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Dam in Continuous Time

by

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Synopsis

This paper discribes a dam model in continuous time. On the assumption that the inflow into the dam has an exponential distribution, we derive the stochastic distribution for the water level. First, we study the fundamental equation of a dam. Next, we consider the dam with infinite capacity which has exponential inflow and a constant (unit) outflow. Finally, we investigate the dam problem under the outflow rule suggested by Holdaway.

Keywords : Moran Dam model, Holdaway's release rule, Stochastic processes

1. Introduction

We will consinder a particular type of storage system which corresponds to the kind of situation which occurs in a dam build to store water[1]. This system consists of three components namely the input, output and the content of a dam. The input is the amount of water that flows into the dam, which depends on some natural phenomena, such as rainfall, melting of snow, stream flows. And these vary from time to time and have a probability distribution[2] in a random manner at every moment, day by day and from year to year. The output is the amount of water to be released from the dam during a certain interval. This component may be deterministic or random. It also may depend on the amount of water in the dam and on the "release rule" which prescribes how and when the water is released[3]. The purpose of the dam is to make the amount released (the output) more uniform then the inflow in some statistical sense; in this connection , it is relevant to study the underlying stochastic process. For a given input and release rule, a special interest to a designer is the varing pattern of contents of the dam. The content of the dam by which we mean the amount of water stored in the dam. Now, we shell study the fundamental equation of a dam model[1, 2, 4]

2. Problem formulation

We consider a dam which can hold at most K units of water which we call the capacity of the dam. Suppose that during a time interval $(t,t+1),(t=0,1,2,3,...)$ X_t units of water flow

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into the dam. Let Z_t units be the amount of water in the dam at the monent t then, it may happen that Z_t+X_t is greater than K and the overflow of amount Z_t+X_t-K is caused. Clearly the distribution of this overflow can be obtained directly from the distribution of X_t and Z_t.

At the end of time interval t the water is released in according to some definite rule. As we are trying to construct the simplest possible model, it is supposed that we release a definite quantity $M < K$ if there is at least amount in the dam, or the amount Z_t+X_t, if $Z_t+X_t < M$, thus leaving finally Z_{t+1}. Specifically, Moran's dam model as follows:

Suppose that during the time interval $(t,t+1)$, $(t=0,1,2,...)$ X_t units of water flow into the dam, any overflow being lost. Provided that the dam is not dry, M units of water flowed at the end of each time interval. Let Z_t denote the amount of water left in the dam just before the input X_t occurs, then Moran formulated the dam equation as follows :

$$Z_{t+1} = \min\{Z_t+X_t, K\} - \min\{Z_t+X_t, M\}, \qquad 0 < M < K \qquad \text{..........(2.1)}$$

The above Eq.(2.1) can be understand as follow :

If $Z_t+X_t > K$ then $\min\{Z_t+X_t,K\}=K$, otherwise $\min\{Z_t+X_t,K\}=Z_t+X_t$,
If $Z_t+X_t > M$ then $\min\{Z_t+X_t,M\}=M$, otherwise $\min\{Z_t+X_t,M\}=Z_t+X_t$,

$$\begin{aligned} Z_{t+1} &= K-M, & Z_t+X_t &> K \\ &= Z_t+X_t-M, & M < Z_t+X_t &< K \\ &= 0 & Z_t+X_t &< M \end{aligned} \qquad \text{..........(2.2)}$$

Thus we can easily see that Z_t, lies in the interval $(0, K-M)$. Now we consider the stationary distribution of dam content for continuous case. By assuming that X_t has a continuous distribution with a probability density which we denote by $f(x)$, $U_t=Z_t+X_t$ also has a continuous probability density which we denote by $g(x)$.

Assuming $f(x)=0$ if $x < 0$, we have

$$g(x) = f(x)\int_0^M g(u)du + \int_M^K g(u) f(x+M-u)du + f(x+K-M)\int_K^\infty g(u)du \qquad \text{......(2.3)}$$

The three terms arising according as

$0 \leqq U_t \leqq M$ (When the dam runs dry),
$M < U_t < K$ (When there is no overflow nor does the dam run dry) , and
$K < U_t$ (When there is an overflow of amount U_t-K).

3. Dam With Exponential Inflow

In this section we will consider a dam of infinite capacity with negative exponent inputs and unit release. Let X_t denote by the inflows during the time interval $(t,t+1)$, $(t=0,1,2,3,...)$, and the amountof water after the release, then we have

$$Z_{t+1} = Z_t+X_t - \min\{Z_t+X_t, 1\} \qquad \text{...... (3.1)}$$

Let us write $U_t = Z_t+X_t$ and define the probability density function of U_t by $g(x)$. and the probability density function of X_t by $f(x)$, then from equation (3.1) the following integral equation is obtain :

$$g(x) = f(x)\int_0^M g(t)dt + \int_M^{M+x} f(M+x-t)g(t)dt \qquad \text{..........(3.2)}$$

