Analytical Integrals of the Original Mindlin's Solutions over a Small Rectangular Area — On Fundamental Solutions to be used for 3-D Elastic Analysis by Point Matching Method —

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Synopsis: This paper executes analytical integrals of the original Mindlin's solutions, which are intended to be used as the fundamental solution in a point matching method in place of the original Mindlin's solutions to avoid a singularity within them and to yield more accurate solutions by the method for 3-D elastic solid or structural problems.

Keywords *: Mindlin's solution, Analytical integral, Fundamental solution, Point matching method, Three-dimensional elastic solid*

1. Introduction

A point matching method is a kind of boundary integral method for 2-D or 3-D structural analyses. The first application of this method to 2-D elastic solid problem was done by Oliveira [1], in which Melan's solution for an infinite elastic plane was used as a fundamental solution. The method can naturally extended to 3-D elastic problem using Kelvin's solution for an infinite elastic solid as a fundamental solution. A point matching method is characterized by satisfying point-wise the conditions prescribed on the boundary planes. Therefore, this method can also be called a boundary point matching method (BPM).

In applying this method, a lot of the known analytical solutions which exactly satisfy the field equation are first superposed and then satisfaction of boundary conditions on the collocation points on boundary planes yields a simultaneous linear equation with the same degree to the number of selected point. Therefore, accuracy of solutions by this method mainly depends on the number of collocation points. From practical viewpoint, however, the number of collocation points as less as possible is more desirable, if the solution obtained has a sufficient accuracy in an engineering sense. It is important, therefore, to find optimal positions of collocation points and optimal type of fundamental solutions to get higher accuracy with less number of collocation points on boundary planes.

In the civil engineering field, a problem of infinite half space bounded by a free surface is often encountered such as soil foundation-structure interaction problems, pipe lines and underground structural problems. For this type of problems, Mindlin's solution [2,3] is naturally better as fundamental solution in a boundary point matching method than Kelvin's solution, because it originally satisfies a free boundary surface [4].

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Another problem to be solved in a boundary point matching method is treatment of singularity of fundamental solution. For this problem, it is effective to use the integral of the original Mindlin's solution as a fundamental solution to remove its singularity.

This paper is mainly devoted to present the analytical integrals of the original Mindlin's solutions regarding a small two-dimensional area, which can be used as a fundamental solution.

2. Boundary Point Matching Method

In a procedure of this method, first, a solid or a structure to be solved is cut off from an infinite half space. Second, arbitrarily distributed forces with an unknown intensity (called adjustment forces in the following) are applied on an auxiliary boundary planes placed exterior to the real boundary plane. Last, a simultaneous linear equation to determine the intensity of adjustment forces is solved to satisfy pointwise boundary conditions on the real boundary planes.

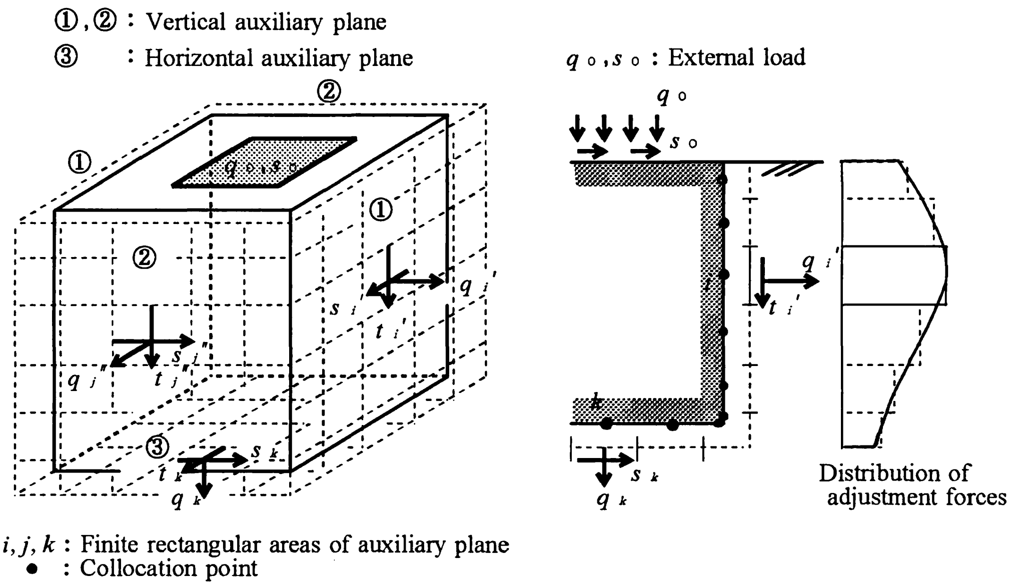


Fig.1 Boundary point matching method

Here, consider a problem of rectangular prism as shown in **Fig.1**. The auxiliary boundary planes are placed exterior near to the real boundary planes. Adjustment forces stepwise distributed on a small area are assumed. These adjustment forces are derived from integrals of the original Mindlin's solutions over a small rectangular area on the auxilially boundary planes. Then, stresses and displacements on the real boundary conditions can be expressed by superposition of the adjustment forces with the unknown intensities and the external load, and simultaneous linear equations to determine those intensities can be obtained as follows:

$$c_m = \sum_i \left[t_i' \times \int \sigma_1 dA' + s_i' \times \int \sigma_2 dA' + q_i' \times \int \sigma_2 dA' \right]_m$$
$$+ \sum_j \left[t_j'' \times \int \sigma_1 dA'' + s_j'' \times \int \sigma_2 dA'' + q_j'' \times \int \sigma_2 dA'' \right]_m$$

$$+\sum_{k}[q_k\times\int\sigma_1 dA+s_k\times\int\sigma_2 dA+t_k\times\int\sigma_2 dA]_m$$

$$+[q_0\times\int(\sigma_1)_{D=0}dA+s_0\times\int(\sigma_2)_{D=0}dA]_m \quad (1)$$

where c_m : a prescribed boundary stress or displacement, σ_1, σ_2 : the stress or displacement in Mindlin's first or second problem respectively, $t_i', s_i', q_i', \cdots\cdots$: the unknown intensities of adjustment force, and q_0, s_0 : the external load intensities.

3. Fundamental Solution

3.1 Characteristics of fundamental solutions

In order to cut off a three dimensional body from an infinite half elastic space, the distributed loads on the auxiliary boundary planes may exert an action like a knife to make prescribed conditions on the real boundary planes. The fundamental solutions are made by a solution for a uniformly distributed stress over a small rectangular area on the auxiliary boundary planes, which can be derived from integrals of the solutions of Mindlin's first and second problems as shown in **Fig.2**. Since a small rectangular area within a three dimensional solid can be yielded in *x-y* plane, *x-z* plane and *y-z* plane respectively in Cartesian coordinates, the relevant integral problems can be classified into 9 types as shown in **Fig.3**, depending on the direction of a force.

