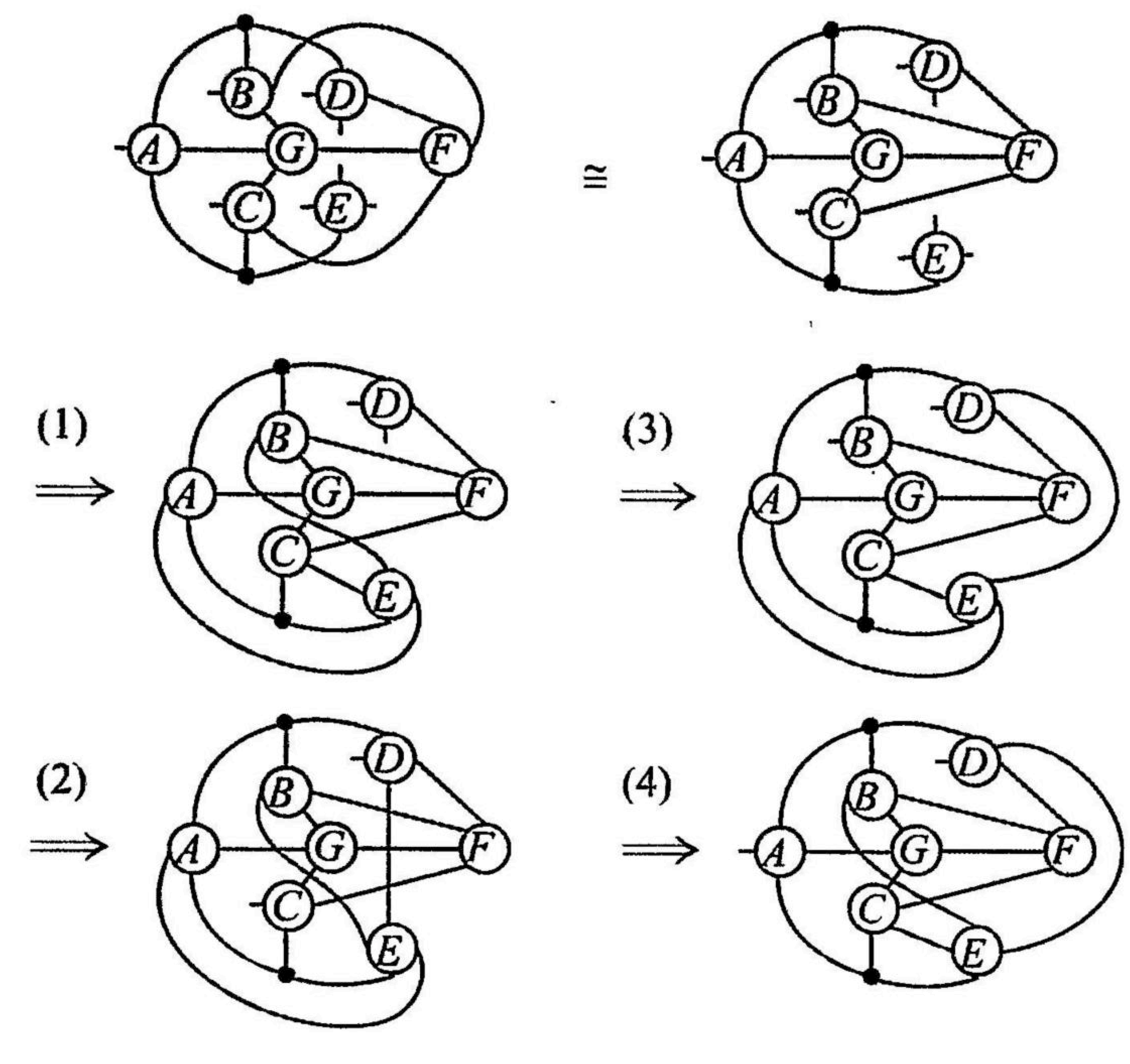


FIG. 3.151. $t = 7$ (c) (iii) (G).

(1) $E \sim A, B, C$. This gives a graph having a loop at D, and so it does not satisfy the condition (P1); see Fig. 3.151.
(2) $E \sim A, B, D$. This gives a graph containing $K_{3,3}$, and so it does not satisfy the condition (P5); see Fig. 3.152.

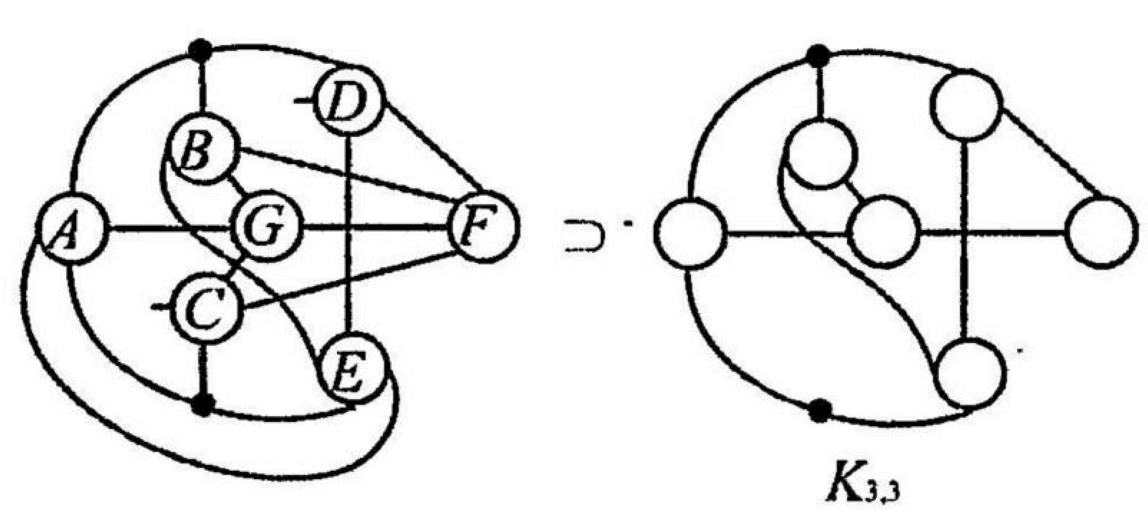


FIG. 3.152. $t = 7$ (c) (iii) (G) (2).

(3) $E \sim A, C, D$. Then $B \sim D$, and we obtain 7^3_*; see Fig. 3.153.

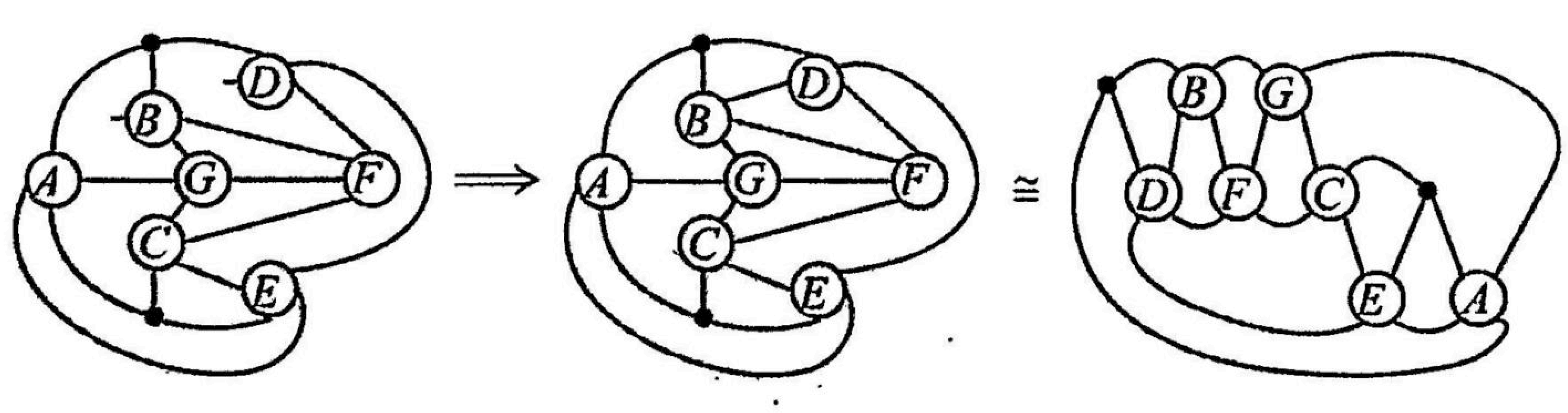


FIG. 3.153. $t = 7$ (c) (iii) (G) (3).

(4) $E \sim B, C, D$. This gives a graph containing $K_{3,3}$, and so it does not satisfy the condition (P5); see Fig. 3.154.

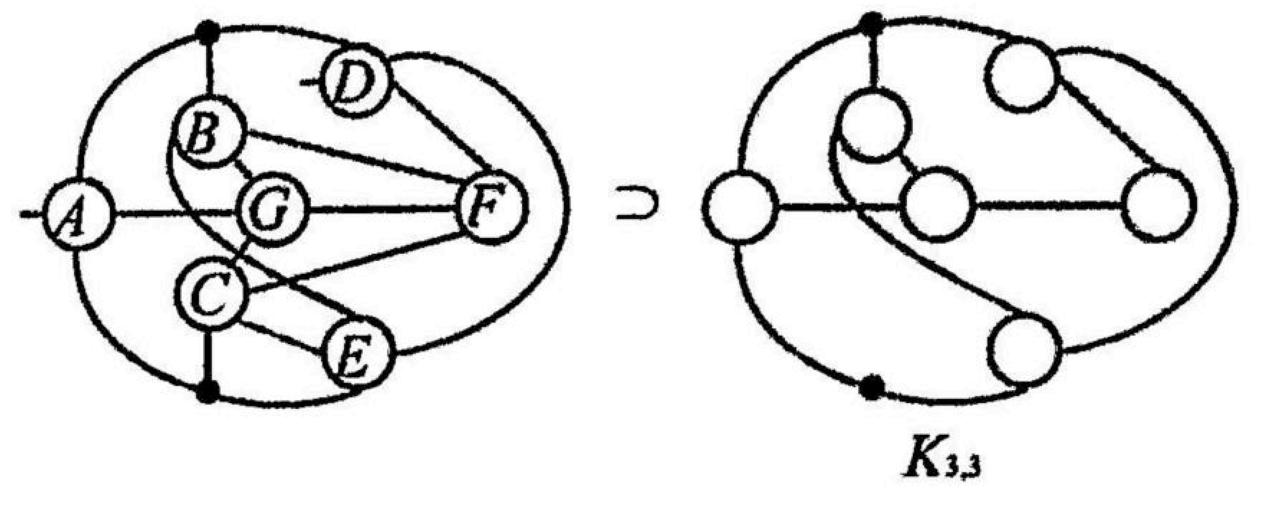


FIG. 3.154. $t = 7$ (c) (iii) (G) (4).

(H) $F \sim B, C, E$. Since D and E are interchangeable in the first figure in Fig. 3.143, case is the same as the case (G).

(I) $F \sim B, D, E$. The vertex E has two remaining hands, so we consider how the of E connect. There are six cases; see Fig. 3.155.

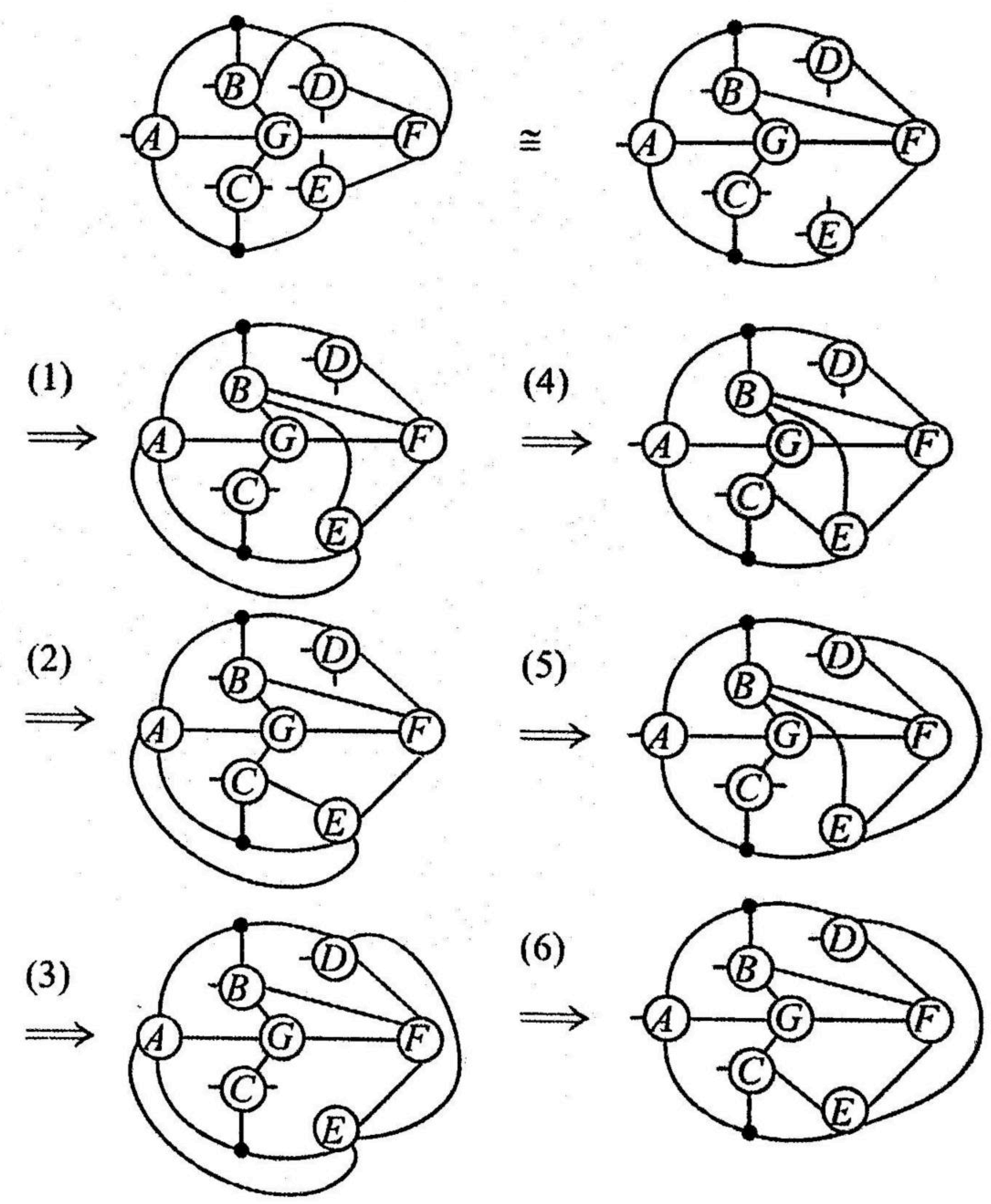


FIG. 3.155. $t = 7$ (c) (iii) (I).

(1) $E \sim A, B$. This gives a graph which does not satisfy the conditions (P1) or (P2); see Fig. 3.155.

(2) $E \sim A, C$. Then $D \sim B, C$. This gives a graph containing $K_{3,3}$, and so it does not satisfy the condition (P5); see Fig. 3.156.

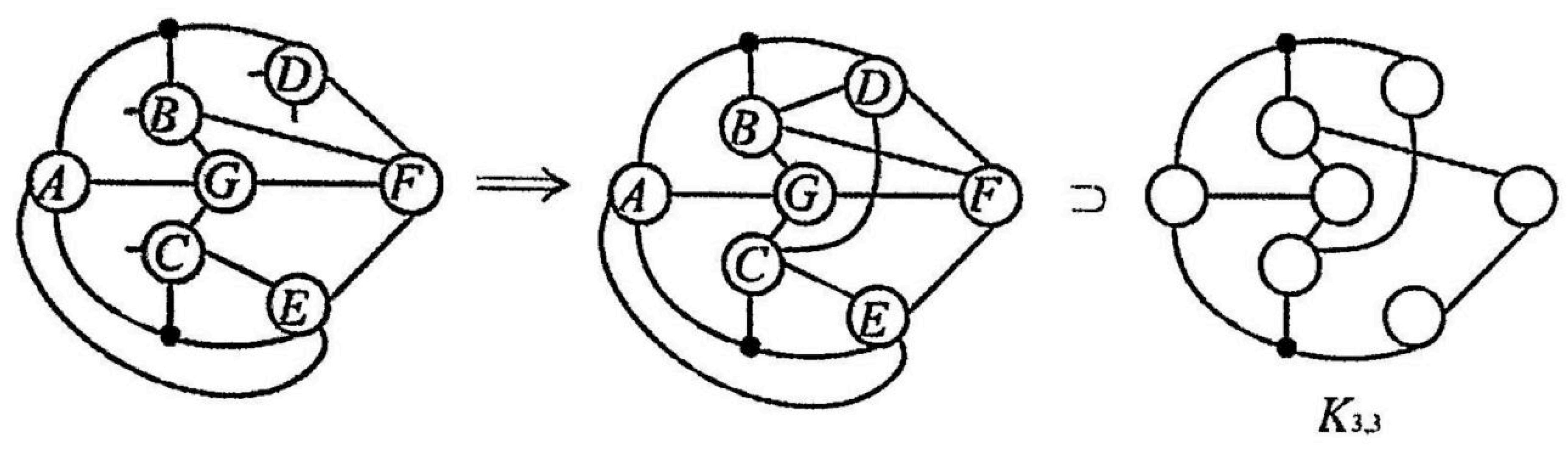


FIG. 3.156. $t = 7$ (c) (iii) (I) (2).

(3) $E \sim A, D$. Then $C \sim B, D$. This gives a graph containing $K_{3,3}$, and so it does not satisfy the condition (P5); see Fig. 3.157.

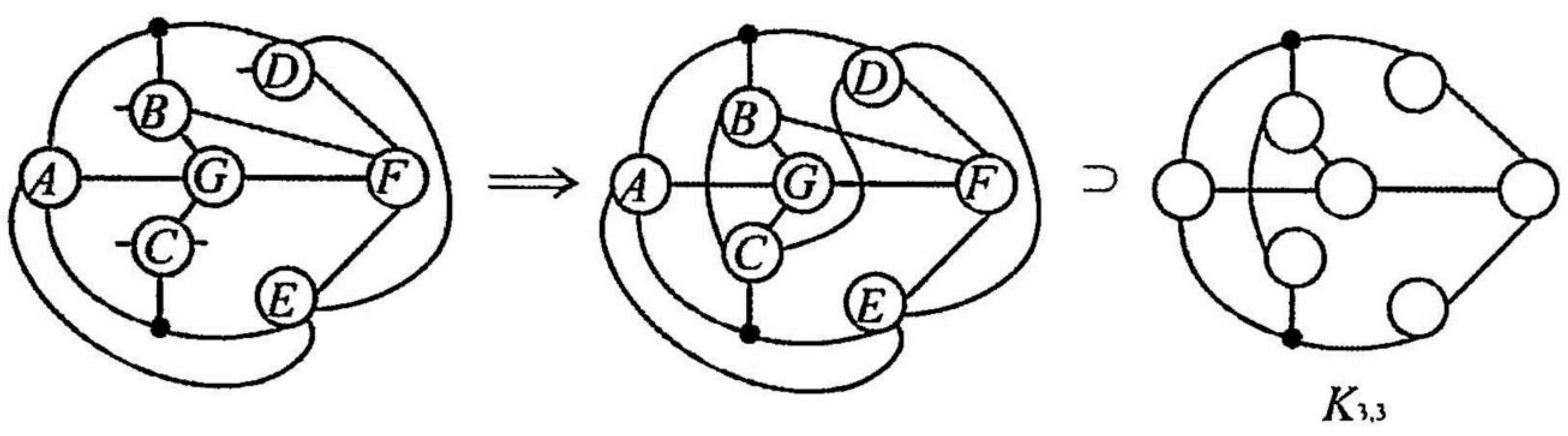


FIG. 3.157. $t = 7$ (c) (iii) (I) (3).

(4) $E \sim B, C$. This gives a graph containing $K_{3,3}$, and so it does not satisfy the condition (P5); see Fig. 3.158.

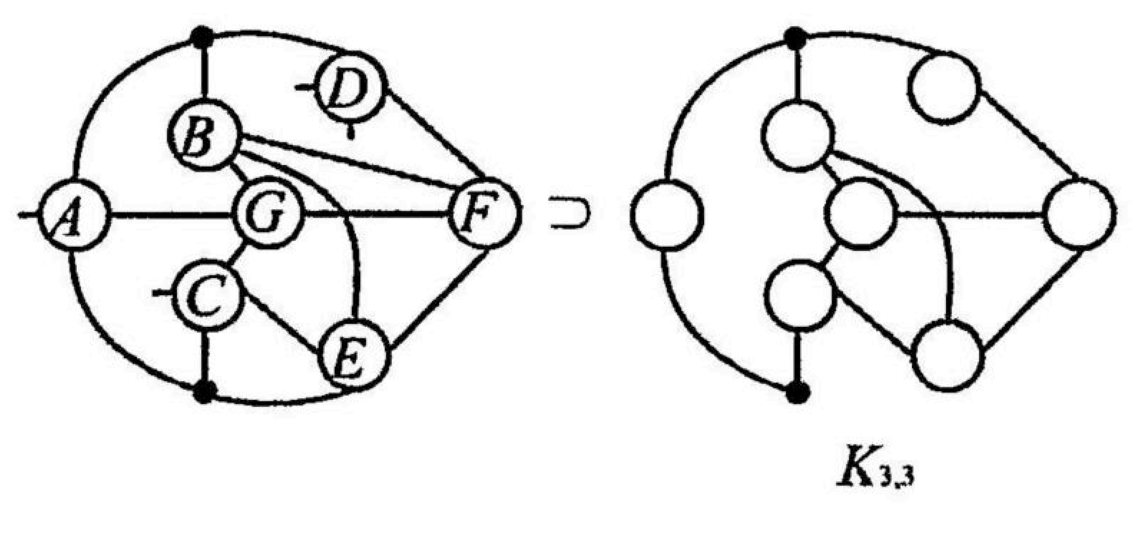


FIG. 3.158. $t = 7$ (c) (iii) (I) (4).

(5) $E \sim B, D$. This gives a graph containing $K_{3,3}$, and so it does not satisfy the condition (P5); see Fig. 3.159.

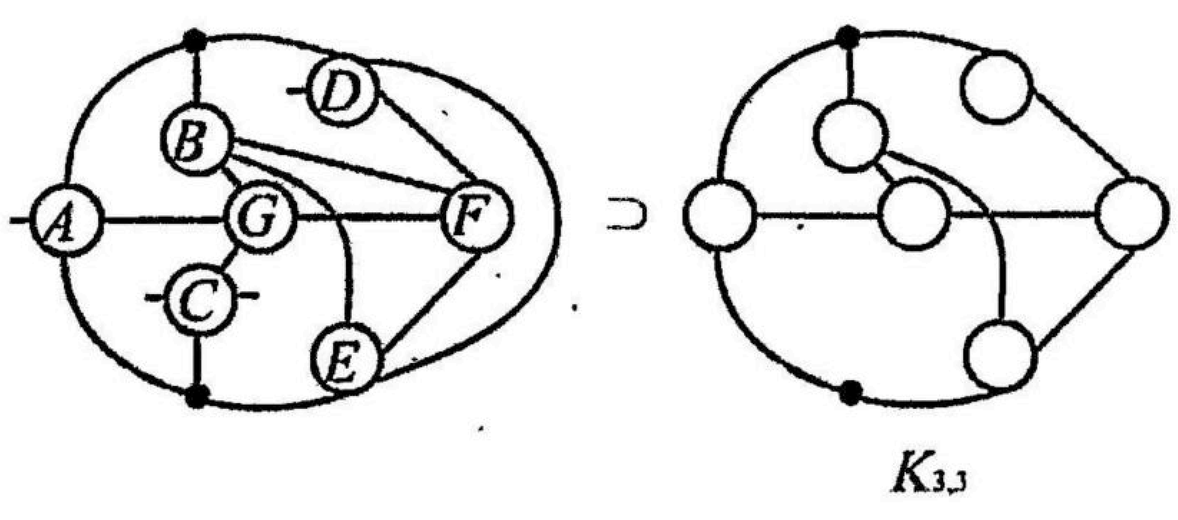


FIG. 3.159. $t = 7$ (c) (iii) (I) (5).

(6) $E \sim C, D$. We have three cases; see Fig. 3.160.

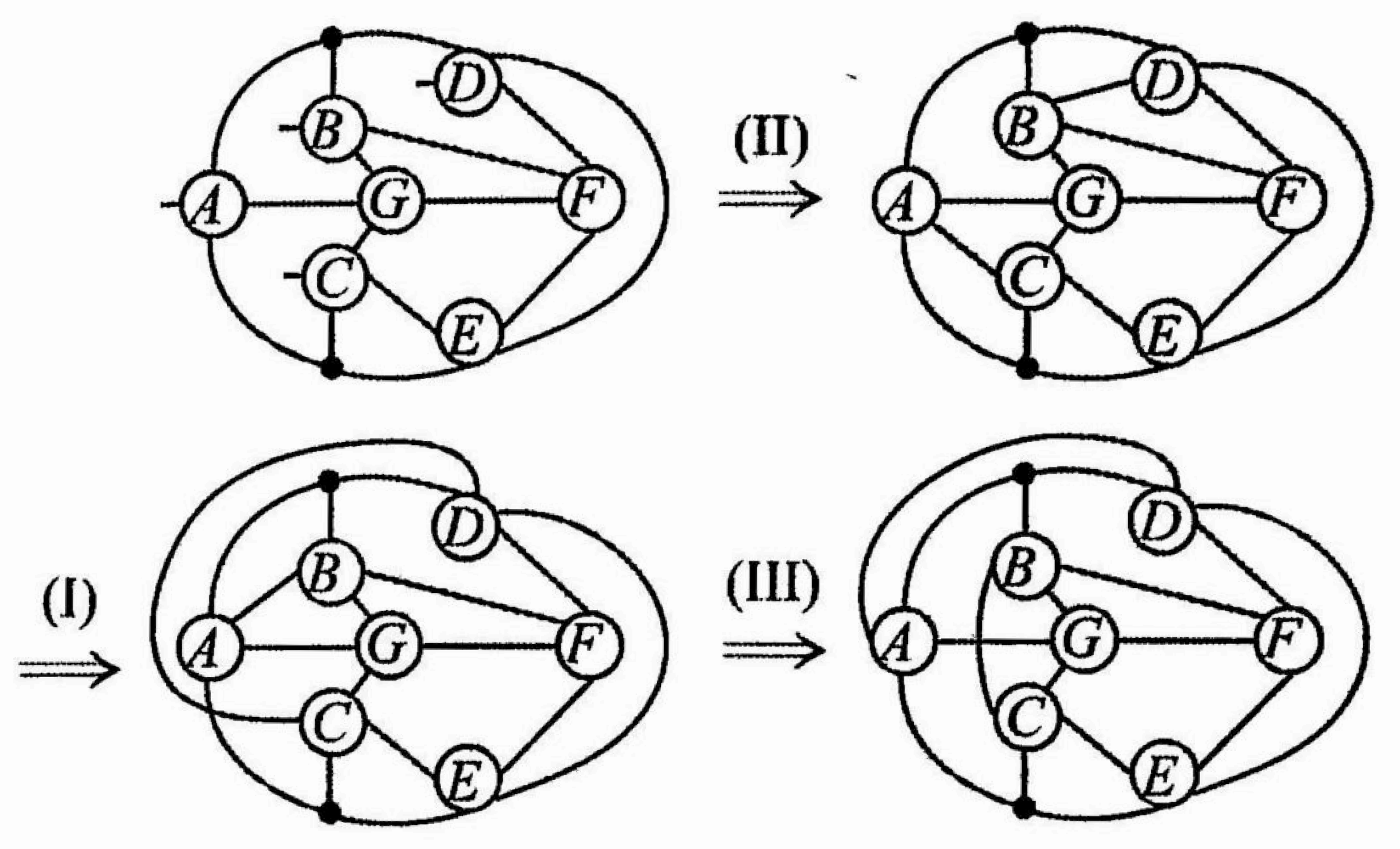


FIG. 3.160. $t = 7$ (c) (iii) (I) (6).

(I) $A \sim B$, $C \sim D$. This gives a graph containing $K_{3,3}$, and so it does not satisfy the condition (P5); see Fig. 3.161.

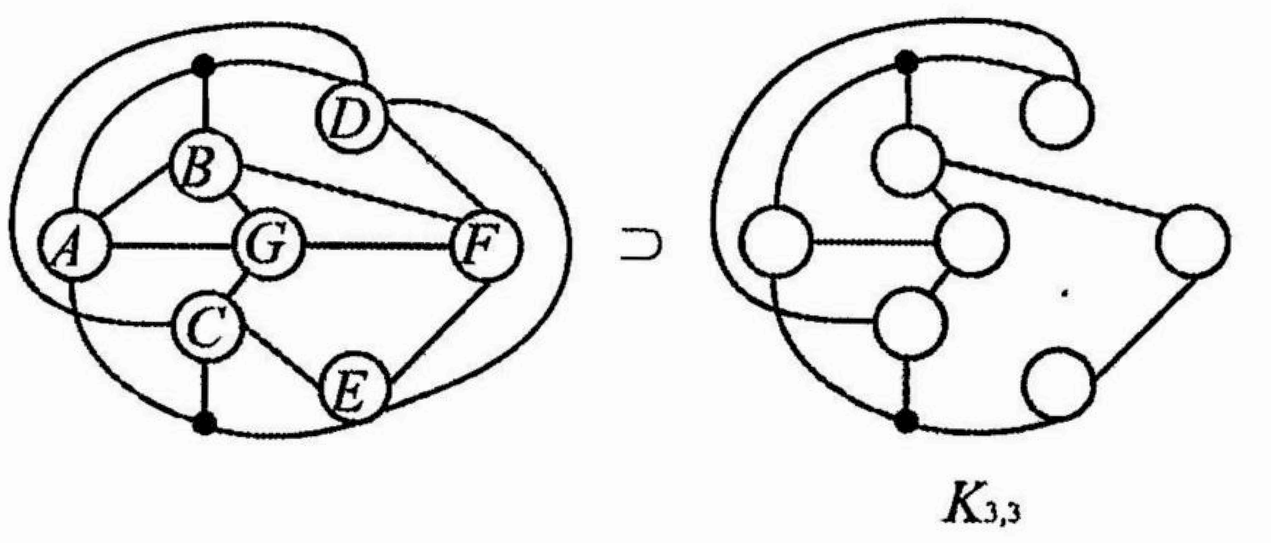


FIG. 3.161. $t = 7$ (c) (iii) (I) (6) (I).

(II) $A \sim C$, $B \sim D$. We obtain 7^4_*; see Fig. 3.162.

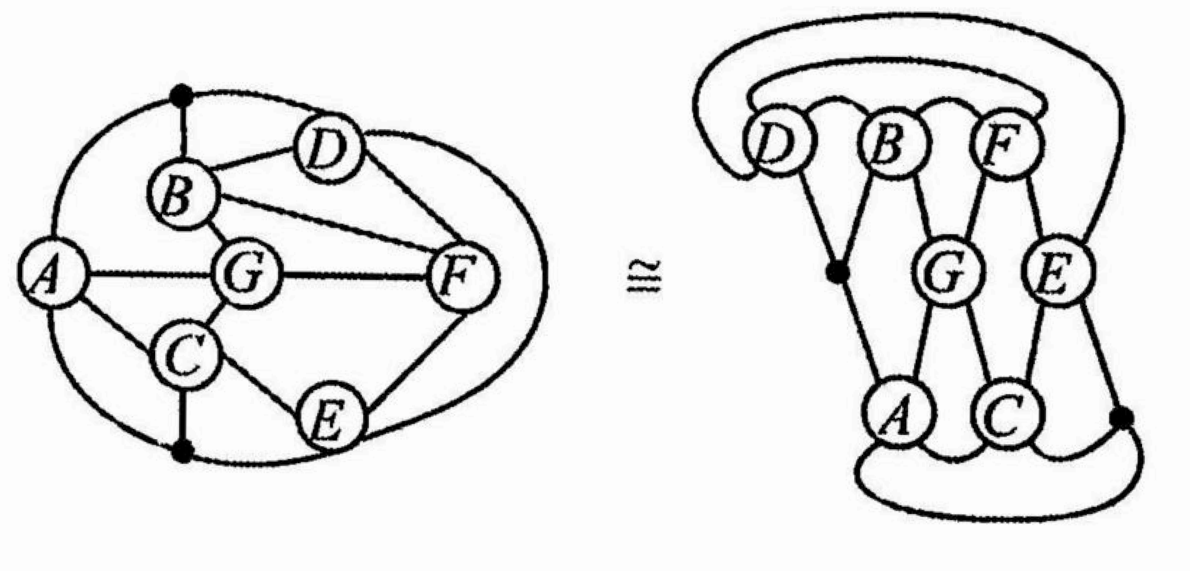


FIG. 3.162. $t = 7$ (c) (iii) (I) (6) (II).

(III) $A \sim D$, $B \sim C$. This gives a graph containing $K_{3,3}$, and so it does not satisfy the condition (P5); see Fig. 3.163.

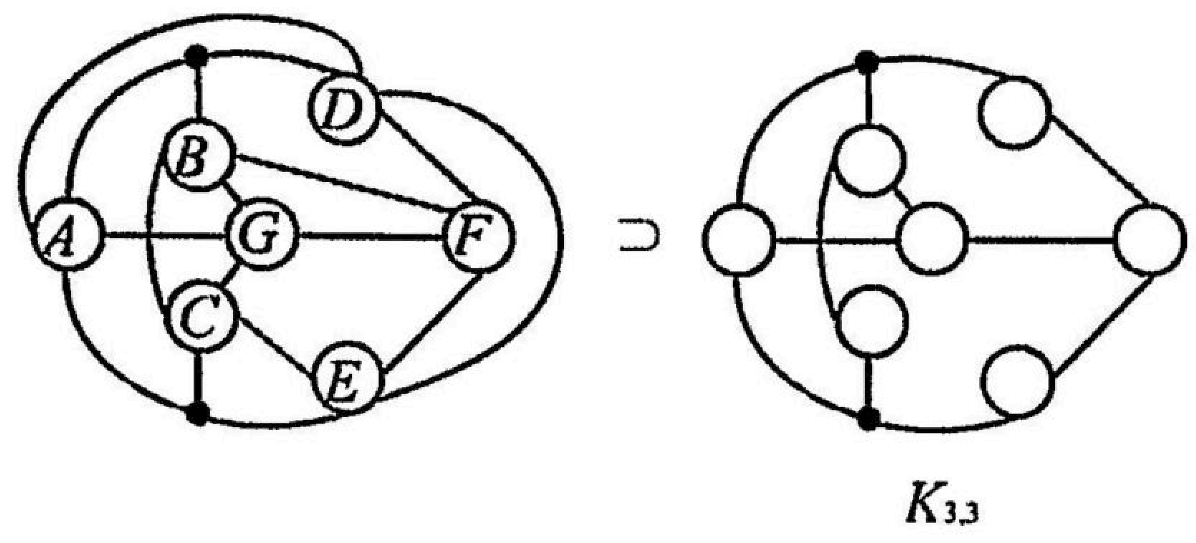


FIG. 3.163. $t = 7$ (c) (iii) (I) (6) (III).

(J) $F \sim C, D, E$. Since B and C are interchangeable in the first figure in Fig. 3.143, this case is the same as the case (I).

(iv) $G \sim A, B, D, E$. Since C and E are interchangeable in the first figure in Fig. 3.135, this case is the same as the case (i).

(v) $G \sim A, B, D, F$. The vertex F has three remaining hands, so we consider how the hands of F connect. There are ten cases; see Fig. 3.164.

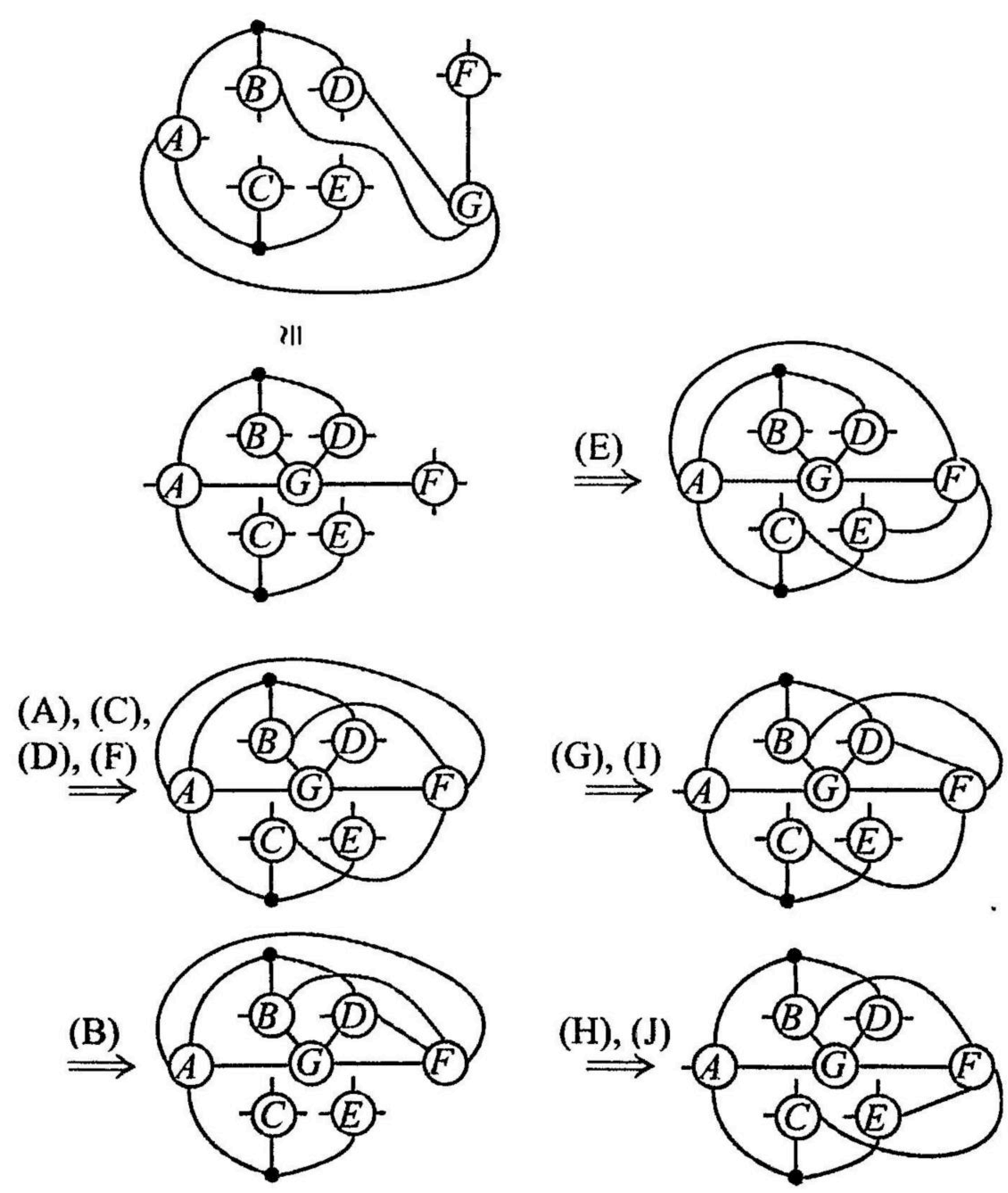


FIG. 3.164. $t = 7$ (c) (v).

(A) $F \sim A, B, C$. Then $E \sim B, C, D$. This gives a graph containing $K_{3,3}$, and so it does not satisfy the condition (P5); see Fig. 3.165.

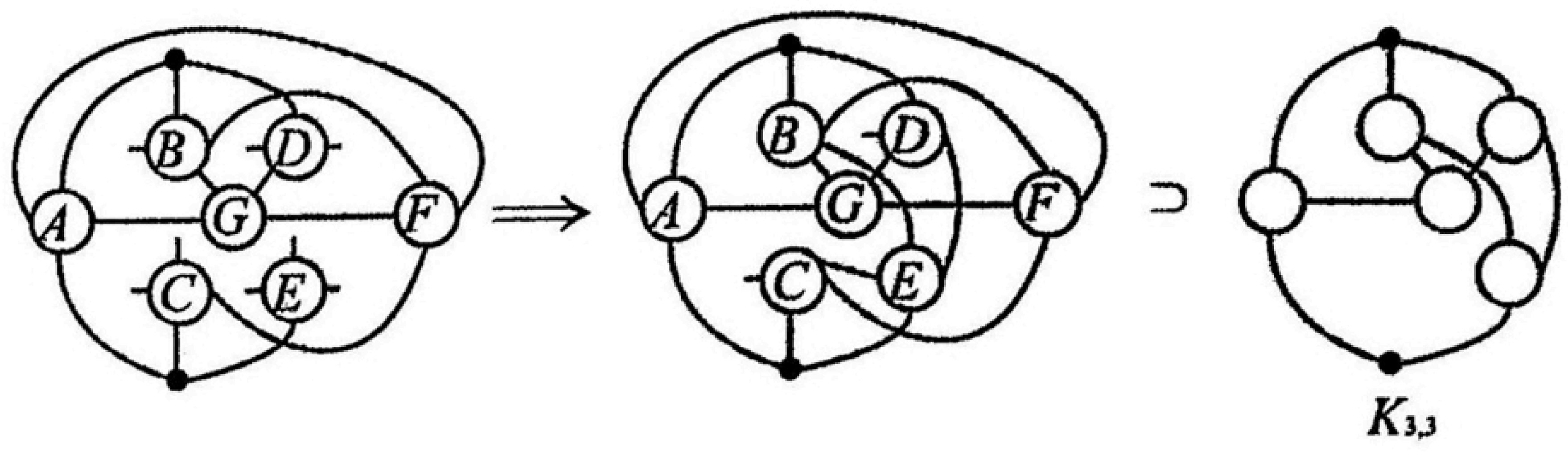


FIG. 3.165. $t = 7$ (c) (v) (A).

(B) $F \sim A, B, D$. This gives a graph containing $K_{3,3}$, and so it does not satisfy condition (P5); see Fig. 3.166.

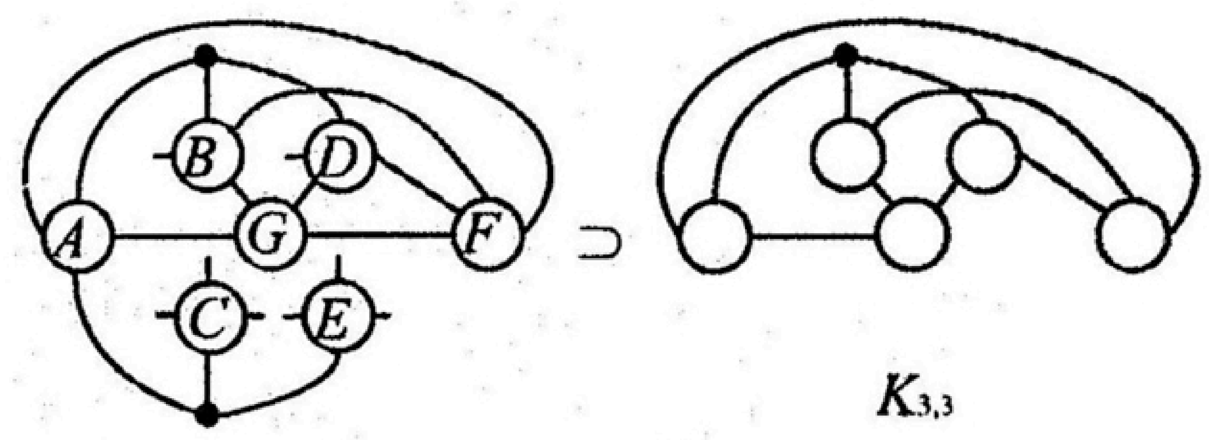


FIG. 3.166. $t = 7$ (c) (v) (B).

(C) $F \sim A, B, E$. Since C and E are interchangeable in the first figure in Fig. 3.164, this case is the same as the case (A).

(D) $F \sim A, C, D$. Since B and D are interchangeable in the first figure in Fig. 3.164, this case is the same as the case (A).

(E) $F \sim A, C, E$. The vertex E has two remaining hands, so we consider how the hands of E connect. There are three cases; see Fig. 3.167.

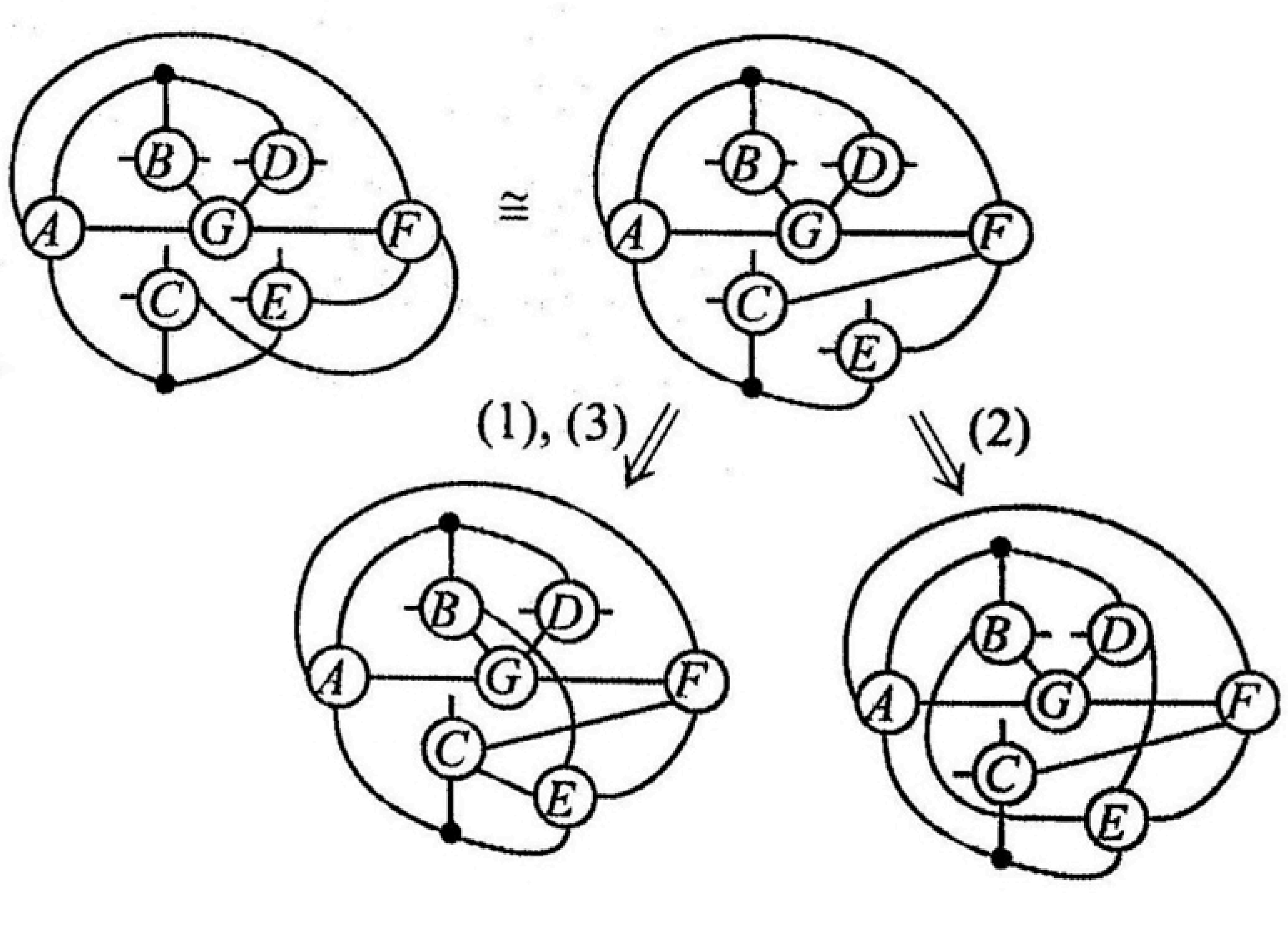


FIG. 3.167. $t = 7$ (c) (v) (E).