In this case it is more convennient to consider the cumulative distribution of X_t and U_t Let us write

$$\text{F(u)} = \int_0^u f(t)dt \quad \text{and} \quad G(u) = \int_0^u g(t)dt$$

Then we obtain from equation (3.2)

$$G(u) = \int_0^u G(M+u-t)dF(t) \qquad \text{.........(3.3)}$$

for M = 1, we have

$$G(u) = \int_0^u G(1+u-t)dF(t) \qquad \text{..........(3.4)}$$

which we can write as follow:

$$H(u-1) = \int_0^u H(u-t)dF(t) \qquad \text{..........(3.5)}$$

Now we will consider the case where the input distribution has the form by Pearson's type III

$$\text{dF(t)} = dF(t) = \frac{a^{n+1}}{n!} t^n e^{-at} \qquad \text{..........(3.6)}$$

with n a non−negative integer. Solving the equation (3.2) , we have obtain the possible solution

$$H(u) = 1 + \sum_{i=0}^{n+1} c_i e^{Z_i u} \qquad \text{..........(3.7)}$$

where is possibly complex. We have

$$1 + \sum_{i=0}^{n+1} c_i e^{Z_i u} = 1 + \frac{c_i a^{n+1} e^{z(u+1)}}{(a+Z_i)^{n+1}} - a^{n+1} e^{-a(u+1)} \sum_{s=0}^{n} \frac{(u+1)^{n-s}}{(n-s)!} \sum_{i=0}^{n+1} \frac{c_i}{(a+Z_i)^{s+1}} \qquad \text{......(3.8)}$$

which implies, $\sum_{i=0}^{n+1} \frac{c_i}{(a+z_i)^{s+1}} = 0$, for $s = 0,1,\ldots..,n$, ...(3.9) which are $n + 1$ equations for the constants $c_1, \ldots\ldots, c_{n+1}$ and $H(u)$ can be shown to be a distribution function and is therefore the required solution.

4. Holdaway's Dam Model in Continuous Time.

Now, we will consider a dam model in which the release rule is governed by the Holdaway's scheme. This kind of dam model is known as Holdaway's dam model. In the following, we investigate this dam model by assuming that the input distribution is negative exponential. Let us Z_t denoted by the content of the dam at a time t just after the release, then the release scheme of Holaday is as follows :

For $D < M < L < K$

(1) If $Z_t + X_t \geqq K + M$, a quantity M is released through the dam and aquantity $Z_t + X_t - K - M$ goes the waste, so that $Z_{t+1} = K$.

(2) If $L + M \leqq Z_t + X_t < K + M$, quantity M is released and the final state is : $Z_{t+1} = Z_t + X_t - M$.

(3) If $L + D \leqq Z_t + X_t < L + M$, a quantity $Z_t + X_t - L$ is released and the final state is $Z_{t+1} = L$.

(4) If $D < Z_t + X_t < L + D$, a quantity D is released and the final state is : $Z_{t+1} = Z_t + X_t - D$

(5) If $Z_t + X_t < D$, $Z_t + X_t$ is released and $Z_{t+1} = 0$

This release rule can be rewritten as follow :

If W_t de note the amount of water to be flow out at the time t, then from the above scheme we have ,

$$W_t = \begin{cases} Z_t + X_t, when Z_t + X_t < D \\ D, when D < Z_t + X_t < L + D \\ Z_t + X_t, when L + D < Z_t + X_t < L - M \\ M, when L + M < Z_t + X_t < K + M \\ M, when Z_t + X_t > K + M, with \\ overflow \end{cases} \qquad \ldots\ldots\ldots\ldots(4.1)$$

By using the release rule , we obtain the following solution system is

(1) for Y < L

$$G(y) = \int_D^{D+Y} G(t) f(Y + D - t) dt, \qquad \ldots\ldots\ldots\ldots(4.2)$$

(2) for L < Y < K

$$G(y) = \int_{D}^{D+L} G(t) f(Y + D - t)dt + \int_{L+M}^{M+Y} G(t) f(Y + M - t)dt, \qquad \text{..........(4.3)}$$

(3) for Y > K

$$G(y) = \int_{D}^{D+L} G(t) f(Y + D - t)dt + \int_{L+M}^{K+M} G(t) f(Y + M - t)dt + F(Y - K), \qquad \text{.....(4.4)}$$

Where $G(y) = pr(Z_t + X_t \le Y), \quad dF(x) = f(x)dx, F(x) = pr(X_t \le x)$.

Conclusion

In this paper, we have considered some problems of dams which have been recently drawn attention by research workers in the theory of storage. Specifically, we have paid our attention on a problem in which release rule is made under Holidaway scheme. By assuming that the input distribution is negative exponential, we have obtained some explicit result for stationary distribution of the dam content. One of the ways to extend this model is to consider the cases in which the input distribution forms a general process such as Person's type III, general independent process and Markovian−type process and so on.

Reference

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