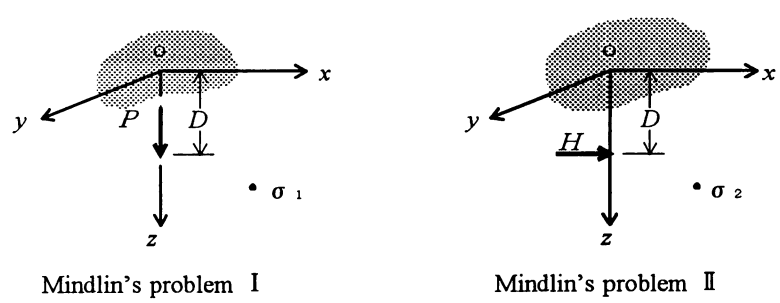


Fig.2 Mindlin's problem

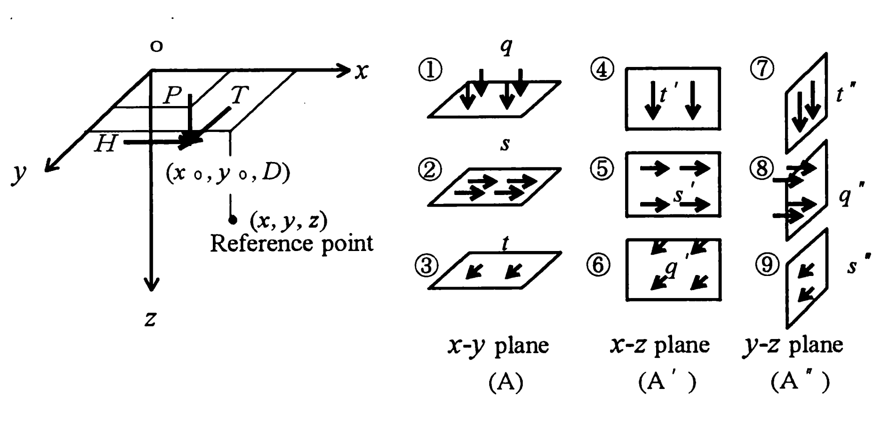


Fig.3 Integral with respect to a rectangular area

The distributed force which acts in the normal and tangential directions to each integral plane is shown in **Fig.3**, and the stress components at any point within elastic solid by them are shown as follows:

①: $q\int\sigma_1 dA$, ②: $s\int\sigma_2 dA$, ③: $t\int\sigma_2' dA$
④: $t'\int\sigma_1 dA'$, ⑤: $s'\int\sigma_2 dA'$, ⑥: $q'\int\sigma_2' dA'$
⑦: $t''\int\sigma_1 dA''$, ⑧: $q''\int\sigma_2 dA''$, ⑨: $s''\int\sigma_2' dA''$

where $q, s, t, \cdots\cdots$: intensities of the distributed loads, and the integral expressions have the following meaning:

$\int\sigma_1 dA$: Integral calculation with respect to the rectangular horizontal plane (A) of a stress components of Mindlin's problems Ⅰ for a unit load.

$\int\sigma_2 dA$: Integral calculation with respect to the rectangular horizontal plane (A) of a stress components of Mindlin's problems Ⅱ for a unit load.

$\int\sigma_2' dA$: Integral calculation with respect to the rectangular horizontal plane (A) of a stress components of Mindlin's problems Ⅱ of y axis directional concentrated force for a unit load.

$\int\sigma_1 dA'$, $\int\sigma_2 dA'$ and $\int\sigma_2' dA'$:
Similar integral calculation with respect to the rectangular vertical plane (A′) of a stress components of Mindlin's problems Ⅰ and Ⅱ.

$\int\sigma_1 dA''$, $\int\sigma_2 dA''$ and $\int\sigma_2' dA''$:
Similar integral calculation with respect to the rectangular vertical plane (A″) of a stress components of Mindlin's problems Ⅰ and Ⅱ.

3.2 Mindlin's solution

Mindlin's solutions are an analytical solution for a concentrated force acting on an inside of semi-infinite solid consisting of homogeneous and isotropic elastic material.

In Cartesian coordinates (x, y, z), the stresses $\sigma(\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{yz}, \tau_{zx})$, the strain $\varepsilon(\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}, \gamma_{yz}, \gamma_{zx})$ and the displacements $\delta(u, v, w)$ are given as,

$$\varepsilon_x=\frac{\partial u}{\partial x},\quad \varepsilon_y=\frac{\partial v}{\partial y},\quad \varepsilon_z=\frac{\partial w}{\partial z},$$
$$\gamma_{xy}=\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x},\quad \gamma_{yz}=\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y},\quad \gamma_{zx}=\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z} \tag{2}$$

$$\varepsilon_x=\frac{1}{E}\{\sigma_x-\nu(\sigma_y+\sigma_z)\},\quad \varepsilon_y=\frac{1}{E}\{\sigma_y-\nu(\sigma_z+\sigma_x)\},\quad \varepsilon_z=\frac{1}{E}\{\sigma_z-\nu(\sigma_x+\sigma_y)\},$$
$$\gamma_{xy}=\frac{1}{G}\tau_{xy},\quad \gamma_{yz}=\frac{1}{G}\tau_{yz},\quad \gamma_{zx}=\frac{1}{G}\tau_{zx} \tag{3}$$

in which E : Young's modulus, ν : Poisson's ratio and G : modulus of elasticity in shear with $G=\dfrac{E}{2(1+\nu)}$.

Figure 4 shows a positive direction of stress components.

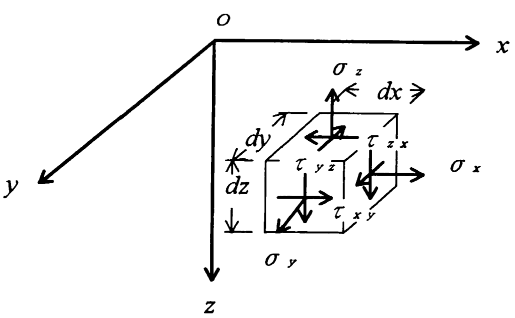
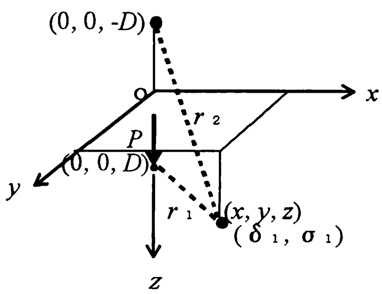


Fig.4 Stress components

(1) The solution of Mindlin's problem I

Taking the position of a vertical force P and the coordinate system as **Fig.5**, the solutions of Mindlin's problem I are given as follows:



$$r_1=[x^2+y^2+(z-D)^2]^{1/2}$$
$$r_2=[x^2+y^2+(z+D)^2]^{1/2}$$

Fig.5 Mindlin's problem I

$$u=\frac{Px}{16\pi(1-\nu)G}\left[\frac{z-D}{r_1^3}+\frac{(3-4\nu)(z-D)}{r_2^3}-\frac{4(1-\nu)(1-2\nu)}{r_2(r_2+z+D)}+\frac{6zD(z+D)}{r_2^5}\right]$$

$$v=\frac{Py}{16\pi(1-\nu)G}\left[\frac{z-D}{r_1^3}+\frac{(3-4\nu)(z-D)}{r_2^3}-\frac{4(1-\nu)(1-2\nu)}{r_2(r_2+z+D)}+\frac{6zD(z+D)}{r_2^5}\right]$$

$$w=\frac{P}{16\pi(1-\nu)G}\left[\frac{3-4\nu}{r_1}+\frac{(z-D)^2}{r_1^3}+\frac{1+4(1-\nu)(1-2\nu)}{r_2}+\frac{(3-4\nu)(z+D)^2-2zD}{r_2^3}\right.$$
$$\left.+\frac{6zD(z+D)^2}{r_2^5}\right]$$

$$\sigma_x=\frac{P}{8\pi(1-\nu)}\left[\frac{(1-2\nu)(z-D)}{r_1^3}-\frac{3x^2(z-D)}{r_1^5}+\frac{(1-2\nu)\{3(z-D)-4\nu(z+D)\}}{r_2^3}\right.$$
$$-\frac{3(3-4\nu)x^2(z-D)-6D(z+D)\{(1-2\nu)z-2\nu D\}}{r_2^5}-\frac{30x^2zD(z+D)}{r_2^7}$$
$$\left.-\frac{4(1-\nu)(1-2\nu)}{r_2(r_2+z+D)}\left(1-\frac{x^2}{r_2(r_2+z+D)}-\frac{x^2}{r_2^2}\right)\right]$$

$$\sigma_y = \frac{P}{8\pi(1-\nu)}\left[\frac{(1-2\nu)(z-D)}{r_1^3} - \frac{3y^2(z-D)}{r_1^5} + \frac{(1-2\nu)\{3(z-D)-4\nu(z+D)\}}{r_2^3} - \frac{3(3-4\nu)y^2(z-D)-6D(z+D)\{(1-2\nu)z-2\nu D\}}{r_2^5} - \frac{30y^2 zD(z+D)}{r_2^7} - \frac{4(1-\nu)(1-2\nu)}{r_2(r_2+z+D)}\left(1-\frac{y^2}{r_2(r_2+z+D)}-\frac{y^2}{r_2^2}\right)\right]$$

$$\sigma_z = \frac{P}{8\pi(1-\nu)}\left[-\frac{(1-2\nu)(z-D)}{r_1^3} - \frac{3(z-D)^3}{r_1^5} + \frac{(1-2\nu)(z-D)}{r_2^3} - \frac{3(3-4\nu)z(z+D)^2-3D(z+D)(5z-D)}{r_2^5} - \frac{30zD(z+D)^3}{r_2^7}\right]$$