(1) $E \sim B, C$. Then $D \sim B, C$. This gives a graph containing $K_{3,3}$, and so it does not satisfy the condition (P5); see Fig. 3.168.

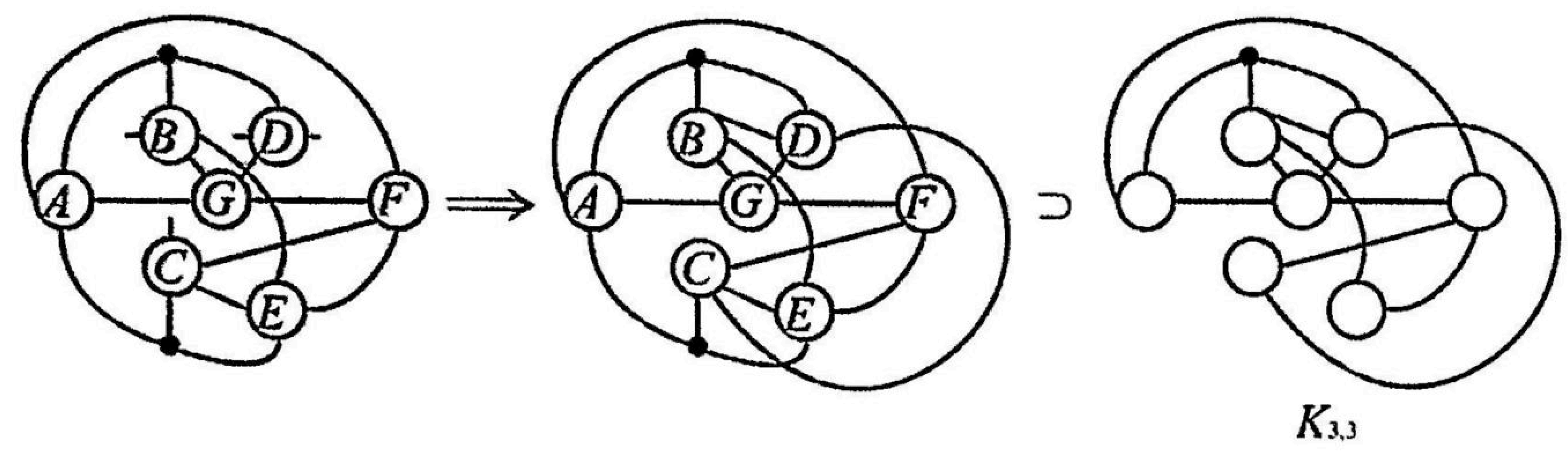


FIG. 3.168. $t = 7$ (c) (v) (E) (1).

(2) $E \sim B, D$. This gives a graph containing $K_{3,3}$, and so it does not satisfy the condition (P5); see Fig. 3.169.

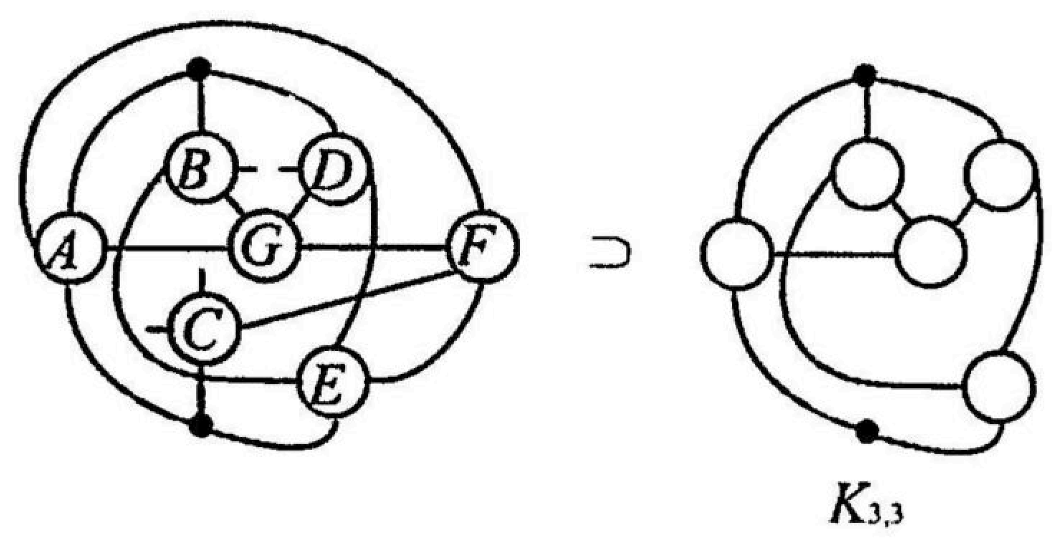


FIG. 3.169. $t = 7$ (c) (v) (E) (2).

(3) $E \sim C, D$. Since B and D are interchangeable in the first figure in Fig. 3.167, this case is the same as the case (1).

(F) $F \sim A, D, E$. Since B and D, C and E are interchangeable in the first figure in Fig. 3.164, this case is the same as the case (A).

(G) $F \sim B, C, D$. This gives a graph containing $K_{3,3}$, and so it does not satisfy the condition (P5); see Fig. 3.170.

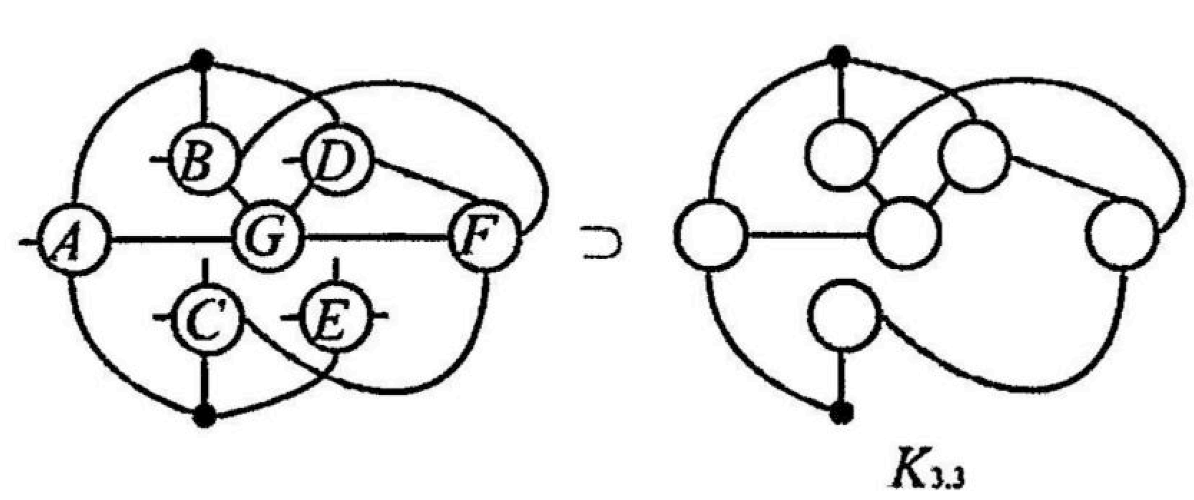


FIG. 3.170. $t = 7$ (c) (v) (G).

(H) $F \sim B, C, E$. The vertex E has two remaining hands, so we consider how the hands of E connect. There are six cases; see Fig. 3.171.

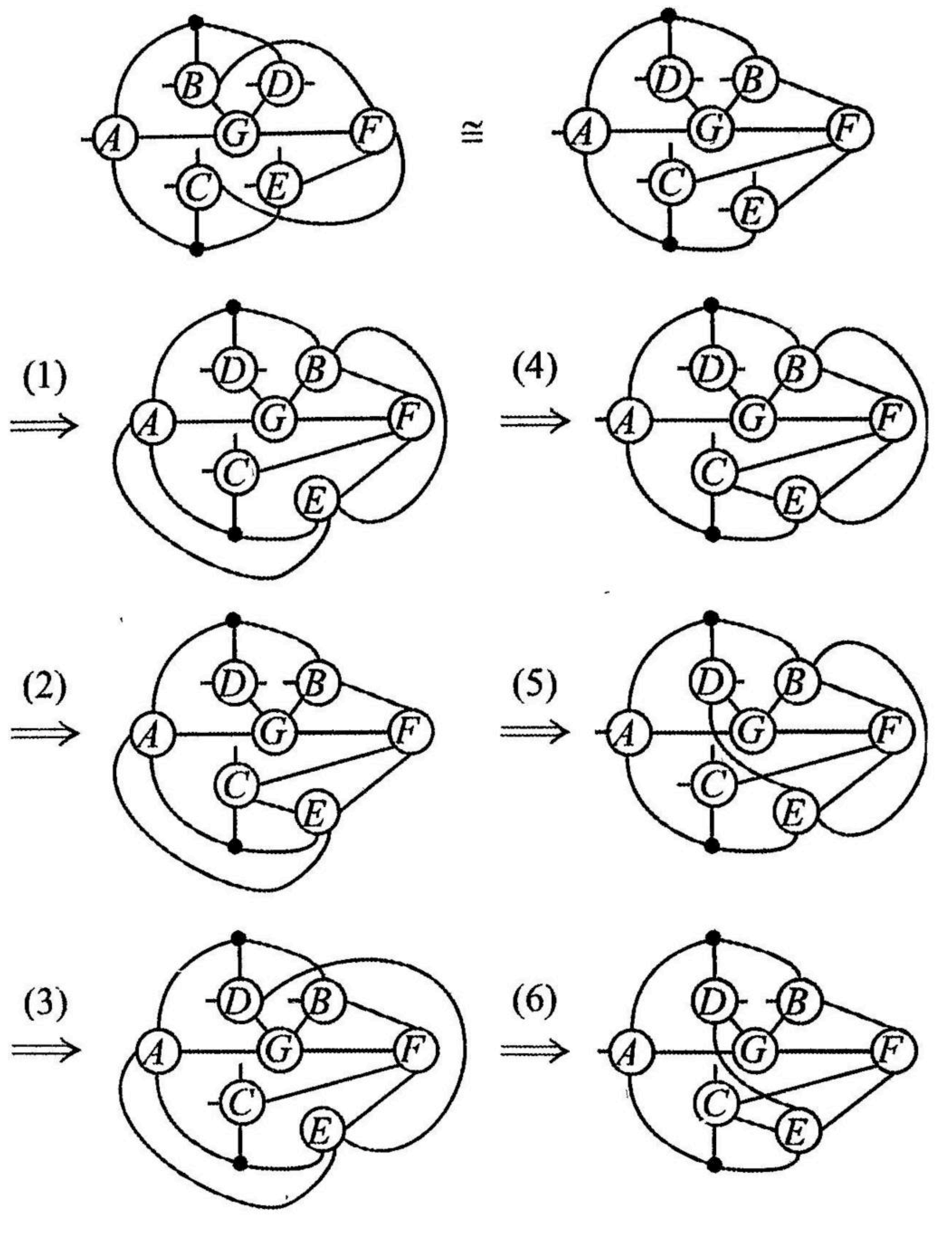


FIG. 3.171. $t = 7$ (c) (v) (H).

(1) $E \sim A, B$. This gives a graph which does not satisfy the conditions (P1) or (P2); see Fig. 3.171.
(2) $E \sim A, C$. Then $D \sim B, C$. This gives a graph containing $K_{3,3}$, and so it does not satisfy the condition (P5); see Fig. 3.172.

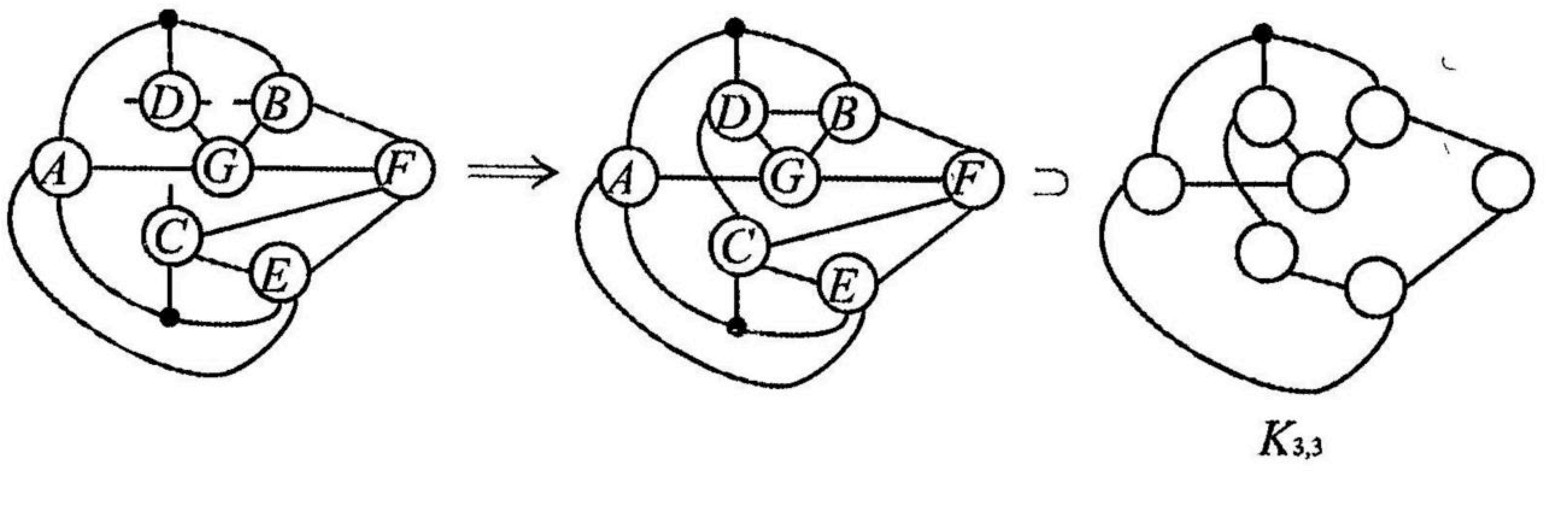


FIG. 3.172. $t = 7$ (c) (v) (H) (2).

(3) $E \sim A, D$. This gives a graph containing $K_{3,3}$, and so it does not satisfy the condition (P5); see Fig. 3.173.

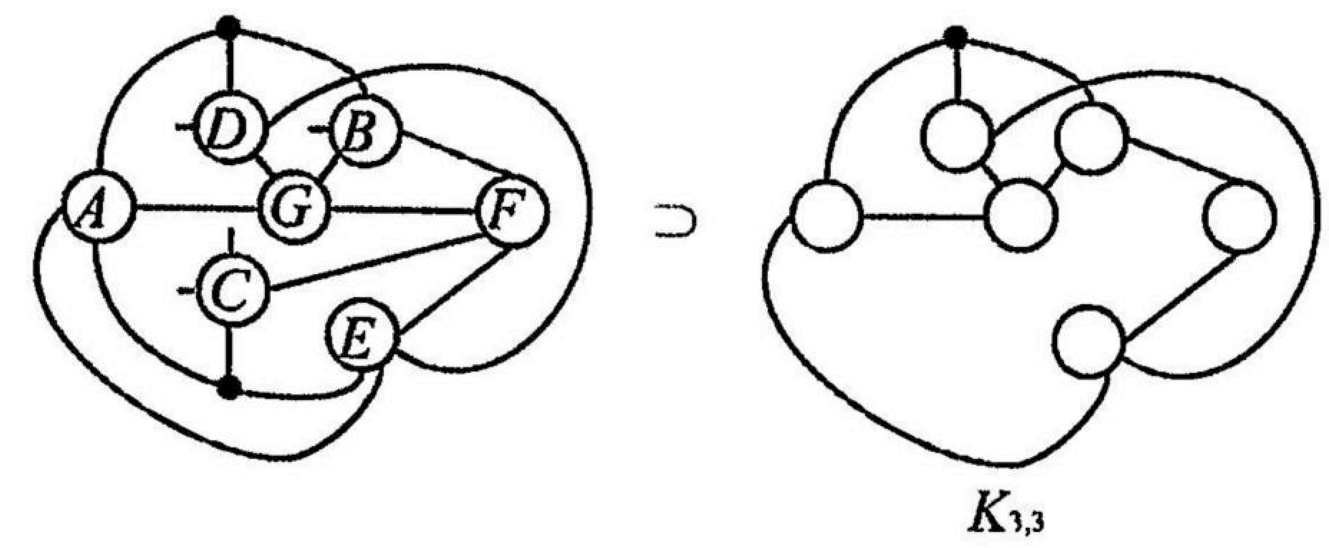


FIG. 3.173. $t = 7$ (c) (v) (H) (3).

(4) $E \sim B, C$. Then $D \sim A, C$. This gives a graph containing $K_{3,3}$, and so it does not satisfy the condition (P5); see Fig. 3.174.

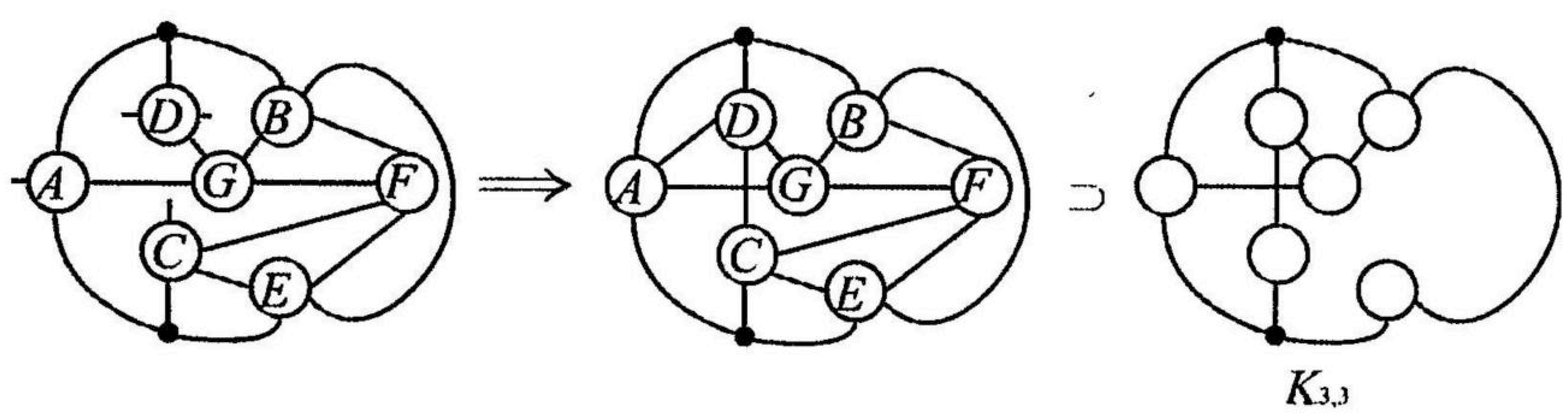


FIG. 3.174. $t = 7$ (c) (v) (H) (4).

(5) $E \sim B, D$. This gives a graph containing $K_{3,3}$, and so it does not satisfy the condition (P5); see Fig. 3.175.

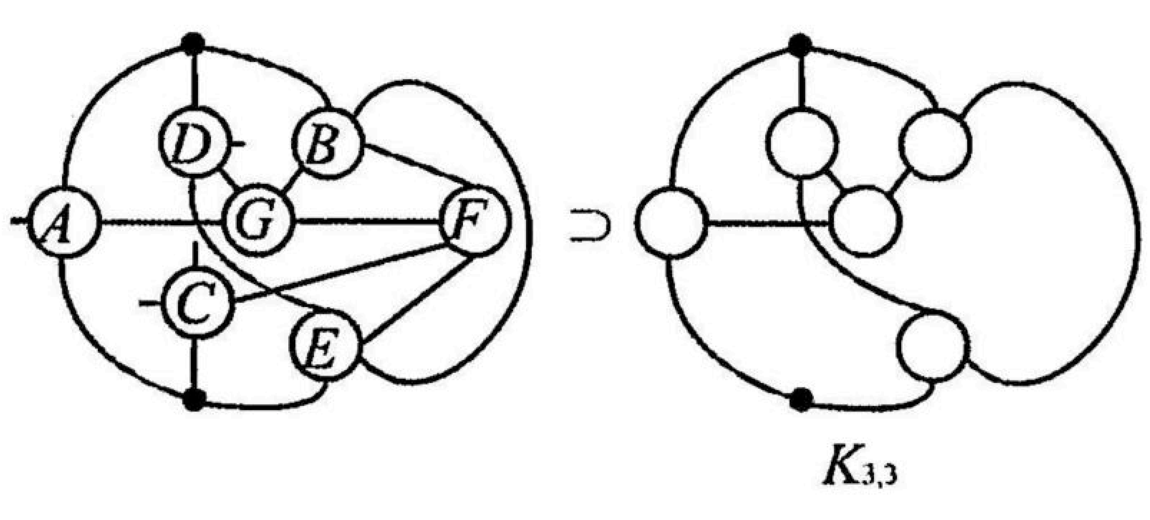


FIG. 3.175. $t = 7$ (c) (v) (H) (5).

(6) $E \sim C, D$. This gives a graph containing $K_{3,3}$, and so it does not satisfy the condition (P5); see Fig. 3.176.

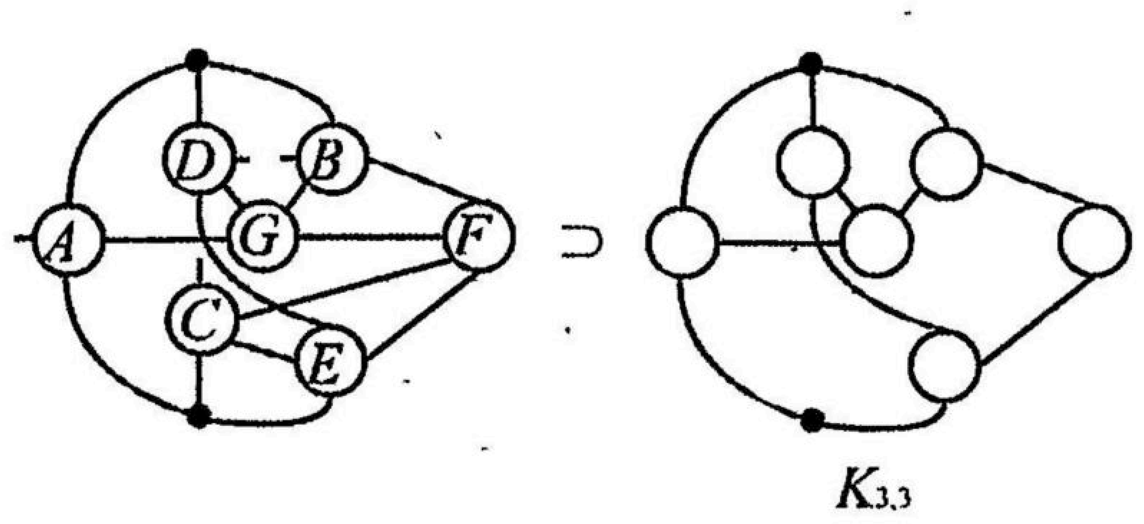


FIG. 3.176. $t = 7$ (c) (v) (H) (6).

(I) $F \sim B, D, E$. Since C and E are interchangeable in the first figure in Fig. 3.164, case is the same as the case (G).

(J) $F \sim C, D, E$. Since B and D are interchangeable in the first figure in Fig. 3.164, case is the same as the case (H).

(vi) $G \sim A, B, E, F$. Since C and E are interchangeable in the first figure in Fig. 3.135, case is the same as the case (iii).

(vii) $G \sim A, C, D, E$. Since B and E are interchangeable in the first figure in Fig. 3.135, case is the same as the case (i).

(viii) $G \sim A, C, D, F$. Since B and D are interchangeable in the first figure in Fig. 3.135, case is the same as the case (iii).

(ix) $G \sim A, C, E, F$. Since B and C, D and E are interchangeable in the first figure in Fig. 3.135, this case is the same as the case (v).

(x) $G \sim A, D, E, F$. Since B and D, C and E are interchangeable in the first figure in Fig. 3.135, this case is the same as the case (iii).

(xi) $G \sim B, C, D, E$. The vertex F has four remaining hands, so we consider how the hands of F connect. There are five cases; see Fig. 3.177.

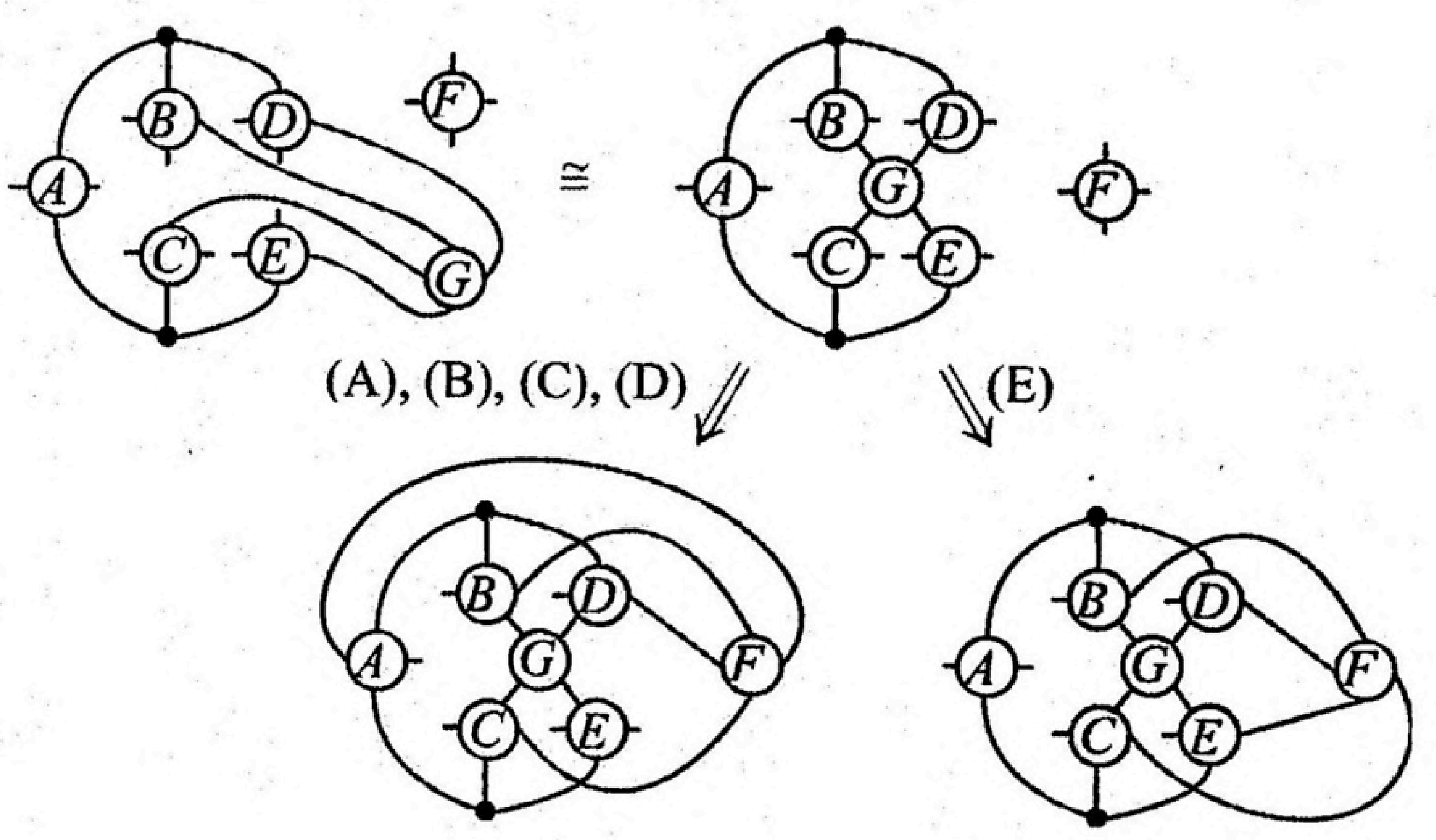


FIG. 3.177. $t = 7$ (c) (xi).

(A) $F \sim A, B, C, D$. This gives a graph containing $K_{3,3}$, and so it does not satisfy the condition (P5); see Fig. 3.178.

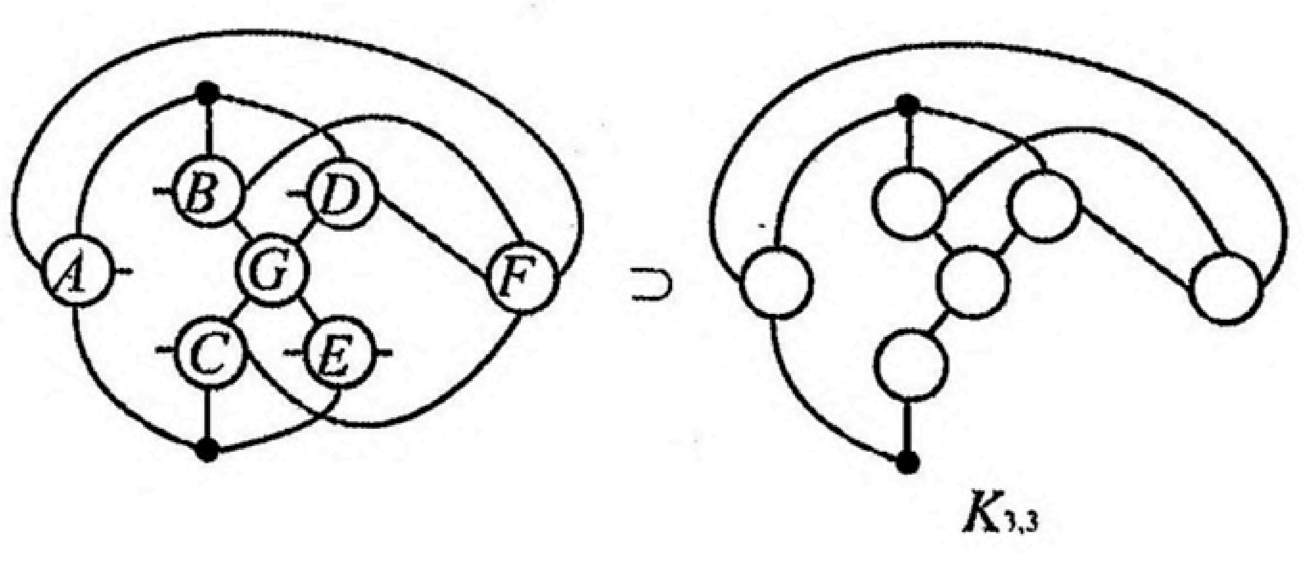


FIG. 3.178. $t = 7$ (c) (xi) (A).

(B) $F \sim A, B, C, E$. Since D and E are interchangeable in the first figure in Fig. 3.177, this case is the same as the case (A).

(C) $F \sim A, B, D, E$. Since C and E are interchangeable in the first figure in Fig. 3.177, this case is the same as the case (A).

(D) $F \sim A, C, D, E$. Since B and E are interchangeable in the first figure in Fig. 3.177, this case is the same as the case (A).

(E) $F \sim B, C, D, E$. This gives a graph containing $K_{3,3}$, and so it does not satisfy the condition (P5); see Fig. 3.179.

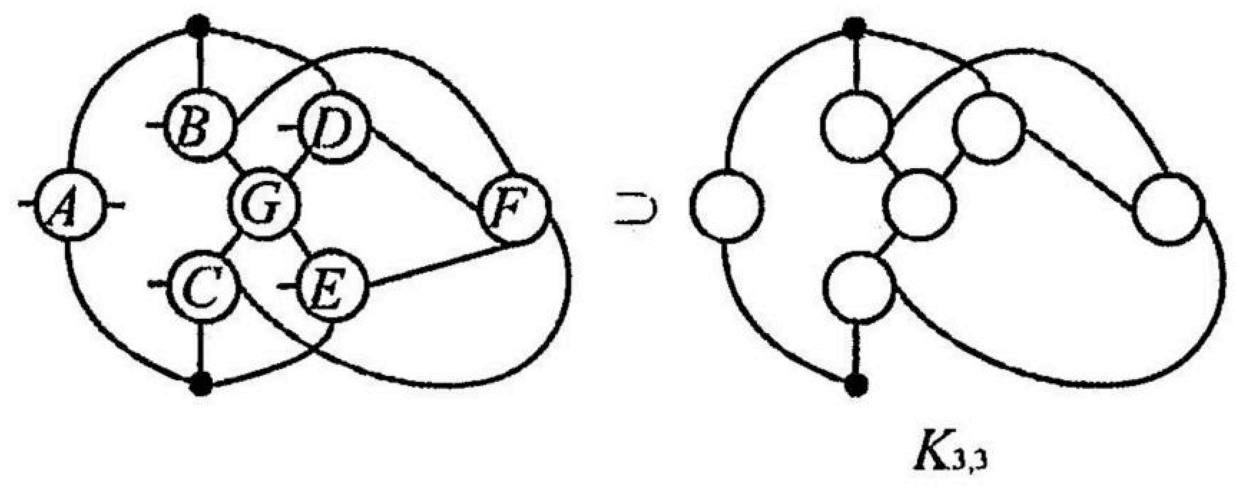


FIG. 3.179. $t = 7$ (c) (xi) (E).

(xii) $G \sim B, C, D, F$. The vertex F has three remaining hands, so we consider how the hands of F connect. There are ten cases; see Fig. 3.180.

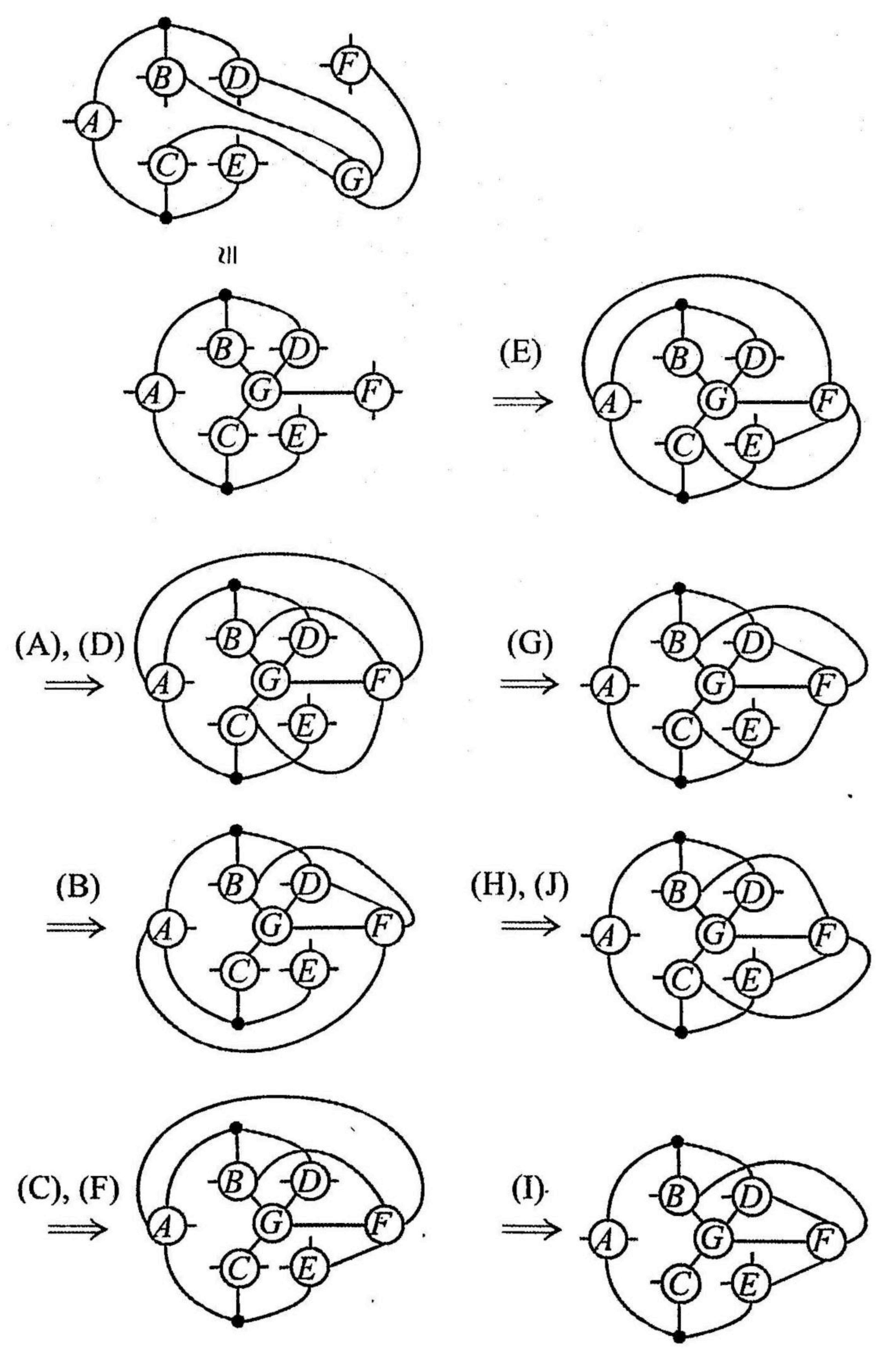


FIG. 3.180. $t = 7$ (c) (xii).

(A) $F \sim A, B, C$. The vertex E has three remaining hands, so we consider how the of E connect. There are four cases; see Fig. 3.181.

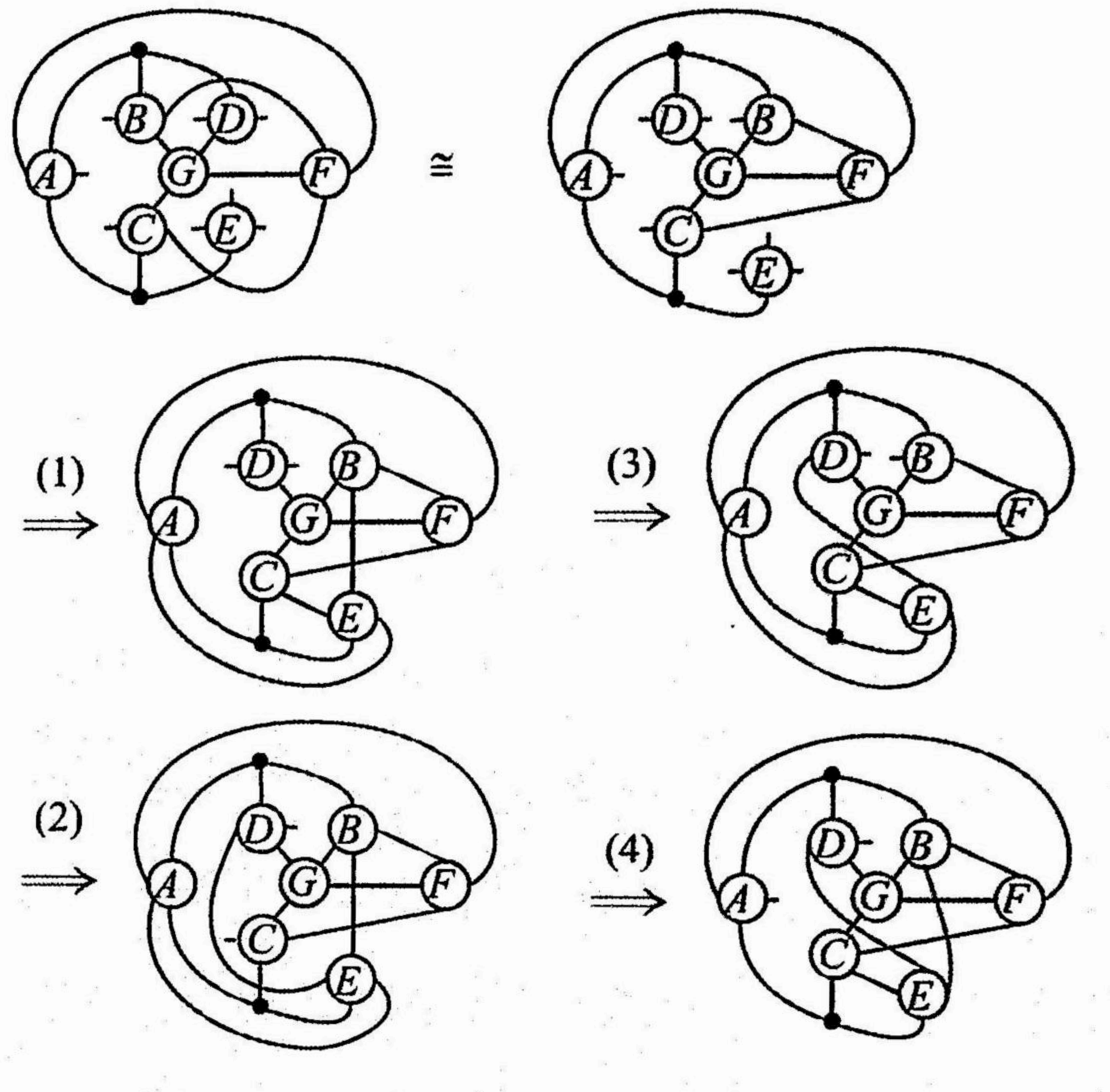


FIG. 3.181. $t = 7$ (c) (xii) (A).

(1) $E \sim A, B, C$. This gives a graph having a loop at D, and so it does not satisfy the condition (P1); see Fig. 3.181.

(2) $E \sim A, B, D$. This gives a graph containing $K_{3,3}$, and so it does not satisfy the condition (P5); see Fig. 3.182.

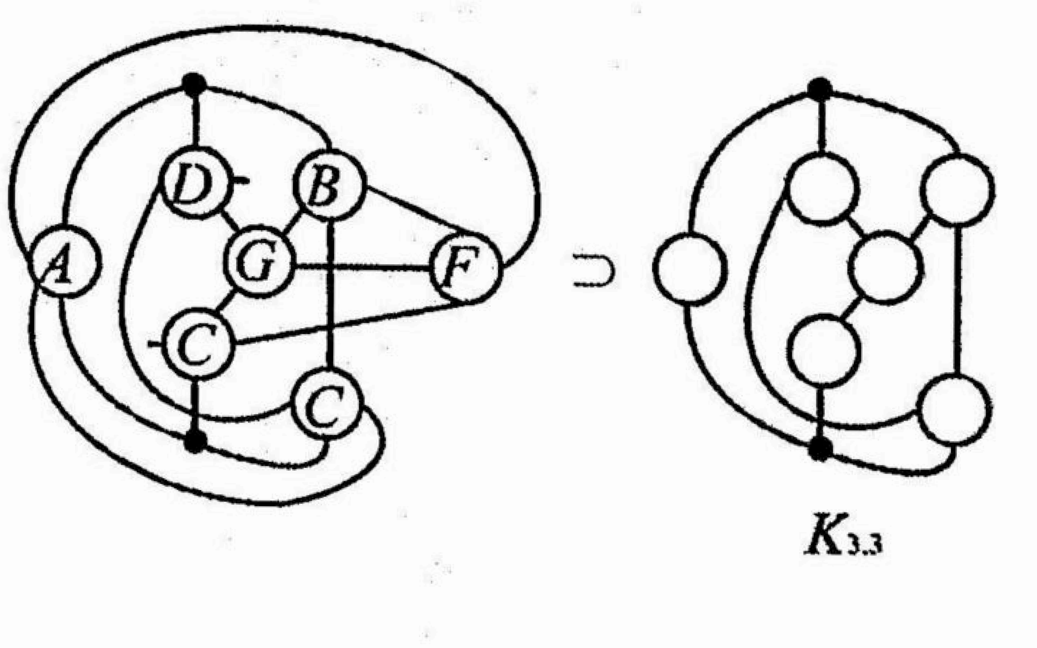


FIG. 3.182. $t = 7$ (c) (xii) (A) (2).

(3) $E \sim A, C, D$. Then $B \sim D$, and we obtain 7^3_*; see Fig. 3.183.