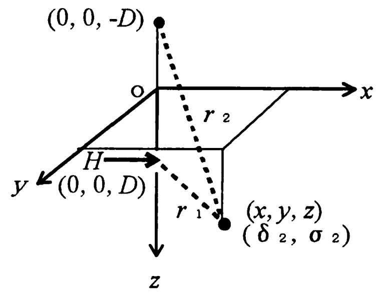
$$\tau_{xy} = \frac{Pxy}{8\pi(1-\nu)}\left[-\frac{3(z-D)}{r_1^5} - \frac{3(3-4\nu)(z-D)}{r_2^5} - \frac{30zD(z+D)}{r_2^7} + \frac{4(1-\nu)(1-2\nu)}{r_2^2(r_2+z+D)}\left(\frac{1}{r_2+z+D}+\frac{1}{r_2}\right)\right]$$

$$\tau_{yz} = \frac{Py}{8\pi(1-\nu)}\left[-\frac{1-2\nu}{r_1^3} - \frac{3(z-D)^2}{r_1^5} + \frac{1-2\nu}{r_2^3} - \frac{3(3-4\nu)z(z+D)-3D(3z+D)}{r_2^5} - \frac{30zD(z+D)^2}{r_2^7}\right]$$

$$\tau_{zx} = \frac{Px}{8\pi(1-\nu)}\left[-\frac{1-2\nu}{r_1^3} - \frac{3(z-D)^2}{r_1^5} + \frac{1-2\nu}{r_2^3} - \frac{3(3-4\nu)z(z+D)-3D(3z+D)}{r_2^5} - \frac{30zD(z+D)^2}{r_2^7}\right] \quad (4)$$

(2) The solution of Mindlin's problem Ⅱ

Taking the position of a horizontal force H and the coordinate system as **Fig.6**, the solutions of Mindlin's problem Ⅱ are given as follows:



$$r_1 = [x^2 + y^2 + (z-D)^2]^{1/2}$$
$$r_2 = [x^2 + y^2 + (z+D)^2]^{1/2}$$

Fig.6 Mindlin's problem Ⅱ

$$u=\frac{H}{16\pi(1-\nu)G}\left[\frac{3-4\nu}{r_1}+\frac{x^2}{r_1^3}+\frac{1}{r_2}+\frac{(3-4\nu)x^2}{r_2^3}+\frac{2zD}{r_2^3}\left(1-\frac{3x^2}{r_2^2}\right)+\frac{4(1-\nu)(1-2\nu)}{r_2+z+D}\left(1-\frac{x^2}{r_2(r_2+z+D)}\right)\right]$$

$$v=\frac{Hxy}{16\pi(1-\nu)G}\left[\frac{1}{r_1^3}+\frac{3-4\nu}{r_2^3}-\frac{6zD}{r_2^5}-\frac{4(1-\nu)(1-2\nu)}{r_2(r_2+z+D)^2}\right]$$

$$w=\frac{Hx}{16\pi(1-\nu)G}\left[\frac{z-D}{r_1^3}+\frac{(3-4\nu)(z-D)}{r_2^3}-\frac{6zD(z+D)}{r_2^5}+\frac{4(1-\nu)(1-2\nu)}{r_2(r_2+z+D)}\right]$$

$$\sigma_x=\frac{Hx}{8\pi(1-\nu)}\left[-\frac{1-2\nu}{r_1^3}-\frac{3x^2}{r_1^5}+\frac{(1-2\nu)(5-4\nu)}{r_2^3}-\frac{3(3-4\nu)x^2}{r_2^5}-\frac{4(1-\nu)(1-2\nu)}{r_2(r_2+z+D)^2}\left(3-\frac{x^2(3r_2+z+D)}{r_2^2(r_2+z+D)}\right)+\frac{6D}{r_2^5}\left(3D-(3-2\nu)(z+D)+\frac{5x^2z}{r_2^2}\right)\right]$$

$$\sigma_y=\frac{Hx}{8\pi(1-\nu)}\left[\frac{1-2\nu}{r_1^3}-\frac{3y^2}{r_1^5}+\frac{(1-2\nu)(3-4\nu)}{r_2^3}-\frac{3(3-4\nu)y^2}{r_2^5}-\frac{4(1-\nu)(1-2\nu)}{r_2(r_2+z+D)^2}\left(1-\frac{y^2(3r_2+z+D)}{r_2^2(r_2+z+D)}\right)+\frac{6D}{r_2^5}\left(D-(1-2\nu)(z+D)+\frac{5y^2z}{r_2^2}\right)\right]$$

$$\sigma_z=\frac{Hx}{8\pi(1-\nu)}\left[\frac{1-2\nu}{r_1^3}-\frac{3(z-D)^2}{r_1^5}-\frac{1-2\nu}{r_2^3}-\frac{3(3-4\nu)(z+D)^2}{r_2^5}+\frac{6D}{r_2^5}\left(D+(1-2\nu)(z+D)+\frac{5z(z+D)^2}{r_2^2}\right)\right]$$

$$\tau_{xy}=\frac{Hy}{8\pi(1-\nu)}\left[-\frac{1-2\nu}{r_1^3}-\frac{3x^2}{r_1^5}+\frac{1-2\nu}{r_2^3}-\frac{3(3-4\nu)x^2}{r_2^5}-\frac{4(1-\nu)(1-2\nu)}{r_2(r_2+z+D)^2}\left(1-\frac{x^2(3r_2+z+D)}{r_2^2(r_2+z+D)}\right)-\frac{6zD}{r_2^5}\left(1-\frac{5x^2}{r_2^2}\right)\right]$$

$$\tau_{yz}=\frac{3Hxy}{8\pi(1-\nu)}\left[-\frac{z-D}{r_1^5}-\frac{(3-4\nu)(z+D)}{r_2^5}+\frac{2D}{r_2^5}\left(1-2\nu+\frac{5z(z+D)}{r_2^2}\right)\right]$$

$$\tau_{zx}=\frac{H}{8\pi(1-\nu)}\left[-\frac{(1-2\nu)(z-D)}{r_1^3}-\frac{3x^2(z-D)}{r_1^5}+\frac{(1-2\nu)(z-D)}{r_2^3}-\frac{3(3-4\nu)x^2(z+D)}{r_2^5}-\frac{6D}{r_2^5}\left(z(z+D)-(1-2\nu)x^2-\frac{5x^2z(z+D)}{r_2^2}\right)\right] \tag{5}$$

3.3 Integral of Mindlin's solutions over a rectangular area

In order to derive solutions for a uniformly distributed load over a small rectangular area from Mindlin's solutions (4) and (5) mentioned above, we first take a coordinate system as **Fig.7**. Then, the integral calculation of Mindlin's solutions are executed by exchanging x by $x-x_0$ and y by $y-y_0$ in expressions (4) and (5) given in the coordinates of **Figs.5** and **6**. From the sake of this, the following formulas about replacement of variables can be utilized.

$$\int_{x_1}^{x_2} f(x - x_o, y - y_o, D)\, dx_o = -\int_{x-x_1}^{x-x_2} f(X, Y, D)\, dX \tag{6}$$

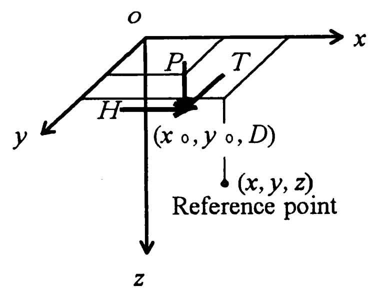


Fig.7 Mindlin's solution

As already mentioned in **Fig.3**, integrals required for the fundamental solutions of boundary point matching method are classified into 9 cases, as shown from ① to ⑨ in **Fig.3**. However, the solutions for a force along y-axis can easily be derived from the solutions for a force along x-axis exchanging corresponding coordinate axes. Therefore, only the five cases for a vertical force P and a horizontal force H along x-axis, namely cases ①, ②, ④, ⑤ and ⑧ in **Fig.3** are enough to describe here the results of integrals.

(1) Integral of the solution of Mindlin's problem I over a horizontal rectangular area

As shown in **Fig.8**, a reference point is placed on z-axis and one corner of a horizontal rectangular loaded area passes through z-axis. From the results of integral using this coordinate system, an integral for an arbitrarily located horizontal rectangular area can be obtained using a superposition principle given by **Fig.9**.

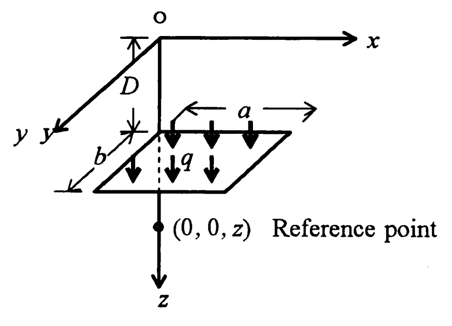


Fig.8 Integral to horizontal rectangular area of Mindlin's problem I

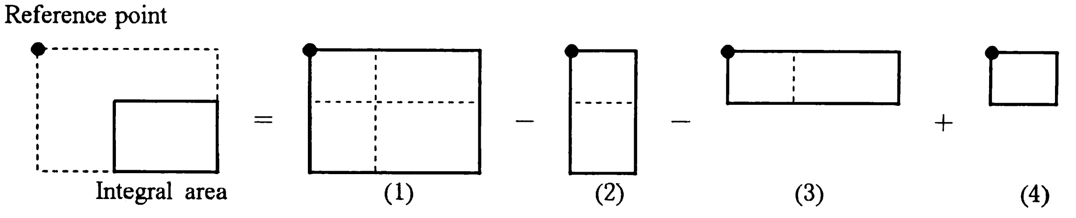


Fig.9 Superposition of loaded area

$$u=\frac{q}{16\pi(1-\nu)G}\Big[(z-D)I_{G1}+\{(3-4\nu)(z-D)+4(1-\nu)(1-2\nu)(z+D)\}I_{G2}+2bzD(z+D)I_{E13}$$
$$+4(1-\nu)(1-2\nu)(bI_{G7}+aI_{F6})\Big]$$

$$v=\frac{q}{16\pi(1-\nu)G}\Big[(z-D)I_{G3}+\{(3-4\nu)(z-D)+4(1-\nu)(1-2\nu)(z+D)\}I_{G4}+2azD(z+D)I_{E14}$$
$$+4(1-\nu)(1-2\nu)(aI_{G5}+bI_{F8})\Big]$$

$$w=\frac{q}{16\pi(1-\nu)G}\Big[(3-4\nu)(bI_{G5}+aI_{G7})+(5-12\nu+8\nu^2)(bI_{G6}+aI_{G8})+\frac{2abzD(A_2{}^2+B_2{}^2)}{A_2{}^2B_2{}^2C_2}$$
$$+\frac{z-D}{2}\left(-2(3-4\nu)I_{F1}+I_{F3}\right)+\frac{z+D}{2}\left(-2(5-12\nu+8\nu^2)I_{F2}+(3-4\nu)I_{F4}\right)\Big]$$

$$\sigma_x=\frac{q}{8\pi(1-\nu)}\Big[-2\nu I_{F1}-2(1-2\nu^2)I_{F2}+2(1-\nu)(1-2\nu)I_{F5}+\frac{ab(z-D)}{A_1{}^2C_1}$$
$$+\frac{ab}{A_2{}^2C_2}\left(3(z+D)-4\nu z-\frac{6D^2}{z+D}-\frac{2r^2zD}{C_2{}^2(z+D)}-\frac{4a^2zD}{A_2{}^2(z+D)}-\frac{4\nu A_2{}^2D}{B_2{}^2}\right)\Big]$$

$$\sigma_y=\frac{q}{8\pi(1-\nu)}\Big[-2\nu I_{F1}-2(1-2\nu^2)I_{F2}-2(1-\nu)(1-2\nu)I_{F5}+\frac{ab(z-D)}{B_1{}^2C_1}$$
$$+\frac{ab}{B_2{}^2C_2}\left(3(z+D)-4\nu z-\frac{6D^2}{z+D}-\frac{2r^2zD}{C_2{}^2(z+D)}-\frac{4b^2zD}{B_2{}^2(z+D)}-\frac{4\nu B_2{}^2D}{A_2{}^2}\right)\Big]$$

$$\sigma_z=\frac{q}{8\pi(1-\nu)}\Big[-2(1-\nu)(I_{F1}+I_{F2})-\frac{ab(z-D)I_{E10}}{C_1}-\frac{ab}{C_2(z+D)}\left(\frac{r^2zD}{3C_2{}^2}+(3-4\nu)z^2\right.$$
$$\left.+2(5-2\nu)zD+D^2\right)I_{E11}+\frac{4abzDI_{E12}}{C_2(z+D)}\Big]$$

$$\tau_{xy}=\frac{q}{8\pi(1-\nu)}\Big[4(1-\nu)(1-2\nu)I_{G10}-(z-D)I_{E1}+(3-4\nu)(z-D)I_{E2}-2zD(z+D)I_{E3}$$
$$+\frac{D}{z+D}\left(5-8\nu-\frac{z-D}{z+D}\right)-\begin{pmatrix}4(1-\nu), & (\text{for } z>D)\\ 2(1-2\nu), & (\text{for } z<D)\end{pmatrix}\Big]$$

$$\tau_{yz}=\frac{q}{8\pi(1-\nu)}\Big[-(1-2\nu)(I_{G3}-I_{G4})+\frac{a}{A_1}-\frac{a(z-D)^2}{B_1{}^2C_1}+\frac{2azD}{A_2{}^3}+\frac{2azD}{B_2{}^2C_2}\left(\frac{2b^2}{B_2{}^2}-\frac{(z-D)^2}{C_2{}^2}\right)$$
$$+a\{(3-4\nu)z+D\}(z+D)\Big]$$

$$\tau_{zx}=\frac{q}{8\pi(1-\nu)}\Big[-(1-2\nu)(I_{G1}-I_{G2})+\frac{b}{B_1}-\frac{b(z-D)^2}{A_1{}^2C_1}+\frac{2bzD}{B_2{}^3}+\frac{2bzD}{A_2{}^2C_2}\left(\frac{2a^2}{A_2{}^2}-\frac{(z-D)^2}{C_2{}^2}\right)$$

$$+ b\{(3-4\nu)z + D\}(z + D) \Bigg] \tag{7}$$

(2) Integral of the solution of Mindlin's problem Ⅱ (in case of a force acting along x-axis)

The case of an uniformly distributed force along x-axis over a horizontal rectangular area as shown in **Fig.10** is dealt with here. Procedure of integral calculation is the same to the before mentioned case. The results obtained are as follows:

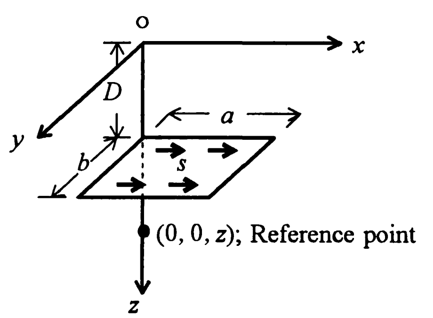


Fig.10 Integral to horizontal rectangular area of Mindlin's problem Ⅱ

$$u = \frac{s}{16\pi(1-\nu)G}\Bigg[-4(1-\nu)\{(z-D)I_{F1} + (z+D)I_{F2} + (1-2\nu)(z+D)I_{F6}\}$$

$$+4(1-\nu)b(I_{G5} + I_{G6}) + (3-4\nu)aI_{G7} + (5-12\nu+8\nu^2)aI_{G8} + \frac{2abzD}{A_2^{\,2}C_2}\Bigg]$$

$$v = \frac{s}{16\pi(1-\nu)G}\Bigg[-C_1 + B_1 + A_1 - |z-D| + (1-8\nu+8\nu^2)(C_2 - B_2 - A_2 + z + D)$$

$$+4(1-\nu)(1-2\nu)(z+D)I_{G10} - 2zD\left(I_{E2} + \frac{1}{z+D}\right)\Bigg]$$

$$w = \frac{s}{16\pi(1-\nu)G}\Bigg[(z-D)I_{G1} + \{(3-4\nu)(z-D) - 4(1-\nu)(1-2\nu)(z+D)\}I_{G2}$$

$$-4(1-\nu)(1-2\nu)(bI_{G9} + aI_{G2}) + 2bzD(z+D)I_{E13}\Bigg]$$

$$\sigma_x = \frac{s}{8\pi(1-\nu)}\Bigg[-(3-2\nu)I_{G1} - (5-6\nu)I_{G2} + 2bD\{z - 2\nu(z+D)\} + 4(1-\nu)(1-2\nu)bI_{E15}$$

$$-a^2 b\left(\frac{1}{A_1^{\,2}C_1} + \frac{3-4\nu}{A_2^{\,2}C_2} + \frac{2zD}{A_2^{\,2}C_2}\left(\frac{2}{A_2^{\,2}} + \frac{1}{C_2^{\,2}}\right)\right)\Bigg]$$

$$\sigma_y = \frac{s}{8\pi(1-\nu)}\Bigg[-2\nu\{I_{G1} + (3-4\nu)I_{G2}\} - 4\nu\, bD(z+D) - 4(1-\nu)(1-2\nu)bI_{E15}$$

$$+ b\{I_{E4}+(3-4\nu)-2zDI_{E6}\}\Bigg]$$

$$\sigma_z=\frac{s}{8\pi(1-\nu)}\left[(1-2\nu)(I_{G1}-I_{G2})+b\{3z^2-D^2-4\nu z(z+D)\}I_{E13}-\frac{b(z-D)^2}{A_1^2C_1}+\frac{b}{B_1}\right.$$
$$\left.-2bzD\left(\frac{2a^2}{A_2^4C_2}-\frac{(z+D)^2}{A_2^2C_2^3}+\frac{1}{B_2^3}\right)\right]$$

$$\tau_{xy}=\frac{s}{8\pi(1-\nu)}\left[(1-\nu)I_{G11}-a\{4(1-\nu)(1-2\nu)I_{E15}-I_{E7}-(3-4\nu)I_{E8}+2zDI_{E9}\}\right]$$

$$\tau_{yz}=\frac{s}{8\pi(1-\nu)}\left[(z-D)\left(I_{E1}+\frac{1}{|z-D|}\right)+\{2(1-2\nu)z+z+D\}\left(I_{E2}+\frac{1}{z+D}\right)\right.$$
$$\left.+2zD(z-D)\left(I_{E3}+\frac{1}{(z+D)^3}\right)\right]$$

$$\tau_{zx}=\frac{s}{8\pi(1-\nu)}\left[-(1-\nu)(I_{F3}+I_{F4})+\frac{ab(z-D)}{A_1^2C_1}\right.$$
$$\left.+\frac{abzD(z+D)}{A_2^2C_2}\left(\frac{3-4\nu}{zD}-\frac{2(1-2\nu)}{z(z+D)}-\frac{6}{A_2^2}+\frac{2b^2}{A_2^2C_2^2}\right)\right] \tag{8}$$

In which, the following notations are used in the above expressions (7) and (8):

$$A_1=[a^2+(z-D)^2]^{1/2},\quad A_2=[a^2+(z+D)^2]^{1/2},\quad B_1=[b^2+(z-D)^2]^{1/2},\quad B_2=[b^2+(z+D)^2]^{1/2},$$
$$C_1=[a^2+b^2+(z-D)^2]^{1/2},\quad C_2=[a^2+b^2+(z+D)^2]^{1/2},\quad r=[a^2+b^2]^{1/2} \tag{9}$$

$$I_{E1}=\frac{1}{C_1}-\frac{1}{B_1}-\frac{1}{A_1},\quad I_{E2}=\frac{1}{C_2}-\frac{1}{B_2}-\frac{1}{A_2},\quad I_{E3}=\frac{1}{C_2^3}-\frac{1}{B_2^3}-\frac{1}{A_2^3},$$

$$I_{E4}=\frac{1}{C_1}-\frac{1}{B_1},\quad I_{E5}=\frac{1}{C_2}-\frac{1}{B_2},\quad I_{E6}=\frac{1}{C_2^3}-\frac{1}{B_2^3},\quad I_{E7}=\frac{1}{C_1}-\frac{1}{A_1},$$

$$I_{E8}=\frac{1}{C_2}-\frac{1}{A_2},\quad I_{E9}=\frac{1}{C_2^3}-\frac{1}{A_2^3},\quad I_{E10}=\frac{1}{A_1^2}+\frac{1}{B_1^2},\quad I_{E11}=\frac{1}{A_2^2}+\frac{1}{B_2^2},$$

$$I_{E12}=\frac{a^2}{A_2^4}+\frac{b^2}{B_2^4},\quad I_{E13}=\frac{1}{B_2(z+D)^2}-\frac{1}{A_2^2C_2},\quad I_{E14}=\frac{1}{A_2(z+D)^2}-\frac{1}{B_2^2C_2},$$

$$I_{E15}=\frac{1}{C_2+z+D}-\frac{1}{B_2+z+D} \tag{10}$$

$$I_{F1} = \pm \sin^{-1}\left(\frac{ab}{A_1 B_1}\right) \text{ (in sign } \pm, \text{ + for } z > D; - \text{ for } z < D), \quad I_{F2} = \sin^{-1}\left(\frac{ab}{A_2 B_2}\right),$$

$$I_{F3} = \pm \left[\sin^{-1}\left(\frac{A_1^2 + aC_1}{A_1(C_1 + a)}\right) - \sin^{-1}\left(\frac{A_1^2 - aC_1}{A_1(C_1 - a)}\right)\right] \text{ (in sign } \pm, \text{ + for } z > D; - \text{ for } z < D),$$

$$I_{F4} = \sin^{-1}\left(\frac{A_2^2 + aC_2}{A_2(C_2 + a)}\right) - \sin^{-1}\left(\frac{A_2^2 - aC_2}{A_2(C_2 - a)}\right), \quad I_{F5} = \sin^{-1}\left(\frac{ab(a^2 - b^2)}{A_2 B_2\{C_2^2 + (2C_2 + z + D)(z + D)\}}\right),$$

$$I_{F6} = \sin^{-1}\left(\frac{b^2 - C_2(C_2 + z + D)}{(C_2 + z + D)A_2}\right) + \frac{\pi}{2}, \quad I_{F7} = \sin^{-1}\left(\frac{a^2 b^2 - C_2^2(z + D)}{A_2^2 B_2^2}\right) + \frac{\pi}{2} \tag{11}$$

$$I_{G1} = \log\frac{|z - D|(C_1 + b)}{A_1(B_1 + b)}, \quad I_{G2} = \log\frac{(z + D)(C_2 + b)}{A_2(B_2 + b)}, \quad I_{G3} = \log\frac{|z - D|(C_1 + a)}{B_1(A_1 + a)},$$

$$I_{G4} = \log\frac{(z + D)(C_2 + a)}{B_2(A_2 + a)}, \quad I_{G5} = \log\frac{C_1 + a}{B_1}, \quad I_{G6} = \log\frac{C_2 + a}{B_2}, \quad I_{G7} = \log\frac{C_1 + b}{A_1},$$

$$I_{G8} = \log\frac{C_2 + b}{A_2}, \quad I_{G9} = \log\frac{C_2 + z + D}{B_2 + z + D}, \quad I_{G10} = \log\frac{(A_2 + z + D)(B_2 + z + D)}{2(z + D)(C_2 + z + D)},$$

$$I_{G11} = \log\frac{(A_1 + a)(A_2 + a)(C_1 - a)(C_2 - a)}{(A_1 - a)(A_2 - a)(C_1 + a)(C_2 + a)} \tag{12}$$

(3) Integral of the solutions of Mindlin's problem I over a vertical rectangular area

In this case, the coordinate system shown in **Fig.11** is used, in which one side of a rectangular loaded area in *x-z* plane coincides with *z*-axis and a reference point is placed in *y-z* plane. The expressions (13) shown in the following are given in a form of indefinite integral along *z*-axis, namely in a form of $f(D)$. Therefore, the required expressions for the case of **Fig.11** must be calculated by $f(D_2) - f(D_1)$. In addition to that, since a reference point is placed in *y-z* plane in these expressions, the expressions for an arbitrary reference points should be obtained using a superposition principle of **Fig.9**.