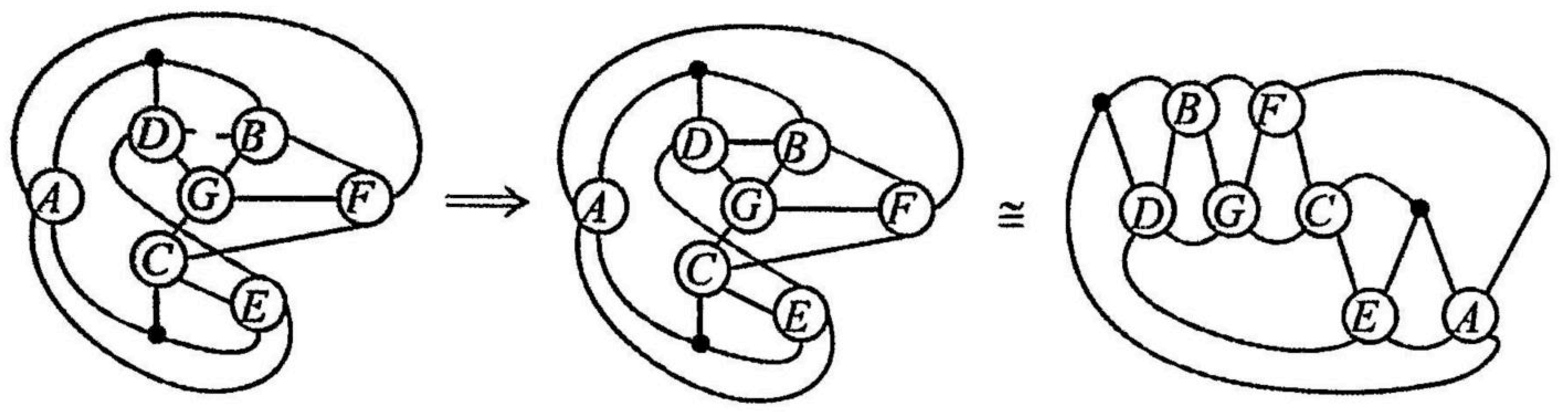


FIG. 3.183. $t = 7$ (c) (xii) (A) (3).

(4) $E \sim B, C, D$. This gives a graph containing $K_{3,3}$, and so it does not satisfy the condition (P5); see Fig. 3.184.

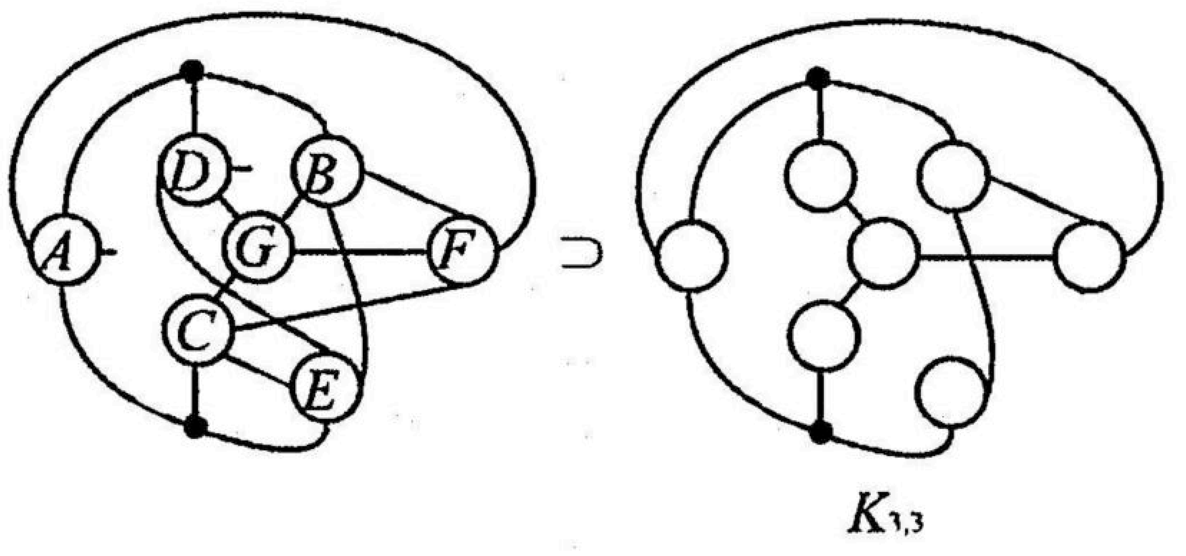


FIG. 3.184. $t = 7$ (c) (xii) (A) (4).

(B) $F \sim A, B, D$. This gives a graph containing $K_{3,3}$, and so it does not satisfy the condition (P5); see Fig. 3.185.

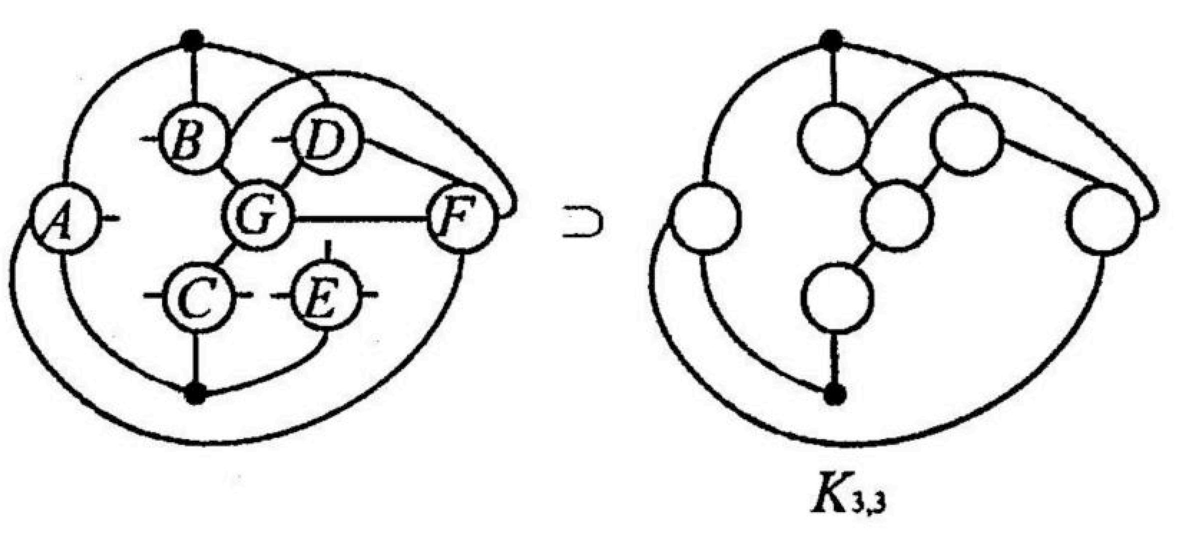


FIG. 3.185. $t = 7$ (c) (xii) (B).

(C) $F \sim A, B, E$. The vertex E has two remaining hands, so we consider how the hands of E connect. There are six cases; see Fig. 3.186.

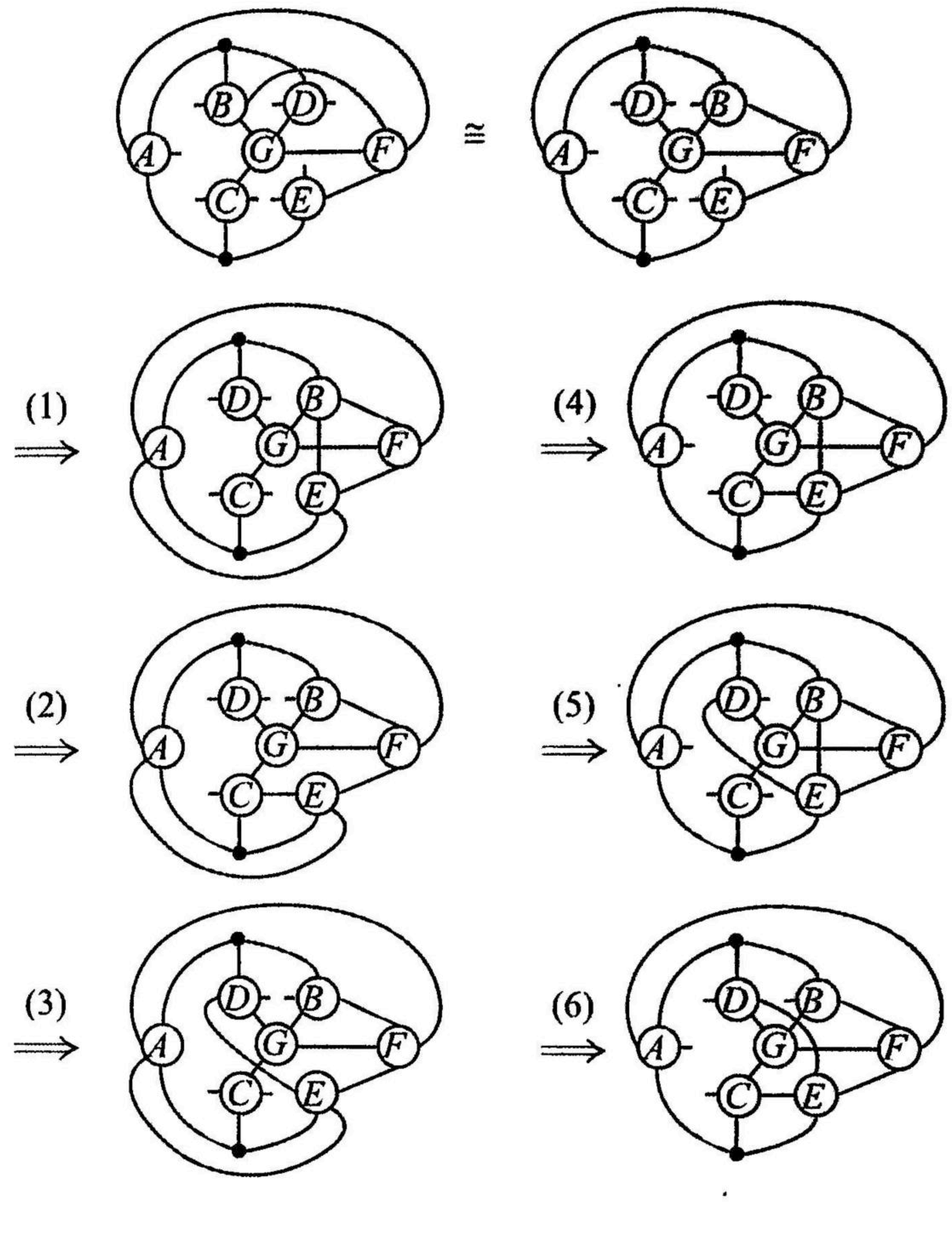


FIG. 3.186. $t = 7$ (c) (xii) (C).

(1) $E \sim A, B$. This gives a graph which does not satisfy the conditions (P1) or (P2); see Fig. 3.186.
(2) $E \sim A, C$. Then $D \sim B, C$, and we obtain 7^4_*; see Fig. 3.187.

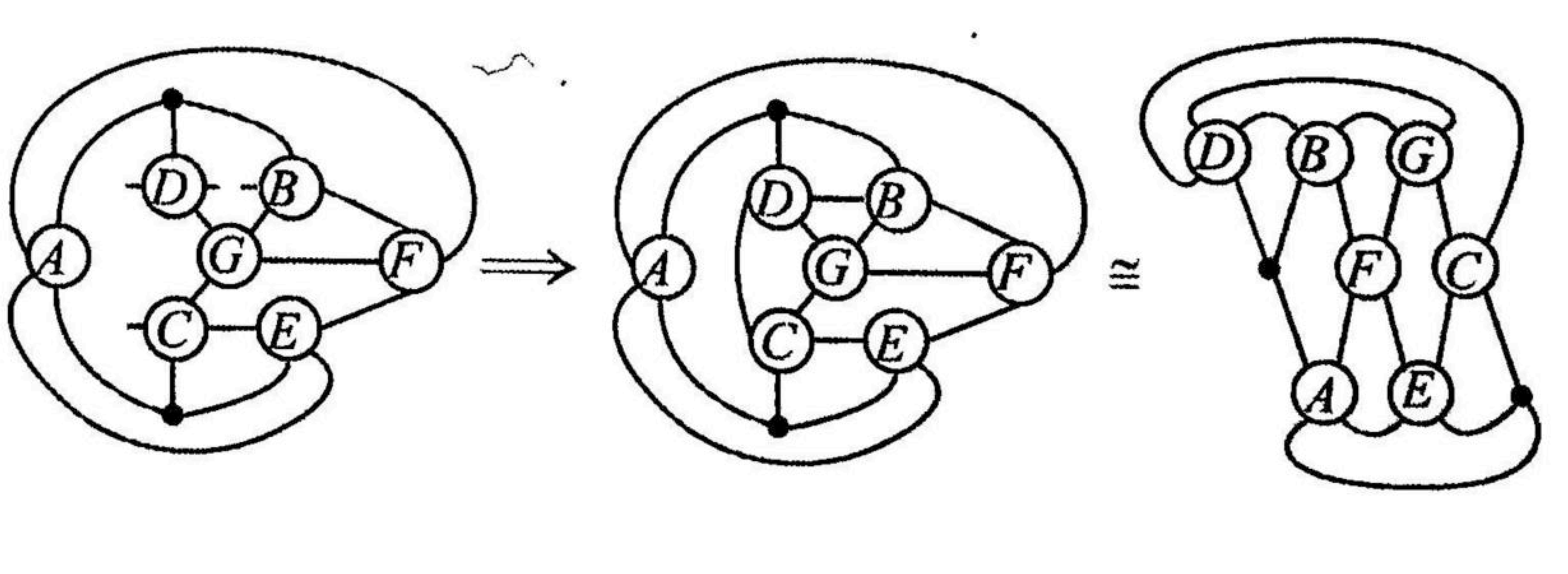


FIG. 3.187. $t = 7$ (c) (xii) (C) (2).

(3) $E \sim A, D$. This gives a graph containing $K_{3,3}$, and so it does not satisfy the condition (P5); see Fig. 3.188.

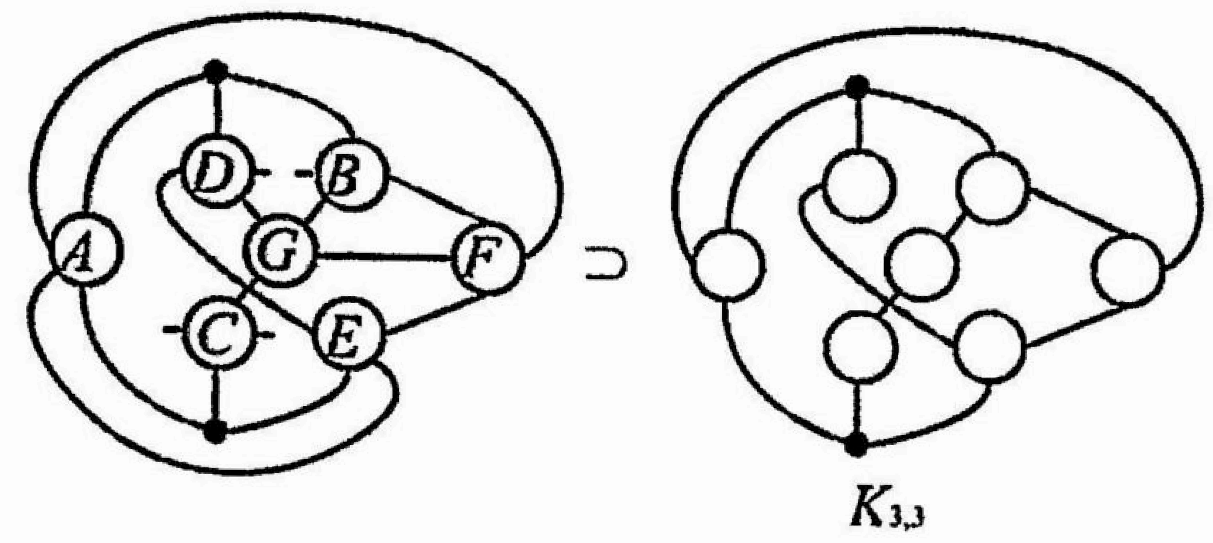


FIG. 3.188. $t = 7$ (c) (xii) (C) (3).

(4) $E \sim B, C$. This gives a graph containing $K_{3,3}$, and so it does not satisfy the condition (P5); see Fig. 3.189.

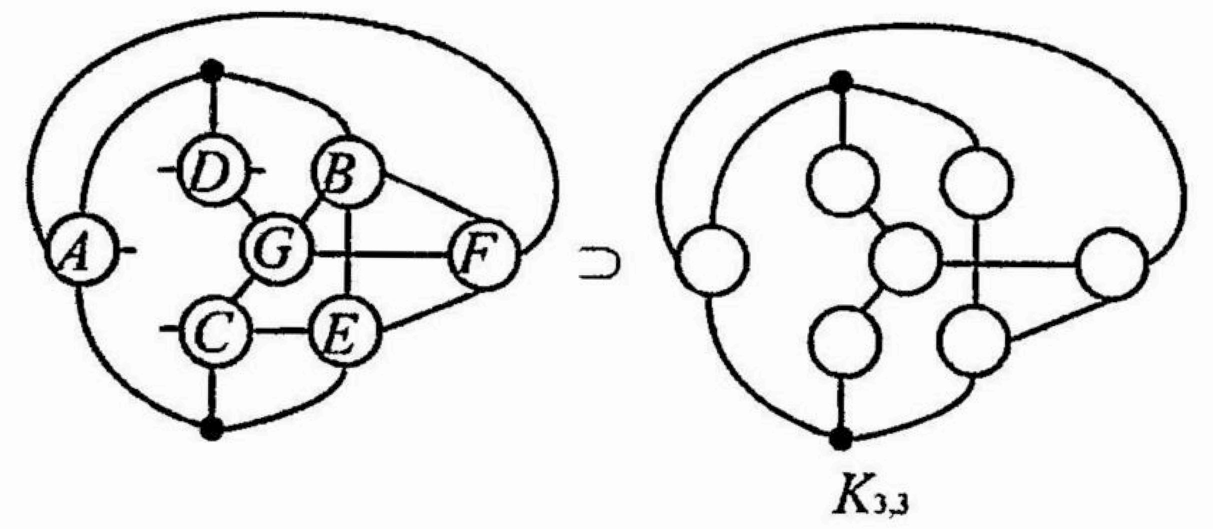


FIG. 3.189. $t = 7$ (c) (xii) (C) (4).

(5) $E \sim B, D$. This gives a graph containing $K_{3,3}$, and so it does not satisfy the condition (P5); see Fig. 3.190.

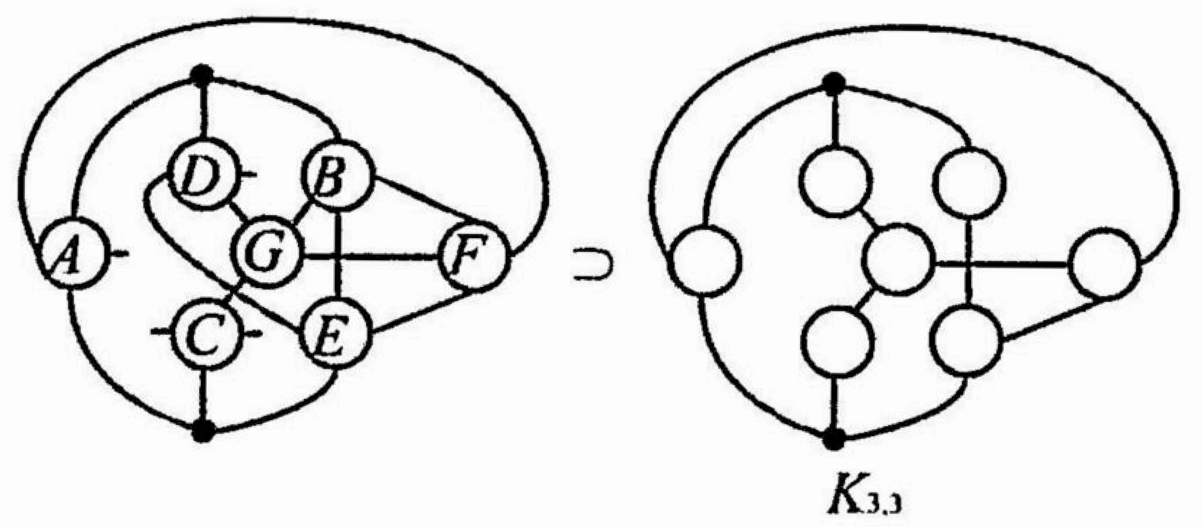


FIG. 3.190. $t = 7$ (c) (xii) (C) (5).

(6) $E \sim C, D$. This gives a graph containing $K_{3,3}$, and so it does not satisfy the condition (P5); see Fig. 3.191.

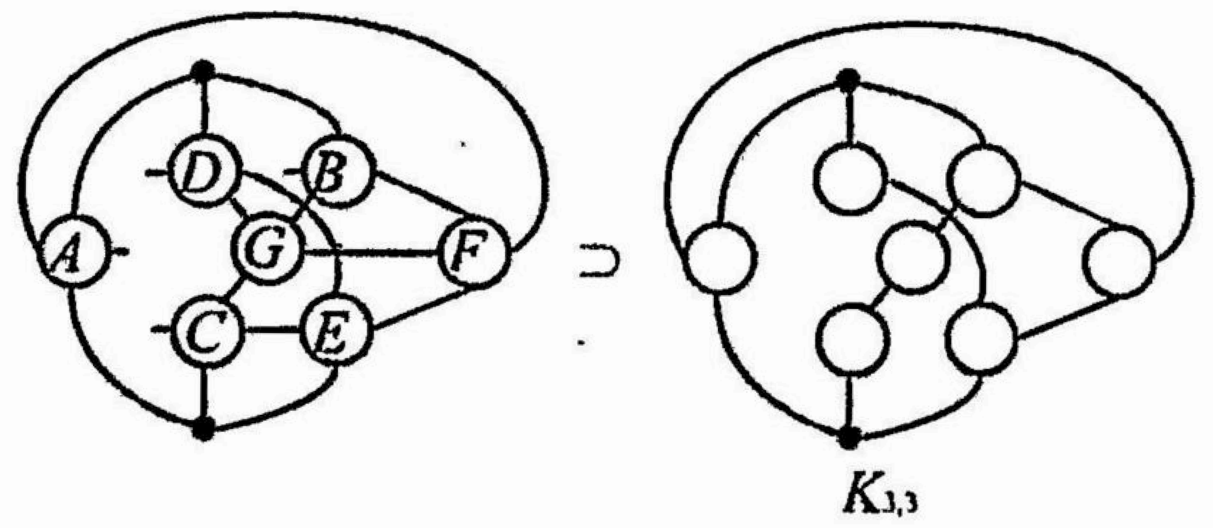


FIG. 3.191. $t = 7$ (c) (xii) (C) (6).

(D) $F \sim A, C, D$. Since B and D are interchangeable in the first figure in Fig. 3.180, case is the same as the case (A).

(E) $F \sim A, C, E$. The vertex E has two remaining hands, so we consider how the of E connect. There are six cases; see Fig. 3.192.

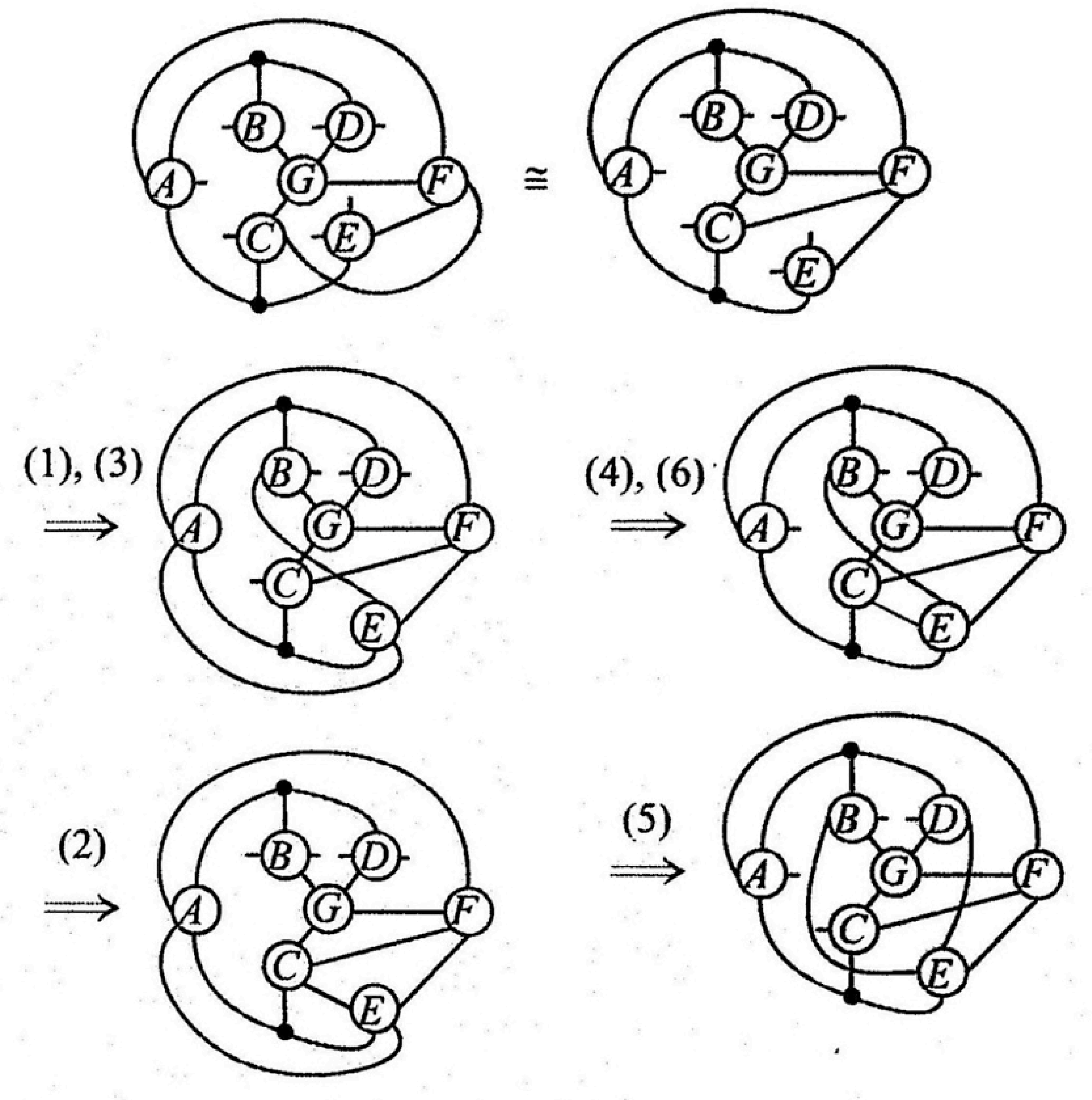


FIG. 3.192. $t = 7$ (c) (xii) (E).

(1) $E \sim A, B$. This gives a graph containing $K_{3,3}$, and so it does not satisfy the condition (P5); see Fig. 3.193.

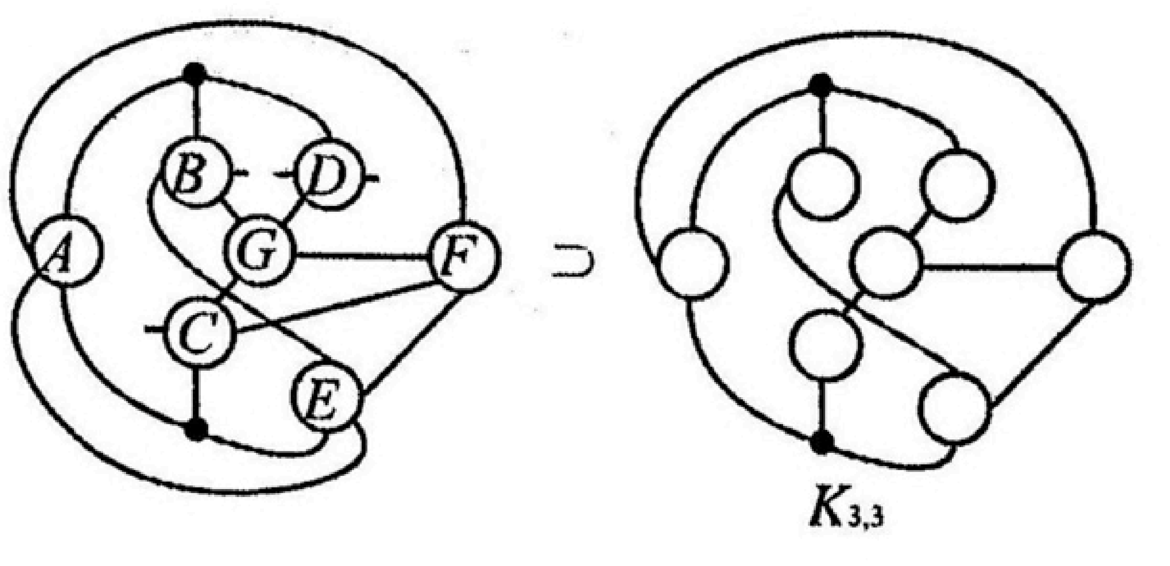


FIG. 3.193. $t = 7$ (c) (xii) (E) (1).

(2) $E \sim A, C$. This gives a graph which does not satisfy the conditions (P1) or (P2); see Fig. 3.192.

(3) $E \sim A, D$. Since B and D are interchangeable in the first figure in Fig. 3.192, this case is the same as the case (1).

(4) $E \sim B, C$. This gives a graph containing $K_{3,3}$, and so it does not satisfy the condition (P5); see Fig. 3.194.

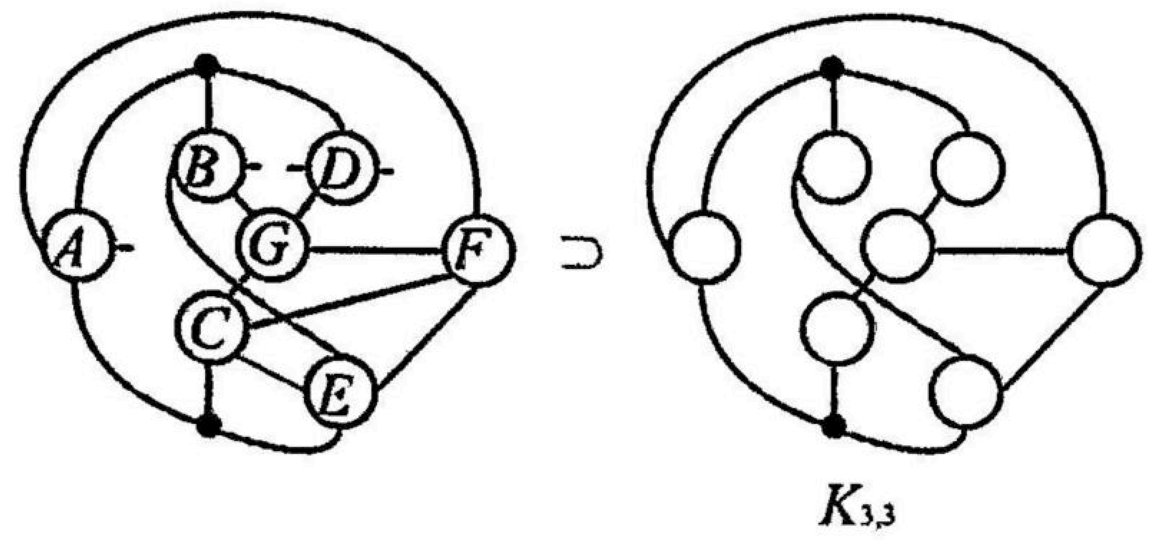


FIG. 3.194. $t = 7$ (c) (xii) (E) (4).

(5) $E \sim B, D$. This gives a graph containing $K_{3,3}$, and so it does not satisfy the condition (P5); see Fig. 3.195.

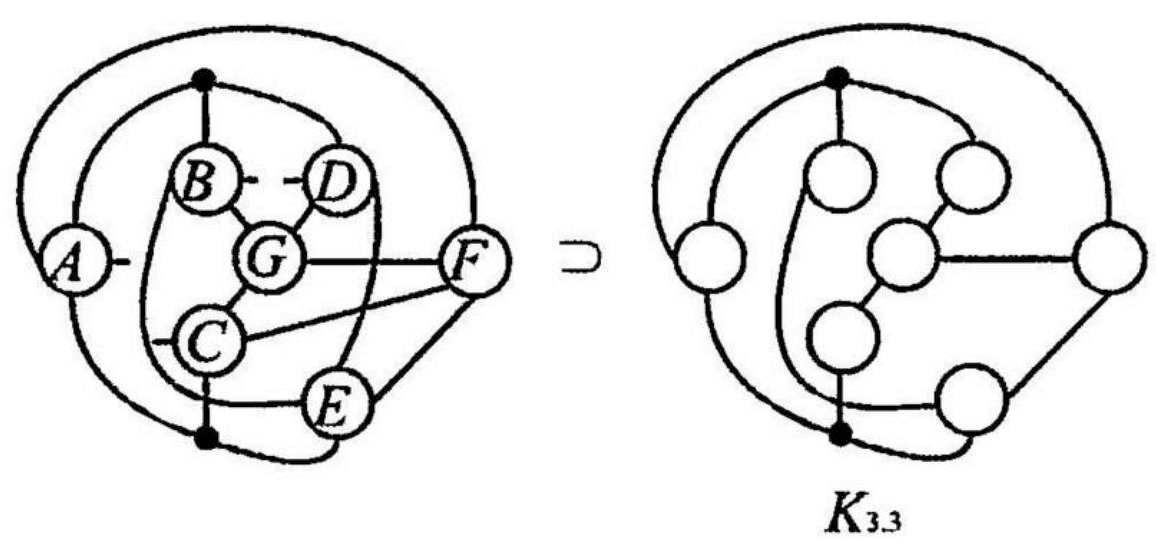


FIG. 3.195. $t = 7$ (c) (xii) (E) (5).

(6) $E \sim C, D$. Since B and D are interchangeable in the first figure in Fig. 3.192, this case is the same as the case (4).

(F) $F \sim A, D, E$. Since B and D are interchangeable in the first figure in Fig. 3.180, this case is the same as the case (C).

(G) $F \sim B, C, D$. This gives a graph containing $K_{3,3}$, and so it does not satisfy the condition (P5); see Fig. 3.196.

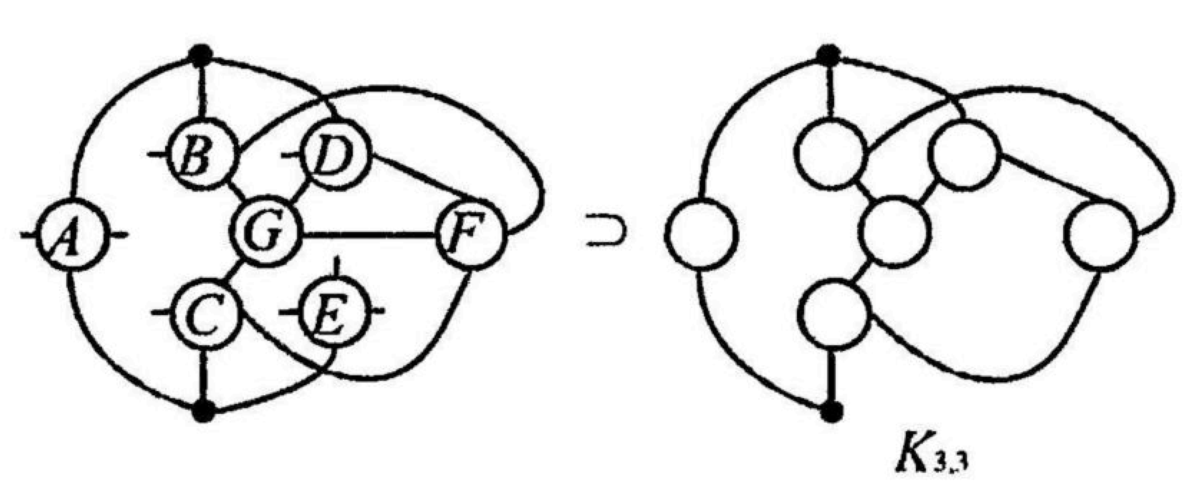


FIG. 3.196. $t = 7$ (c) (xii) (G).

(H) $F \sim B, C, E$. The vertex E has two remaining hands, so we consider how the hands of E connect. There are six cases; see Fig. 3.197.

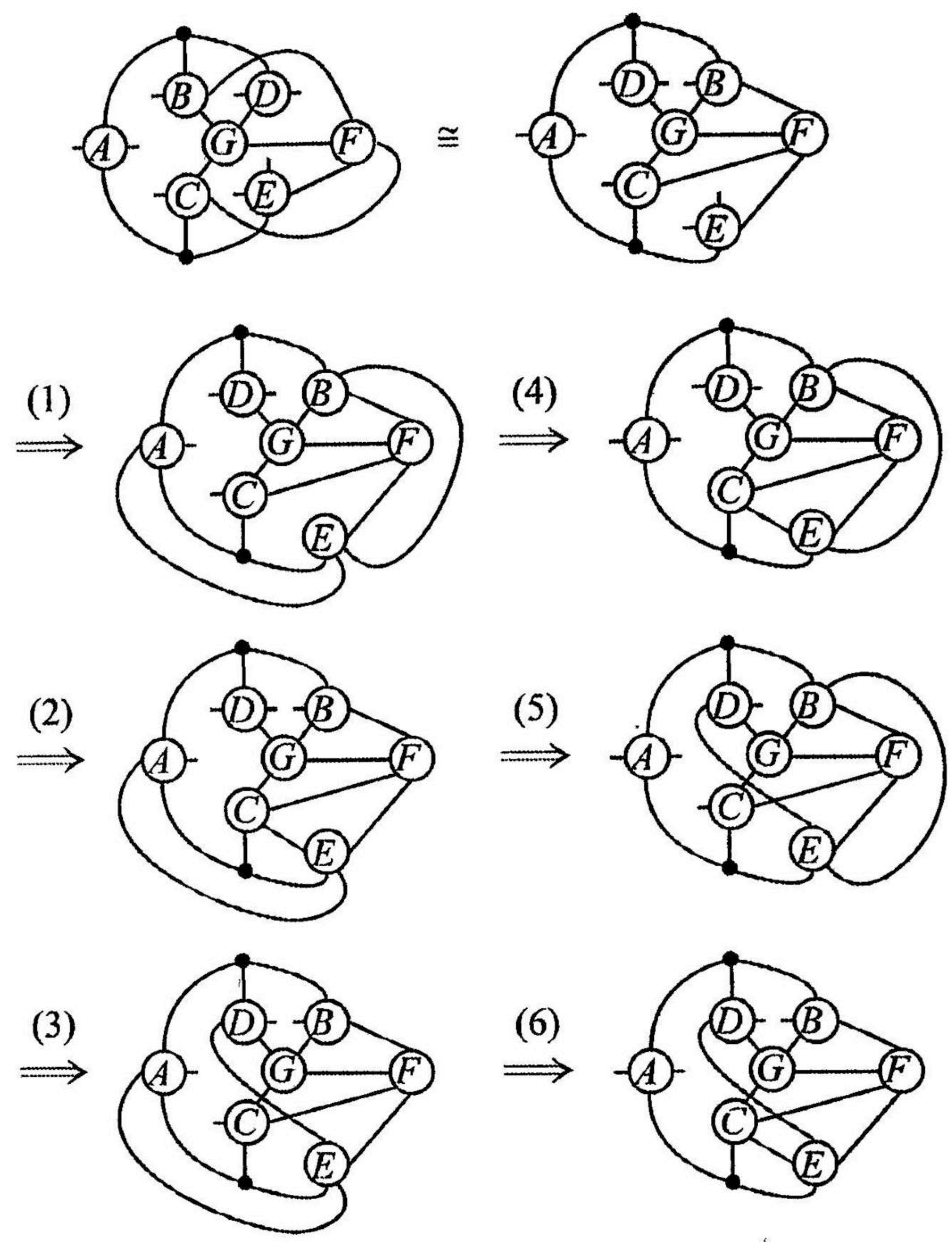


FIG. 3.197. $t = 7$ (c) (xii) (H).

(1) $E \sim A, B$. Then $D \sim A, C$, and we obtain 7^5_*; see Fig. 3.198.

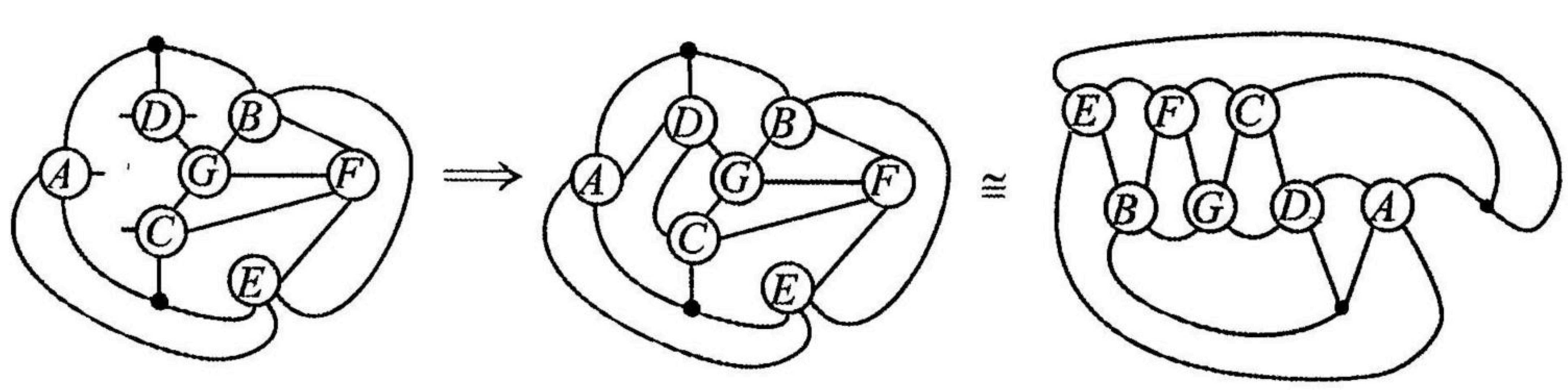


FIG. 3.198. $t = 7$ (c) (xii) (H) (1).

(2) $E \sim A, C$. Then $D \sim A, B$, and we obtain 7^6_*; see Fig. 3.199.

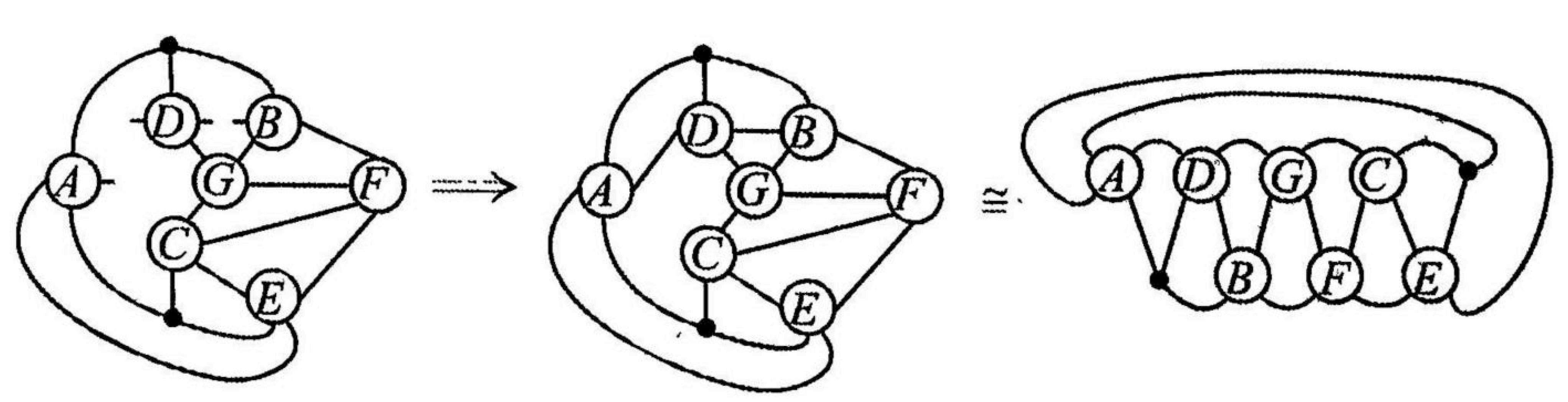


FIG. 3.199. $t = 7$ (c) (xii) (H) (2).

(3) $E \sim A, D$. This gives a graph containing $K_{3,3}$, and so it does not satisfy the condition (P5); see Fig. 3.200.

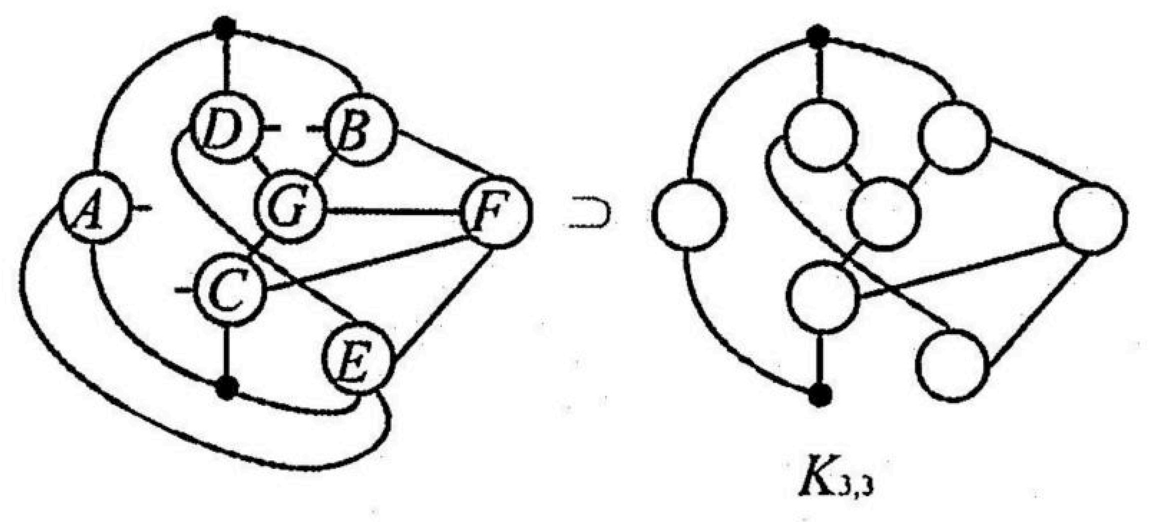


FIG. 3.200. $t = 7$ (c) (xii) (H) (3).

(4) $E \sim B, C$. This gives a graph which does not satisfy the conditions (P1) or (P2); see Fig. 3.197.

(5) $E \sim B, D$. This gives a graph containing $K_{3,3}$, and so it does not satisfy the condition (P5); see Fig. 3.201.