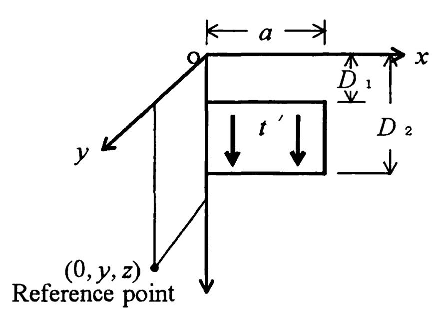


Fig.11 Integral of Mindlin's problem I over a vertical rectangular area

$$u=\frac{t'}{16\pi G(1-\nu)}\Big[B_1-C_1+\{(3-4\nu)+4(1-\nu)(1-2\nu)\}(B_2-C_2)-2zDI_{E1}$$

$$+\{8(1-\nu)z+4(1-\nu)(1-2\nu)(z+D)\}I_{G6}\Big]$$

$$v=\frac{t'}{16\pi G(1-\nu)}\Big[\frac{2ayzD}{B_2{}^2C_2}-yI_{G1}-\{(3-4\nu)+4(1-\nu)(1-2\nu)\}yI_{G2}$$

$$-4(1-\nu)(1-2\nu)\{zI_{F6}-(z+D)I_{F7}\}\Big]$$

$$w=\frac{t'}{16\pi G(1-\nu)}\Big[\frac{2azD(z+D)}{C_2B_2{}^2}+(3-4\nu)(z-D)I_{G1}+\{3z-(5-12\nu+8\nu^2)(z+D)\}I_{G2}$$

$$+4(1-\nu)a\{I_{G3}-2(1-\nu)I_{G4}\}-\frac{y}{2}\left((7-8\nu)I_{F1}+I_{F3}+\frac{2z^2I_{F6}}{y^2}\right)$$

$$+\frac{y}{2}\left(2(5-12\nu+8\nu^2)I_{F2}+\left(3-4\nu-\frac{2z^2}{y^2}\right)(I_{F2}-I_{F4})\right)\Big]$$

$$\sigma_x=\frac{t'}{8\pi(1-\nu)}\Big[-\frac{2a^3zD}{B_2{}^2C_2{}^3}-\frac{4(1-\nu)(1-2\nu)a}{C_2+z+D}-\frac{a}{C_2}\left(3-4\nu-8(1-\nu)\frac{z}{z+D}\right)$$

$$+\frac{2aD\{(1-2\nu)(z+D)-D\}}{B_2{}^2C_2}-\frac{8(1-\nu)azC_2}{(z+D)r^2}-(1-\nu)I_{F5}-(1-3\nu+4\nu^2)I_{F6}$$

$$-4(1-\nu)(1+2\nu)I_{F7}\pm\left(\frac{a}{r}-\frac{a}{C_1}\right)\ \text{(in sign }\pm,\ +\text{ for } z>D;\ -\text{ for } z<D)\Big]$$

$$\sigma_y=\frac{t'}{8\pi(1-\nu)}\Big[\frac{ay^2}{B_1{}^2C_1}+\frac{ay^2}{B_2{}^2C_2}\left(13-4\nu-\frac{2(9-4\nu)z+10D}{z+D}\right)+\frac{2ay^2D\{z-2\nu(z+D)\}}{B_2{}^2C_2}$$

$$+\frac{8(1-\nu)azC_2}{r^2(z+D)}-\frac{2ay^2zD(3C_2{}^2-a^2)}{B_2{}^4C_2{}^3}+\frac{4(1-\nu)(1-2\nu)a}{C_2+z+D}-\nu I_{F5}$$

$$-(1-4\nu)I_{F6}\Big]$$

$$\sigma_z=\frac{t'}{8\pi(1-\nu)}\Big[\frac{a(z-D)^2}{B_1{}^2C_1}+\frac{a\{5(z+D)^2-6D^2+4z^2-2(7-2\nu)(z+D)z\}}{B_2{}^2C_2}$$

$$-\frac{2a(z+D)^2zD(3C_2{}^2-a^2)}{B_2{}^4C_2{}^3}-\nu(I_{F5}-I_{F6})\Big]$$

$$\tau_{xy}=\frac{t'y}{8\pi(1-\nu)}\Big[-I_{E1}+\frac{(5-4\nu)z-(3-4\nu)DI_{E2}}{z+D}+2zDI_{E3}-4(1-\nu)(1-2\nu)I_{E8}$$

$$-\frac{8(1-\nu)z}{z+D}\left(\frac{C_2}{r^2}-\frac{B_2}{y^2}\right)\Big]$$

$$\tau_{yz}=\frac{t'}{8\pi(1-\nu)}\left[(1-\nu)(I_{G1}-I_{G2})+\frac{(z-D)}{B_1{}^2C_1}-\frac{ay\{2(3-2\nu)z-(z+D)\}}{B_2{}^2C_2}\right.$$
$$\left.-\frac{2a(z+D)zD(3C_2{}^2-a^2)}{B_2{}^4C_2{}^3}\right]$$

$$\tau_{zx}=\frac{t'}{8\pi(1-\nu)}\left[-(1-\nu)I_{G4}-(z-D)I_{E1}+\{(5-6\nu)z-D\}I_{E2}+2(z+D)zDI_{E3}\right] \tag{13}$$

(4) Integral of the solutions of Mindlin's problem II over a vertical rectangular area (in case of forces acting in *x-z* plane)

In this case, **Fig.12** is used as a coordinate system. Procedure of integral to obtain the required expressions is the same to previous case (3). Expressions corresponding to (13) are shown as follows.

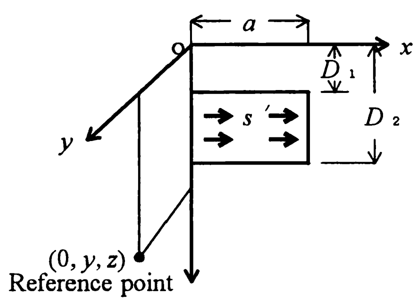


Fig.12 Integral of Mindlin's problem II over a vertical rectangular area

$$u=\frac{s'}{16\pi G(1-\nu)}\left[\frac{2az}{C_2}\left(1+\frac{z(z+D)}{r^2}\right)+\frac{(1-\nu)(1-2\nu)r^2}{(C_2+z+D)^2}+4(1-\nu)\{(z-D)I_{G1}-(z+D)I_{G2}\}\right.$$
$$\left.+a\{(3-4\nu)I_{G3}-(3-6\nu+4\nu^2)I_{G4}\}-4(1-\nu)y(I_{F1}-I_{F2})\right]$$

$$v=\frac{s'y}{16\pi G(1-\nu)}\left[2zI_{E2}+2z^2(z+D)I_{E5}+(1-\nu)(1-2\nu)\left(\frac{r^2}{(C_2+z+D)^2}-\frac{y^2}{(B_2+z+D)^2}\right)\right.$$
$$\left.-I_{G5}+(1+2\nu-4\nu^2)I_{G6}\right]$$