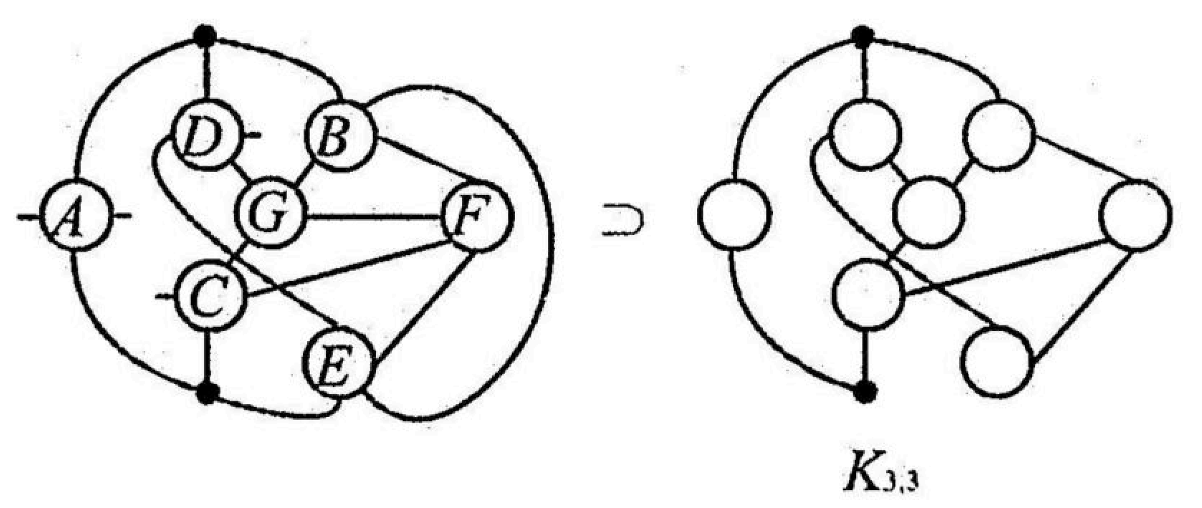


FIG. 3.201. $t = 7$ (c) (xii) (H) (5).

(6) $E \sim C, D$. This gives a graph containing $K_{3,3}$, and so it does not satisfy the condition (P5); see Fig. 3.202.

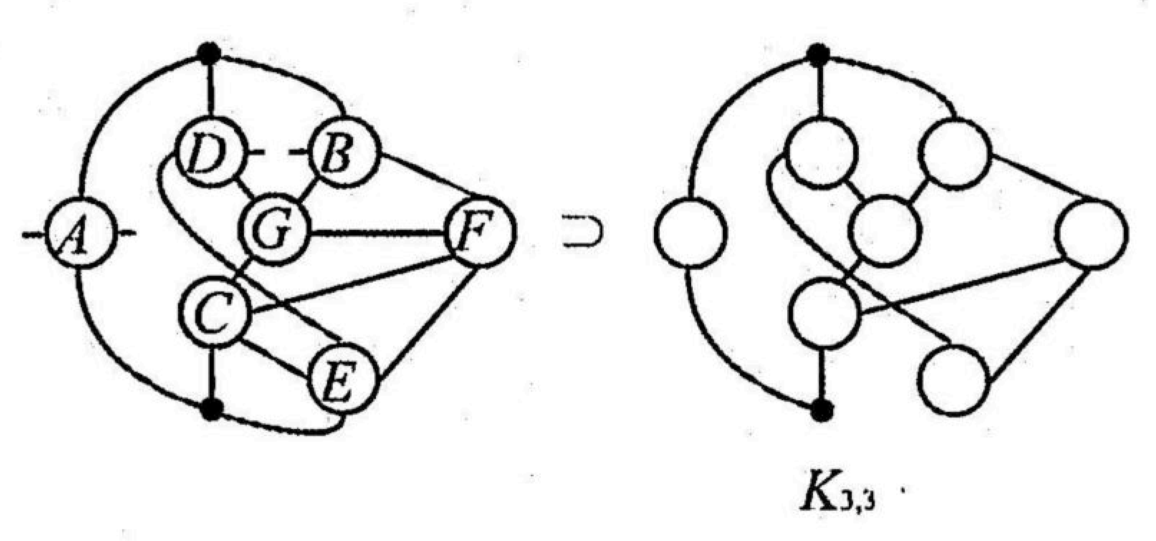


FIG. 3.202. $t = 7$ (c) (xii) (H) (6).

(I) $F \sim B, D, E$. This gives a graph containing $K_{3,3}$, and so it does not satisfy the condition (P5); see Fig. 3.203.

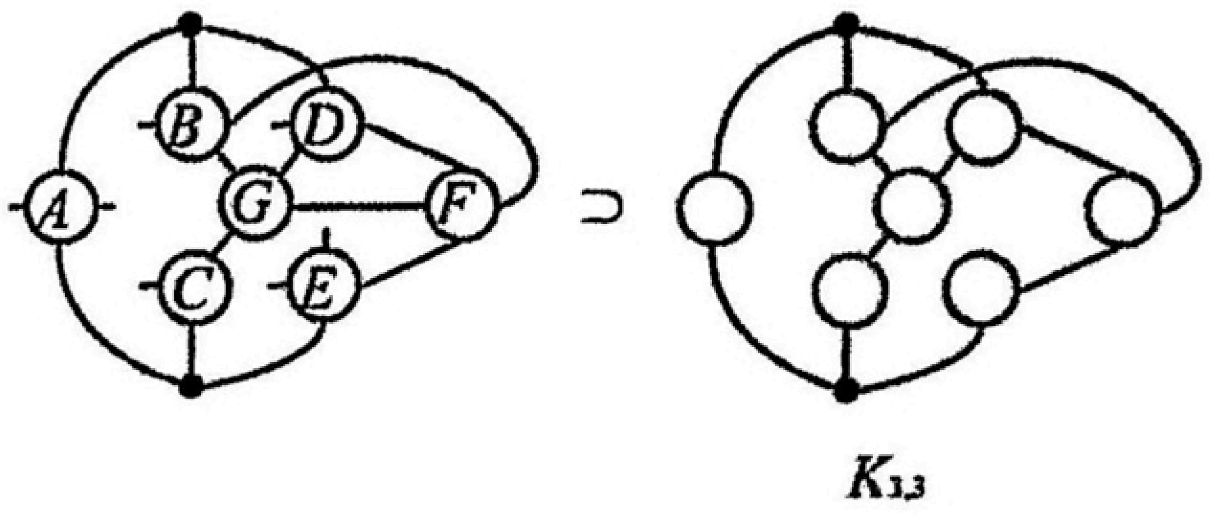


FIG. 3.203. $t = 7$ (c) (xii) (I).

(J) $F \sim C, D, E$. Since B and D are interchangeable in the first figure in Fig. 3.180, this case is the same as the case (H).

(xiii) $G \sim B, C, E, F$. Since D and E are interchangeable in the first figure in Fig. 3.135, this case is the same as the case (xii).

(xiv) $G \sim B, D, E, F$. Since C and E are interchangeable in the first figure in Fig. 3.135, this case is the same as the case (xii).

(xv) $G \sim C, D, E, F$. Since B and E are interchangeable in the first figure in Fig. 3.135, this case is the same as the case (xii).

Pattern (d). The vertex G has four remaining hands, so we consider how the hands of G connect. There are fifteen cases; see Fig. 3.204.

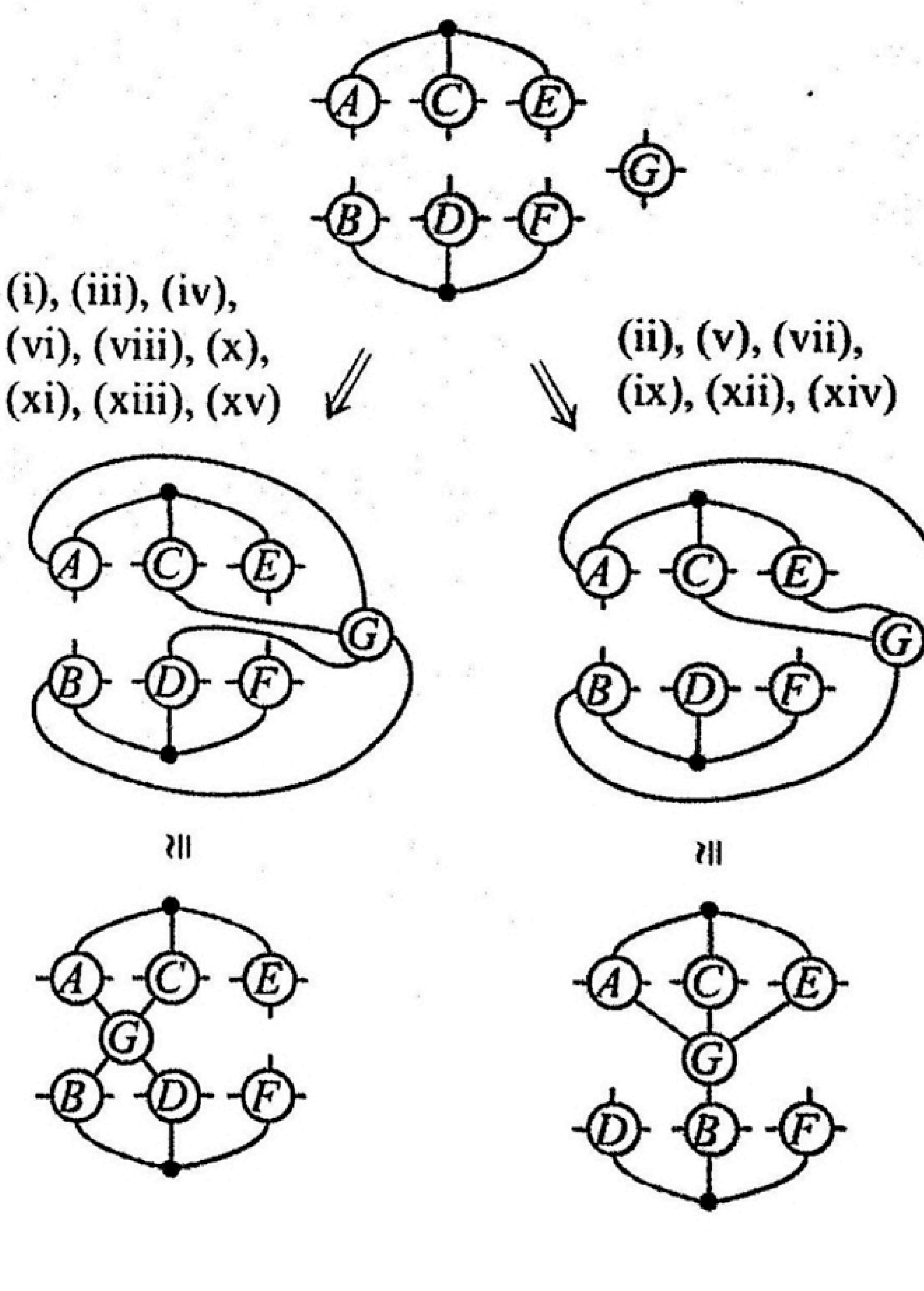


FIG. 3.204. $t = 7$ (d).

(i) $G \sim A, B, C, D$. The vertex F has three remaining hands, so we consider how the hands of F connect. There are ten cases; see Fig. 3.205.

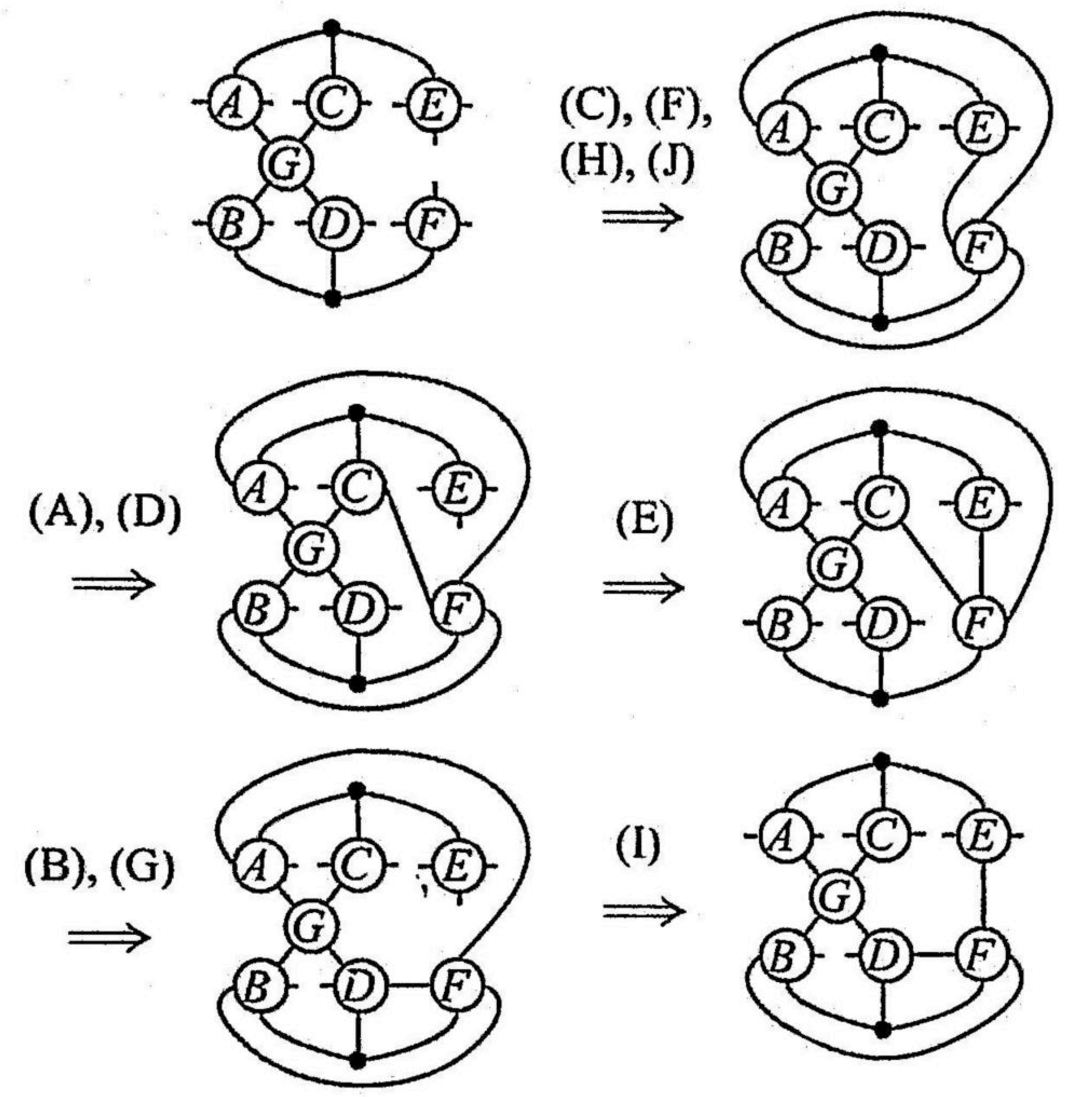


FIG. 3.205. $t = 7$ (d) (i).

(A) $F \sim A, B, C$. The vertex E has three remaining hands, so we consider how the hands of E connect. There are four cases; see Fig. 3.206.

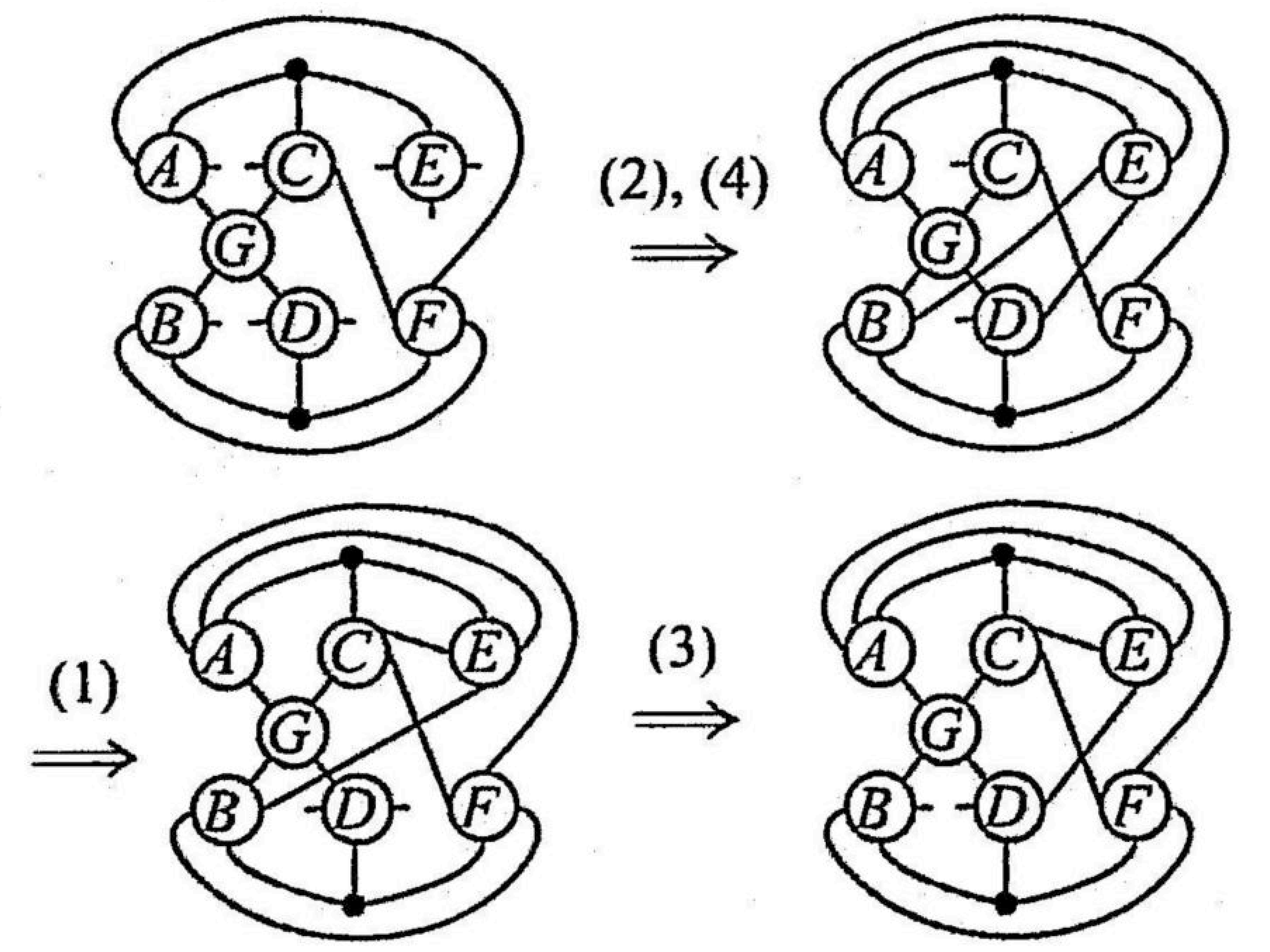


FIG. 3.206. $t = 7$ (d) (i) (A).

(1) $E \sim A, B, C$. This gives a graph having a loop at D, and so it does not satisfy the condition (P1); see Fig. 3.206.
(2) $E \sim A, B, D$. This gives a graph containing $K_{3,3}$, and so it does not satisfy the condition (P5); see Fig. 3.207.

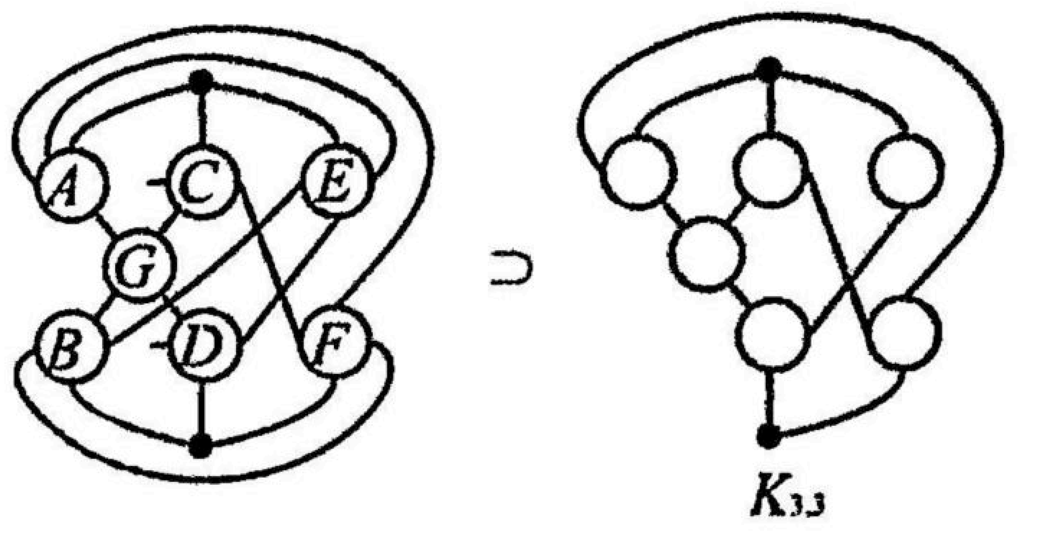


FIG. 3.207. $t = 7$ (d) (i) (A) (2).

(3) $E \sim A, C, D$. This gives a graph containing $K_{3,3}$, and so it does not satisfy the condition (P5); see Fig. 3.208.

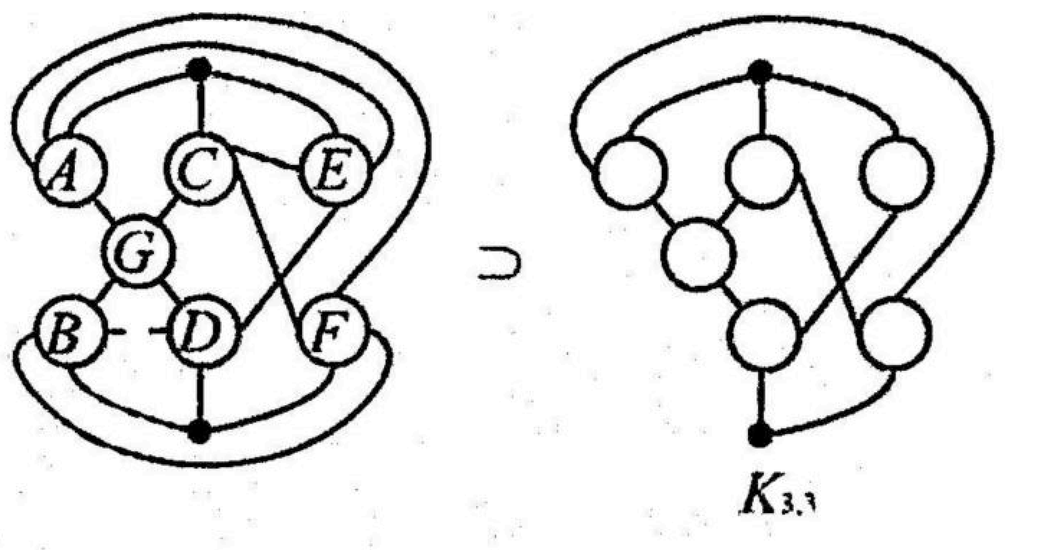


FIG. 3.208. $t = 7$ (d) (i) (A) (3).

(4) $E \sim B, C, D$. Since A and C are interchangeable in the first figure in Fig. 3.206, this case is the same as the case (2).

(B) $F \sim A, B, D$. The vertex E has three remaining hands, so we consider how the hands of E connect. There are four cases; see Fig. 3.209.

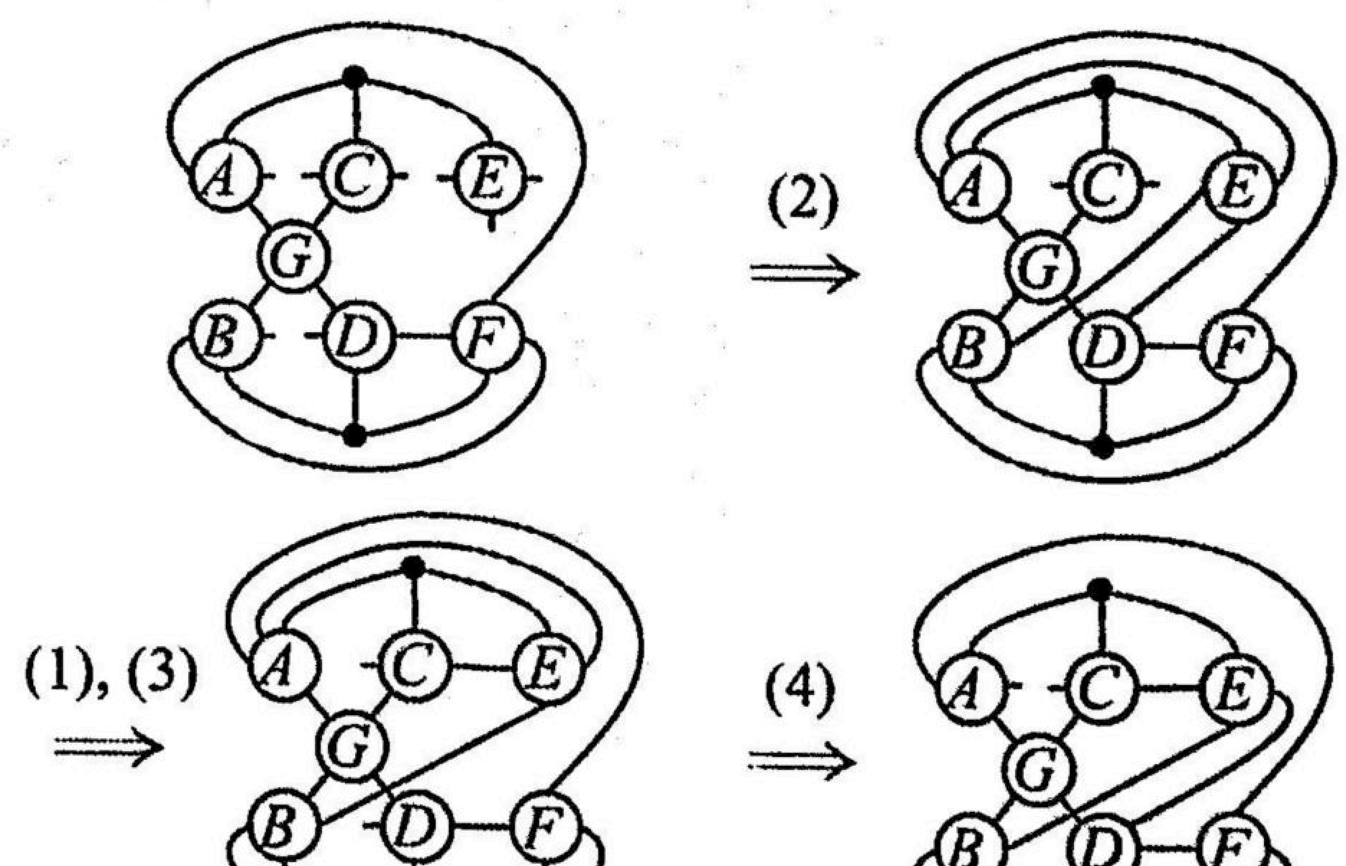


FIG. 3.209. $t = 7$ (d) (i) (B).

(1) $E \sim A, B, C$. Then $C \sim D$. This gives a graph containing $K_{3,3}$, and so it does not satisfy the condition (P5); see Fig. 3.210.

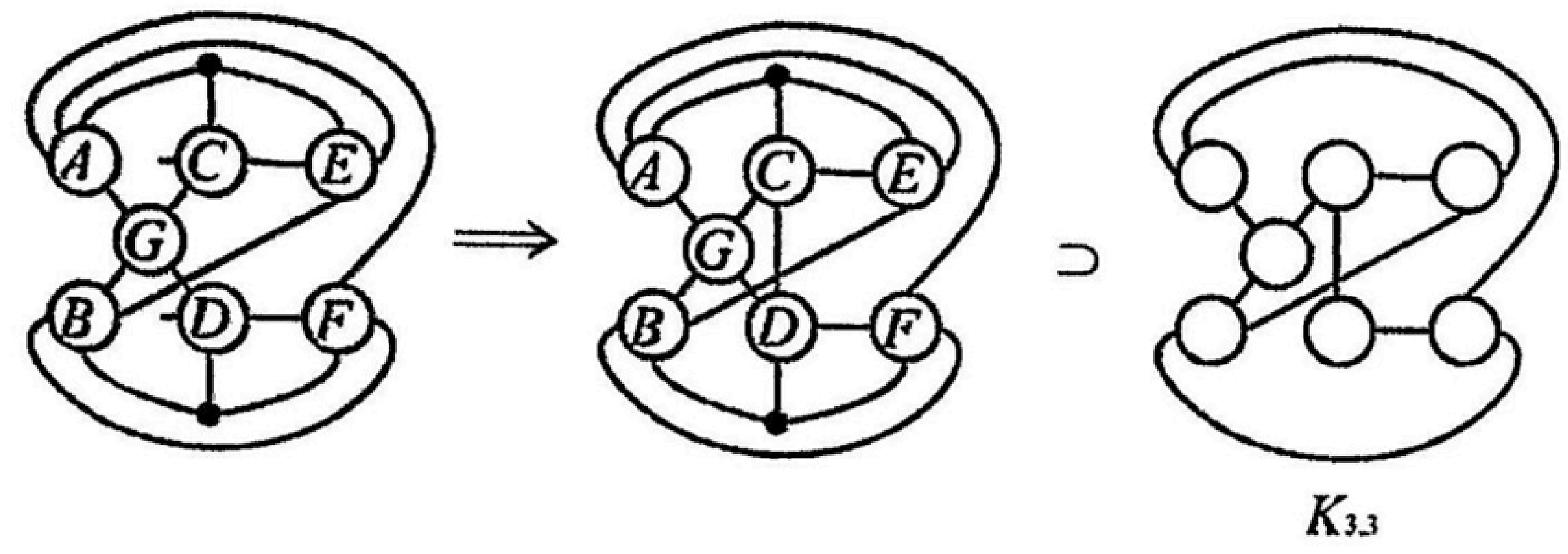


FIG. 3.210. $t = 7$ (d) (i) (B) (1).

(2) $E \sim A, B, D$. This gives a graph having a loop at D, and so it does not satisfy the condition (P1); see Fig. 3.209.

(3) $E \sim A, C, D$. Since B and D are interchangeable in the first figure in Fig. 3.209, this case is the same as the case (1).

(4) $E \sim B, C, D$. This gives a graph containing $K_{3,3}$, and so it does not satisfy the condition (P5); see Fig. 3.211.

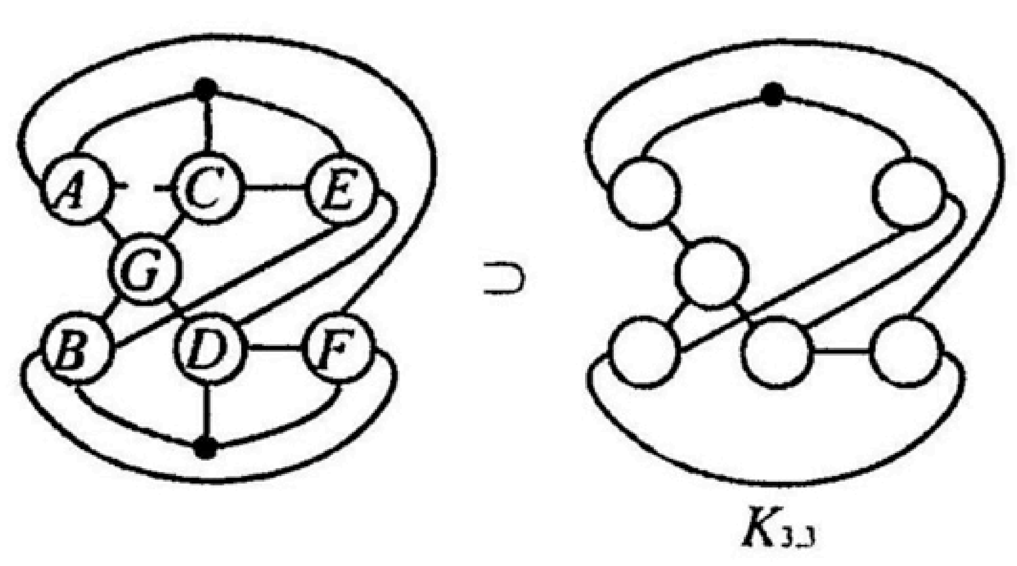


FIG. 3.211. $t = 7$ (d) (i) (B) (4).

(C) $F \sim A, B, E$. The vertex E has two remaining hands, so we consider how the hands of E connect. There are six cases; see Fig. 3.212.

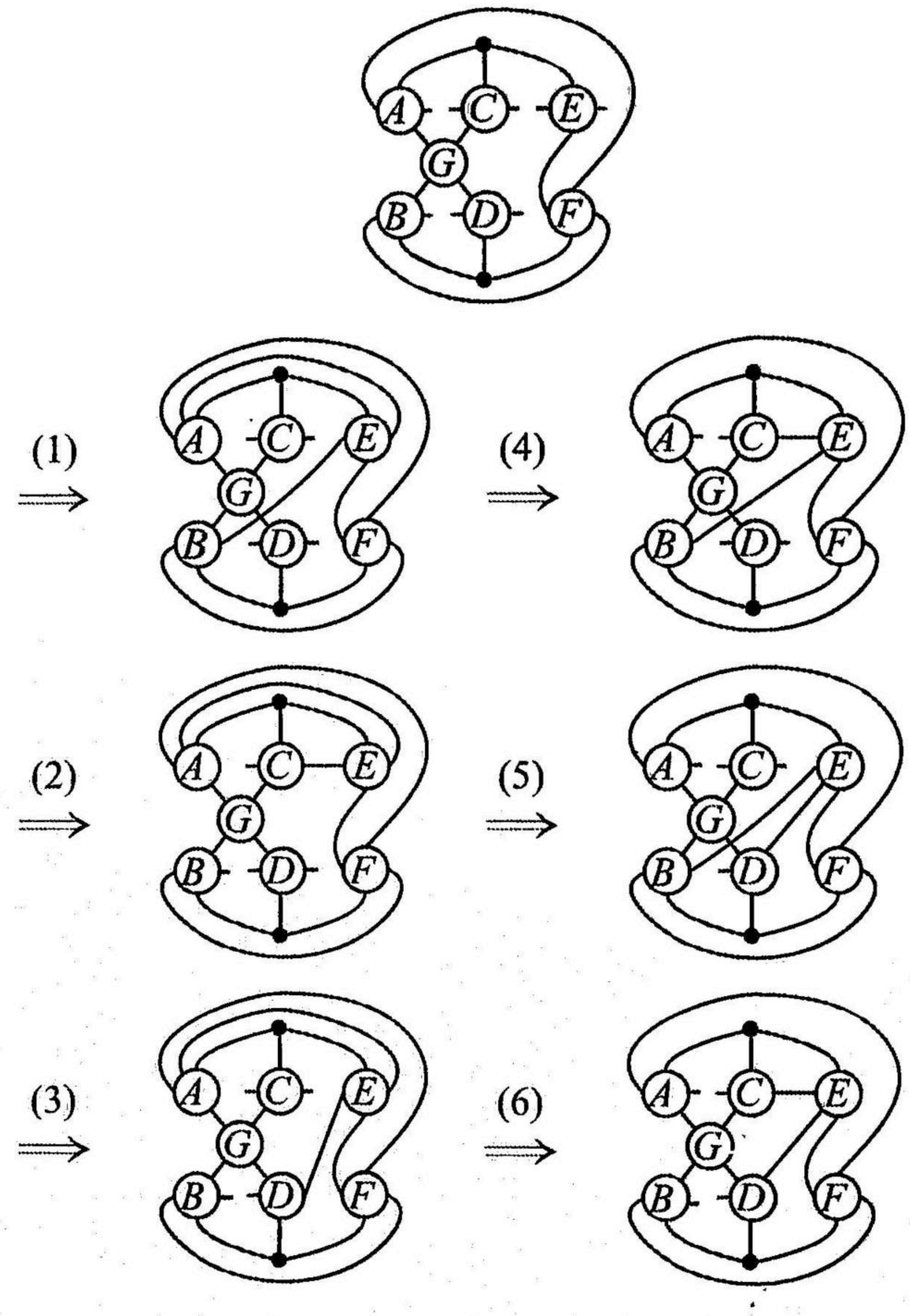


FIG. 3.212. $t = 7$ (d) (i) (C).

(1) $E \sim A, B$. This gives a graph which does not satisfy the conditions (P1) or (P2); see Fig. 3.212.
(2) $E \sim A, C$. Then $D \sim B, C$, and we obtain 7^7_*; see Fig. 3.213.

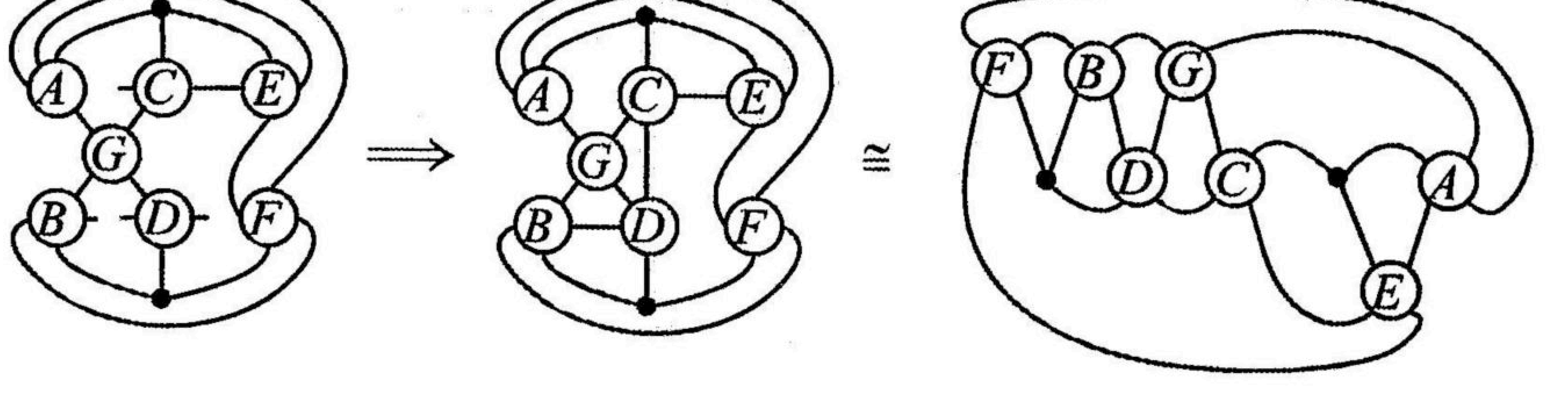


FIG. 3.213. $t = 7$ (d) (i) (C) (2).

(3) $E \sim A, D$. Then $C \sim B, D$. This gives a graph containing $K_{3,3}$, and so it does not satisfy the condition (P5); see Fig. 3.214.

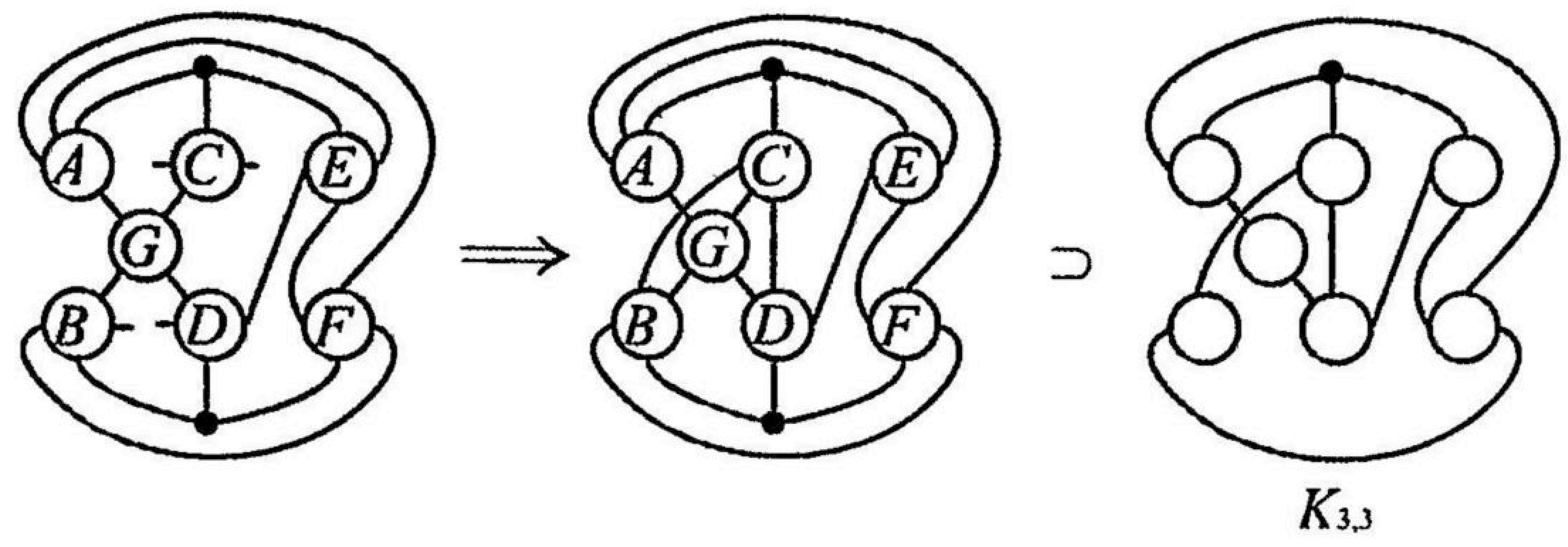


FIG. 3.214. $t = 7$ (d) (i) (C) (3).

(4) $E \sim B, C$. Then $D \sim A, C$. This gives a graph containing $K_{3,3}$, and so it does not satisfy the condition (P5); see Fig. 3.215.

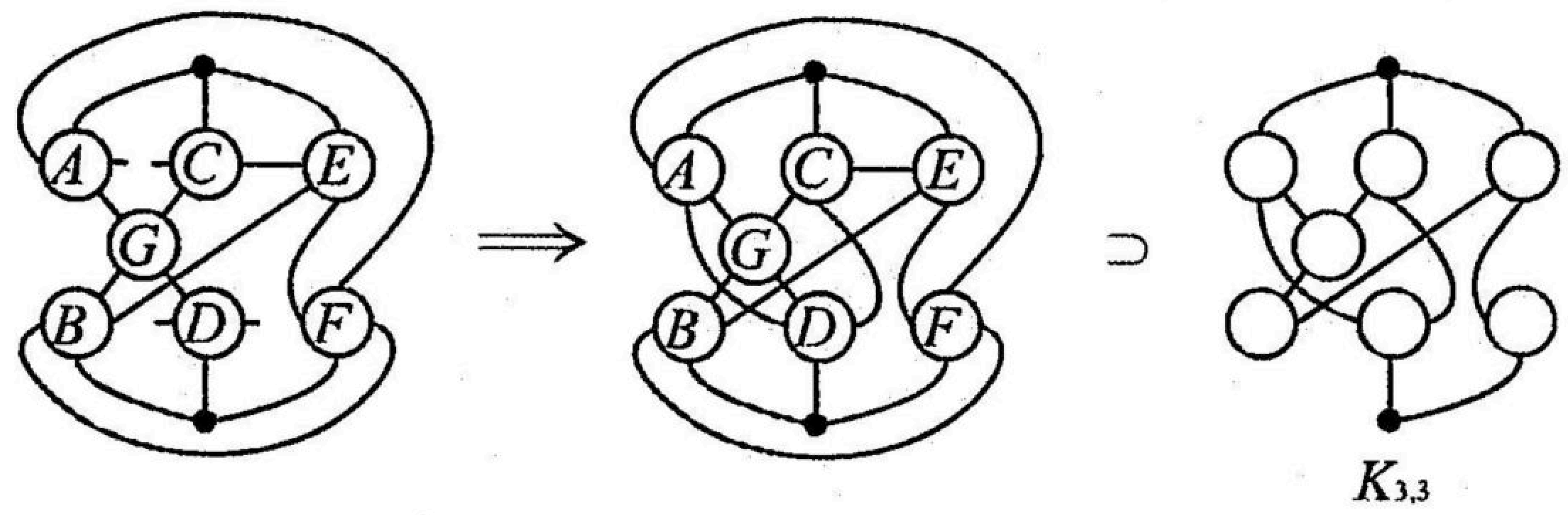


FIG. 3.215. $t = 7$ (d) (i) (C) (4).

(5) $E \sim B, D$. This gives a graph containing $K_{3,3}$, and so it does not satisfy the condition (P5); see Fig. 3.216.

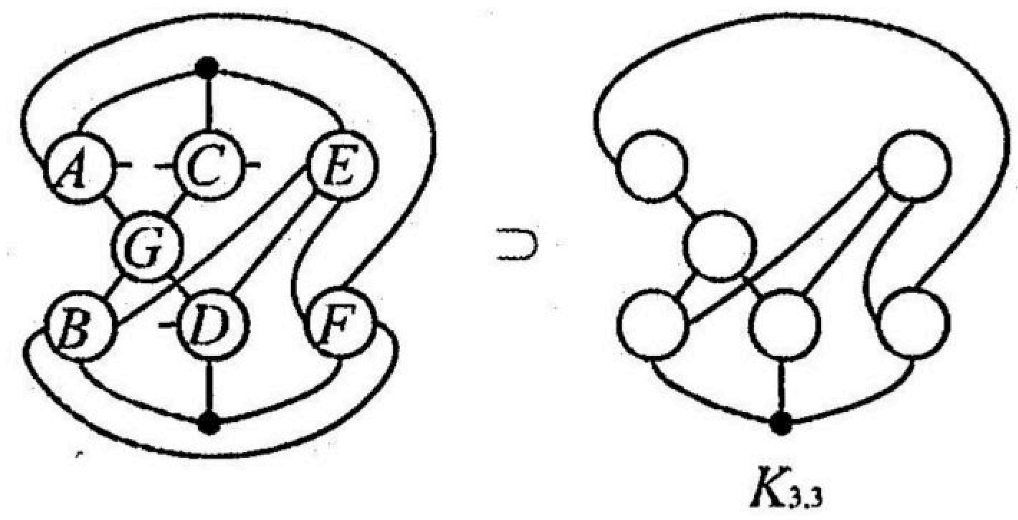


FIG. 3.216. $t = 7$ (d) (i) (C) (5).

(6) $E \sim C, D$. We have three cases; see Fig. 3.217.

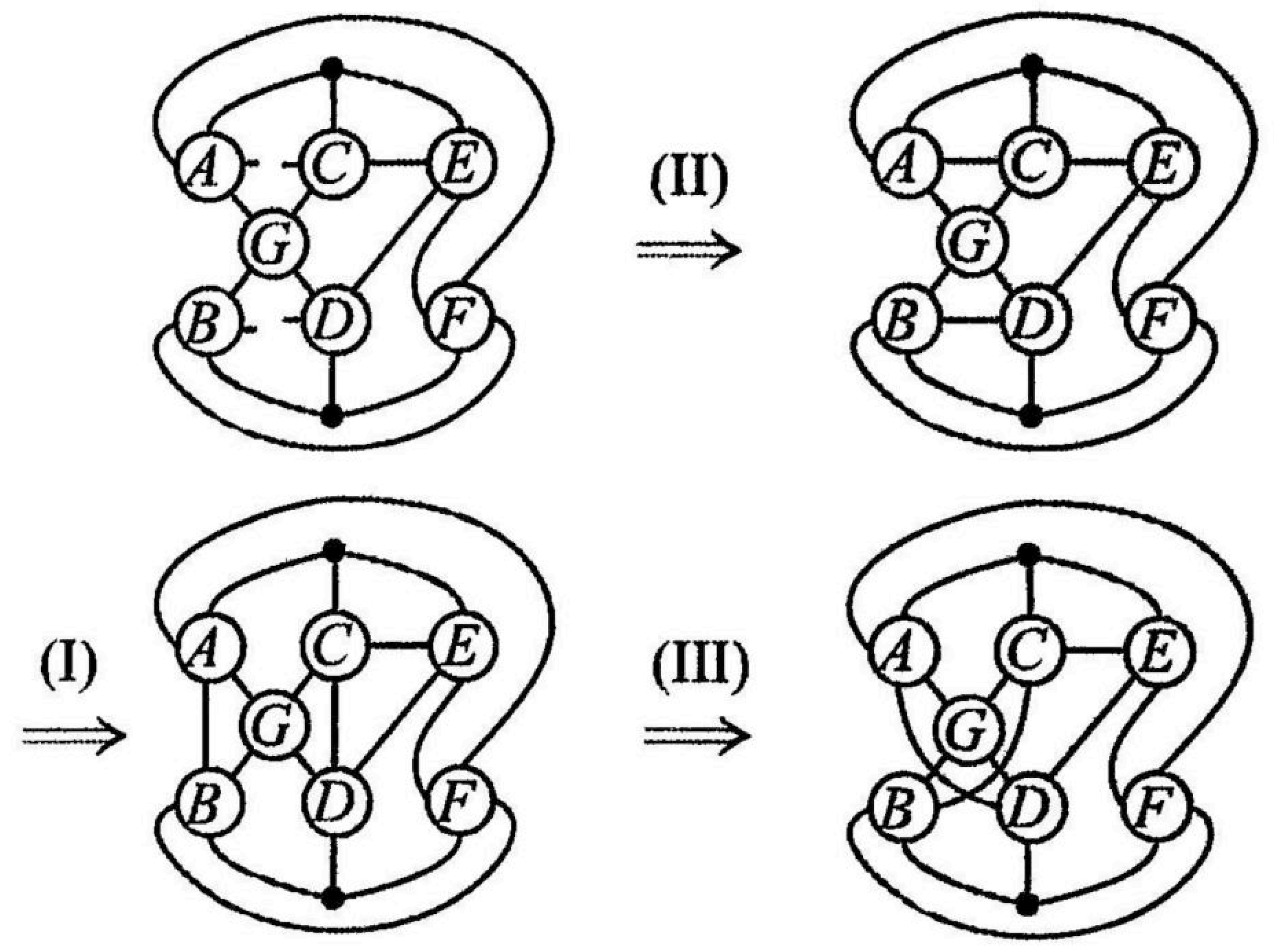


FIG. 3.217. $t = 7$ (d) (i) (C) (6).

(I) $A \sim B, C \sim D$. We obtain 7^8_*; see Fig. 3.218.

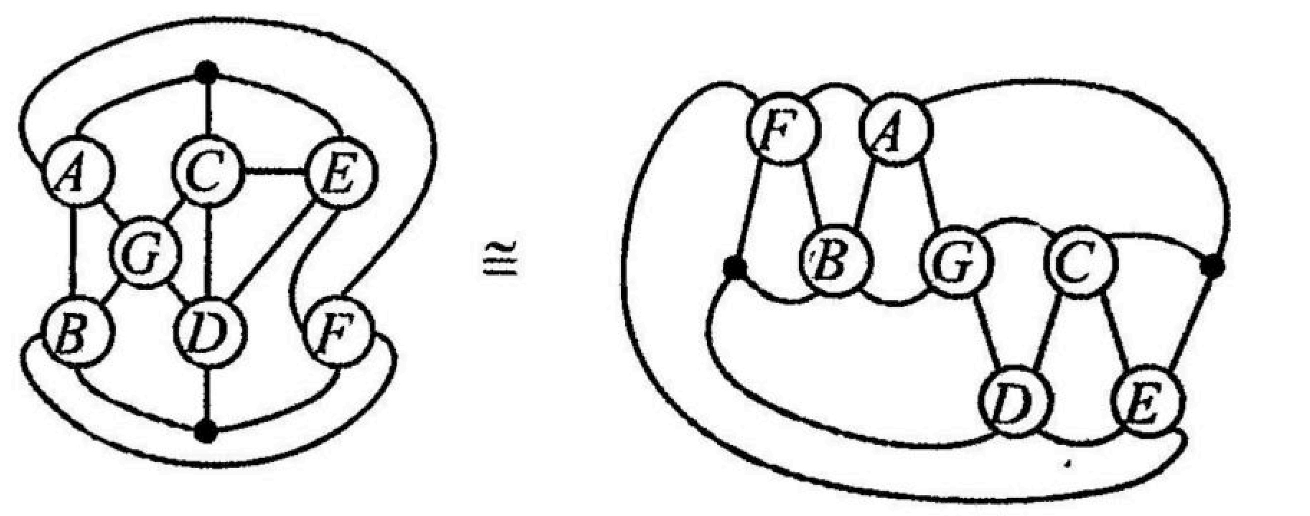


FIG. 3.218. $t = 7$ (d) (i) (C) (6) (I).

(II) $A \sim C, B \sim D$. We obtain 7^9_*; see Fig. 3.219.

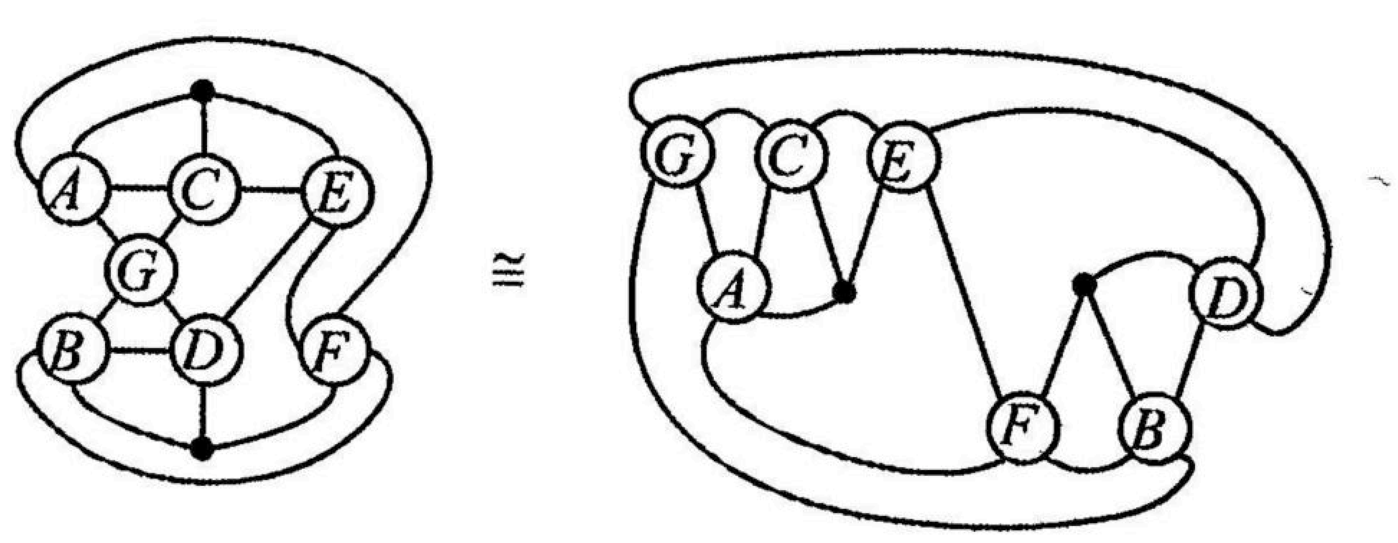


FIG. 3.219. $t = 7$ (d) (i) (C) (6) (II).

(III) $A \sim D$, $B \sim C$. This gives a graph containing $K_{3,3}$, and so it does not satisfy the condition (P5); see Fig. 3.220.

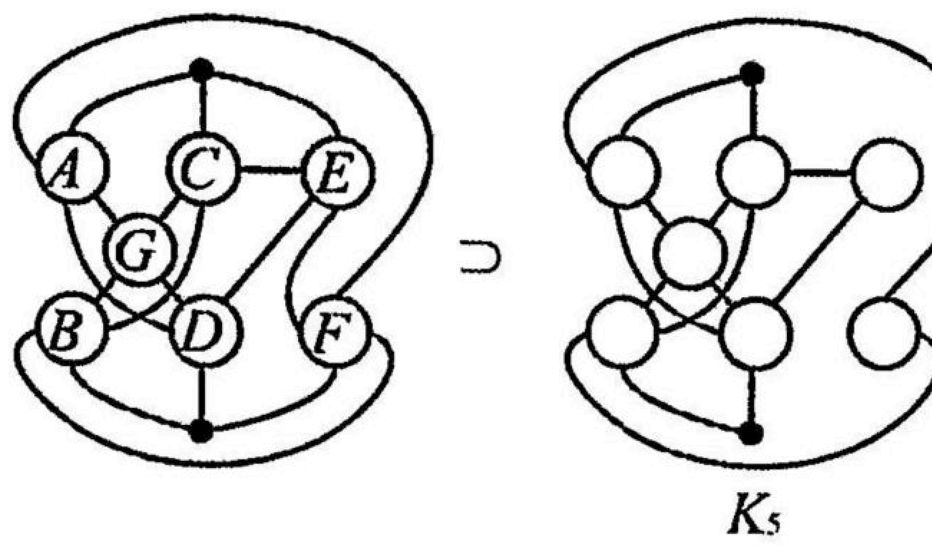


FIG. 3.220. $t = 7$ (d) (i) (C) (6) (III).

(D) $F \sim A, C, D$. Since B and D are interchangeable in the first figure in Fig. 3.205, this case is the same as the case (A).

(E) $F \sim A, C, E$. The vertex E has two remaining hands, so we consider how the hands of E connect. There are six cases; see Fig. 3.221.

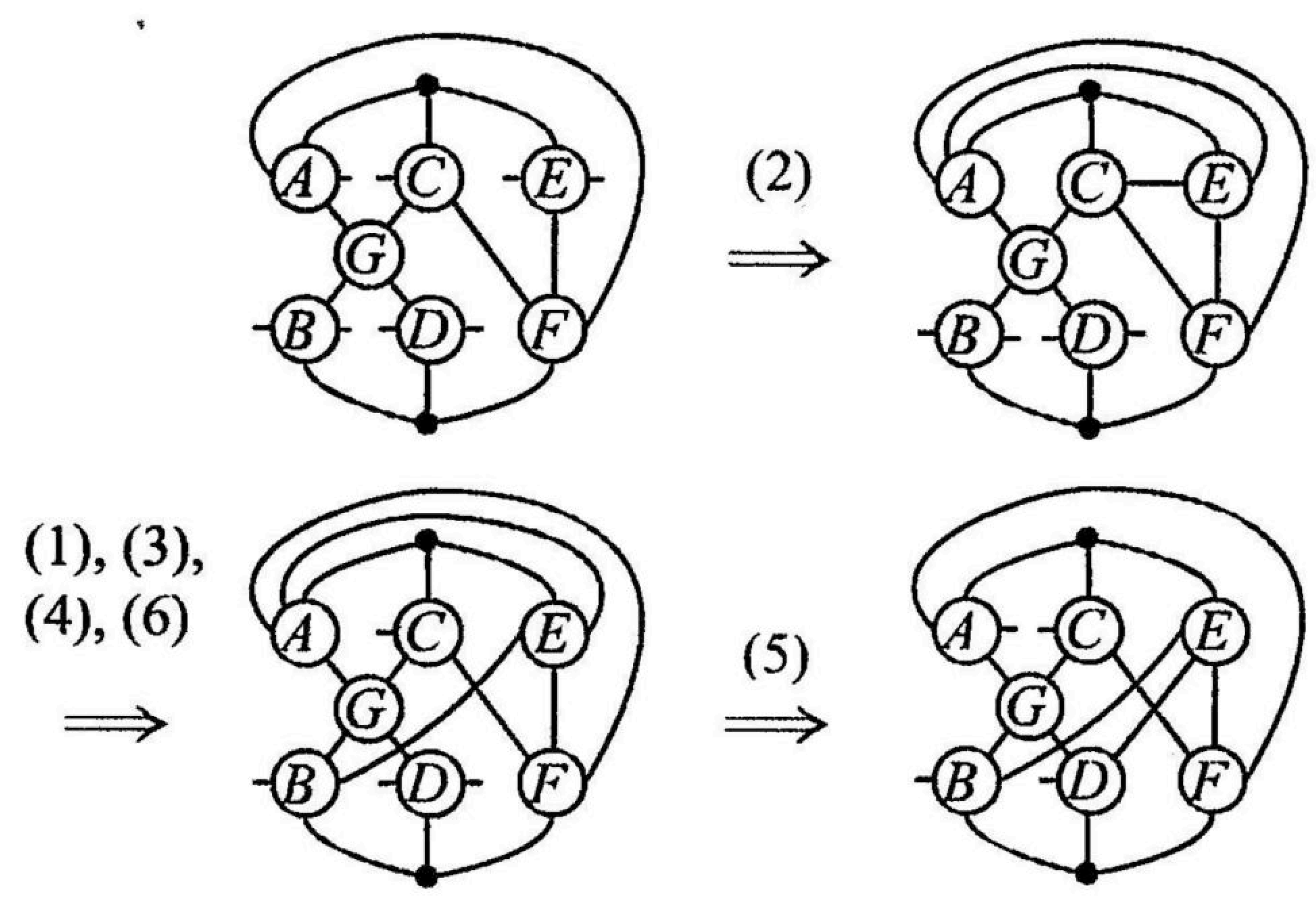


FIG. 3.221. $t = 7$ (d) (i) (E).

(1) $E \sim A, B$. This gives a graph containing $K_{3,3}$, and so it does not satisfy the condition (P5); see Fig. 3.222.

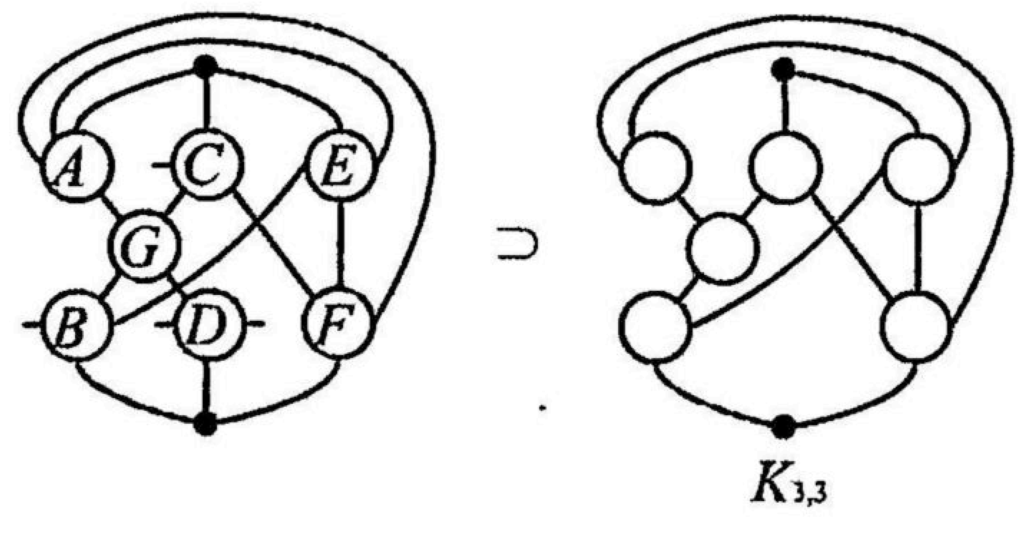


FIG. 3.222. $t = 7$ (d) (i) (E) (1).

(2) $E \sim A, C$. This gives a graph which does not satisfy the conditions (P1) or (P2); see Fig. 3.221.

(3) $E \sim A, D$. Since B and D are interchangeable in the first figure in Fig. 3.221, this case is the same as the case (1).

(4) $E \sim B, C$. Since A and C are interchangeable in the first figure in Fig. 3.22, this case is the same as the case (1).

(5) $E \sim B, D$. This gives a graph containing $K_{3,3}$, and so it does not satisfy condition (P5); see Fig. 3.223.

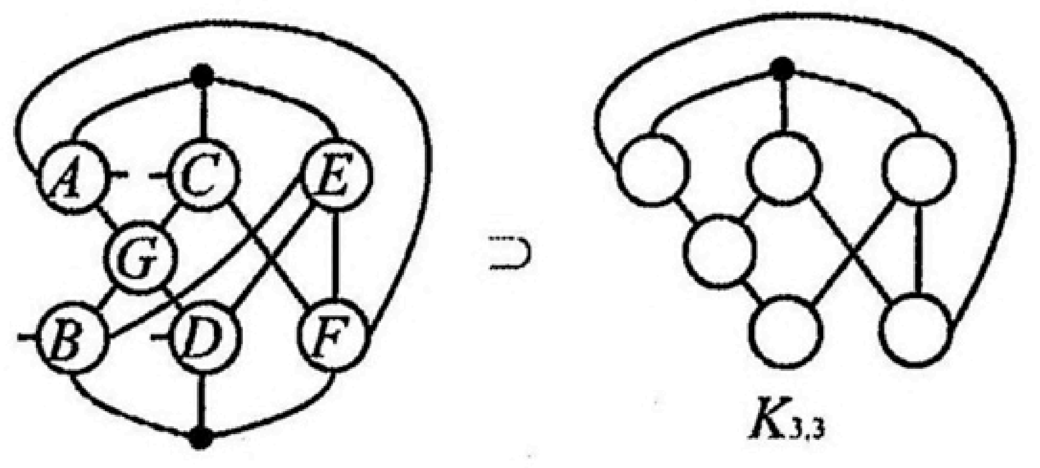


FIG. 3.223. $t = 7$ (d) (i) (E) (5).

(6) $E \sim C, D$. Since A and C, B and D are interchangeable in the first figure in Fig. 3.221, this case is the same as the case (1).

(F) $F \sim A, D, E$. Since B and D are interchangeable in the first figure in Fig. 3.205, this case is the same as the case (C).

(G) $F \sim B, C, D$. Since A and C are interchangeable in the first figure in Fig. 3.205, this case is the same as the case (B).

(H) $F \sim B, C, E$. Since A and C are interchangeable in the first figure in Fig. 3.205, this case is the same as the case (C).

(I) $F \sim B, D, E$. The vertex E has two remaining hands, so we consider how the hands of E connect. There are six cases; see Fig. 3.224.

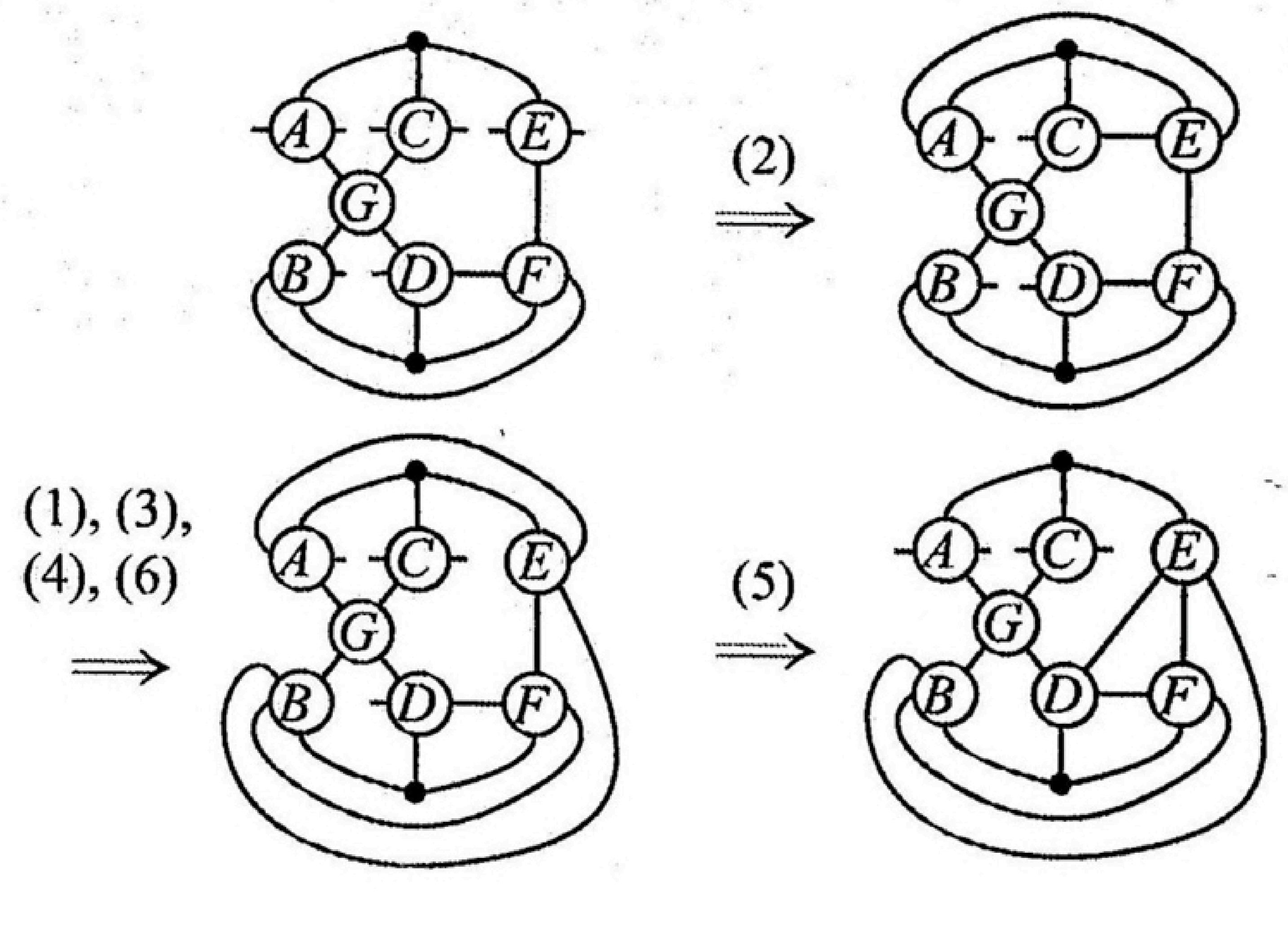


FIG. 3.224. $t = 7$ (d) (i) (I).

(1) $E \sim A, B$. Then $C \sim A, D$, and we obtain 7^7_*; see Fig. 3.225.

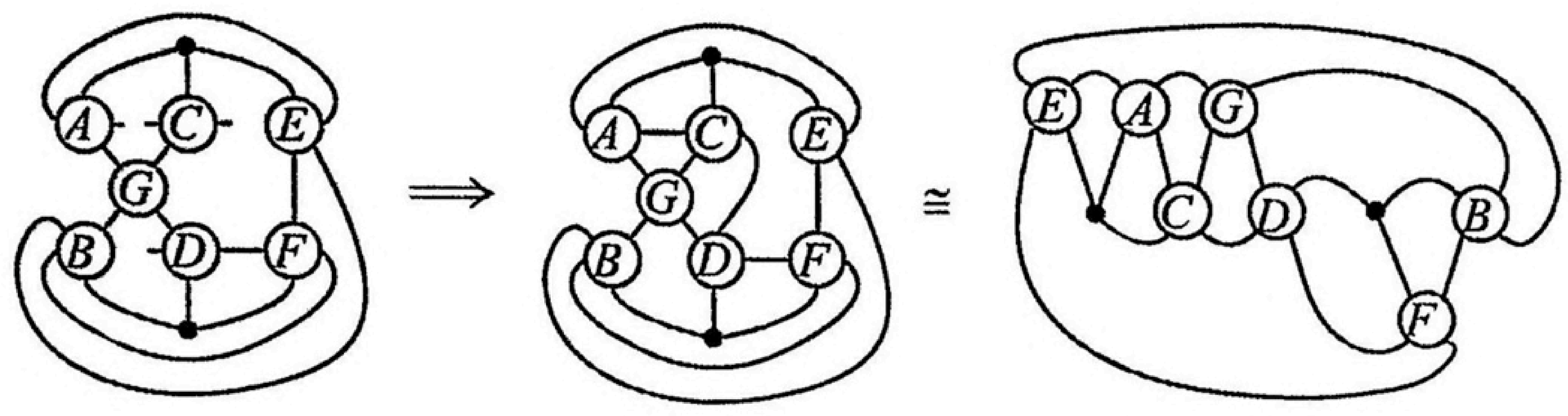


FIG. 3.225. $t = 7$ (d) (i) (I) (1).

(2) $E \sim A, C$. We have three cases; see Fig. 3.226.

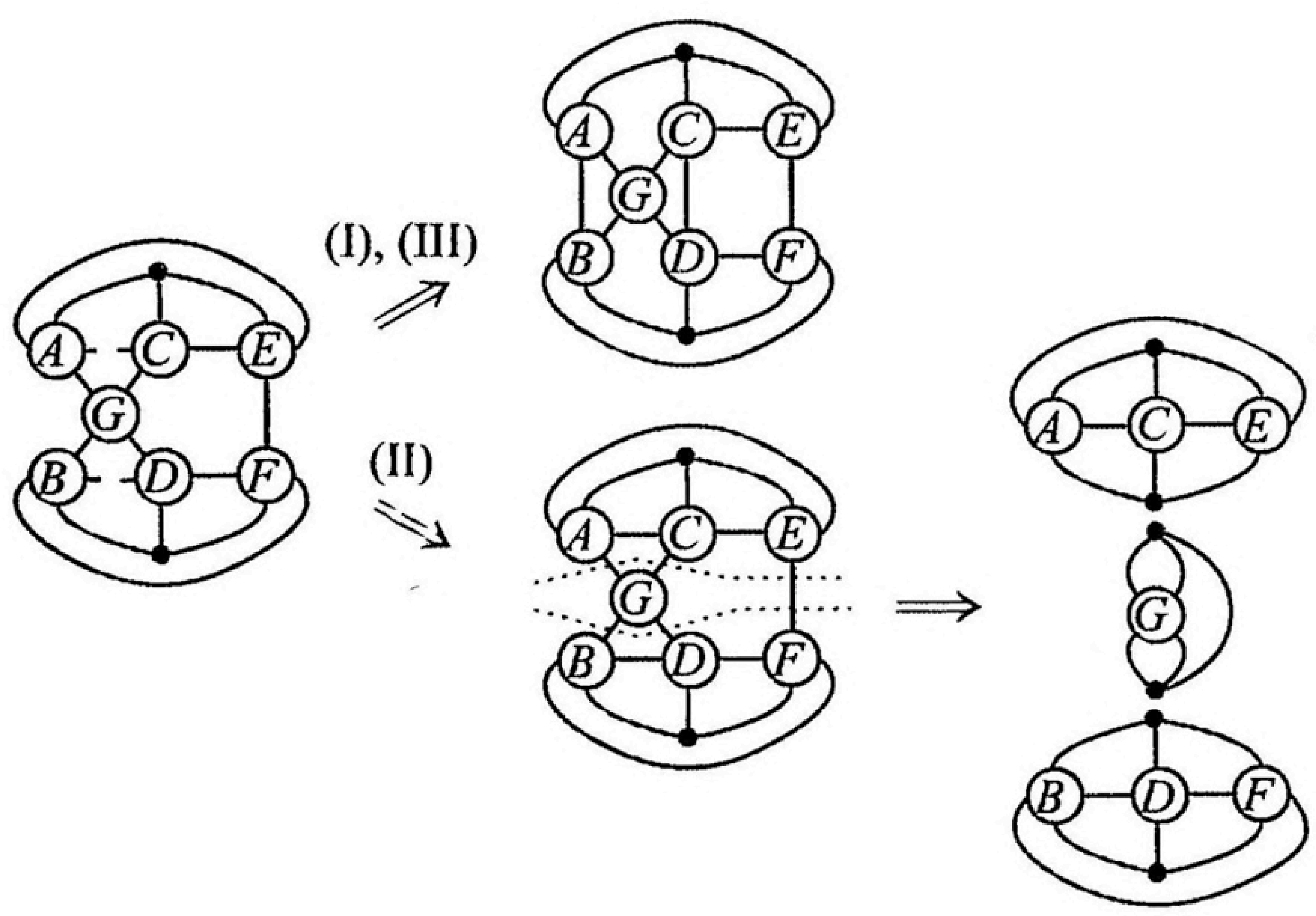


FIG. 3.226. $t = 7$ (d) (i) (I) (2).

(I) $A \sim B, C \sim D$. We obtain 7_*^{10}.
(II) $A \sim C, B \sim D$. This gives a nonprime θ-polyhedron.
(III) $A \sim D, B \sim C$. Since A and C are interchangeable in the first figure in Fig. 3.226, this case is the same as the case (I).

(3) $E \sim A, D$. Since B and D are interchangeable in the first figure in Fig. 3.224, this case is the same as the case (1).

(4) $E \sim B, C$. Since A and C are interchangeable in the first figure in Fig. 3.224, this case is the same as the case (1).

(5) $E \sim B, D$. This gives a graph which does not satisfy the conditions (P1) or (P2); see Fig. 3.224.

(6) $E \sim C, D$. Since A and C, B and D are interchangeable in the first figure in Fig. 3.224, this case is the same as the case (1).

(J) $F \sim C, D, E$. Since A and C, B and D are interchangeable in the first figure in Fig. 3.205, this case is the same as the case (C).

(ii) $G \sim A, B, C, E$. The vertex F has three remaining hands, so we consider how the hands of F connect. There are ten cases; see Fig. 3.227.

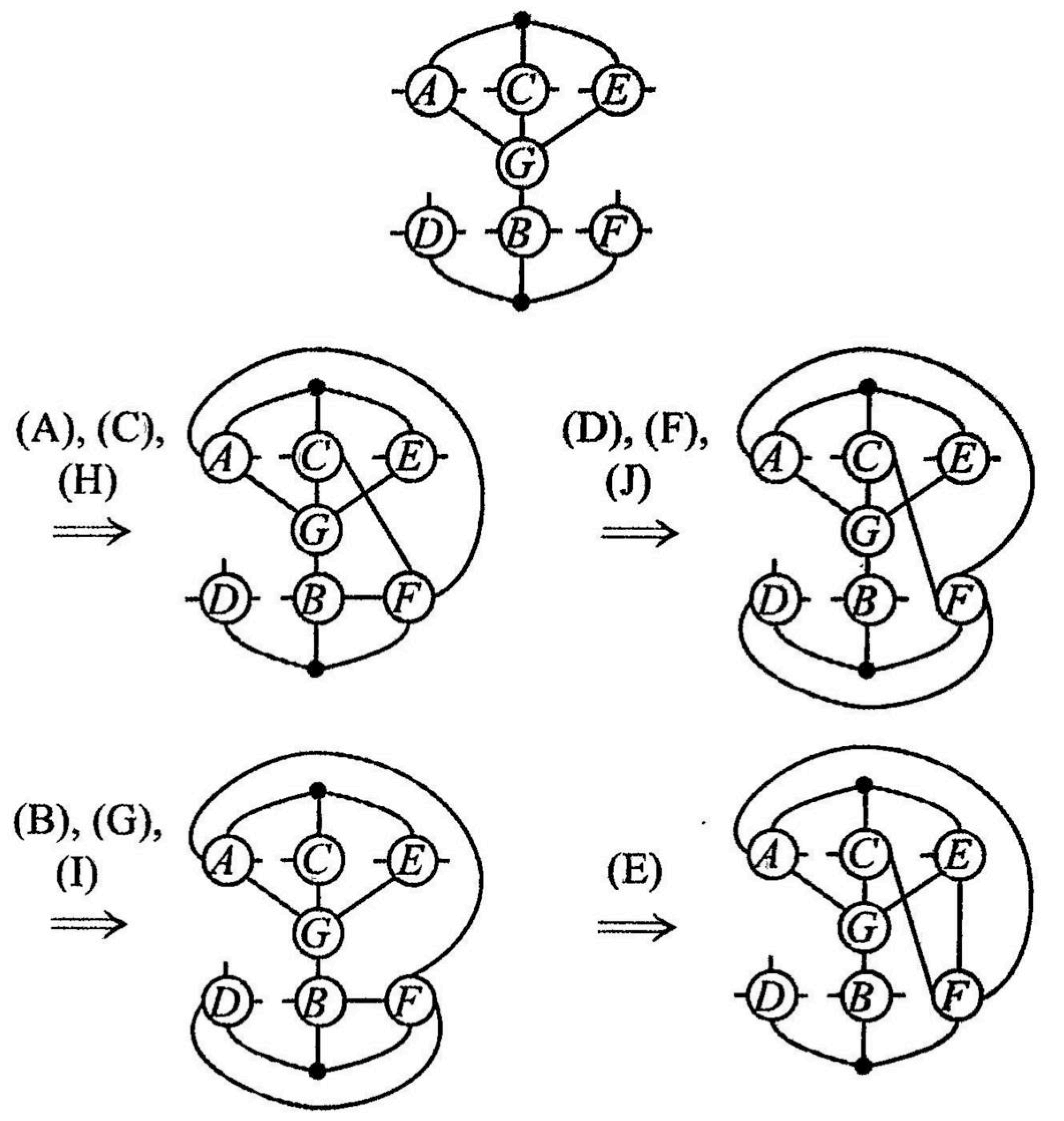


FIG. 3.227. $t = 7$ (d) (ii).

(A) $F \sim A, B, C$. The vertex D has three remaining hands, so we consider how the hands of D connect. There are four cases; see Fig. 3.228.

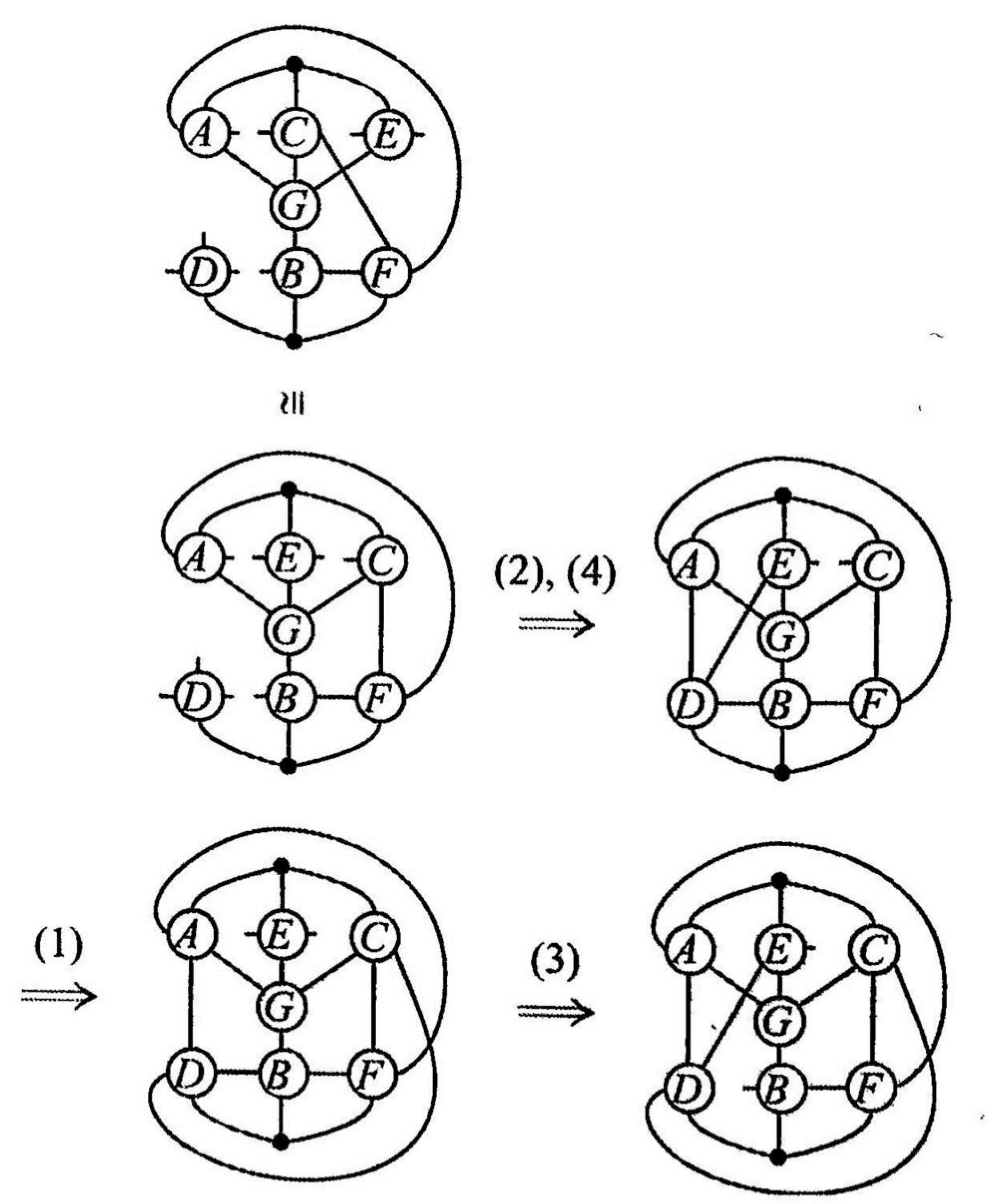


FIG. 3.228. $t = 7$ (d) (ii) (A).

(1) $D \sim A, B, C$. This gives a graph having a loop at E, and so it does not satisfy the condition (P1); see Fig. 3.228.

(2) $D \sim A, B, E$. This gives a graph containing $K_{3,3}$, and so it does not satisfy the condition (P5); see Fig. 3.229.

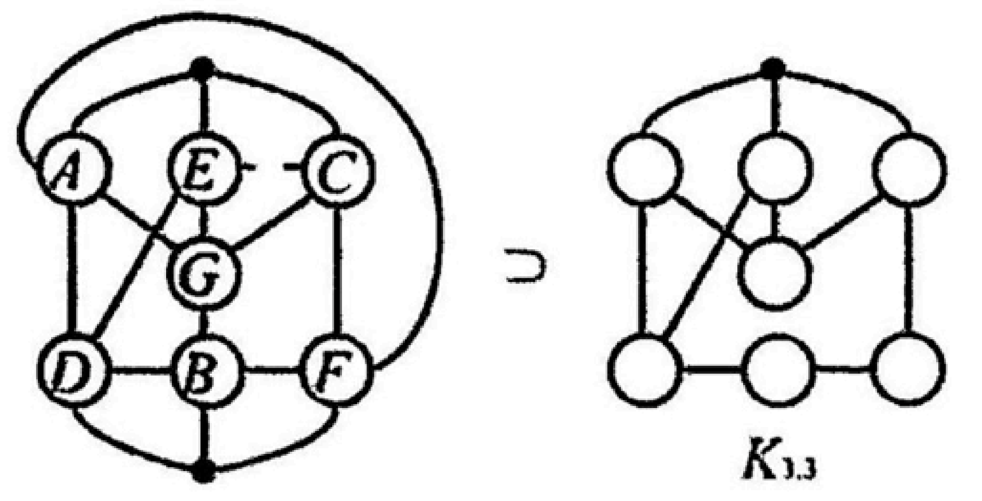


FIG. 3.229. $t = 7$ (d) (ii) (A) (2).

(3) $D \sim A, C, E$. This gives a graph containing $K_{3,3}$, and so it does not satisfy the condition (P5); see Fig. 3.230.

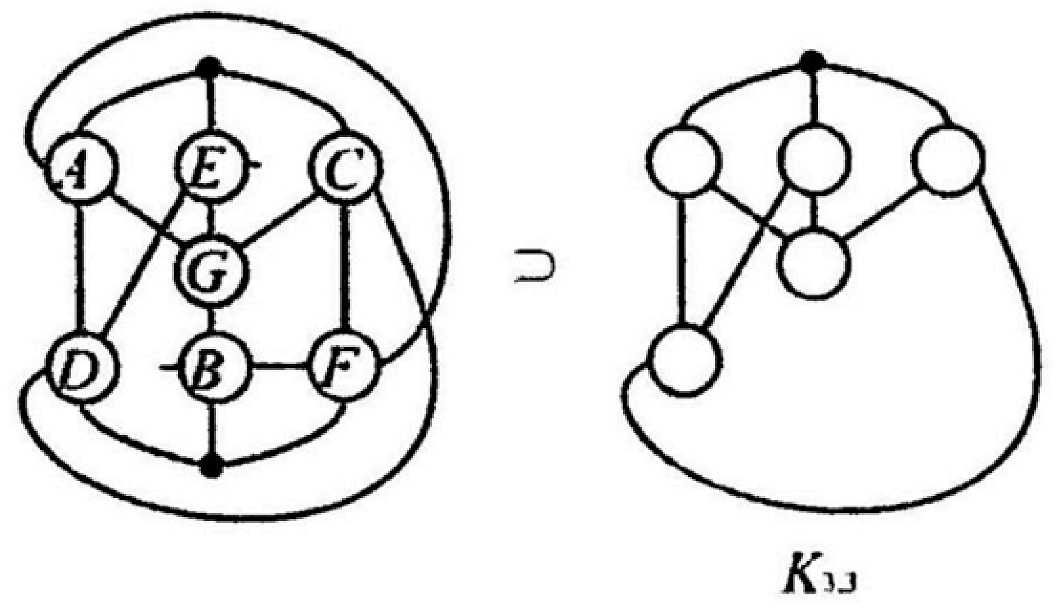


FIG. 3.230. $t = 7$ (d) (ii) (A) (3).

(4) $D \sim B, C, E$. Since A and C are interchangeable in the first figure in Fig. 3.228, this case is the same as the case (2).

(B) $F \sim A, B, D$. The vertex D has two remaining hands, so we consider how the hands of D connect. There are six cases; see Fig. 3.231.

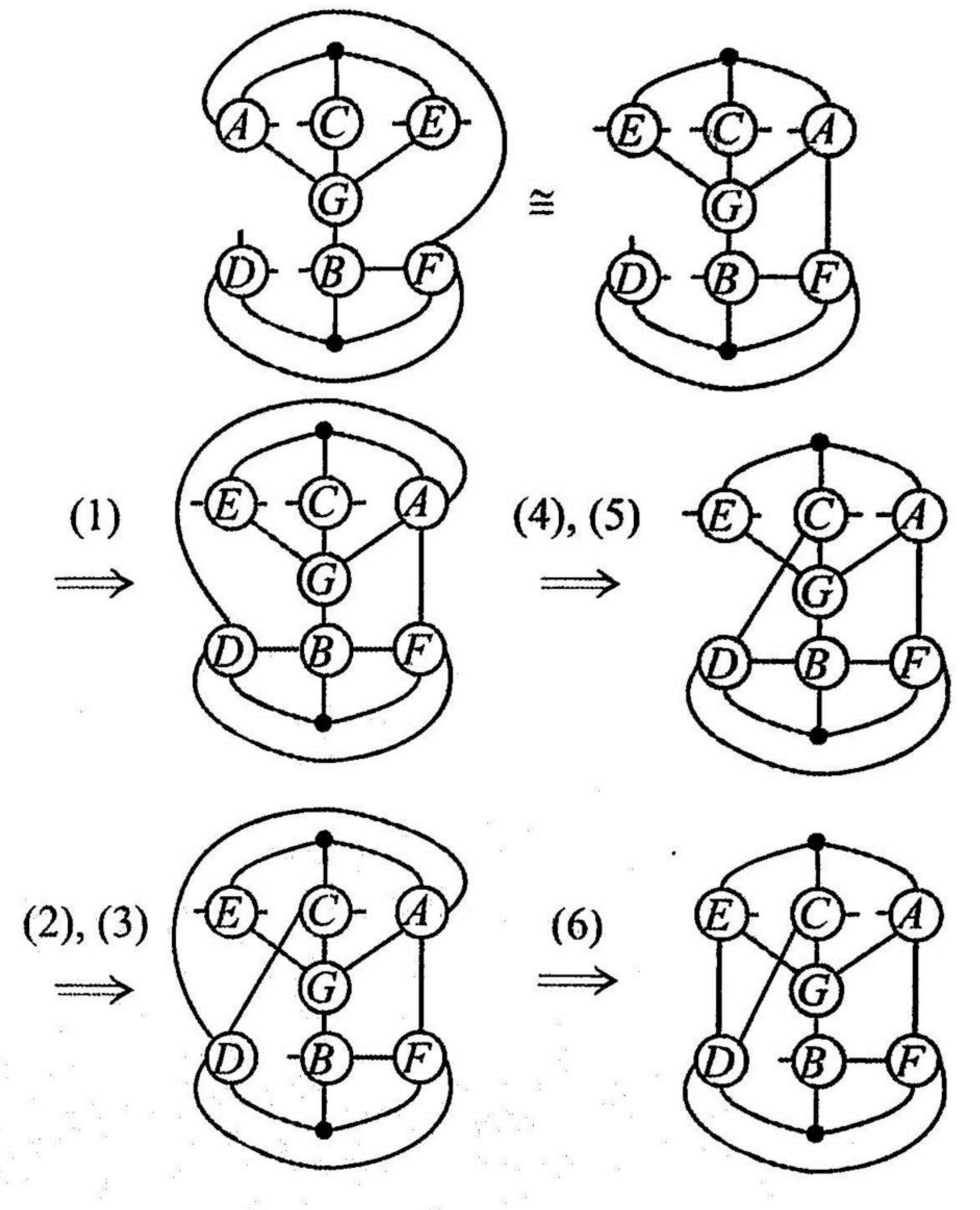


FIG. 3.231. $t = 7$ (d) (ii) (B).