$$w=\frac{s'}{16\pi G(1-\nu)}\left[B_1-C_1+\{3-4\nu-4(1-\nu)(1-2\nu)\}(B_2-C_2)+2zDI_{E2}\right.$$
$$\left.+4(1-2\nu)\{z-(1-\nu)(z+D)\}I_{G6}\right]$$

$$\sigma_x=\frac{s'}{8\pi(1-\nu)}\left[-(1-3\nu)+2(1-\nu)(1-2\nu)\left(\frac{C_2(C_2+z+D)-x^2}{(C_2+z+D)^2}-\frac{B_2}{B_2+z+D}\right)\right.$$

$$+\frac{(z-D)x^2}{r^2C_1}-2x^2z\left(\frac{D}{C_2^3(z+D)}+\frac{zC_2}{r^4(z+D)}\right)-\{(3D+(1-4\nu)z\}I_{E2}$$

$$\left.+(3-4\nu)y^2(z+D)I_{E5}+2y^2z^2(z+D)I_{E6}+(3-2\nu)I_{G5}-(1-2\nu)(3+2\nu)I_{G6}\right]$$

$$\sigma_y=\frac{s'}{8\pi(1-\nu)}\left[\left[\begin{cases}(2-3\nu),\ (\text{for } z>D)\\(-3\nu),\ (\text{for } z<D)\end{cases}-2(1-\nu)(1-2\nu)\left(\frac{C_2(C_2+z+D)-x^2}{(C_2+z+D)^2}-\frac{B_2}{B_2+z+D}\right)\right.\right.$$

$$-\frac{4\nu x^2D}{r^2C_1}+2zI_{E2}+z^2(z+D)I_{E5}-y^2\{3(z+D)-4\nu z\}I_{E5}+y^2(z+D)I_{E4}$$

$$\left.-2y^2z\{I_{E3}+z(z+D)(I_{E7}+2I_{E6})\}-(1-2\nu)I_{G5}+(1+4\nu^2)I_{G6}\right]$$

$$\sigma_z=\frac{s'}{8\pi(1-\nu)}\left[-(z-D)I_{E1}-\{z+D-2(1-2\nu)z\}I_{E2}+2zD(z+D)I_{E3}+2\nu(I_{G5}+I_{G6})\right]$$

$$\tau_{xy}=\frac{s'}{8\pi(1-\nu)}\left[-(1-\nu)I_{F5}-\frac{2ay(1-\nu)(1-2\nu)}{(C_2+z+D)^2}+\frac{ay}{r^2}\left(\frac{z-D}{C_1}-\left(3-4\nu+\frac{2z^2}{r^2}\right)\frac{z+D}{C_2}\right)\right.$$

$$\left.-\frac{2ay}{z+D}\left(\frac{z^2C_2}{r^4}+\frac{zD}{C_2^3}\right)\right]$$

$$\tau_{yz}=\frac{s'y}{8\pi(1-\nu)}\left[-I_{E1}+I_{E2}+4\nu z(z+D)I_{E5}+2zDI_{E3}\right]$$

$$\tau_{zx}=\frac{s'}{8\pi(1-\nu)}\left[2(1-\nu)(I_{G1}-I_{G2})-\frac{2azD}{B_2^2C_2}\left(1-\frac{a^2}{C_2^2}\right)-\frac{a}{C_1}+\frac{a}{C_2}\left(1+\frac{4\nu z(z+D)}{r^2}\right)\right] \quad (14)$$

In which, the following notations are used in the above expressions (13) and (14):

$$B_1=[y^2+(z-D)^2]^{1/2}\ ,\quad B_2=[y^2+(z+D)^2]^{1/2}\ ,\quad C_1=[a^2+y^2+(z-D)^2]^{1/2}\ ,$$

$$C_2=[a^2+y^2+(z+D)^2]^{1/2}\ ,\quad r=[a^2+y^2]^{1/2} \quad (15)$$

$$I_{E1}=\frac{1}{C_1}-\frac{1}{B_1}\ ,\quad I_{E2}=\frac{1}{C_2}-\frac{1}{B_2}\ ,\quad I_{E3}=\frac{1}{C_2^3}-\frac{1}{B_2^3}\ ,\quad I_{E4}=\frac{1}{C_1r^2}-\frac{1}{B_1y^2}\ ,$$

$$I_{E5}=\frac{1}{C_2r^2}-\frac{1}{B_2y^2}\ ,\quad I_{E6}=\frac{1}{C_2r^4}-\frac{1}{B_2y^4}\ ,\quad I_{E7}=\frac{1}{C_2^3r^2}-\frac{1}{B_2^3y^2}\ ,$$

$$I_{E8}=\frac{1}{C_2+z+D}-\frac{1}{B_2+z+D} \quad (16)$$

$$I_{F1}=\pm\left[\sin^{-1}\left(\frac{aC_1+r^2}{r(C_1+a)}\right)-\sin^{-1}\left(\frac{y}{B_1}\right)\right]\quad \text{(in sign } \pm,\ + \text{ for } z>D;\ - \text{ for } z<D\text{)},$$

$$I_{F2}=\sin^{-1}\left(\frac{aC_2+r^2}{r(C_2+a)}\right)-\sin^{-1}\left(\frac{y}{B_2}\right),$$

$$I_{F3}=\pm\left[\sin^{-1}\left(\frac{aC_1-r^2}{r(C_1-a)}\right)+\sin^{-1}\left(\frac{y}{B_1}\right)\right]\quad \text{(in sign } \pm,\ + \text{ for } z>D;\ - \text{ for } z<D\text{)},$$

$$I_{F4}=\sin^{-1}\left(\frac{aC_2-r^2}{r(C_2-a)}\right)+\sin^{-1}\left(\frac{y}{B_2}\right),$$

$$I_{F5}=\pm\left[\sin^{-1}\left(\frac{a^2(z-D)^2-y^2C_1^2}{r^2B_1^2}\right)+\frac{\pi}{2}\right]\quad \text{(in sign } \pm,\ + \text{ for } z>D;\ - \text{ for } z<D\text{)},$$

$$I_{F6}=\sin^{-1}\left(\frac{a^2(z+D)^2-y^2C_2^2}{r^2B_2^2}\right)+\frac{\pi}{2},\qquad I_{F7}=\sin^{-1}\left(\frac{a^2-C_2(C_2+z+D)}{(C_2+z+D)B_2}\right)+\frac{\pi}{2}\tag{17}$$

$$I_{G1}=\log\frac{C_1+a}{B_1},\qquad I_{G2}=\log\frac{C_2+a}{B_2},$$

$$I_{G3}=\pm\log\frac{C_1+|z-D|}{r}\quad \text{(in sign } \pm,\ + \text{ for } z>D;\ - \text{ for } z<D\text{)},\qquad I_{G4}=\log\frac{C_2+z+D}{r},$$

$$I_{G5}=\pm\log\frac{(C_1+|z-D|)y}{(B_1+z-D)r}\quad \text{(in sign } \pm,\ + \text{ for } z>D;\ - \text{ for } z<D\text{)},$$

$$I_{G6}=\log\frac{C_2+z+D}{B_2+z+D}\tag{18}$$

(5) Integral of the solutions of Mindlin's problem II over a vertical rectangular area (in case of forces acting in y-z plane)

In this case, the coordinate system as shown in **Fig.13** is used. Procedure of integral calculation is the same to previous case. Expressions corresponding to (14) are shown as follows.

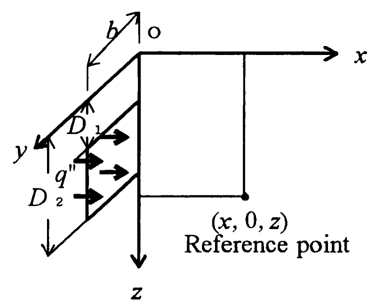


Fig.13 Integral of Mindlin's problem II over a vertical rectangular area

$$u=\frac{q'}{16\pi G(1-\nu)}\left[-2b\left(\frac{C_2 z^2}{(z+D)r^2}-\frac{(1-\nu)(1-2\nu)(z+D)}{C_2+z+D}\right)-\frac{2x^2 bzD}{A_2^2 C_2(z+D)}+(3-4\nu)xI_{F1}\right.$$

$$-\{1+4(1-\nu)(1-2\nu)\}xI_{F2}+\frac{x\{I_{F3}-(3-4\nu)I_{F4}\}}{2}$$

$$\left.-\{1+4(1-\nu)(1-2\nu)\}\{bI_{G4}+(z+D)I_{G6}\}+(3-4\nu)(z-D)I_{G1}+2zI_{G2}\right]$$

$$v=\frac{q'x}{16\pi G(1-\nu)}\left[2zI_{E2}+2z^2(z+D)I_{E4}+(1-\nu)(1-2\nu)\left(\frac{r^2}{(C_2+z+D)^2}-\frac{x^2}{(A_2+z+D)^2}\right)\right.$$

$$\left.-I_{G5}+(1+2\nu-4\nu^2)I_{G6}\right]$$

$$w=\frac{q'}{16\pi G(1-\nu)}\left[-\frac{2bxzD}{A_2^2 C_2}-xI_{G1}-\{(3-4\nu)-4(1-\nu)(1-2\nu)\}xI_{G2}\right.$$

$$\left.-2(1-2\nu)\{zI_{F4}-2(1-\nu)(z+D)I_{F2}\}\right]$$

$$\sigma_x=\frac{q'}{8\pi(1-\nu)}\left[-bx\left(\frac{(z-D)(C_1^2+x^2)}{r^2 C_1 A_1^2}-\frac{(3-4\nu)(z+D)(C_2^2+x^2)}{r^2 C_2 A_2^2}-\frac{6zD(z+D)}{A_2^4 C_2}\right.\right.$$

$$+\frac{2x^2 b^2 Dz}{A_2^4 C_2^3(z+D)}-\frac{4\nu D}{A_2^2 C_2}-\frac{2x^2 z^2}{A_2^2 C_2(z+D)^3}+\frac{2z^2 C_2}{r^2(z+D)^3}-\frac{4z^2 C_2}{r^4(z+D)}$$

$$\left.\left.+\frac{2(1-\nu)(1-2\nu)}{(C_2+z+D)^2}\right)-(1-\nu)I_{F3}-(1-3\nu+4\nu^2)I_{F4}-4(1-\nu)(1-2\nu)I_{F2}\right]$$

$$\sigma_y=\frac{q'}{8\pi(1-\nu)}\left[bx\left(\frac{z-D}{r^2 C_1}-\frac{(3-4\nu)(z+D)}{r^2 C_2}+\frac{2\nu D}{A_2^2 C_2}-\frac{zD}{C_2^3(z+D)}-\frac{z^2 C_2}{r^4(z+D)}\right.\right.$$

$$\left.\left.-\frac{z^2(z+D)}{r^4 C_2}-\frac{2(1-\nu)(1-2\nu)}{(C_2+z+D)^2}\right)-\nu\{I_{F3}-(1-4\nu)I_{F4}\}\right]$$

$$\sigma_z=\frac{q'}{8\pi(1-\nu)}\left[\frac{bx(z-D)}{A_1^2 C_1}+\frac{bx}{A_2^2 C_2}\left(z+D+4(1+\nu)z-\frac{6x^2 z}{A_2^2}-\frac{6z^2(z+D)}{A_2^2}-\frac{2bzD(z+D)}{A_2^2 C_2^2}\right)\right.$$

$$\left.-\nu(I_{F3}+I_{F4})\right]$$

$$\tau_{xy}=\frac{q'y}{8\pi(1-\nu)}\left[\frac{b^2(z-D)}{r^2 C_1}-\frac{(3-4\nu)b^2(z+D)}{r^2 C_2}+2zI_{E2}-2x^2 zI_{E3}-(1-2\nu)\{I_{G5}+(1-2\nu)I_{G6}\}\right.$$

$$+2z^2(z+D)\left(\frac{1}{r^2 C_2}-\frac{x^2}{r^2 C_2^3}-\frac{2x^2}{r^4 C_2}+\frac{1}{A_2^3}+\frac{1}{x^2 A_2}\right)$$

$$+2(1-\nu)(1-2\nu)\left(\frac{b^2}{(C_2+z+D)^2}-\frac{C_2}{C_2+z+D}+\frac{A_2}{A_2+z+D}\right)\Bigg]$$

$$\tau_{yz}=\frac{q'x}{8\pi(1-\nu)}\Big[-I_{E1}+I_{E2}+2zDI_{E3}+4\nu z(z+D)I_{E4}\Big]$$

$$\tau_{zx}=\frac{q'}{8\pi(1-\nu)}\Bigg[\frac{bx^2}{A_1^2C_1}-\frac{bx^2}{A_2^2C_2}-\frac{4\nu bz}{z+D}\left(\frac{C_2}{r^2}-\frac{x^2}{A_2^2C_2}\right)-\frac{2bzD}{A_2^2C_2}\left(1-\frac{3x^2}{A_2^2}-\frac{b^2x^2}{A_2^2C_2^2}\right)+(1-2\nu)(I_{G1}-I_{G2})\Bigg] \quad (19)$$

In which, the following notations are used in the above expression (19):

$$A_1=[x^2+(z-D)^2]^{1/2}, \quad A_2=[x^2+(z+D)^2]^{1/2}, \quad C_1=[x^2+b^2+(z-D)^2]^{1/2},$$

$$C_2=[x^2+b^2+(z+D)^2]^{1/2}, \quad r=[x^2+b^2]^{1/2} \quad (20)$$

$$I_{E1}=\frac{1}{C_1}-\frac{1}{A_1}, \quad I_{E2}=\frac{1}{C_2}-\frac{1}{A_2}, \quad I_{E3}=\frac{1}{C_2^3}-\frac{1}{A_2^3}, \quad I_{E4}=\frac{1}{C_2r^2}-\frac{1}{A_2x^2} \quad (21)$$

$$I_{F1}=\pm\left[\sin^{-1}\left(\frac{b^2-C_1(C_1+|z-D|)}{A_1(C_1+|z-D|)}\right)+\sin^{-1}\left(\frac{x}{r}\right)\right] \quad \text{(in sign }\pm,\ +\text{ for } z>D;\ -\text{ for } z<D),$$

$$I_{F2}=\sin^{-1}\left(\frac{b^2-C_2(C_2+z+D)}{A_2(C_2+z+D)}\right)+\frac{\pi}{2},$$

$$I_{F3}=\pm\left[\sin^{-1}\left(\frac{b^2(z-D)^2-C_1^2x^2}{r^2A_1^2}\right)+\frac{\pi}{2}\right] \quad \text{(in sign }\pm,\ +\text{ for } z>D;\ -\text{ for } z<D),$$

$$I_{F4}=\sin^{-1}\left(\frac{b^2(z+D)^2-C_2^2x^2}{r^2A_2^2}\right)+\frac{\pi}{2} \quad (22)$$

$$I_{G1}=\log\frac{C_1+b}{A_1}, \quad I_{G2}=\log\frac{C_2+b}{A_2},$$

$$I_{G3}=\pm\log\frac{C_1+|z-D|}{B_1} \quad \text{(in sign }\pm,\ +\text{ for } z>D;\ -\text{ for } z<D), \quad I_{G4}=\log\frac{C_2+z+D}{B_2},$$

$$I_{G5}=\pm\log\frac{(C_1+z-D)r}{(A_1+z-D)x} \quad \text{(in sign }\pm,\ +\text{ for } z>D;\ -\text{ for } z<D),$$

$$I_{G6}=\log\frac{C_2+z+D}{A_2+z+D} \quad (23)$$

4. Concluding Remarks

To avoid a singularity of a fundamental solution to be used in a boundary point matching method and to get more accurate solutions by the method for 3-D elastic analysis of solids or structures, analytical integrals of the original Mindlin's solutions over a small rectangular area within an infinite half space were executed. Since the resultant integral expressions may be rather complicated but analytically exact, they are useful as not only a fundamental solution of boundary point matching method but also an elementary solution available for a specific 3-D elastic problem in place of the original Mindlin's solutions.

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