(1) $D \sim A, B$. This gives a graph which does not satisfy the conditions (P1) or (P2); see Fig. 3.231.
(2) $D \sim A, C$. Then $E \sim B, C$. This gives a graph containing $K_{3,3}$, and so it does not satisfy the condition (P5); see Fig. 3.232.

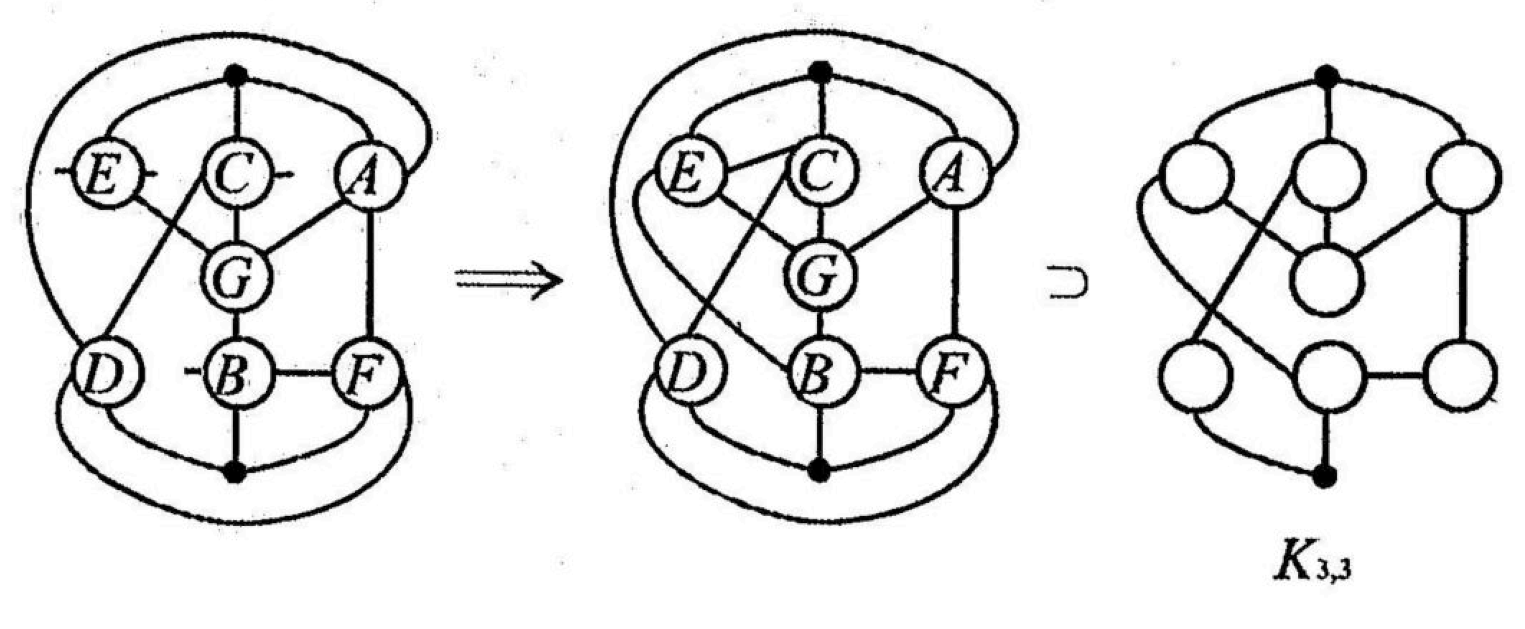


FIG. 3.232. $t = 7$ (d) (ii) (B) (2).

(3) $D \sim A, E$. Since C and E are interchangeable in the first figure in Fig. 3.231, this case is the same as the case (2).
(4) $D \sim B, C$. Then $E \sim A, C$. This gives a nonprime θ-polyhedron; see Fig. 3.233.

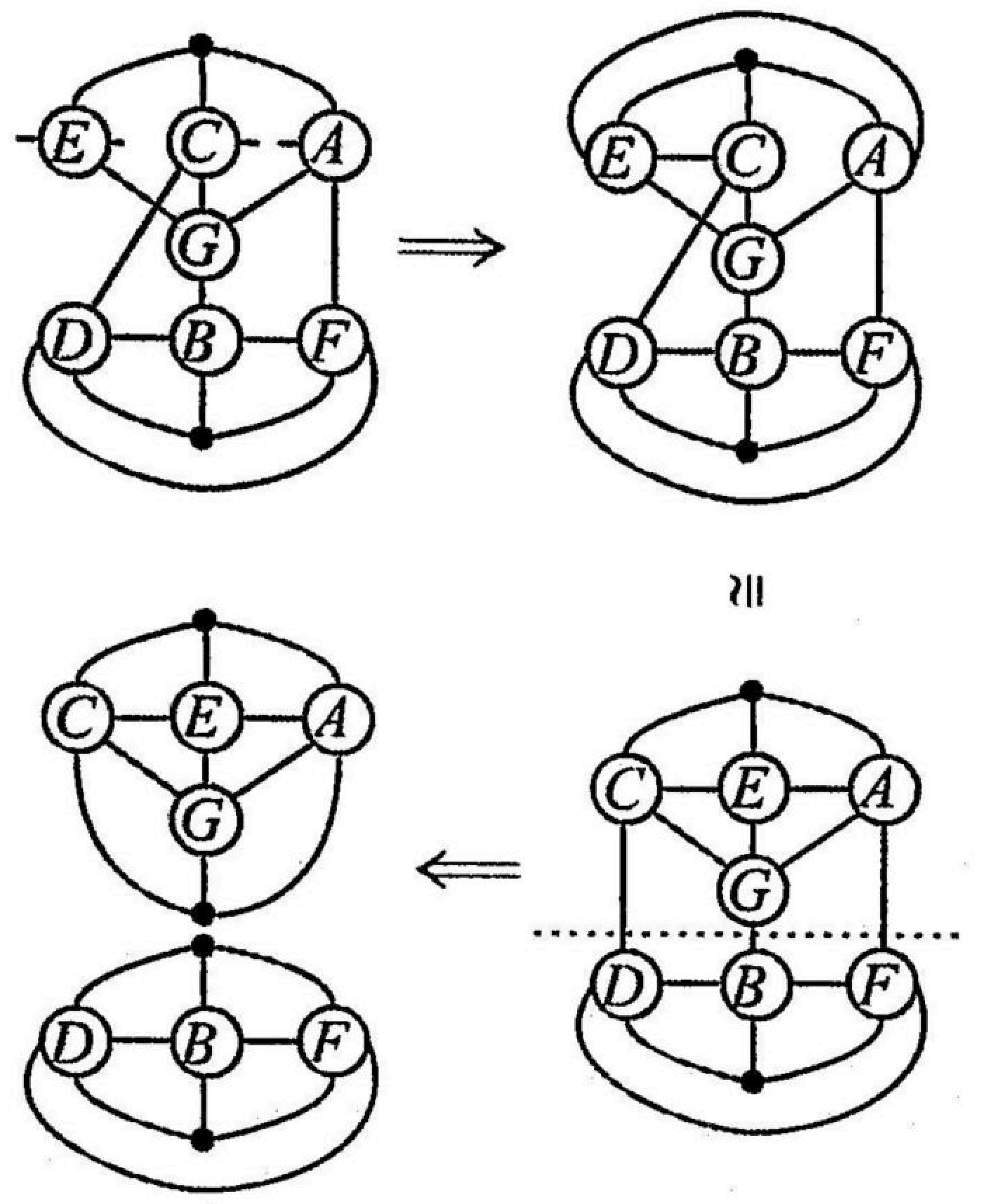


FIG. 3.233. $t = 7$ (d) (ii) (B) (4).

(5) $D \sim B, E$. Since C and E are interchangeable in the first figure in Fig. 3.231, this case is the same as the case (4).

(6) $D \sim C, E$. This gives a graph containing $K_{3,3}$, and so it does not satisfy the condition (P5); see Fig. 3.234.

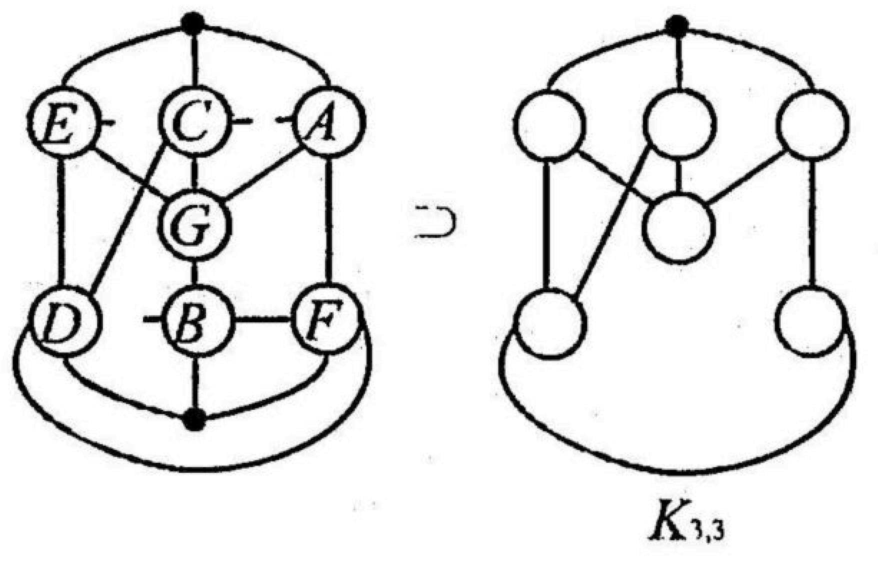


FIG. 3.234. $t = 7$ (d) (ii) (B) (6).

(C) $F \sim A, B, E$. Since C and E are interchangeable in the first figure in Fig. 3.227, this case is the same as the case (A).

(D) $F \sim A, C, D$. The vertex D has two remaining hands, so we consider how the hands of D connect. There are six cases; see Fig. 3.235.

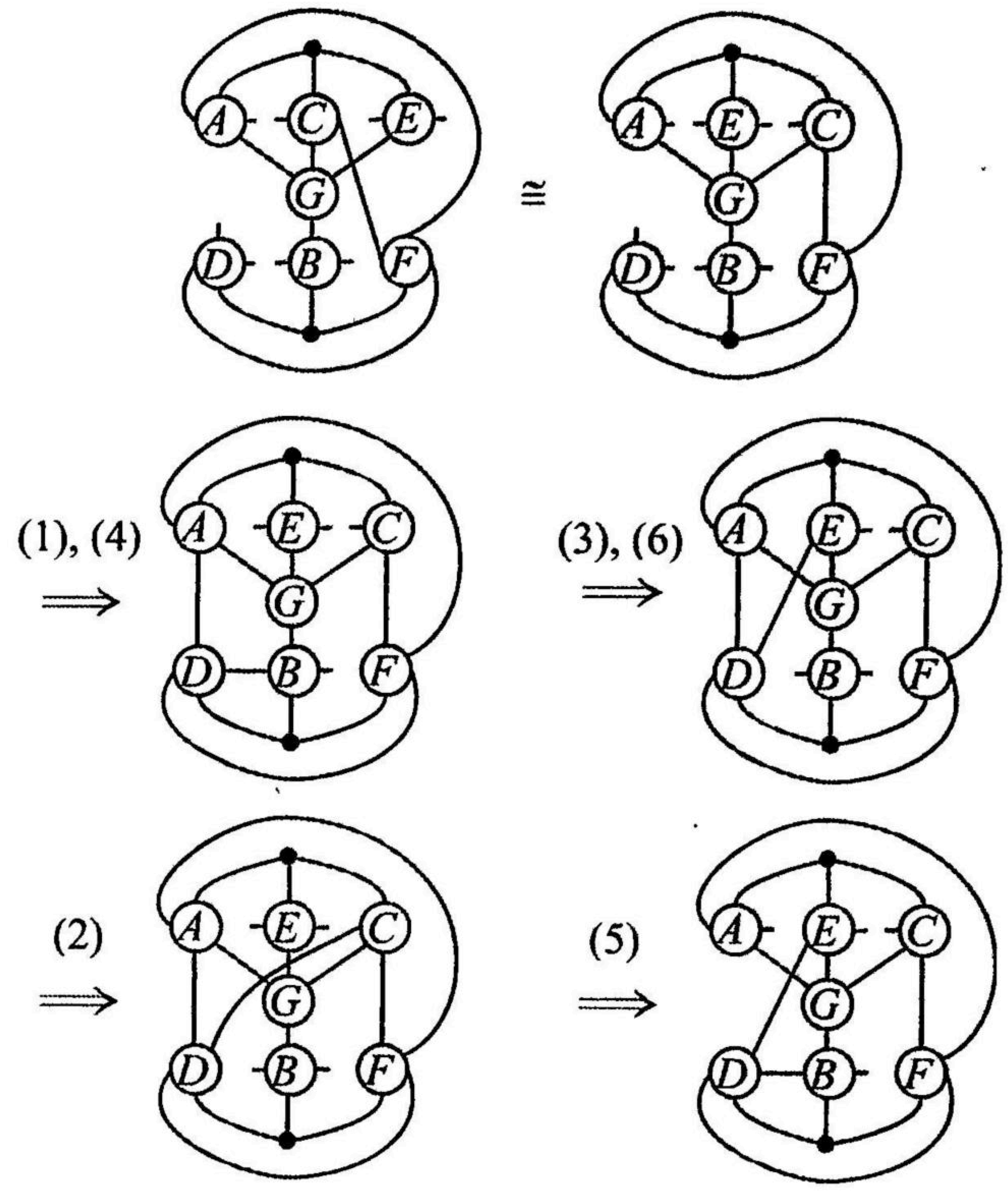


FIG. 3.235. $t = 7$ (d) (ii) (D).

(1) $D \sim A, B$. Then $E \sim B, C$. This gives a graph containing $K_{3,3}$, and so it does not satisfy the condition (P5); see Fig. 3.236.

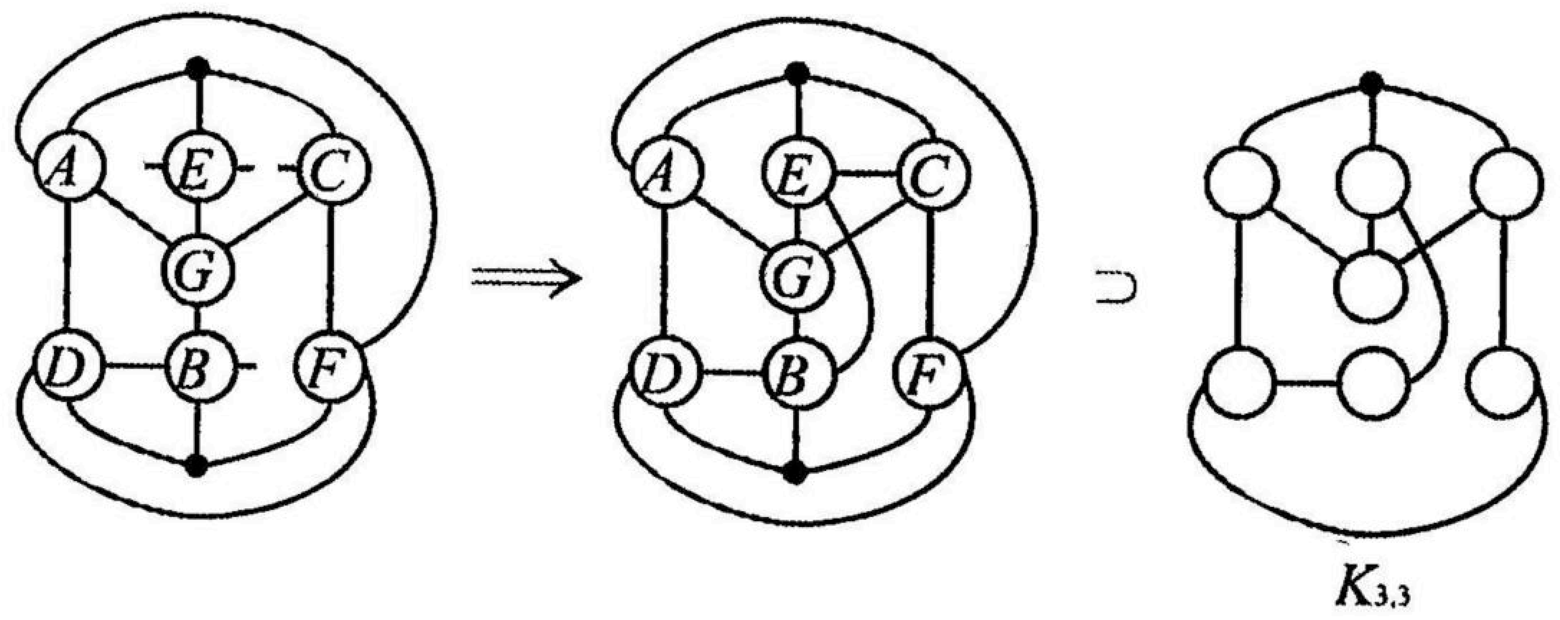


FIG. 3.236. $t = 7$ (d) (ii) (D) (1).

(2) $D \sim A, C$. This gives a graph which does not satisfy the conditions (P1) or (P2); see Fig. 3.235.

(3) $D \sim A, E$. This gives a graph containing $K_{3,3}$, and so it does not satisfy the condition (P5); see Fig. 3.237.

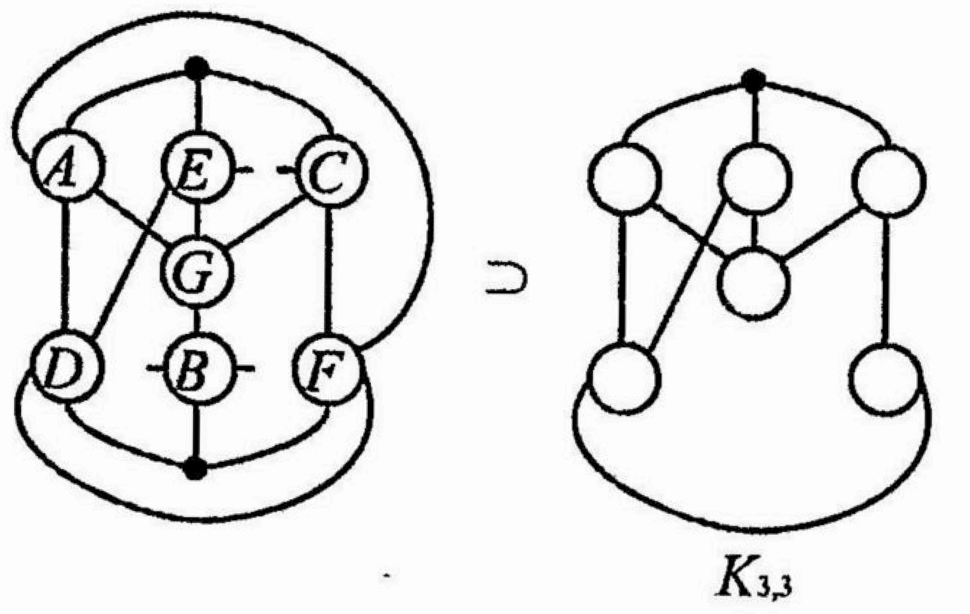


FIG. 3.237. $t = 7$ (d) (ii) (D) (3).

(4) $D \sim B, C$. Since A and C are interchangeable in the first figure in Fig. 3.235, this case is the same as the case (1).
(5) $D \sim B, E$. This gives a graph containing $K_{3,3}$, and so it does not satisfy the condition (P5); see Fig. 3.238.

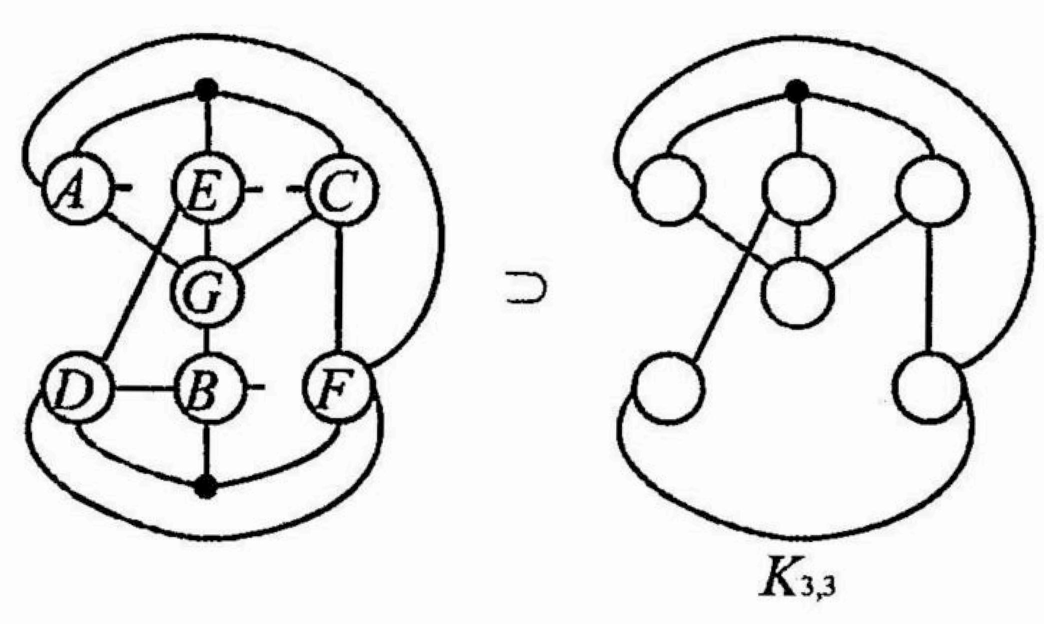


FIG. 3.238. $t = 7$ (d) (ii) (D) (5).

(6) $D \sim C, E$. Since A and C are interchangeable in the first figure in Fig. 3.235, this case is the same as the case (3).
(E) $F \sim A, C, E$. This gives a graph containing $K_{3,3}$, and so it does not satisfy the condition (P5); see Fig. 3.239.

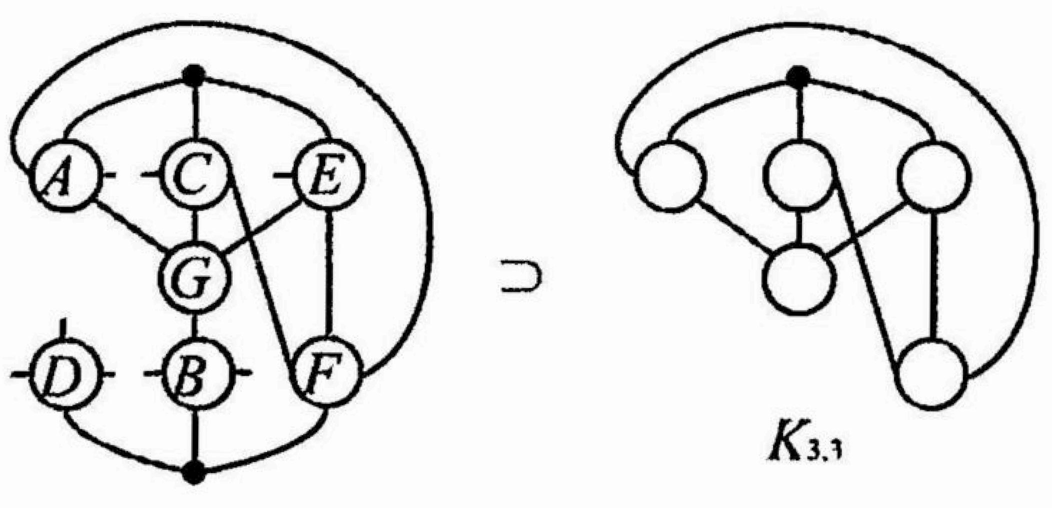


FIG. 3.239. $t = 7$ (d) (ii) (E).

(F) $F \sim A, D, E$. Since C and E are interchangeable in the first figure in Fig. 3.227, this case is the same as the case (D).
(G) $F \sim B, C, D$. Since A and C are interchangeable in the first figure in Fig. 3.227, this case is the same as the case (B).
(H) $F \sim B, C, E$. Since A and E are interchangeable in the first figure in Fig. 3.227, this case is the same as the case (A).

(I) $F \sim B, D, E$. Since A and E are interchangeable in the first figure in Fig. 3.227, case is the same as the case (B).

(J) $F \sim C, D, E$. Since A and E are interchangeable in the first figure in Fig. 3.227, case is the same as the case (D).

(iii) $G \sim A, B, C, F$. Since D and F are interchangeable in the first figure in Fig. 3.204, case is the same as the case (i).

(iv) $G \sim A, B, D, E$. Since C and E are interchangeable in the first figure in Fig. 3.204, case is the same as the case (i).

(v) $G \sim A, B, D, F$. Since C and D, E and F are interchangeable in the first figure in Fig. 3.204, this case is the same as the case (ii).

(vi) $G \sim A, B, E, F$. Since C and E, D and F are interchangeable in the first figure in Fig. 3.204, this case is the same as the case (i).

(vii) $G \sim A, C, D, E$. Since B and D are interchangeable in the first figure in Fig. 3.204, this case is the same as the case (ii).

(viii) $G \sim A, C, D, F$. Since B and F are interchangeable in the first figure in Fig. 3.204, this case is the same as the case (i).

(ix) $G \sim A, C, E, F$. Since B and F are interchangeable in the first figure in Fig. 3.204, this case is the same as the case (ii).

(x) $G \sim A, D, E, F$. Since C and E, B and F are interchangeable in the first figure in Fig. 3.204, this case is the same as the case (i).

(xi) $G \sim B, C, D, E$. Since A and D are interchangeable in the first figure in Fig. 3.204, this case is the same as the case (ii).

(xii) $G \sim B, C, D, F$. Since A and D, E and F are interchangeable in the first figure in Fig. 3.204, this case is the same as the case (ii).

(xiii) $G \sim B, C, E, F$. Since A and E, D and F are interchangeable in the first figure in Fig. 3.204, this case is the same as the case (i).

(xiv) $G \sim B, D, E, F$. Since A and F, C and D are interchangeable in the first figure in Fig. 3.204, this case is the same as the case (ii).

(xv) $G \sim C, D, E, F$. Since A and E, B and F are interchangeable in the first figure in Fig. 3.204, this case is the same as the case (i).

This completes the proof. □

CHAPTER 4

Conclusion

From Lemma 2.5, Theorems 3.6 and 3.10, we can obtain all the prime θ-curves with up to seven crossings, which are the exactly same ones as in Litherland's; see Table 1.1. Litherland showed that these θ-curves are mutually distinct by investigating the *Alexander polynomial* ([19]) and constituent knots. We show this using the *Yamada polynomial* ([38]), which will be explained in Chapter 5; see Table 5.1.

In the same way, we also obtain all the prime handcuff graphs with up to seven crossings. We can show that handcuff graphs in Table 1.2 are mutually distinct by investigating their constituent links and the Yamada polynomial; see Table 5.2.

1. θ-curve

We give an enumeration of θ-curve with up to seven crossings by using our notation; see Table 4.1. Knots in the second column correspond to Rolfsen's knot table [31], and θ-curves in the last column correspond to Litherland's table [20]. $\overline{K}$ and $\overline{\Theta}$ denote mirror images of K and Θ, respectively.

EXAMPLE 4.1. The θ-curve diagram as in Fig. 4.1(a) is denoted by $4^1_* 2\ 1\ 0.1.1.\bar{2}\ 0$. Its constituent knots are $5_2, 3_1, 0_1$. This θ-curve and 7_{22} are ambient isotopic (see Fig. 4.1).

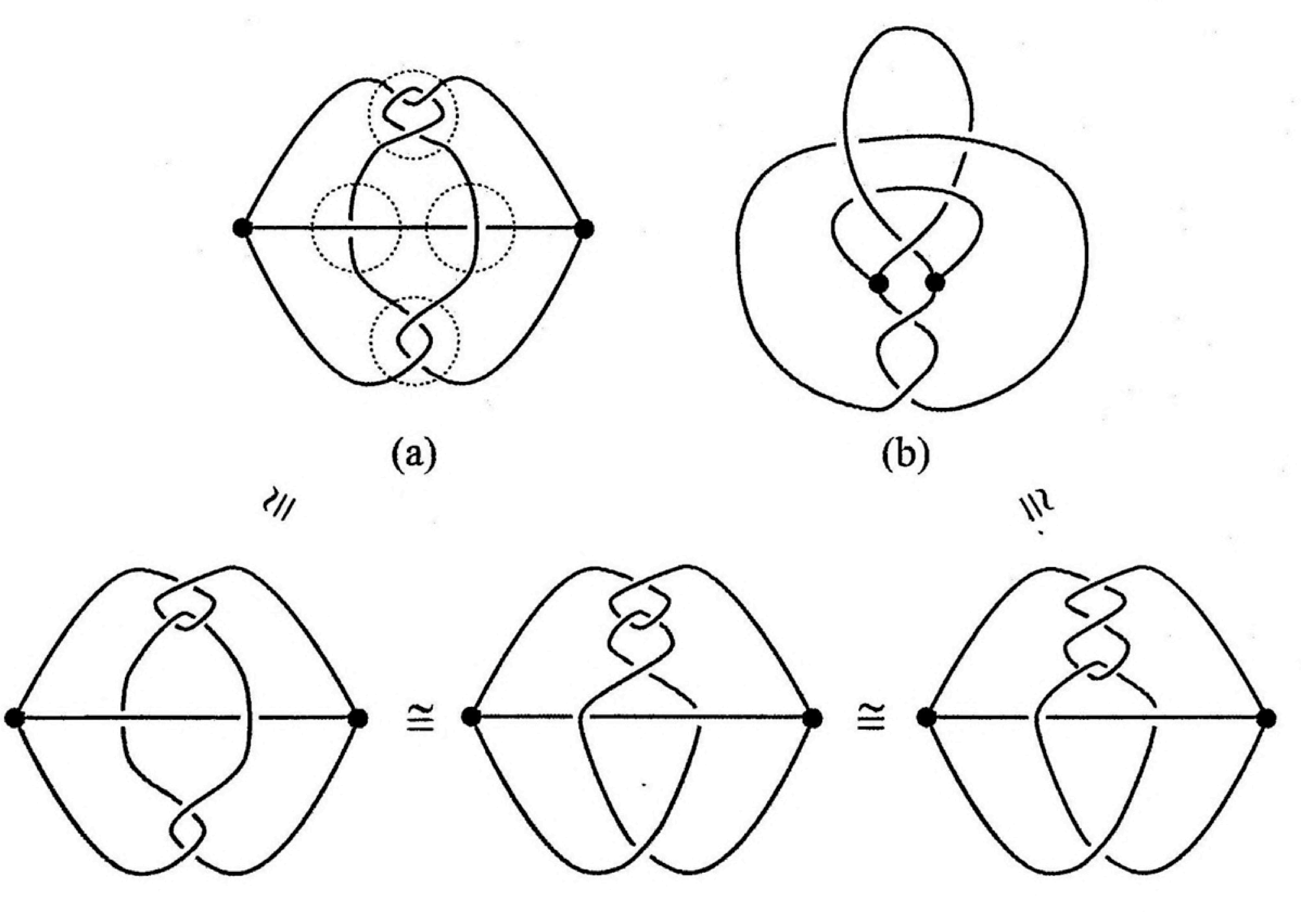


FIG. 4.1. (a) $4^1_* 2\ 1\ 0.1.1.\bar{2}\ 0$, (b) 7_{22}.

TABLE 4.1. θ-curves with up to seven crossings.

notation	constituent knot	θ-curve
$1^1_\times\ 3$	$3_1, 0, 0$	$\overline{3_1}$
$1^1_\times\ 2\,2$	$4_1, 0, 0$	4_1
$1^1_\times\ 5$	$5_1, 0, 0$	5_3
$1^1_\times\ 3\,2$	$5_2, 0, 0$	5_6
$1^1_\times\ 2\,3$	$\overline{5_2}, 0, 0$	$\overline{5_5}$
$1^1_\times\ 3, 2$	$\overline{5_1}, \overline{3_1}, 0$	$\overline{5_4}$
$1^1_\times\ 2\,1, 2$	$5_2, 3_1, 0$	5_7
$3^1_*\ 2.2.\bar{1}$	$\overline{3_1}, 0, 0$	5_2
$4^1_*\ 2.1.1.1$	$\overline{3_1}, 0, 0$	5_2
$4^1_*\ 2\,0.1.1.1$	$0, 0, 0$	5_1
$1^1_\times\ 4\,2$	$6_1, 0, 0$	6_5
$1^1_\times\ 2\,4$	$\overline{6_1}, 0, 0$	$\overline{6_6}$
$1^1_\times\ 3\,1\,2$	$6_2, 0, 0$	6_9
$1^1_\times\ 2\,1\,3$	$6_2, 0, 0$	6_{10}
$1^1_\times\ 2\,1\,1\,2$	$6_3, 0, 0$	$\overline{6_{14}}$
$1^1_\times\ 2\,2, 2$	$6_1, 4_1, 0$	6_8
$1^1_\times\ 2\,1\,1, 2$	$\overline{6_2}, 4_1, 0$	$\overline{6_{13}}$
$1^1_\times\ 3, 2\,1$	$\overline{6_1}, 0, 0$	$\overline{6_7}$
$1^1_\times\ 3, 2+$	$\overline{6_2}, \overline{3_1}, 0$	$\overline{6_{12}}$
$1^1_\times\ 2\,1, 2+$	$6_3, 3_1, 0$	$\overline{6_{16}}$
$3^1_*\ 3.2.\bar{1}$	$4_1, 3_1, 0$	6_4
$3^1_*\ 2.2.2$	$0, 0, 0$	5_1
$3^1_*\ 2.2.\bar{2}$	$0, 0, 0$	6_1
$4^1_*\ 2\,1.1.1.1$	$0, 0, 0$	6_1
$4^1_*\ 2\,1\,0.1.1.1$	$\overline{3_1}, 0, 0$	6_2
$4^1_*\ \bar{2}\,\bar{1}\,0.1.1.1$	$4_1, \overline{3_1}, 0$	$\overline{6_4}$
$4^1_*\ 2.2\,0.1.1$	$4_1, 3_1, 0$	6_4
$4^1_*\ 2.1.1.2\,0$	$4_1, \overline{3_1}, 0$	$\overline{6_3}$
$4^1_*\ 2.1.1.\bar{2}\,0$	$4_1, 3_1, 0$	6_4
$4^1_*\ 2\,0.2\,0.1.1$	$\overline{3_1}, 0, 0$	6_2
$4^1_*\ \bar{2}\,0.2\,0.1.1$	$0, 0, 0$	$\overline{6_1}$
$5^1_\times\ 2.1.1.1.1$	$6_3, 0, 0$	6_{15}
$5^1_\times\ 2\,0.1.1.1.1$	$\overline{6_2}, 0, 0$	$\overline{6_{11}}$
$5^1_*\ 2\,0.1.1.1.1$	$0, 0, 0$	$\overline{6_1}$
$5^1_*\ 1.1.1.1.2\,0$	$\overline{5_2}, \overline{3_1}, 0$	$\overline{5_7}$
$6^2_*\ 1.1.1.1.1.1$	$0, 0, 0$	$\overline{6_1}$
$6^4_*\ 1.1.1.1.1.1$	$0, 0, 0$	$\overline{5_1}$

TABLE 4.1. θ-curves with up to seven crossings (continued).

notation	constituent knot	θ-curve
$1^1_\times\ 7$	$7_1, 0, 0$	7_{25}
$1^1_\times\ 5\,2$	$7_2, 0, 0$	$\overline{7_{29}}$
$1^1_\times\ 4\,3$	$7_3, 0, 0$	7_{33}
$1^1_\times\ 3\,4$	$\overline{7_3}, 0, 0$	$\overline{7_{34}}$
$1^1_\times\ 2\,5$	$\overline{7_2}, 0, 0$	7_{28}
$1^1_\times\ 3\,2\,2$	$7_5, 0, 0$	$\overline{7_{43}}$
$1^1_\times\ 3\,1\,3$	$7_4, 0, 0$	7_{38}
$1^1_\times\ 2\,2\,3$	$7_5, 0, 0$	$\overline{7_{44}}$
$1^1_\times\ 2\,2\,1\,2$	$7_6, 0, 0$	$\overline{7_{53}}$
$1^1_\times\ 2\,1\,2\,2$	$\overline{7_6}, 0, 0$	7_{50}
$1^1_\times\ 2\,1\,1\,1\,2$	$7_7, 0, 0$	7_{59}
$1^1_\times\ 5, 2$	$\overline{7_1}, \overline{5_1}, 0$	$\overline{7_{27}}$
$1^1_\times\ 4\,1, 2$	$\overline{7_3}, 5_1, 0$	$\overline{7_{36}}$
$1^1_\times\ 3\,2, 2$	$7_3, \overline{5_2}, 0$	7_{37}
$1^1_\times\ 2\,3, 2$	$7_2, 5_2, 0$	$\overline{7_{32}}$
$1^1_\times\ 3\,1\,1, 2$	$\overline{7_4}, 5_2, 0$	$\overline{7_{42}}$
$1^1_\times\ 2\,2\,1, 2$	$\overline{7_5}, \overline{5_2}, 0$	7_{49}
$1^1_\times\ (3, 2), 2$	$7_5, 5_1, 0$	$\overline{7_{48}}$
$1^1_\times\ (3, 2), \bar{2}$	$5_1, 5_2, 0$	7_{18}
$1^1_\times\ (3, \bar{2}), 2$	$6_2, 0, 0$	6_{11}
$1^1_\times\ (3, \bar{2}), \bar{2}$	$6_3, 0, 0$	6_{15}
$1^1_\times\ (2\,1, 2), 2$	$\overline{7_6}, \overline{5_2}, 0$	7_{58}
$1^1_\times\ (2\,1, 2), \bar{2}$	$\overline{5_1}, \overline{5_2}, 0$	$\overline{7_{18}}$
$1^1_\times\ (2\,1, \bar{2}), 2$	$6_3, 0, 0$	$\overline{6_{15}}$
$1^1_\times\ (2\,1, \bar{2}), \bar{2}$	$\overline{6_2}, 0, 0$	$\overline{6_{11}}$
$1^1_\times\ 4, 3$	$\overline{7_1}, \overline{3_1}, 0$	$\overline{7_{26}}$
$1^1_\times\ 3\,1, 3$	$7_3, \overline{3_1}, 0$	7_{35}
$1^1_\times\ 2\,2, 3$	$7_2, 0, 0$	$\overline{7_{30}}$
$1^1_\times\ 4, 2\,1$	$7_2, 3_1, 0$	$\overline{7_{31}}$
$1^1_\times\ 3\,1, 2\,1$	$7_5, 3_1, 0$	$\overline{7_{45}}$
$1^1_\times\ 2\,1\,1, 2\,1$	$\overline{7_6}, 0, 0$	7_{54}
$1^1_\times\ 2\,2, 2+$	$\overline{7_6}, 4_1, 0$	7_{57}
$1^1_\times\ 2\,2, \bar{2}-$	$6_2, 4_1, 0$	6_{13}
$1^1_\times\ 2\,1\,1, 2+$	$\overline{7_7}, 4_1, 0$	$\overline{7_{65}}$
$1^1_\times\ 2\,1\,1, \bar{2}-$	$\overline{6_1}, 4_1, 0$	$\overline{6_8}$
$1^1_\times\ 3, 3+$	$\overline{7_4}, 0, 0$	$\overline{7_{39}}$
$1^1_\times\ 3, \bar{3}-$	$\overline{6_1}, 0, 0$	$\overline{6_7}$
$1^1_\times\ 2\,1, 2\,1+$	$7_7, 0, 0$	7_{62}
$1^1_\times\ 2\,1, 2\,1-$	$\overline{3_1}, 0, 0$	5_2
$1^1_\times\ 2\,1, \bar{2}\,\bar{1}+$	$6_1, 0, 0$	6_7
$1^1_\times\ 2\,1, \bar{2}\,\bar{1}-$	$\overline{6_1}, 0, 0$	$\overline{6_7}$
$1^1_\times\ 3, 2++$	$\overline{7_5}, \overline{3_1}, 0$	7_{46}
$1^1_\times\ 3, \bar{2}--$	$6_3, \overline{3_1}, 0$	6_{16}
$1^1_\times\ 2\,1, 2++$	$7_6, 3_1, 0$	$\overline{7_{56}}$
$1^1_\times\ 2\,1, \bar{2}--$	$6_2, 3_1, 0$	6_{12}

TABLE 4.1. θ-curves with up to seven crossings (continued).

notation	constituent knot	θ-curve
$3^1_*\ (3,2).1.\bar{1}$	$\overline{6_2}, 0, 0$	$\overline{6_{11}}$
$3^1_*\ (3,\bar{2}).1.\bar{1}$	$\overline{5_1}, \overline{5_2}, 0$	$\overline{7_{18}}$
$3^1_*\ (2\,1,2).1.\bar{1}$	$6_3, 0, 0$	6_{15}
$3^1_*\ (2\,1,\bar{2}).1.\bar{1}$	$5_1, 5_2, 0$	7_{18}
$3^1_*\ 4.2.\bar{1}$	$\overline{5_2}, 0, 0$	$\overline{7_{20}}$
$3^1_*\ 3\,1.2.\bar{1}$	$\overline{5_1}, 0, 0$	7_{15}
$3^1_*\ 2\,1\,1.2.\bar{1}$	$5_2, 3_1, 0$	7_{22}
$3^1_*\ 2\,1\,1\,0.\bar{2}.1$	$\overline{6_2}, \overline{3_1}, 0$	$\overline{6_{12}}$
$3^1_*\ 3.2\,1.\bar{1}$	$5_2, 4_1, 0$	$\overline{7_{24}}$
$3^1_*\ 2\,1.2\,1.\bar{1}$	$\overline{5_1}, 0, 0$	7_{17}
$3^1_*\ 3.2.2$	$\overline{3_1}, 0, 0$	6_2
$3^1_*\ 3.2.\bar{2}$	$4_1, 0, 0$	$\overline{7_{13}}$
$3^1_*\ 3.\bar{2}.2$	$4_1, 0, 0$	$\overline{7_{13}}$
$3^1_*\ 3.\bar{2}.\bar{2}$	$3_1, 0, 0$	$\overline{7_6}$
$3^1_*\ 3\,0.\bar{2}.2$	$4_1, \overline{3_1}, 0$	$\overline{6_4}$
$3^1_*\ 3\,0.\bar{2}.\bar{2}$	$3_1, \overline{3_1}, 0$	7_8
$3^1_*\ 2\,1.\bar{2}\,0.2\,0$	$\overline{3_1}, 0, 0$	6_2
$3^1_*\ 2\,1.\bar{2}\,0.\bar{2}\,0$	$\overline{3_1}, 0, 0$	7_7
$4^1_*\ 4.1.1.1$	$3_1, \overline{3_1}, 0$	$\overline{7_8}$
$4^1_*\ 4\,0.1.1.1$	$\overline{3_1}, 0, 0$	7_5
$4^1_*\ \bar{4}\,0.1.1.1$	$5_1, 0, 0$	$\overline{7_{15}}$
$4^1_*\ 3\,1.1.1.1$	$\overline{3_1}, 0, 0$	7_6
$4^1_*\ 3\,1\,0.1.1.1$	$0, 0, 0$	$\overline{7_1}$
$4^1_*\ \bar{3}\,\bar{1}\,0.1.1.1$	$5_2, 0, 0$	7_{20}
$4^1_*\ 2\,2.1.1.1$	$\overline{3_1}, 0, 0$	7_7
$4^1_*\ 2\,2\,0.1.1.1$	$3_1, \overline{3_1}, 0$	7_{10}
$4^1_*\ \bar{2}\,\bar{2}\,0.1.1.1$	$\overline{5_2}, \overline{3_1}, 0$	$\overline{7_{23}}$
$4^1_*\ 2\,1.2\,0.1.1$	$4_1, 0, 0$	$\overline{7_{13}}$
$4^1_*\ 2\,1\,0.2\,0.1.1$	$4_1, 4_1, 0$	7_{14}
$4^1_*\ \bar{2}\,\bar{1}\,0.2\,0.1.1$	$4_1, 0, 0$	7_{13}
$4^1_*\ 3.1.1.2$	$5_1, 0, 0$	7_{16}
$4^1_*\ 3\,0.1.1.2$	$5_2, 4_1, 0$	7_{23}
$4^1_*\ \bar{3}\,0.1.1.2$	$\overline{5_2}, 4_1, 0$	$\overline{7_{24}}$
$4^1_*\ 2\,1.1.1.2$	$\overline{5_2}, 0, 0$	$\overline{7_{21}}$
$4^1_*\ 3.1.1.2\,0$	$4_1, 0, 0$	7_{11}
$4^1_*\ 3.1.1.\bar{2}\,0$	$\overline{5_2}, 0, 0$	$\overline{7_{20}}$
$4^1_*\ 3\,0.1.1.2\,0$	$0, 0, 0$	$\overline{7_2}$
$4^1_*\ 3\,0.1.1.\bar{2}\,0$	$\overline{5_1}, 0, 0$	7_{15}
$4^1_*\ \bar{3}\,0.1.1.2\,0$	$5_1, 0, 0$	$\overline{7_{17}}$
$4^1_*\ 2\,1\,0.1.1.2\,0$	$4_1, 0, 0$	7_{12}
$4^1_*\ 2\,1\,0.1.1.\bar{2}\,0$	$5_2, 3_1, 0$	7_{22}
$4^1_*\ \bar{2}\,\bar{1}\,0.1.1.2\,0$	$\overline{5_2}, 4_1, 0$	$\overline{7_{24}}$
$4^1_*\ 2.3.1.1$	$\overline{5_1}, 0, 0$	7_{15}
$4^1_*\ 2.3\,0.1.1$	$\overline{5_2}, 0, 0$	$\overline{7_{20}}$
$4^1_*\ 2.2\,1.1.1$	$5_2, 3_1, 0$	7_{22}

TABLE 4.1. θ-curves with up to seven crossings (continued).

notation	constituent knot	θ-curve
$4^1_* \; 2\;0.3.1.1$	$\overline{3_1}, 0, 0$	7_5
$4^1_* \; \bar{2}\;0.3.1.1$	$3_1, \overline{3_1}, 0$	7_8
$4^1_* \; 2\;0.3\;0.1.1$	$0, 0, 0$	$\overline{7_1}$
$4^1_* \; \bar{2}\;0.3\;0.1.1$	$3_1, 0, 0$	$\overline{7_6}$
$4^1_* \; 2\;0.2\;1.1.1$	$3_1, \overline{3_1}, 0$	7_{10}
$4^1_* \; \bar{2}\;0.2\;1.1.1$	$3_1, 0, 0$	$\overline{7_7}$
$4^1_* \; 2.2.2.1$	$5_2, 4_1, 0$	7_{24}
$4^1_* \; 2.2.2\;0.1$	$\overline{5_1}, 0, 0$	7_{17}
$4^1_* \; \bar{2}.2.2\;0.1$	$\overline{5_1}, 0, 0$	7_{17}
$4^1_* \; 2.2.1.2$	$\overline{5_1}, \overline{5_2}, 0$	$\overline{7_{18}}$
$4^1_* \; 2.2.1.2\;0$	$5_2, 0, 0$	7_{19}
$4^1_* \; 2.2.1.\bar{2}\;0$	$4_1, 0, 0$	7_{13}
$4^1_* \; 2\;0.2\;0.2.1$	$4_1, 4_1, 0$	7_{14}
$4^1_* \; \bar{2}\;0.2\;0.2.1$	$\overline{3_1}, \overline{3_1}, 0$	7_9
$5^1_\times \; 2\;1.1.1.1.1$	$\overline{7_7}, 0, 0$	$\overline{7_{63}}$
$5^1_\times \; \bar{2}\;\bar{1}.1.1.1.1$	$\overline{6_1}, 4_1, 0$	$\overline{6_8}$
$5^1_\times \; 2\;1\;0.1.1.1.1$	$\overline{7_6}, 0, 0$	$\overline{7_{51}}$
$5^1_\times \; 2.1.1.2.1$	$7_7, 0, 0$	7_{60}
$5^1_\times \; 2.1.1.\bar{2}.1$	$6_2, 0, 0$	6_{11}
$5^1_\times \; \bar{2}.1.1.\bar{2}.1$	$5_1, 0, 0$	7_{16}
$5^1_\times \; 2.1.1.1.2$	$7_7, \overline{3_1}, 0$	7_{64}
$5^1_\times \; 2.1.1.1.\bar{2}$	$6_2, 4_1, 0$	6_{13}
$5^1_\times \; 2.2\;0.1.1.1$	$\overline{7_6}, 0, 0$	7_{52}
$5^1_\times \; 2.\bar{2}\;0.1.1.1$	$\overline{5_1}, 0, 0$	$\overline{7_{16}}$
$5^1_\times \; \bar{2}.2\;0.1.1.1$	$6_3, 0, 0$	$\overline{6_{15}}$
$5^1_\times \; \bar{2}.\bar{2}\;0.1.1.1$	$\overline{6_2}, 0, 0$	$\overline{6_{11}}$
$5^1_\times \; 2.1.2\;0.1.1$	$7_6, 3_1, 0$	$\overline{7_{55}}$
$5^1_\times \; 2\;0.1.2\;0.1.1$	$\overline{7_5}, \overline{3_1}, 0$	7_{47}
$5^1_\times \; 2\;0.1.1.2\;0.1$	$\overline{7_4}, 3_1, 0$	$\overline{7_{41}}$
$5^1_\times \; 2\;0.1.1.1.2\;0$	$\overline{7_4}, 3_1, 0$	$\overline{7_{40}}$
$5^1_* \; \bar{1}.\bar{3}.1.1.1$	$\overline{5_1}, \overline{5_2}, 0$	$\overline{7_{18}}$
$5^1_* \; 1.1.1.1.\bar{3}$	$5_1, 5_2, 0$	7_{18}
$5^1_* \; 3\;0.1.1.1.1$	$4_1, 0, 0$	7_{13}
$5^1_* \; \bar{3}\;0.1.1.1.1$	$4_1, 0, 0$	$\overline{7_{13}}$
$5^1_* \; \bar{1}.3\;0.1.1.1$	$6_1, 4_1, 0$	6_8
$5^1_* \; 1.1.1.1.3\;0$	$\overline{6_1}, 4_1, 0$	$\overline{6_8}$
$5^1_* \; \bar{1}.2\;1.1.1.1$	$6_3, 0, 0$	$\overline{6_{15}}$
$5^1_* \; 1.1.1.1.\bar{2}\;\bar{1}$	$\overline{5_1}, \overline{5_2}, 0$	$\overline{7_{18}}$

TABLE 4.1. θ-curves with up to seven crossings (continued).

notation	constituent knot	θ-curve
$5^1_*\ \bar{2}.\bar{2}.1.1.1$	$\overline{5_1}, 0, 0$	$\overline{7_{16}}$
$5^1_*\ \bar{2}.\bar{1}.\bar{1}.\bar{1}.2$	$\overline{5_1}, 0, 0$	$\overline{7_{16}}$
$5^1_*\ \bar{1}.\bar{2}.1.2.1$	$5_1, 0, 0$	7_{16}
$5^1_*\ \bar{1}.\bar{2}.1.1.2$	$\overline{5_1}, \overline{5_2}, 0$	$\overline{7_{18}}$
$5^1_*\ \bar{1}.\bar{2}.\bar{1}.\bar{1}.2$	$\overline{5_1}, \overline{5_2}, 0$	$\overline{7_{18}}$
$5^1_*\ \bar{1}.\bar{1}.\bar{2}.\bar{1}.2$	$5_1, 0, 0$	7_{16}
$5^1_*\ \bar{2}.2\ 0.1.1.1$	$4_1, 3_1, 0$	6_3
$5^1_*\ 2.1.1.1.2\ 0$	$4_1, \overline{3_1}, 0$	$\overline{6_3}$
$5^1_*\ \bar{1}.\bar{2}.1.2\ 0.1$	$\overline{5_2}, 0, 0$	$\overline{7_{21}}$
$5^1_*\ \bar{1}.\bar{2}.1.\bar{2}\ 0.1$	$4_1, 3_1, 0$	6_3
$5^1_*\ 1.2.1.1.2\ 0$	$6_2, 4_1, 0$	6_{13}
$5^1_*\ \bar{1}.\bar{2}.1.1.2\ 0$	$5_1, 5_2, 0$	7_{18}
$5^1_*\ 1.1.1.2.2\ 0$	$\overline{6_1}, 0, 0$	$\overline{6_7}$
$5^1_*\ 2\ 0.2.1.1.1$	$\overline{3_1}, \overline{3_1}, 0$	7_9
$5^1_*\ 2\ 0.\bar{2}.1.1.1$	$4_1, \overline{3_1}, 0$	$\overline{6_3}$
$5^1_*\ \bar{2}\ 0.2.1.1.1$	$0, 0, 0$	7_4
$5^1_*\ \bar{2}\ 0.\bar{2}.1.1.1$	$5_2, 0, 0$	7_{21}
$5^1_*\ 2\ 0.1.2.1.1$	$4_1, 0, 0$	7_{13}
$5^1_*\ \bar{2}\ 0.1.2.1.1$	$\overline{3_1}, 0, 0$	7_6
$5^1_*\ 2\ 0.1.1.2.1$	$3_1, 0, 0$	$\overline{7_6}$
$5^1_*\ \bar{2}\ 0.1.1.2.1$	$4_1, 0, 0$	$\overline{7_{13}}$
$5^1_*\ 2\ 0.1.1.1.2$	$0, 0, 0$	$\overline{7_4}$
$5^1_*\ 2\ 0.1.1.1.\bar{2}$	$\overline{5_2}, 0, 0$	$\overline{7_{21}}$
$5^1_*\ \bar{2}\ 0.1.1.1.2$	$3_1, 3_1, 0$	$\overline{7_9}$
$5^1_*\ \bar{2}\ 0.1.1.1.\bar{2}$	$4_1, 3_1, 0$	6_3
$5^1_*\ \bar{1}.2\ 0.2.1.1$	$6_1, 0, 0$	6_7
$5^1_*\ \bar{1}.2\ 0.1.1.2$	$\overline{6_2}, 4_1, 0$	$\overline{6_{13}}$
$5^1_*\ \bar{1}.\bar{2}\ 0.\bar{1}.\bar{1}.2$	$5_1, 5_2, 0$	7_{18}
$5^1_*\ \bar{1}.\bar{1}.2\ 0.\bar{1}.2$	$4_1, 3_1, 0$	6_3
$5^1_*\ \bar{1}.\bar{1}.\bar{2}\ 0.\bar{1}.2$	$\overline{5_2}, 0, 0$	$\overline{7_{21}}$
$5^1_*\ \bar{1}.2\ 0.1.2\ 0.1$	$\overline{6_2}, 0, 0$	$\overline{6_{11}}$
$5^1_*\ \bar{1}.2\ 0.1.\bar{2}\ 0.1$	$\overline{5_1}, 0, 0$	$\overline{7_{16}}$
$5^1_*\ \bar{1}.\bar{1}.2\ 0.2\ 0.1$	$0, 0, 0$	6_1
$5^1_*\ \bar{1}.\bar{1}.2\ 0.\bar{2}\ 0.1$	$3_1, 0, 0$	$\overline{6_2}$
$5^1_*\ \bar{1}.\bar{1}.\bar{2}\ 0.2\ 0.1$	$\overline{5_1}, 0, 0$	$\overline{7_{16}}$
$5^1_*\ \bar{1}.\bar{1}.\bar{2}\ 0.\bar{2}\ 0.1$	$0, 0, 0$	6_1
$5^1_*\ 1.1.2\ 0.1.2\ 0$	$6_2, 0, 0$	6_{11}
$5^1_*\ 1.1.\bar{2}\ 0.1.2\ 0$	$5_1, 0, 0$	7_{16}

TABLE 4.1. θ-curves with up to seven crossings (continued).

notation	constituent knot	θ-curve
$6^1_* \; 1.2.1.1.1.1$	$0, 0, 0$	7_2
$6^1_* \; 1.2\,0.1.1.1.1$	$4_1, 0, 0$	7_{12}
$6^1_* \; 1.\bar{2}\,0.1.1.1.1$	$4_1, 3_1, 0$	6_3
$6^2_* \; 1.1.1.2.1.1$	$3_1, 0, 0$	$\overline{7_6}$
$6^2_* \; 2\,0.1.1.1.1.1$	$0, 0, 0$	$\overline{7_3}$
$6^2_* \; \bar{2}\,0.1.1.1.1.1$	$0, 0, 0$	$\overline{7_4}$
$6^2_* \; 1.2\,0.1.1.1.1$	$4_1, 0, 0$	7_{13}
$6^2_* \; 1.1.2\,0.1.1.1$	$\overline{3_1}, \overline{3_1}, 0$	7_9
$6^2_* \; 1.1.\bar{2}\,0.1.1.1$	$4_1, \overline{3_1}, 0$	$\overline{6_3}$
$6^3_* \; 2.1.\bar{1}.1.1.1$	$3_1, 3_1, 0$	$\overline{3_1}\#_3\overline{3_1}$
$6^3_* \; 2.1.1.1.\bar{1}.\bar{1}$	$4_1, 3_1, 0$	6_4
$6^3_* \; 1.2.\bar{1}.1.1.1$	$4_1, 3_1, 0$	6_4
$6^3_* \; 1.2.1.1.\bar{1}.\bar{1}$	$3_1, 3_1, 0$	$\overline{3_1}\#_3\overline{3_1}$
$6^3_* \; 1.1.2.1.1.1$	$6_3, 0, 0$	6_{15}
$6^3_* \; 1.1.\bar{2}.1.1.1$	$5_2, 0, 0$	7_{21}
$6^3_* \; 1.1.2.1.\bar{1}.\bar{1}$	$0, 0, 0$	$\overline{6_1}$
$6^3_* \; 1.1.\bar{2}.1.\bar{1}.\bar{1}$	$5_1, 0, 0$	7_{16}
$6^3_* \; 1.1.1.1.2.\bar{1}$	$6_3, 0, 0$	6_{15}
$6^3_* \; 1.1.1.1.\bar{2}.\bar{1}$	$5_2, 0, 0$	7_{21}
$6^3_* \; 1.1.\bar{1}.1.2.1$	$0, 0, 0$	$\overline{6_1}$
$6^3_* \; 1.1.\bar{1}.1.\bar{2}.1$	$5_1, 0, 0$	7_{16}
$6^3_* \; 2\,0.1.1.1.\bar{1}.\bar{1}$	$\overline{3_1}, 0, 0$	6_2
$6^3_* \; 2\,0.1.\bar{1}.1.1.1$	$3_1, \overline{3_1}, 0$	$3_1\#_3\overline{3_1}$
$6^3_* \; 2.\bar{1}.1.\bar{1}.\bar{1}.\bar{1}$	$\overline{3_1}, 0, 0$	3_1
$6^3_* \; 2.\bar{1}.\bar{1}.\bar{1}.1.1$	$0, 0, 0$	6_1
$6^3_* \; 1.2\,0.1.1.\bar{1}.\bar{1}$	$3_1, \overline{3_1}, 0$	$3_1\#_3\overline{3_1}$
$6^3_* \; 1.2\,0.\bar{1}.1.1.1$	$\overline{3_1}, 0, 0$	6_2
$6^3_* \; 1.\bar{2}\,0.1.1.\bar{1}.\bar{1}$	$3_1, 0, 0$	$\overline{3_1}$
$6^3_* \; 1.\bar{2}\,0.\bar{1}.1.1.1$	$0, 0, 0$	$\overline{6_1}$
$6^3_* \; 1.1.2\,0.1.\bar{1}.\bar{1}$	$4_1, \overline{3_1}, 0$	$\overline{6_3}$
$6^3_* \; 1.1.\bar{2}\,0.1.1.1$	$0, 0, 0$	6_1
$6^3_* \; 1.1.1.1.\bar{2}\,0.\bar{1}$	$0, 0, 0$	6_1
$6^3_* \; 1.1.\bar{1}.1.\bar{2}\,0.1$	$4_1, \overline{3_1}, 0$	$\overline{6_3}$
$6^4_* \; 2\,0.1.1.1.1.1$	$3_1, 0, 0$	$\overline{6_2}$
$6^4_* \; 1.2\,0.1.1.1.1$	$4_1, \overline{3_1}, 0$	$\overline{6_3}$
$6^4_* \; 1.1.2\,0.1.1.1$	$0, 0, 0$	$\overline{6_1}$
$7^4_\times \; 1.1.1.1.1.1.1$	$7_7, 0, 0$	7_{61}
$7^3_* \; 1.1.1.1.1.1.1$	$6_3, 0, 0$	$\overline{6_{15}}$
$7^4_* \; 1.1.1.1.1.1.1$	$0, 0, 0$	7_4
$7^4_* \; 1.\bar{1}.1.\bar{1}.\bar{1}.1.\bar{1}$	$4_1, \overline{3_1}, 0$	$\overline{6_3}$
$7^9_* \; 1.1.1.1.1.1.1$	$\overline{3_1}, 0, 0$	6_2

Furthermore, we show the primeness of these θ-curves by using the following proposition ([**20**]) originally due to Thurston. We can apply this since each θ-curve in the table has a trivial constituent knot.

PROPOSITION 4.2. *Suppose a θ-curve Θ contains a trivial constituent knot. Take the 2-f branched cover over this trivial knot; the two lifts of the remaining edge give a knot $\tilde{K}$ in S^3. Θ is prime if and only if $\tilde{K}$ is prime.*

EXAMPLE 4.3. For the θ-curve 7_{41}, we obtain the knot 8_{18} (see Fig. 4.2).

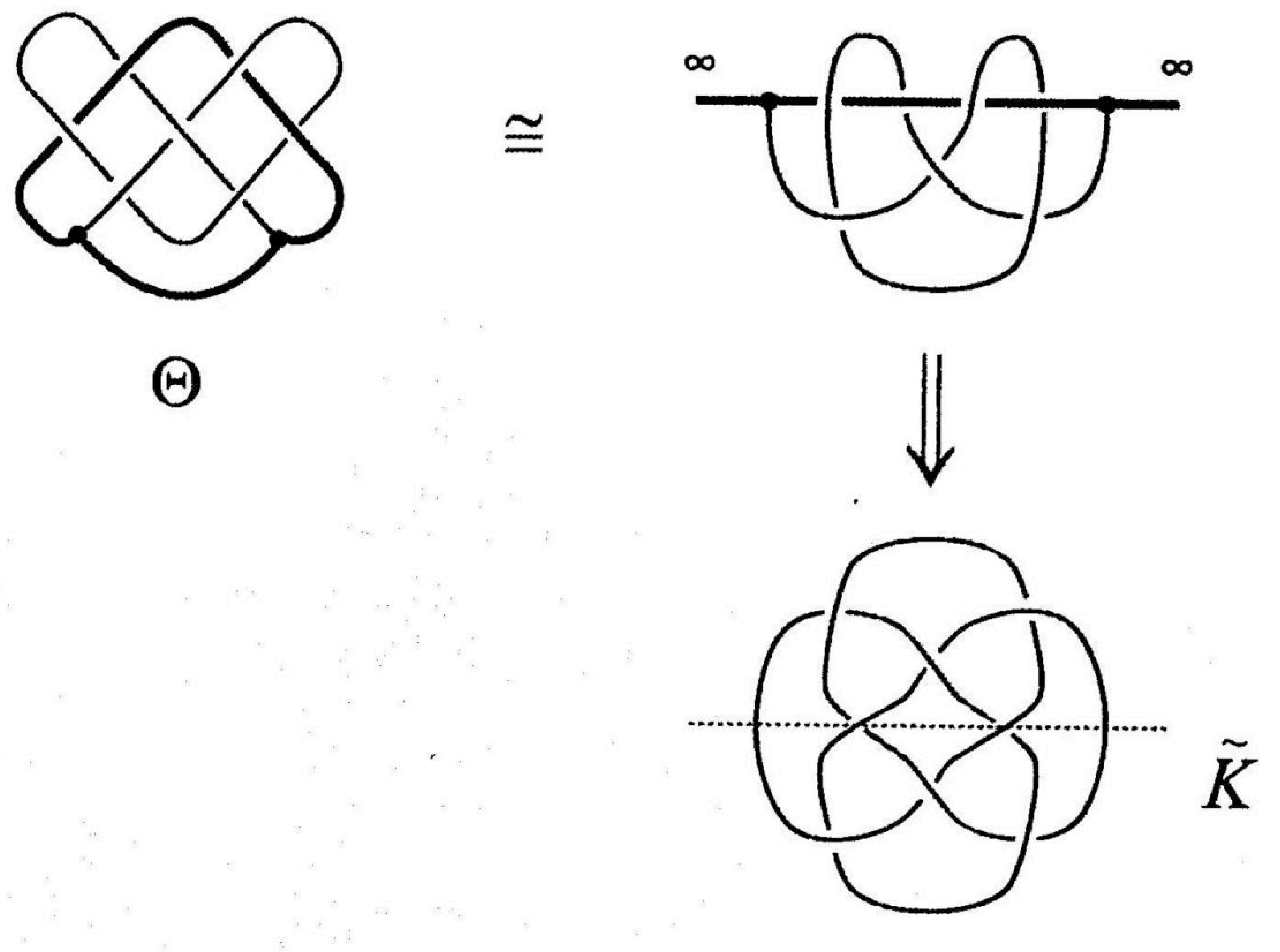


FIG. 4.2. The θ-curve 7_{41} is prime.

2. Handcuff graph

In the same way, we also obtain all prime handcuff graph diagrams. We give an enumeration of handcuff graphs with up to seven crossings by using our notation. Links in the second column correspond to Rolfsen's knot table ([31]), and handcuff graphs in the last column correspond to Table 1.2. A link $\overline{L}$ and a handcuff graph $\overline{\Phi}$ denote mirror images of L and Φ, respectively. Moreover, $\#_n$ $(n = 2, 3)$ denotes an order n vertex connected sum ([37]), and 3_1^θ (resp. 4_1^θ) denotes the θ-curve 3_1 (resp. 4_1) (Table 1.1).

TABLE 4.2. Handcuff graphs with up to seven crossings.

notation	constituent link	handcuff graph
$1^1_\times\ 2$	2^2_1	2_1
$1^1_\times\ 4$	4^2_1	4_1
$1^1_\times\ 2\,1\,2$	5^2_1	5_1
$1^1_\times\ 6$	6^2_1	6_5
$1^1_\times\ 3\,3$	6^2_2	6_7
$1^1_\times\ 2\,2\,2$	6^2_3	6_8
$1^1_\times\ 3,3$	$\overline{6^2_1}$	6_6
$1^1_\times\ 2\,1,2\,1$	6^2_3	6_9
$3^1_*\ 2\,0.\bar{2}.1$	$\overline{4^2_1}$	4_1
$3^1_*\ 2\,0.2.\bar{1}$	0^2_1	$2_1\#_3 2_1$
$3^1_*\ \bar{2}\,0.\bar{2}\,0.2$	2^2_1	6_2
$3^1_*\ 2\,1\,0.2\,0.\bar{1}$	4^2_1	$\overline{6_4}$
$4^1_*\ 3.1.1.1$	2^2_1	6_2
$4^1_*\ 3\,0.1.1.1$	$\overline{2^2_1}$	6_3
$4^1_*\ \bar{3}\,0.1.1.1$	4^2_1	$\overline{6_4}$
$4^1_*\ 2.2.1.1$	$\overline{4^2_1}$	6_4
$4^1_*\ 2\,0.2.1.1$	$\overline{2^2_1}$	6_3
$4^1_*\ \bar{2}\,0.2.1.1$	$\overline{2^2_1}$	$\overline{6_2}$
$4^1_*\ 2\,0.1.1.2\,0$	0^2_1	6_1
$4^1_*\ 2\,0.1.1.\bar{2}\,0$	$\overline{4^2_1}$	6_4
$6^1_*\ 1.1.1.1.1.1$	0^2_1	6_1
$6^3_*\ 1.1.1.1.1.1$	5^2_1	5_1
$6^3_*\ 1.1.1.1.\bar{1}.\bar{1}$	2^2_1	$2_1\#_3 3^\theta_1$
$6^3_*\ 1.1.\bar{1}.1.1.1$	2^2_1	$2_1\#_3 3^\theta_1$
$1^1_\times\ 4\,1\,2$	7^2_1	7_{18}
$1^1_\times\ 2\,1\,4$	7^2_1	7_{19}
$1^1_\times\ 2\,3\,2$	7^2_3	7_{26}
$1^1_\times\ 3\,1\,1\,2$	7^2_2	7_{22}
$1^1_\times\ 2\,1\,1\,3$	7^2_2	7_{23}
$1^1_\times\ 2\,1\,1,3$	$\overline{7^2_1}$	7_{20}
$1^1_\times\ 2\,2,2\,1$	7^2_3	7_{21}
$1^1_\times\ 3,2\,1+$	7^2_2	7_{24}
$1^1_\times\ (3,2)\ 2$	$\overline{7^2_4}$	7_{28}
$1^1_\times\ (3,2)\ \bar{2}$	$\overline{7^2_7}$	7_{35}
$1^1_\times\ (3,\bar{2})\ \bar{2}$	7^2_8	$\overline{7_{36}}$
$1^1_\times\ (2\,1,2)\ 2$	$\overline{7^2_5}$	7_{30}
$1^1_\times\ (2\,1,2)\ \bar{2}$	$\overline{7^2_8}$	7_{36}

TABLE 4.2. Handcuff graphs with up to seven crossings (continued).

notation	constituent link	handcuff graph
$3^1_*\ 2\ 2.2.\bar{1}$	$\overline{5^2_1}$	7_{14}
$3^1_*\ 2\ 1.2.2$	2^2_1	6_3
$3^1_*\ 2\ 1.2.\bar{2}$	2^2_1	7_7
$3^1_*\ 2\ 1.\bar{2}.2$	2^2_1	7_7
$3^1_*\ 2\ 1.\bar{2}.\bar{2}$	2^2_1	$\overline{7_3}$
$3^1_*\ 3.\bar{2}\ 0.2$	4^2_1	$\overline{6_4}$
$3^1_*\ 3\ 0.2\ 0.\bar{2}$	$\overline{4^2_1}$	$\overline{7_{10}}$
$3^1_*\ 3\ 0.\bar{2}\ 0.2$	4^2_1	$\overline{6_4}$
$3^1_*\ 2\ 1\ 0.2\ 0.\bar{2}$	2^2_1	$\overline{7_5}$
$4^1_*\ 2\ 1\ 1.1.1.1$	2^2_1	7_3
$4^1_*\ 2\ 1\ 1\ 0.1.1.1$	2^2_1	7_4
$4^1_*\ \bar{2}\ \bar{1}\ \bar{1}\ 0.1.1.1$	5^2_1	$\overline{7_{14}}$
$4^1_*\ 3.2\ 0.1.1$	4^2_1	7_{10}
$4^1_*\ 3\ 0.2\ 0.1.1$	$\overline{4^2_1}$	7_{11}
$4^1_*\ \bar{3}\ 0.2\ 0.1.1$	2^2_1	$\overline{7_7}$
$4^1_*\ 2\ 1.2.1.1$	2^2_1	7_7
$4^1_*\ 2\ 1\ 0.2.1.1$	$\overline{4^2_1}$	7_{11}
$4^1_*\ \bar{2}\ \bar{1}\ 0.2.1.1$	$\overline{4^2_1}$	$\overline{7_{10}}$
$4^1_*\ 3.2.1.1$	2^2_1	7_5
$4^1_*\ 3\ 0.2.1.1$	2^2_1	7_6
$4^1_*\ \bar{3}\ 0.2.1.1$	2^2_1	$\overline{7_5}$
$4^1_*\ 2\ 1.1.1.2\ 0$	$\overline{4^2_1}$	7_{12}
$4^1_*\ 2\ 1.1.1.\bar{2}\ 0$	$\overline{5^2_1}$	7_{14}
$4^1_*\ 2.2\ 1\ 0.1.1$	$\overline{5^2_1}$	7_{14}
$4^1_*\ 2\ 0.2\ 1\ 0.1.1$	2^2_1	7_4
$4^1_*\ \bar{2}\ 0.2\ 1\ 0.1.1$	2^2_1	$\overline{7_3}$
$4^1_*\ 2\ 0.2.2.1$	$\overline{4^2_1}$	7_{11}
$4^1_*\ \bar{2}\ 0.2.2.1$	0^2_1	7_2
$4^1_*\ 2\ 0.2.2\ 0.1$	2^2_1	7_6
$4^1_*\ \bar{2}\ 0.2.2\ 0.1$	2^2_1	7_9
$4^1_*\ 2.2\ 0.1.2\ 0$	5^2_1	7_{15}
$4^1_*\ 2.2\ 0.1.\bar{2}\ 0$	4^2_1	7_{10}
$4^1_*\ 2\ 0.2.1.2\ 0$	2^2_1	7_8
$4^1_*\ 2\ 0.2.1.\bar{2}\ 0$	2^2_1	7_7
$4^1_*\ \bar{2}\ 0.2.1.2\ 0$	2^2_1	$\overline{7_5}$
$4^1_*\ \bar{2}\ 0.2.1.\bar{2}\ 0$	2^2_1	$2_1 \#_3 \overline{3^\theta_1}$

TABLE 4.2. Handcuff graphs with up to seven crossings (continued).

notation	constituent link	handcuff graph
$5^1_\times\ 2.2.1.1.1$	$\overline{7^2_5}$	7_{31}
$5^1_\times\ 2.\bar{2}.1.1.1$	$\overline{7^2_8}$	7_{36}
$5^1_\times\ \bar{2}.\bar{2}.1.1.1$	7^2_7	$\overline{7_{35}}$
$5^1_\times\ 2.1.2.1.1$	7^2_5	7_{32}
$5^1_\times\ 2.1.\bar{2}.1.1$	7^2_8	$\overline{7_{36}}$
$5^1_\times\ \bar{2}.1.2.1.1$	$\overline{6^2_3}$	$\overline{6_9}$
$5^1_\times\ \bar{2}.1.\bar{2}.1.1$	$\overline{5^2_1}$	$\overline{5_1}$
$5^1_\times\ 2.1.1.2\ 0.1$	7^2_2	7_{25}
$5^1_\times\ \bar{2}.1.1.2\ 0.1$	6^2_2	7_{17}
$5^1_\times\ 2\ 0.1.2.1.1$	7^2_4	7_{29}
$5^1_\times\ 2\ 0.1.\bar{2}.1.1$	7^2_7	$\overline{7_{35}}$
$5^1_\times\ 2\ 0.2\ 0.1.1.\bar{1}$	$\overline{7^2_1}$	7_{21}
$5^1_\times\ \bar{2}\ 0.\bar{2}\ 0.1.1.1$	6^2_2	7_{17}
$5^1_*\ 2\ 1\ 0.1.1.1.1$	2^2_1	$\overline{7_7}$
$5^1_*\ \bar{2}\ \bar{1}\ 0.1.1.1.1$	2^2_1	7_7
$5^1_*\ 2\ 0.2\ 0.1.1.1$	0^2_1	7_2
$5^1_*\ \bar{2}\ 0.2\ 0.1.1.1$	4^2_1	$\overline{7_{13}}$
$5^1_*\ 2\ 0.1.2\ 0.1.1$	2^2_1	$\overline{7_7}$
$5^1_*\ \bar{2}\ 0.1.2\ 0.1.1$	2^2_1	7_3
$5^1_*\ 2\ 0.1.\bar{2}\ 0.1.1$	2^2_1	$2_1\#_3\overline{3^\theta_1}$
$5^1_*\ \bar{2}\ 0.1.\bar{2}\ 0.1.1$	2^2_1	6_2
$5^1_*\ 2\ 0.1.1.2\ 0.1$	2^2_1	$\overline{7_3}$
$5^1_*\ \bar{2}\ 0.1.1.2\ 0.1$	2^2_1	7_7
$5^1_*\ 2\ 0.1.1.\bar{2}\ 0.1$	2^2_1	$\overline{6_2}$
$5^1_*\ \bar{2}\ 0.1.1.\bar{2}\ 0.1$	2^2_1	$2_1\#_3\overline{3^\theta_1}$
$5^1_*\ 2\ 0.1.1.1.2\ 0$	$\overline{4^2_1}$	7_{13}
$5^1_*\ \bar{2}\ 0.1.1.1.2\ 0$	0^2_1	$\overline{7_2}$
$5^1_*\ \bar{1}.2\ 0.2\ 0.1.1$	6^2_3	6_9
$5^1_*\ \bar{1}.\bar{2}\ 0.2\ 0.1.1$	$\overline{4^2_1}$	6_4
$5^1_*\ \bar{1}.2\ 0.\bar{2}\ 0.1.1$	5^2_1	5_1
$5^1_*\ \bar{1}.\bar{2}\ 0.\bar{2}\ 0.1.1$	$\overline{5^2_1}$	$\overline{5_1}$
$5^1_*\ 1.1.1.2\ 0.2\ 0$	$\overline{6^2_3}$	$\overline{6_9}$
$5^1_*\ 1.1.1.\bar{2}\ 0.2\ 0$	$\overline{5^2_1}$	$\overline{5_1}$
$5^1_*\ 1.1.1.2\ 0.\bar{2}\ 0$	4^2_1	$\overline{6_4}$
$5^1_*\ 1.1.1.\bar{2}\ 0.\bar{2}\ 0$	5^2_1	5_1
$5^1_*\ 1.1.2.1.2\ 0$	6^2_2	7_{17}
$5^1_*\ \bar{1}.2\ 0.1.2.1$	6^2_2	7_{17}
$6^1_\times\ 2.1.1.1.1.1$	7^2_6	7_{33}
$6^1_\times\ 2.1.1.1.1.\bar{1}$	2^2_1	$2_1\#_3\overline{3^\theta_1}$
$6^1_\times\ \bar{2}.1.1.1.1.1$	$\overline{7^2_8}$	7_{36}
$6^1_\times\ \bar{2}.1.1.1.1.\bar{1}$	$2^2_1\#_2 3_1$	$2_1\#_3\overline{3^\theta_1}$

TABLE 4.2. Handcuff graphs with up to seven crossings (continued).

notation	constituent link	handcuff graph
$6^1_*\ 2.1.1.1.1.1$	0^2_1	7_1
$6^1_*\ \bar{2}.1.1.1.1.1$	0^2_1	7_2
$6^1_*\ 1.1.2.1.1.1$	2^2_1	7_8
$6^1_*\ 2\,0.1.1.1.1.1$	$\overline{5^2_1}$	7_{16}
$6^1_*\ 1.1.2\,0.1.1.1$	2^2_1	$\overline{7_8}$
$6^1_*\ 1.1.\bar{2}\,0.1.1.1$	2^2_1	6_2
$6^2_*\ 1.2.1.1.1.1$	2^2_1	$\overline{7_7}$
$6^2_*\ 1.1.2.1.1.1$	0^2_1	7_2
$6^2_*\ 1.1.1.2\,0.1.1$	2^2_1	$\overline{7_3}$
$6^2_*\ 1.1.1.\bar{2}\,0.1.1$	2^2_1	$\overline{6_2}$
$6^3_*\ 1.1.1.1.1.2$	0^2_1	$\overline{7_2}$
$6^3_*\ 1.1.1.1.\bar{1}.\bar{2}$	2^2_1	$2_1\#_3 4^\theta_1$
$6^3_*\ 1.1.\bar{1}.1.1.2$	0^2_1	$2_1\#_3 4_1$
$6^3_*\ 1.1.\bar{1}.1.\bar{1}.\bar{2}$	2^2_1	$2_1\#_3 \overline{3^\theta_1}$
$6^3_*\ 1.1.\bar{1}.2\,0.1.1$	2^2_1	$2_1\#_3 \overline{4^\theta_1}$
$6^3_*\ 1.1.1.2\,0.\bar{1}.\bar{1}$	2^2_1	$2_1\#_3 \overline{4^\theta_1}$
$6^3_*\ 1.1.1.1.1.\bar{2}\,0$	0^2_1	$\overline{7_2}$
$6^3_*\ 1.1.1.1.\bar{1}.\bar{2}\,0$	0^2_1	$2_1\#_3 4_1$
$6^3_*\ 1.1.\bar{1}.1.1.2\,0$	2^2_1	$2_1\#_3 4^\theta_1$
$6^3_*\ 1.1.\bar{1}.1.\bar{1}.2\,0$	2^2_1	$2_1\#_3 \overline{3^\theta_1}$
$6^4_*\ 2.1.1.1.1.1$	2^2_1	$\overline{6_3}$
$6^4_*\ \bar{2}.1.1.1.1.1$	2^2_1	2_1
$6^4_*\ 2.1.\bar{1}.1.1.1$	2^2_1	$2_1\#_3 \overline{3^\theta_1}$
$6^4_*\ 2.\bar{1}.1.1.1.1$	4^2_1	4_1
$6^4_*\ \bar{2}.1.\bar{1}.1.1.1$	2^2_1	$2_1\#_3 3^\theta_1$
$6^4_*\ \bar{2}.\bar{1}.1.1.1.1$	0^2_1	trivial
$6^4_*\ 2.\bar{1}.\bar{1}.1.1.1$	$\overline{4^2_1}$	6_4
$6^4_*\ \bar{2}.\bar{1}.\bar{1}.1.1.1$	$\overline{5^2_1}$	$\overline{5_1}$
$6^4_*\ 1.2.1.1.1.1$	0^2_1	6_1
$6^4_*\ \bar{1}.2.\bar{1}.1.1.1$	4^2_1	$\overline{6_4}$
$6^4_*\ 1.\bar{2}.1.1.1.1$	4^2_1	$\overline{6_2}$
$6^4_*\ \bar{1}.\bar{2}.\bar{1}.1.1.1$	5^2_1	5_1
$7^3_\times\ 1.1.1.1.1.1.1$	$\overline{7^2_6}$	7_{34}
$7^3_\times\ 1.1.1.1.\bar{1}.\bar{1}.\bar{1}$	7^2_7	$\overline{7_{35}}$
$7^3_\times\ 1.1.\bar{1}.\bar{1}.1.\bar{1}.1$	7^2_7	$\overline{7_{35}}$
$7^3_\times\ 1.1.\bar{1}.\bar{1}.\bar{1}.\bar{1}.\bar{1}$	$\overline{7^2_8}$	7_{36}
$7^2_*\ 1.1.1.1.1.1.1$	0^2_1	$\overline{7_1}$
$7^2_*\ \bar{1}.\bar{1}.1.1.1.1.1$	2^2_1	$2_1\#_3 \overline{3^\theta_1}$
$7^2_*\ \bar{1}.\bar{1}.\bar{1}.1.1.1.1$	2^2_1	$\overline{7_9}$
$7^2_*\ \bar{1}.\bar{1}.\bar{1}.1.1.\bar{1}.1$	0^2_1	$2_1\#_3 2_1\#_3 2_1$
$7^7_*\ 1.1.1.1.1.\bar{1}.\bar{1}$	$\overline{5^2_1}$	$\overline{5_1}$
$7^7_*\ 1.1.1.1.\bar{1}.1.1$	$\overline{4^2_1}$	6_4

CHAPTER 5

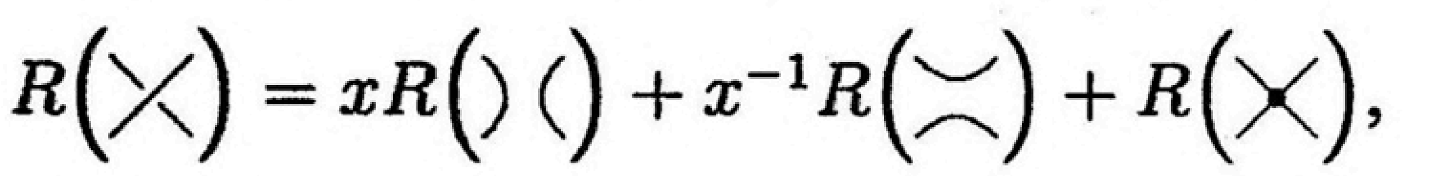
Yamada Polynomial

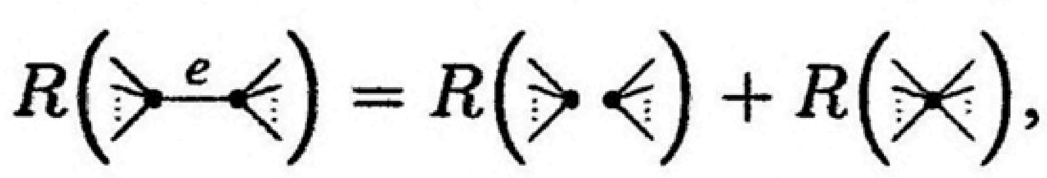
In this chapter, we introduce the polynomial invariants due to Yamada ([38]) in order to classify θ-curves in Table 1.1 and handcuff graphs in Table 1.2 up to ambient isotopy. This chapter is organized as follows: In Section 1, we review the Yamada polynomial. In Section 2, we calculate the Yamada polynomials of θ-curves in Table 1.1. In Section 3, we calculate the Yamada polynomials of handcuff graphs in Table 1.2. In Section 4, we mention the chirality of these spatial graphs.

1. Definition

DEFINITION 5.1. Let γ be a spatial graph diagram. Then $R(\gamma) \in \mathbf{Z}[x^{\pm 1}]$ is defined by the following recursive formulas:

(Y1) $R(\emptyset) = 1$, where $\emptyset$ denotes the empty graph,

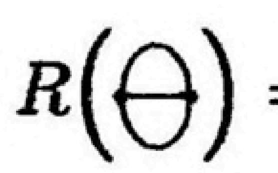
(Y2) 

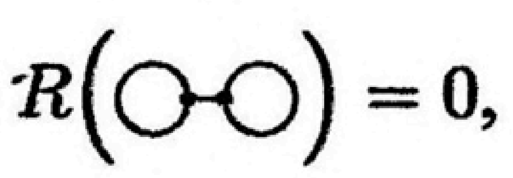
(Y3) , where e is a nonloop edge,

(Y4) $R(\gamma_1 \sqcup \gamma_2) = R(\gamma_1)R(\gamma_2)$, where $\gamma_1 \sqcup \gamma_2$ denotes the disjoint union of spatial graph diagrams γ_1 and γ_2,

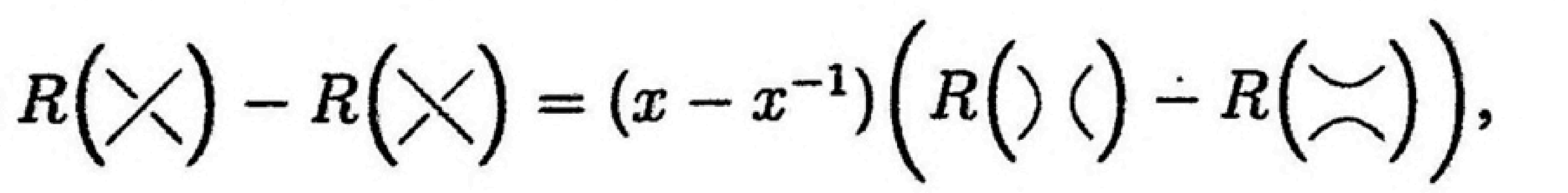
(Y5) $R(B_n) = -(-x - 1 - x^{-1})^n$, where B_n is the n-leafed bouquet.
In particular, $R(\bullet) = R(B_0) = -1$, $= R(B_1) = x + 1 + x^{-1}$.

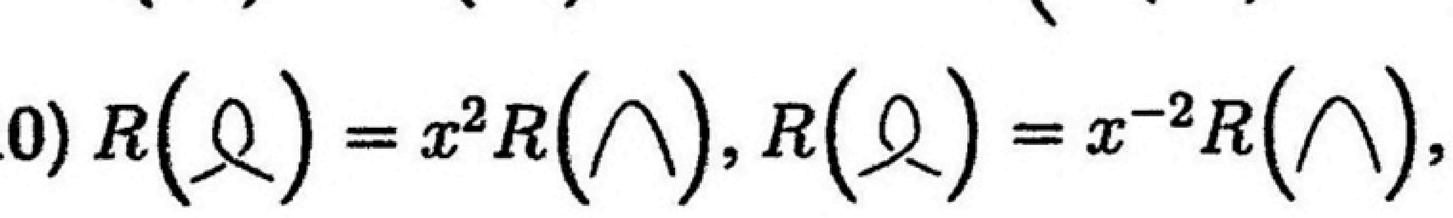
For a spatial 3-valent graph diagram γ, we easily calculate $R(\gamma)$ by using the following properties:

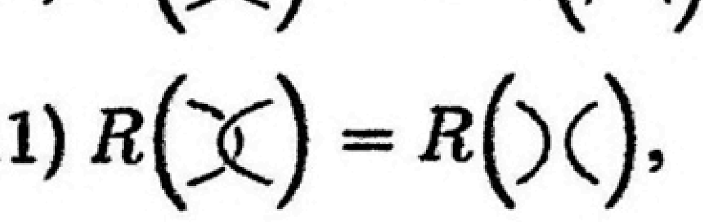
(Y6)  $= -(x + 1 + x^{-1})(x + x^{-1}) = -x^2 - x - 2 - x^{-1} - x^{-2}$,

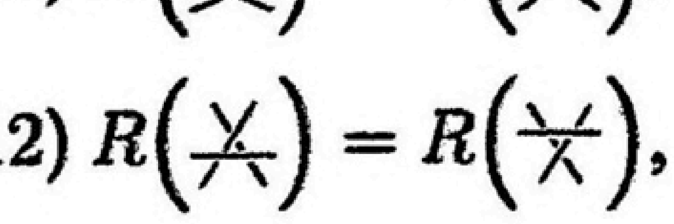
(Y7) 

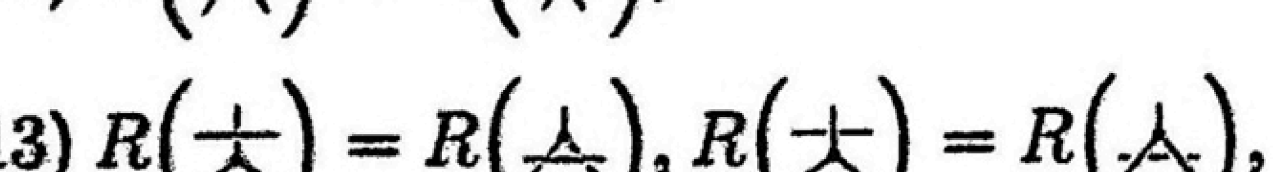
(Y8) $R(\gamma' \sqcup \bigcirc) = (x + 1 + x^{-1})R(\gamma')$ for an arbitrary spatial graph diagram γ',

(Y9) 

(Y10) 

(Y11) 

(Y12) 

(Y13) 

(Y14) $R(\text{diagram}) = -xR(-\!\!\prec)$, $R(\text{diagram}) = -x^{-1}R(-\!\!\prec)$.

We call $R(g)$ the *Yamada polynomial*. From Lemma 1.3, Yamada gave the following theorem ([**38**]).

THEOREM 5.2. *Let Γ be a spatial graph whose maximum degree is less than 4, and γ be a diagram of Γ. Then $R(\gamma)$ is an ambient isotopy invariant of γ up to multiplying $(-x)^n$ for some integer n.*

Here, we mention the influence of the order-2 and order-3 vertex connected sum.

PROPOSITION 5.3. *Let γ_1 and γ_2 be spatial graph diagrams. Then*
(1) $R(\gamma_1 \#_2 \gamma_2) = R(\gamma_1)R(\gamma_2)/(x + 1 + x^{-1})$,
(2) $R(\gamma_1 \#_3 \gamma_2) = R(\gamma_1)R(\gamma_2)/(-x^2 - x - 2 - x^{-1} - x^{-2})$.

Moreover, Yamada defined another polynomial invariant of a θ-curve in [**38**]. First, we review the *twisting number* of a knot diagram. For a knot diagram k, we fix an orientation. Put $t(k) = \sum_c \operatorname{sign}(c)$, where c ranges over all crossings of k and $\operatorname{sign}(\text{positive crossing}) = +1$, $\operatorname{sign}(\text{negative crossing}) = -1$. Here, we note that $t(k)$ does not depend on the choice of the orientation of k. Second, let θ be a θ-curve diagram, and θ_{ij} the constituent knot diagram of θ. Then the *twisting number* of θ is defined by $t(\theta) = \sum_{i<j} t(\theta_{ij})/2$. Finally, we put $S(\Theta) = (-x)^{-2t(\theta)}R(\theta)$.

PROPOSITION 5.4. *Let Θ be a θ-curve. Then $S(\Theta)$ is an ambient isotopy invariant of Θ.*

We can also compute the Yamada polynomial of a handcuff graph by the following way. In [**39**], Yamada introduced a polynomial invariant for a 3-regular spatial graph Γ with some good weight ω, which is the linear combination of the bracket polynomial. First, we mention the *Temperley-Lieb algebra*. The mth Temperley-Lieb algebra V_m is an algebra over the field of complex numbers $\mathbf{C}$ generated by the elements $1_m, \varepsilon_1, \ldots, \varepsilon_{m-1}$ and the following relations:

(TL1) $\varepsilon_i\varepsilon_j = \varepsilon_j\varepsilon_i$ if $|i-j| \geq 2$,

(TL2) $\varepsilon_i^2 = \delta\varepsilon_i$ where $\delta = -q - q^{-1}$ and $q \in \mathbf{C}$,

(TL3) $\varepsilon_i\varepsilon_{i\pm1}\varepsilon_i = \varepsilon_i$.

The elements 1_m and ε_i diagrammatically appear as in Fig. 5.1.

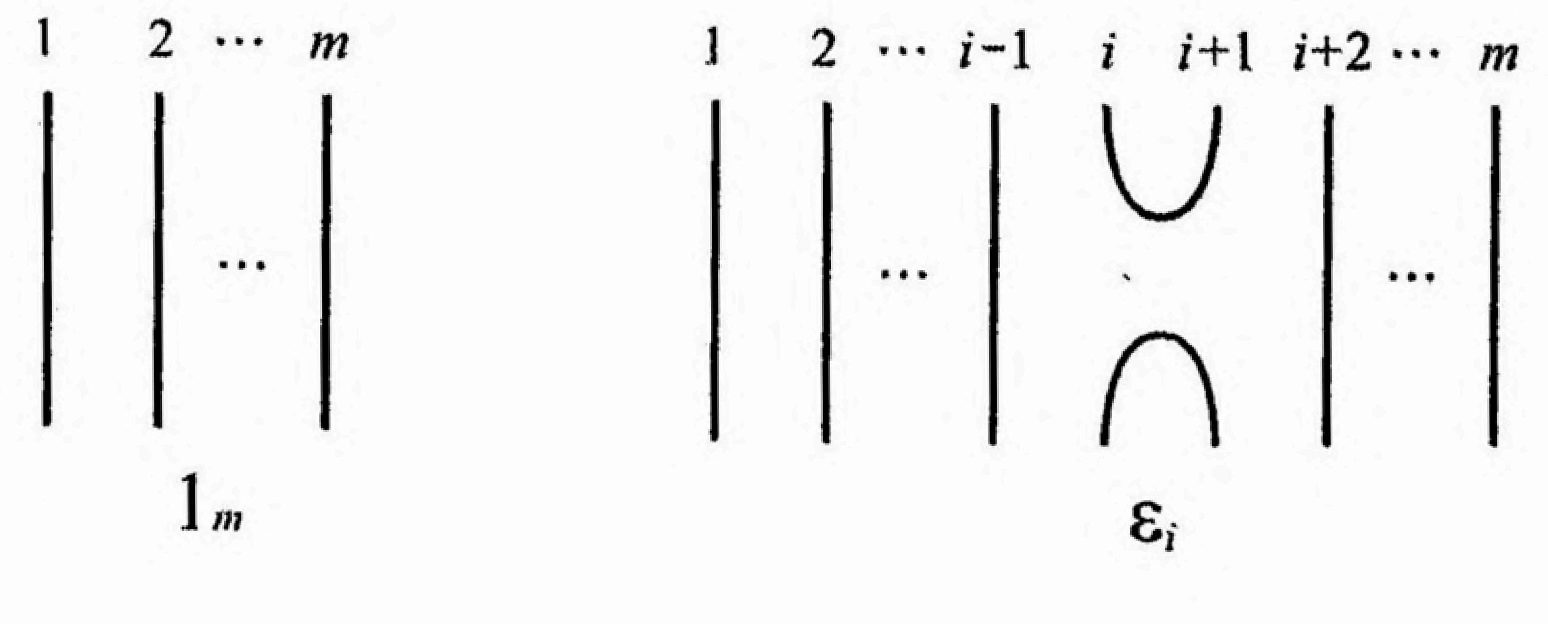


FIG. 5.1. The elements 1_m and ε_i.

For any non-negative integer m, we set

$$[m] = \frac{q^m - q^{-m}}{q - q^{-1}}.$$

Then we define elements g_n and f_n of V_m by

$$g_n = \frac{1}{[n]} \sum_{j=1}^{n} [j] \varepsilon_j \dots \varepsilon_{n-1},$$
$$f_n = g_n g_{n-1} \dots g_1.$$

The elements g_n and f_n hold the following recursive formulas,

$$g_{n+1} = \frac{[n]}{[n+1]} g_n \varepsilon_n + 1,$$
$$f_{n+1} = g_{n+1} f_n.$$

By induction, it follows easily that $g_n - 1$ and $f_n - 1$ are elements in the proper subalgebra $\mathfrak{A}(\varepsilon_1, \dots, \varepsilon_{n-1})$ generated by $\varepsilon_1, \dots, \varepsilon_{n-1}$. So that, g_n and f_n are commute with $\varepsilon_{n+1}, \dots, \varepsilon_{m-1}$.

The following lemma is given by Wenzl ([34]), Lickorish ([18]), and Yamada ([39]).

LEMMA 5.5. *f_n is the unique element in V_m which is generated by $1, \varepsilon_1, \varepsilon_2, \dots, \varepsilon_{n-1}$ and holds the following:*

(E1) $\varepsilon_i f_n = f_n \varepsilon_i = 0$, *for all* $i \leq n - 1$,

(E2) $f_n^2 = f_n$.

We call f_n the *magic knitting of degree n*.

Let Γ be a 3-regular graph and $\omega : E(\Gamma) \to \mathbf{Z}_+$ be a positive integer valued map. We say ω is a *good weight* on Γ if the following conditions are satisfied. For each vertex of Γ, if e_1, e_2, and e_3 are the edges incident with the vertex, then

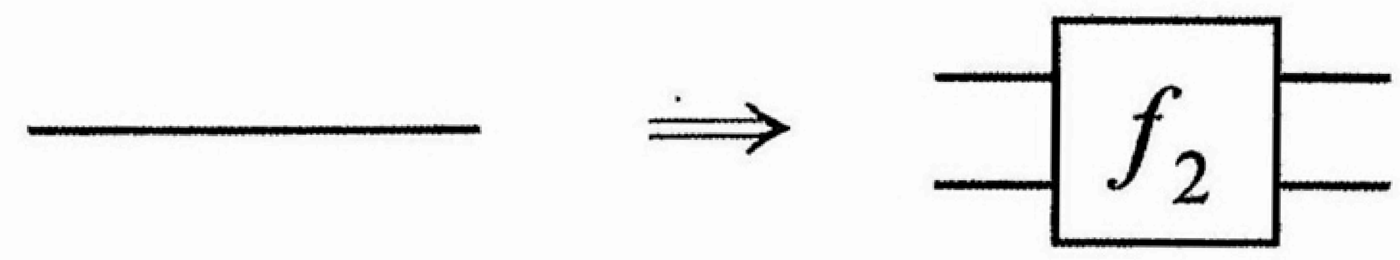
(V1) $\omega(e_1) + \omega(e_2) + \omega(e_3) \in \mathbf{Z}_2$,

(V2) $|\omega(e_1) - \omega(e_2)| \leq \omega(e_3) \leq \omega(e_1) + \omega(e_2)$.

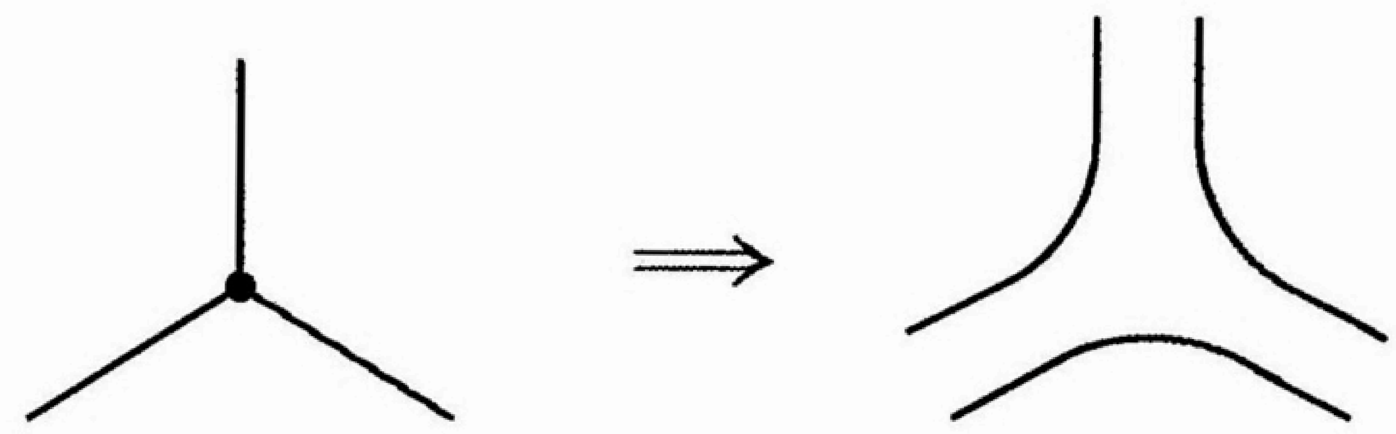
From now on, we consider the case $\omega(e) = 2$ for each edge e. Let ω be a good weight on a spatial 3-regular graph Γ and γ be a diagram of Γ. Then γ^ω is defined as the linear combination of link diagrams derived from γ:

(W1) Parallelize each edge e by the weight $\omega(e)$ and immerse the magic knitting of degree $\omega(e)$ as in Fig. 5.2(a).

(W2) Connect those parallelized edges at each vertex as in Fig. 5.2(b).



(a)



(b)

FIG. 5.2. How to make γ^ω.

The bracket polynomial $\langle l\rangle \in \mathbf{Z}[A^{\pm 1}]$ of a non-oriented link diagram l is defined by the following formulas (originally introduced in [13]):

(B1) $\langle \bigcirc \rangle = -A^2 - A^{-2}$,

(B2) $\langle l \sqcup \bigcirc \rangle = (-A^2 - A^{-2})\langle l \rangle$,

(B3) 

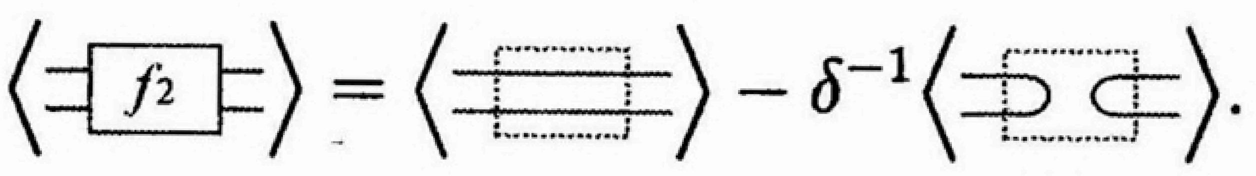
Yamada [39] defined $\langle \gamma^\omega \rangle$ by the linear extension of the bracket polynomial, and gave a relation between $\langle \gamma^\omega \rangle$ and $R(\gamma)$.

LEMMA 5.6. *Let γ be a diagram of a spatial 3-regular graph Γ with the good weight defined $\omega(e) = 2$ for each edge e. Then*

$$\langle \gamma^\omega \rangle(A) = (-\delta)^{\chi(\Gamma)} R(\gamma)(A^4),$$

where $\delta = -A^2 - A^{-2}$ and $\chi(\Gamma)$ the Euler characteristic of Γ.

REMARK 5.7. The magic knitting of degree 2 is decomposed as follows:

(1) 

2. θ-curve

We assume Γ is a θ-curve Θ. Then we can describe $\langle \gamma^\omega \rangle$ concretely. Since $\chi(\Theta) = -1$, we obtain the following proposition.

PROPOSITION 5.8. *For a θ-curve diagram θ and its constituent knot diagram θ_{ij}, let $\theta^{(2)}$ be the link diagram obtained from θ by parallelizing each edge, and $\theta^{(2)}_{ij}$ be the link diagram obtained from θ_{ij} by $(2,0)$-cabling. Then*

$$R(\theta)(x) = -\delta\, \langle \theta^{(2)} \rangle (x^{1/4}) + \sum_{i<j} \langle \theta^{(2)}_{ij} \rangle (x^{1/4}) - 2,$$

where $\delta = -x^{1/2} - x^{-1/2}$.

PROOF. From Lemma 5.6, we obtain the following equation:

$$R(\theta)(x) = (-\delta)\langle\theta^\omega\rangle(x^{1/4}), \tag{2}$$

where $\delta = -x^{1/2} - x^{-1/2}$. We can describe $\langle\theta^\omega\rangle$ concretely by using the equation (1):

$$\begin{aligned} \langle\theta^\omega\rangle &= \langle\theta^{(2)}\rangle - \delta^{-1}\sum_{i<j}\langle\theta^{(2)}_{ij}\rangle + 3\delta^{-2}\langle\bigcirc\rangle - \delta^{-3}\langle\bigcirc\sqcup\bigcirc\rangle \\ &= \langle\theta^{(2)}\rangle - \delta^{-1}\sum_{i<j}\langle\theta^{(2)}_{ij}\rangle + 3\delta^{-1} - \delta^{-1} \\ &= \langle\theta^{(2)}\rangle - \delta^{-1}\sum_{i<j}\langle\theta^{(2)}_{ij}\rangle + 2\delta^{-1}. \end{aligned}$$

Then we replace $\langle\theta^\omega\rangle$ in the equation (2) with the above. □

By using Proposition 5.8 and KNOT program [**16**], we compute the Yamada polynomial of all the prime θ-curves with up to seven crossings immediately; Table 1.1. Moreover, we calculate $S(\Theta)$. For example, the entry for 3_1 appears as follows:

3_1 {-10} (1 0 0 0 -1 0 -1 0 -1 -1 -1 -1 -1),

which means that $S(3_1) = x^{-10} - x^{-6} - x^{-4} - x^{-2} - x^{-1} - 1 - x - x^2$.

TABLE 5.1. Yamada polynomial of θ-curve with up to seven crossings.

Θ	$S(\Theta)$
3_1	{-10} (1 0 0 0 -1 0 -1 0 -1 -1 -1 -1 -1)
4_1	{-9} (1 0 -1 0 -1 -1 0 -1 0 -1 0 0 -1 0 0 -1)
5_1	{-12} (-1 1 2 -1 1 0 -2 -1 -2 0 -1 0 1 -1 -1 1 -1 -1)
5_2	{-10} (1 1 0 0 0 -2 -2 -1 -2 0 -1 1 0 0 1 -1 -1)
5_3	{-2} (-1 -1 -1 -1 -1 0 0 0 0 0 -1 0 -1 0 -1 0 1 0 1)
5_4	{0} (-1 -1 -1 -1 -2 -1 -1 -1 0 1 0 1 0 1 0 0 1 -1)
5_5	{-2} (-1 0 -1 -2 0 -1 -2 0 -1 1 0 1 1 -1 0 0 -1 0 1)
5_6	{-4} (-1 0 0 -2 0 0 -1 1 -1 0 -2 -1 0 -1 0 1 0 0 1)
5_7	{0} (-2 -1 0 -2 -1 0 -2 0 -1 1 0 0 2 0 0 1 0 -1)
6_1	{-11} (-1 0 2 -1 -1 2 -1 -1 1 -1 0 -2 0 -1 -3 1 0 -2 1 1)
6_2	{-14} (1 -1 -2 2 0 -2 3 1 -1 1 -1 0 -3 -1 0 -4 0 2 -2 0 1)
6_3	{-6} (-2 0 2 -2 0 1 -2 0 -1 1 -1 -1 1 -2 -2 1 0 -1 1 1)
6_4	{-6} (-1 -1 0 0 -1 1 1 -1 0 0 -2 -1 -2 0 -1 0 2 -1 0 1)
6_5	{-11} (1 0 0 1 -2 -1 0 -2 0 -1 0 -1 -1 1 0 0 1 0 -1 0 0 -1)
6_6	{-6} (-1 0 -1 -1 1 -1 -1 1 -1 0 -1 0 0 -1 1 0 -1 0 0 -1 0 1)
6_7	{-7} (1 -1 -1 1 -2 -2 0 -2 0 0 2 1 -1 1 -1 -2 0 0 -1 0 1)
6_8	{-8} (-1 1 0 -2 1 0 -2 0 -1 0 -1 0 1 -2 0 1 -1 0 1 0 -1)
6_9	{-9} (1 0 -1 1 0 -2 0 -1 -3 -1 0 -1 1 0 2 -1 -1 2 -2 0 1 -1)
6_{10}	{-6} (-1 0 1 -1 -1 1 -2 -2 0 -1 -1 1 0 1 -1 1 1 -3 1 0 -1 1)
6_{11}	{-7} (1 0 -2 0 1 -3 -1 1 -2 -1 1 -1 0 -1 2 0 -2 3 -1 -2 1)
6_{12}	{-4} (-1 0 0 -2 -1 1 -2 -1 1 -1 -1 1 0 1 -1 1 0 -3 2 0 -1 1)
6_{13}	{-8} (-1 0 1 -1 -1 2 -1 -2 1 -1 -2 0 -1 0 -2 1 2 -2 2 1 -2)
6_{14}	{-10} (1 -1 -1 2 -2 -1 3 -1 -1 1 -2 -1 -3 0 0 -3 2 1 -2 2 1 -1)
6_{15}	{-8} (2 0 -3 2 0 -4 1 0 -3 0 -1 1 -2 0 3 -3 0 3 -2 -1 1)
6_{16}	{-13} (-1 1 1 -2 2 2 -3 1 1 -2 0 -2 0 -3 -2 2 -2 -1 3 -1 -1 1)
7_1	{-7} (-1 0 1 -3 1 3 -4 0 2 -3 0 -1 1 -2 -2 3 -2 -2 3 0 -2 2 1 -1)
7_2	{-12} (-1 1 3 -2 -1 4 -3 -4 3 -2 -3 1 -1 0 -3 3 2 -5 3 2 -4 1 1 -1)
7_3	{-14} (-1 0 3 -2 -3 6 0 -5 4 0 -4 0 -1 0 -4 2 4 -6 1 4 -5 -1 2)
7_4	{-7} (1 0 -3 1 3 -5 -1 3 -4 -2 2 -1 0 -1 3 -1 -4 5 -1 -4 3 1 -1)
7_5	{-18} (-1 1 0 -1 4 -1 -1 2 -1 -1 -2 0 -1 -2 1 1 -2 1 1 -2 -1 1 -1 -1)
7_6	{-12} (1 0 -1 3 0 -4 2 -1 -3 1 -1 0 -3 0 1 -3 1 3 -2 0 2 -1 -1)
7_7	{-18} (-1 0 2 0 -1 2 1 -3 1 0 -2 0 -1 1 -2 1 2 -3 -1 1 -2 -1)
7_8	{-9} (-1 1 0 -1 1 0 1 -1 0 -1 -2 -1 -1 -2 -1 0 0 0 1 1)
7_9	{-16} (-1 -1 3 0 -3 4 2 -4 2 1 -3 0 -1 1 -4 -1 3 -5 -1 4 -2 -1 1)
7_{10}	{-10} (1 -1 0 4 -3 -2 5 -4 -4 1 -3 -1 -2 3 1 -3 4 1 -5 2 2 -3 0 1)
7_{11}	{-13} (1 -1 0 3 -2 -1 1 -2 -1 -1 1 -1 -2 1 -1 -2 2 1 -1 0 1 -1 -1)
7_{12}	{-14} (1 -2 -2 6 -2 -4 7 -2 -4 2 -1 -1 -3 2 1 -6 3 4 -6 1 3 -3 -1 1)
7_{13}	{-9} (-1 0 2 -3 -1 4 -2 -1 2 -2 -1 -2 1 -1 -3 3 0 -3 2 1 -2 0 1)
7_{14}	{-10} (-1 1 2 -3 -1 4 -4 -3 4 -2 -2 2 0 0 -3 2 0 -6 4 2 -4 2 1 -1)
7_{15}	{-16} (1 0 0 2 0 -2 2 -1 -3 0 -2 0 -2 1 1 -2 1 1 -2 0 1 -1 -1)
7_{16}	{-4} (-1 -1 0 0 -1 0 1 -1 -1 0 -2 0 -1 1 -1 -1 1 -1 0 1 1)
7_{17}	{-20} (1 -1 -1 3 -1 -3 3 -1 -2 2 1 2 -1 1 0 -5 0 1 -4 0 1 -1 -1)
7_{18}	{0} (-1 -1 0 0 -3 -1 0 -3 -1 1 -1 1 0 2 0 0 3 -1 -1 1 0 -1)
7_{19}	{-8} (-1 0 2 -2 -2 4 -1 -4 3 0 -3 1 -1 0 -4 1 2 -5 2 3 -3 0 2)
7_{20}	{-8} (-1 0 0 -2 2 1 -1 1 -1 0 -2 0 0 -2 -1 0 -2 -1 1 0 0 1 1)

TABLE 5.1. Yamada polynomial of θ-curve with up to seven crossings (continued).

Θ	$S(\Theta)$
7_{21}	{-4} (-1 -1 1 0 -3 1 1 -3 0 1 -2 0 -1 1 -2 -1 3 -2 0 3 0 -1)
7_{22}	{-2} (-1 0 1 -2 -2 2 -3 -4 1 -1 -2 2 1 2 -1 2 2 -4 2 1 -3 0 1)
7_{23}	{-6} (-2 1 1 -4 2 1 -3 1 0 1 -2 0 1 -3 -1 2 -2 -2 2 0 -1 1 1)
7_{24}	{-6} (-1 0 1 -1 -2 1 -1 -3 2 1 -1 1 -1 0 -3 0 1 -3 1 2 -1 0 1)
7_{25}	{-2} (-1 -1 -1 -1 -1 0 0 0 0 0 0 0 0 0 -1 0 -1 0 -1 0 0 0 1 0 1)
7_{26}	{0} (-1 -1 -1 -1 -2 -1 0 0 0 1 0 -1 0 0 -1 1 0 1 -1 0 1 -1 1)
7_{27}	{2} (-1 -1 -1 -1 -2 -1 -1 -1 -1 0 0 0 1 1 0 1 0 1 0 1 1 -1 0 -1)
7_{28}	{-22} (1 0 -1 0 0 -1 0 1 0 0 1 0 0 1 0 0 -2 0 -1 -2 0 -1 -1 0 -1)
7_{29}	{-18} (1 0 0 1 0 -1 0 0 -1 0 0 -1 -1 0 -1 0 -1 1 -1 -1 1 -1 0 0 -1)
7_{30}	{-20} (1 0 -1 0 1 -1 -1 2 0 -1 2 0 -2 0 -1 0 -2 1 1 -3 0 0 -2)
7_{31}	{-21} (-1 0 1 0 0 2 1 -1 0 0 -2 0 1 -1 0 -1 1 -2 -1 1 -2 -1 0 -1)
7_{32}	{-23} (-1 0 1 0 -1 1 1 -1 1 2 -1 0 1 -1 0 -1 1 -2 -2 1 -2 -2 0 -1)
7_{33}	{-22} (1 0 0 1 -1 -2 2 -1 -1 2 -1 1 -1 1 0 -2 1 0 -2 0 0 -2 -1 0 -1)
7_{34}	{-20} (1 0 1 1 -1 0 1 -3 -1 1 -2 1 -1 1 -1 -1 2 -1 -1 1 -1 -2 0 -1)
7_{35}	{-21} (-1 1 1 -1 1 3 -2 0 1 -3 0 -1 2 -1 -1 2 -2 -2 1 -1 -2 0 0 -1)
7_{36}	{-25} (-1 0 0 -1 1 2 -2 2 2 -1 2 0 1 -2 -1 1 -2 -1 1 -2 -2 0 -1 -2)
7_{37}	{-23} (-1 1 0 -2 1 2 -2 2 3 -1 1 -1 1 -2 -1 2 -3 -2 1 -2 -2 0 0 -1)
7_{38}	{-20} (1 0 -1 1 1 -2 1 1 -2 0 1 -1 -1 1 -1 0 -2 2 -1 -3 2 -2 -1 1 -1)
7_{39}	{-18} (1 0 0 2 1 -2 1 0 -4 -1 0 -2 -1 2 0 1 -1 2 -2 -3 3 -2 -1 1 -1)
7_{40}	{-22} (1 0 -2 0 2 -3 -1 4 0 -1 4 1 -2 1 -1 -1 -5 1 1 -5 2 1 -3)
7_{41}	{-21} (-1 0 2 0 -2 3 1 -3 1 2 -2 0 2 -2 -1 -2 2 -3 -2 4 -3 -2 1 -1)
7_{42}	{-23} (-1 0 1 -1 -1 3 0 -1 3 1 -2 1 1 -2 0 -1 1 -4 -1 2 -4 0 1 -2)
7_{43}	{-20} (1 -1 0 3 -2 -1 4 -3 -1 2 -2 0 -2 1 -1 -3 3 1 -3 2 0 -3 0 0 -1)
7_{44}	{-22} (1 -1 0 2 -3 0 4 -3 0 2 -2 0 -1 2 0 -2 3 -1 -4 2 -1 -3 0 0 -1)
7_{45}	{-23} (-1 1 1 -3 2 3 -4 3 2 -3 0 -1 1 -2 0 4 -2 -2 3 -3 -3 1 -1 -2)
7_{46}	{-24} (1 -1 -1 3 -2 -3 4 -2 -1 3 0 1 -1 2 1 -3 2 1 -5 0 0 -3 -1 0 -1)
7_{47}	{-21} (-1 2 1 -4 2 3 -5 2 3 -2 1 0 2 -3 -2 3 -4 -3 4 -2 -3 1 0 -1)
7_{48}	{-25} (-1 1 0 -3 3 2 -5 3 2 -2 2 1 2 -2 0 3 -4 -2 3 -4 -3 1 -1 -2)
7_{49}	{-23} (-2 1 3 -3 0 4 -4 0 3 -1 1 0 3 -1 -3 3 -2 -5 2 -1 -3 0 0 -1)
7_{50}	{-18} (1 -1 -1 3 -3 -2 5 -3 0 4 -1 0 -2 1 -1 -4 2 0 -5 2 1 -3 1 1 -1)
7_{51}	{-16} (1 -2 -1 5 -3 -3 7 -2 -3 3 -1 -1 -3 2 0 -6 3 2 -6 2 3 -3 -1 1)
7_{52}	{-15} (-1 2 2 -5 2 5 -6 1 3 -4 -1 -1 2 -3 -2 5 -3 -4 5 -1 -4 2 1 -1)
7_{53}	{-14} (1 -1 0 3 -3 0 4 -5 -1 2 -3 0 -1 2 -1 -3 3 -1 -4 3 0 -3 2 1 -1)
7_{54}	{-14} (2 -1 -3 4 -1 -5 5 0 -3 3 0 0 -4 0 1 -6 1 4 -4 0 3 -2 -1 1)
7_{55}	{-19} (-1 1 1 -4 3 4 -6 3 3 -3 1 0 1 -4 -2 3 -4 -3 5 -2 -3 3 0 -2)
7_{56}	{-17} (-1 1 0 -2 4 0 -4 5 0 -3 2 -1 0 -3 0 1 -5 0 3 -4 0 3 -2 -1 1)
7_{57}	{-16} (1 -1 0 3 -4 -1 5 -4 -1 3 -2 0 -1 2 -1 -4 3 -1 -5 3 1 -3 1 1 -1)
7_{58}	{-19} (-2 1 3 -4 1 6 -5 0 4 -2 0 -1 2 -3 -4 4 -2 -5 4 0 -4 1 1 -1)
7_{59}	{-13} (1 -1 -2 3 -1 -4 5 0 -4 4 0 -3 0 -1 1 -4 0 4 -6 0 4 -4 1 2 -1)
7_{60}	{-10} (-2 0 4 -3 -3 7 -2 -5 5 -1 -4 1 0 0 -5 2 3 -8 3 5 -5 1 2 -1)
7_{61}	{-14} (-1 1 3 -4 -2 7 -4 -5 6 -2 -3 3 1 0 -4 3 1 -9 3 3 -7 2 3 -1)
7_{62}	{-10} (-1 1 1 -4 1 4 -5 1 5 -4 -1 1 -3 -2 -3 3 -1 -4 7 0 -4 4 -1 -2 1)
7_{63}	{-15} (1 -1 -2 4 0 -6 5 2 -7 3 2 -3 0 0 2 -5 -1 6 -6 -2 6 -4 -2 2)
7_{64}	{-15} (2 0 -4 3 2 -7 2 4 -6 1 4 -1 0 -1 3 -5 -5 6 -4 -4 6 -1 -2 1)
7_{65}	{-12} (-1 1 1 -3 2 3 -6 1 4 -5 0 2 -2 -1 -2 3 -2 -4 6 -2 -4 5 -1 -2 1)

3. Handcuff graph

We assume Γ is a handcuff graph Φ. Then we can describe $\langle\gamma^\omega\rangle$ concretely. Since $\chi(\Phi) = -1$, we obtain the following proposition.

PROPOSITION 5.9. *For a handcuff graph diagram ϕ and its constituent link diagram ϕ_{12}, let $\phi^{(2)}$ be the link diagram obtained from ϕ by parallelizing each edge, and $\phi_{12}^{(2)}$ be the link diagram obtained from ϕ_{12} by $(2,0)$-cabling. Then*

$$R(\phi)(x) = -\delta\,\langle\phi^{(2)}\rangle(x^{1/4}) + \langle\phi_{12}^{(2)}\rangle(x^{1/4}),$$

where $\delta = -x^{1/2} - x^{-1/2}$.

PROOF. From Lemma 5.6, we obtain the following equation:

$$R(\phi)(x) = (-\delta)\langle\phi^\omega\rangle(x^{1/4}), \tag{3}$$

where $\delta = -x^{1/2} - x^{-1/2}$. Assume $\phi_{12} = \phi_1 \cup \phi_2$, then we can describe $\langle\phi^\omega\rangle$ concretely by using the equation (1):

$$\begin{aligned}\langle\phi^\omega\rangle &= \langle\phi^{(2)}\rangle - \delta^{-1}(\langle\phi_{12}^{(2)}\rangle + \langle\phi_1^{(2)}\rangle + \langle\phi_2^{(2)}\rangle)\\ &\quad +\delta^{-2}(\langle\phi_1^{(2)} \sqcup \bigcirc\rangle + \langle\phi_2^{(2)} \sqcup \bigcirc\rangle + \langle\bigcirc\rangle) - \delta^{-3}\langle\bigcirc \sqcup \bigcirc\rangle\\ &= \langle\phi^{(2)}\rangle - \delta^{-1}\langle\phi_{12}^{(2)}\rangle.\end{aligned}$$

Then we replace $\langle\phi^\omega\rangle$ in the equation (3) with the above. □

By using Proposition 5.9 and KNOT program **[16]**, we compute the Yamada polynomial of all the prime handcuff graphs with up to seven crossings immediately; Table 1.2. For example, the entry for 2_1 appears as follows:

2_1 {-4} (1 1 1 1 0 0 -1 -1 -1 -1),

which means that $R(2_1) = x^{-4} + x^{-3} + x^{-2} + x^{-1} - x^2 - x^3 - x^4 - x^5$. From Theorem 5.2, we note that there is ambiguity for degrees and coefficients of the Yamada polynomial.

TABLE 5.2. The Yamada polynomial of handcuff graphs with up to seven crossings.

Φ	$R(\phi)$
2_1	{-4} (1 1 1 1 0 0 -1 -1 -1 -1)
4_1	{-6} (1 1 1 1 1 0 0 0 -1 0 -1 0 -1 -1 0 -1)
5_1	{-8} (1 0 -1 1 0 -1 1 0 -1 0 -1 0 -1 0 2 -1 1 1 -1)
6_1	{-10} (1 -1 -3 2 0 -3 3 1 -1 1 0 1 -1 1 3 -3 2 -3 -1 1)
6_2	{-9} (1 1 0 0 0 -1 -2 0 -1 -1 0 -1 0 -1 1 1 0 2 1)
6_3	{-11} (1 -1 0 3 -2 0 1 -2 -1 -2 0 -1 -1 2 0 -1 2 1 -1 1 1)
6_4	{-10} (1 0 -1 1 1 -1 2 2 0 1 -1 0 -2 -1 1 -2 0 1 -1 -1)
6_5	{-8} (1 1 1 1 1 0 0 0 0 0 0 0 -1 0 -1 0 -1 0 0 -1 0 -1)
6_6	{-10} (2 0 1 1 -1 0 -1 1 0 1 2 1 0 0 -1 -2 -1 -1 -1 -1)
6_7	{-10} (1 0 0 2 1 -1 2 0 -1 1 -1 1 -1 1 1 -2 0 0 -2 -1 0 -1)
6_8	{-10} (1 0 0 3 0 -1 3 -1 -1 1 -1 0 -2 0 0 -3 1 1 -2 1 1 -1)
6_9	{-9} (-2 -1 1 -3 -2 2 -2 0 3 1 2 0 2 0 -3 2 0 -2 2 1 -1)
7_1	{-12} (1 -2 -1 6 -4 -4 7 -4 -4 3 0 0 -1 4 2 -5 5 3 -7 2 3 -4 -1 1)
7_2	{-10} (1 2 -2 -1 3 -3 -1 2 -1 0 -1 1 -1 -2 3 0 -2 2 1 -2 0 1)
7_3	{-11} (1 0 -2 2 2 -3 1 3 -3 0 2 -1 0 -1 2 -2 -3 4 -2 -2 3 0 -1)
7_4	{-12} (1 -2 -2 5 -2 -3 7 -1 -2 3 0 0 -2 2 1 -6 2 2 -6 1 3 -2 0 1)
7_5	{-11} (1 0 -1 1 1 -1 0 2 -2 -1 0 -2 -1 -2 1 -1 -1 3 0 0 2 1)
7_6	{-13} (1 -1 1 2 -4 2 2 -2 1 0 0 -3 -2 0 -3 -1 3 0 -1 3 1 -1 1 1)
7_7	{-12} (1 -1 -2 3 -1 -4 3 -1 -3 1 0 1 -1 2 3 -3 2 3 -3 0 2 -1 -1)
7_8	{-12} (1 -1 -1 5 -2 -4 6 -2 -3 3 0 0 -2 2 1 -5 3 3 -5 1 3 -3 -1 1)
7_9	{-9} (1 1 -1 0 0 -1 0 0 1 0 0 1 -1 0 1 0 0 0 0 -1 -1)
7_{10}	{-10} (1 1 0 0 1 0 -1 1 1 -1 1 1 -1 0 -1 0 -2 -1 1 -2 0 1)
7_{11}	{-12} (1 -1 0 4 -2 -2 5 -2 -2 3 0 1 -1 2 0 -5 2 0 -5 1 2 -2 0 1)
7_{12}	{-11} (1 0 -3 2 3 -4 1 5 -3 0 3 -1 0 -1 3 -2 -4 5 -2 -4 3 0 -2)
7_{13}	{-11} (1 1 -2 -1 2 -2 -3 3 -1 -2 1 -1 0 -2 2 1 -3 3 2 -2 1 2)
7_{14}	{-11} (1 0 -2 0 1 -3 -1 3 -2 0 2 0 1 0 3 0 -2 3 -1 -3 1 0 -1)
7_{15}	{-10} (1 1 -1 0 2 -1 -2 2 0 -3 1 0 -2 0 0 1 -2 1 3 -3 1 2 -1)
7_{16}	{-11} (1 -1 -3 2 2 -4 2 5 -4 0 2 -2 -1 -1 3 -2 -2 6 -2 -3 4 0 -2)
7_{17}	{-9} (-2 -1 1 -1 -2 2 -1 -2 1 -1 1 -1 2 1 -2 2 1 -1 1 2)
7_{18}	{-10} (1 0 -1 1 0 -2 0 0 -2 0 1 -1 0 1 0 0 -1 2 0 -1 3 -1 0 1 -1)
7_{19}	{-10} (1 0 -1 1 1 -1 1 2 -1 0 1 -1 -1 0 -1 -1 -2 1 0 -2 3 0 0 1 -1)
7_{20}	{-12} (-1 2 0 -3 3 -1 -2 2 1 1 0 2 1 -3 0 1 -3 0 2 -1 -1 1 0 -1)
7_{21}	{-11} (2 -2 -2 4 -3 -2 2 -1 -1 -1 2 0 -2 3 1 -3 2 2 -2 0 2 0 -1)
7_{22}	{-12} (1 -1 -1 3 -2 -2 4 -3 -2 2 -2 0 -1 2 1 -3 3 1 -3 2 2 -2 1 1 -1)
7_{23}	{-11} (1 -1 -2 2 0 -3 3 2 -3 3 1 -1 1 0 2 -3 -1 3 -4 -1 3 -2 0 1 -1)
7_{24}	{-12} (1 -1 0 3 -3 -1 5 -2 4 -1 0 -2 0 -2 -5 3 0 -3 3 2 -2 1 1 -1)
7_{25}	{-11} (-1 2 1 -5 2 3 -6 1 3 -3 0 0 2 -2 -1 6 -3 -2 5 -1 -3 2 1 -1)
7_{26}	{-12} (1 0 0 2 -2 -1 3 -3 -1 2 -2 0 -1 1 0 -2 3 0 -2 2 1 -2 1 1 -1)
7_{27}	{-11} (-1 1 1 -3 2 3 -4 1 2 -3 0 0 2 -1 -1 4 -3 -2 3 -1 -2 2 1 -1)
7_{28}	{-10} (1 0 -1 2 1 -2 1 1 -3 -1 1 -2 -1 1 0 0 -1 3 -2 4 -1 -1 1 -1)
7_{29}	{-12} (-1 2 -1 -3 5 -1 -2 3 0 -1 -2 1 0 -3 2 2 -3 1 3 -2 -1 2 0 -1)

TABLE 5.2. The Yamada polynomial of handcuff graphs with up to seven crossings (continued).

Φ	$R(\phi)$
7_{30}	{-11} (1 -1 -2 3 -1 -4 4 1 -4 4 2 -1 2 1 3 -3 0 4 -6 -1 3 -4 -1 1 -1)
7_{31}	{-10} (-2 0 4 -3 -2 6 -1 -3 6 1 -2 2 0 0 -5 2 3 -7 3 3 -5 0 1 -1)
7_{32}	{-11} (-2 1 2 -6 0 4 -7 0 4 -2 1 1 4 -1 -2 6 -2 -4 5 0 -4 2 1 -1)
7_{33}	{-11} (1 -2 -2 5 -2 -5 7 0 -6 5 1 -3 1 1 2 -5 1 6 -8 1 6 -5 0 2 -1)
7_{34}	{-11} (-2 3 4 -8 2 6 -10 0 5 -3 0 2 5 -2 -3 8 -4 -7 7 -1 -6 3 2 -1)
7_{35}	{-11} (-1 -1 -1 -1 -1 -1 0 0 0 1 1 1 2 1 1 0 0 0 -1)
7_{36}	{-12} (-1 0 1 0 0 1 1 -1 1 0 -1 0 -1 0 -2 0 1 -1 1 1)

4. Chirality

In this section, we mention the chirality of θ-cures and handcuff graphs. A spatial graph Γ is said to be *achiral* if Γ is isotopic to its mirror image, and Γ is said to be *chiral* if Γ is not achiral. In **[38]**, S. Yamada also gave the following proposition:

PROPOSITION 5.10 ([**38**]). *Let* $\overline{\gamma}$ *be the mirror image of a spatial graph diagram* γ. *Then* $R(\overline{\gamma})(x) = R(\gamma)(x^{-1})$.

From Theorem 5.2 and Proposition 5.10, we can conclude all the θ-curves in Table 1.1 and almost all handcuff graphs in Table 1.2 are chiral.

For example, we consider the case of the handcuff graph 4_1. $R(4_1) = x^{-6} + x^{-5} + x^{-4} + x^{-3} + x^{-2} - x^2 - x^4 - x^6 - x^7 - x^9$. So $R(\overline{4_1}) = -x^{-9} - x^{-7} - x^{-6} - x^{-4} - x^{-2} + x^2 + x^3 + x^4 + x^5 + x^6$. Then, for arbitrary integer n, $R(\overline{4_1}) \neq (-x)^n R(4_1)$. Thus the handcuff graph 4_1 is chiral.

On the other hand, we prove that the handcuff graphs 2_1, 6_1, and 7_{17} are achiral by deformations as in Fig. 5.3.

Therefore we obtain the following:

THEOREM 5.11. *There exist only three achiral handcuff graphs in Table* 1.2 : 2_1, 6_1, *and* 7_{17}.

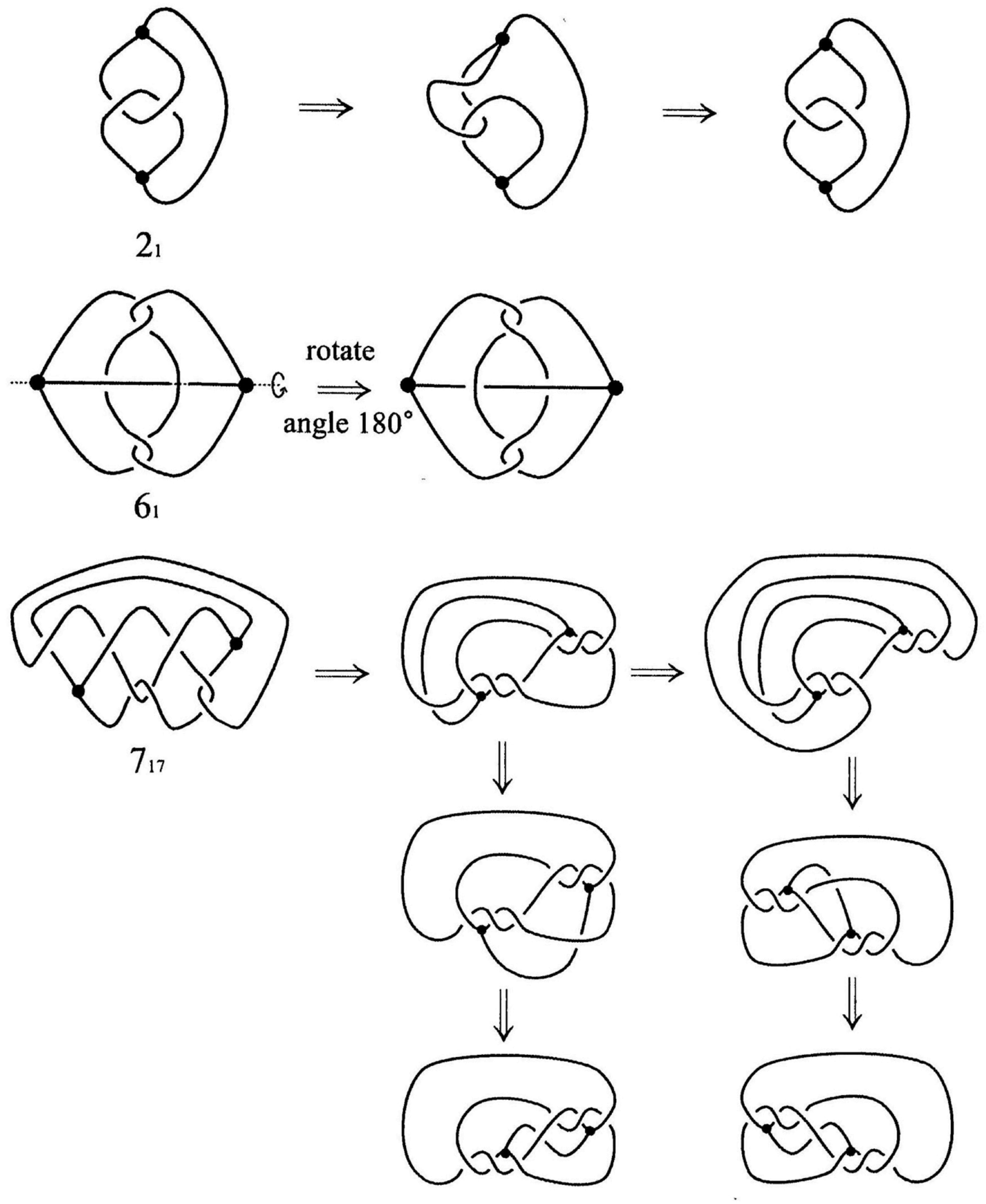


FIG. 5.3. The handcuff graphs 2_1, 6_1, and 7_{17} are achiral.

Open problems

Finally, we give some open problems.

PROBLEM 5.12. Are all of the handcuff graphs in Table 1.2 prime?

We hope so, but we do not know how to conclude the primeness of a handcuff graph (cf. Proposition 4.2). When making Table 1.2, we omit handcuff graphs which can be decomposed obviously. More generally, we give the following.

PROBLEM 5.13. Develop the way of decision whether a spatial 3-valent graph is prime or not.

From Proposition 5.3, the Yamada polynomial of a nonprime θ-curve (or a handcuff graph with admissible sphere II or III) can be factorized: it has the factors which are the Yamada polynomial of other spatial graph diagram. However, we must not conclude the primeness of a spatial 3-valent graph by using this fact. There may exist nontrivial spatial graphs with the "trivial" Yamada polynomial.

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