A Simple Numerical Method for Biaxial Bending Moment-Curvature Relations of Reinforced Concrete Column Sections

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Synopsis

Proposed in this paper is a simple numerical method for computing the bending moment - curvature relations for reinforced concrete rectangular cross section of bridge piers under biaxial bending action, which is essentially required at their seismic design considering a horizontal force due to earthquake in arbitrary direction. The cross section is divided by a number of rectangular finite areas of concrete and reinforcing bar areas. And then numerical integration technique is developed to evaluate the stress-strain relation for each rectangular finite area divided. Numerical results are presented to show validity of the present method with satisfactory accuracy compared with those by the method recommended in the Japanese seismic design code [1].

Key Words: Reinforced concrete, Elasto-plasticity, Biaxial bending, M-ϕ relations, Numerical Method

1. Introduction

The current Japanese seismic design code for bridge structures [1] treats that the seismic force for bridge piers should be considered under horizontal actions individually parallel and perpendicular to the direction of bridge axis. It is, however, natural and rational to evaluate interactively those actions. The bending moment - curvature relation, i.e. $M$-$\phi$ relation for cross sections of bridge pier necessary in the seismic design is also shown to be evaluated numerically using the method of multi-layered model [1] that essentially requires a sufficient number of divided layers resulting in complicated computational effort.

In this paper, therefore, we propose a simple numerical method for computing $M$-$\phi$ relations for reinforced concrete rectangular cross section of bridge piers under a biaxial bending action. The cross section is divided by a number of rectangular finite areas of concrete and reinforcing bar areas. And then numerical integration technique is developed to evaluate the stress-strain relation for each rectangular finite area divided. Their validity is discussed based upon the numerical results obtained comparing with those by the method recommended in the code.

2. Outline of Existing Estimation Method

In the seismic design code [1], fifty-layered model is recommended to evaluate $M$-$\phi$ relations for reinforced concrete rectangular cross section of bridge piers under a horizontal action, namely uni-axial bending. In case of the action in arbitrary direction, for example biaxial bending, the code has not given an obvious evaluation method for $M$-$\phi$ relations.

Chen and co-workers [2], [3] have developed the portioning model for cross sections in biaxial bending as an extension of the layered model. They have obtained various kinds of the $M$-$\phi$ curves relating axial compression and biaxial bending moments for a standard concrete cross section specified by ACI code, however, their models are sophisticated for practical design work, so that more simplified methods are earnestly awaited.

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3. Simplified Estimation Method

3.1 Required accuracy

In case of uni-axial bending action, the $M-\phi$ relations obtained by the proposed method should not be exceed 10% deviation from those by the evaluation method recommended in the code [1] with respect to the following essential view points: initial yielding point of reinforcement, peak strength point and maximum deformation capacity. Furthermore, in case of biaxial bending action, the same accuracy as the former should also be guaranteed.

3.2 Numerical idealization

Divided layered number of fifty in the vertical direction as described in the code as shown in Fig. 1(a) is decreased to three as shown in Fig. 2, in which same subdivision in the horizontal direction is also shown. The area of cover concrete located outsides of reinforcements is ignored because the area becomes ineffective owing to cracking or peeling off even before the initial yielding point of reinforcement. The yield of reinforcement is the first-coming essential phase on the $M-\phi$ relations.

![Layered model for uni-axial bending](image1)

![Finite element model for biaxial bending](image2)

Fig. 1. Layered Model for Cross Section

![Model for Proposed Method](image3)

Fig. 2. Model for Proposed Method
3.3 Numerical integration technique

The stress-strain relations of cross section are evaluated numerically by dividing the cross section into the several rectangular finite elements shown in Fig. 2. Number of finite elements \( N_c \) considered is 9 (3 \( \times \) 3) for the section. In figure, the mark \( \bigcirc \) present the calculation points the \((x_0, y_0)\) for numerical integration that are located at the gravity center of each finite element, and those except central finite element are defined by

\[
(x, y) = \alpha \sqrt{\frac{A_t}{A}} (x_0, y_0)
\]  
(1)

where \( \alpha \) is coefficient to be taken as 0.8, which is valid in case of the ratio of cross sectional height to breadth being 1.0 to 2.0; \( A \) is the effective area surrounded by reinforcement arrangement for each finite element; \( A_0 \) is the whole area including cover concrete area for each finite element; and \((x_0, y_0)\) are the location at the gravity center of each finite element.

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**Fig. 3. Flow Chart for Numerical Calculation**

START

- Setting curvature: \( \phi = \phi + \Delta \phi \)

- Assumption of neutral axis

- Setting of coordinates of sampling points for numerical integration

- Calculation of strain at integration points by using Bernoulli's assumption, the strain are given by Eq. (2)

  \[
  \varepsilon_{ij} = \varepsilon_c \varepsilon_{ij} \frac{a}{a}
  \]

  \( \sigma_{ij} = \sigma_c (\varepsilon_{ij}) \), \( \sigma_{ij} = \sigma_c (\varepsilon_{ij}) \)

- Calculation of stress at sampling points by using the stress-strain relations, the stresses were given by Eq. (3)

  \[
  \sigma_{ij} = \sigma_c (E_{ij})
  \]

- Integration to calculate for normal force

  \[
  N = \sum_{i=1}^{n} \sum_{j=1}^{m} \sigma_{ij} \Delta A_j + \sum_{k=1}^{l} \sigma_{ik} \Delta A_k
  \]

- Judgement

  Integrated Normal Force:

  - External Normal Force

  - yes

  Integration to calculate for bending moment

  \[
  M_x = \sum_{i=1}^{n} \sum_{j=1}^{m} \sigma_{ij} y_i \Delta A_j + \sum_{k=1}^{l} \sigma_{ik} y_k \Delta A_k
  \]

  \[
  M_y = \sum_{i=1}^{n} \sum_{j=1}^{m} \sigma_{ij} x_i \Delta A_j + \sum_{k=1}^{l} \sigma_{ik} x_k \Delta A_k
  \]

END
3.4 Numerical procedure

The elasto-plastic behavior of RC cross section members is formulated on the basis of the following assumptions:

1. The strain distribution across the section is assumed to be linear, varying in proportion to the distance from the neutral axis, i.e., it is assumed that the plane section before bending remains the plane after the application of loads.
2. The strain and stress in are computed at the divided sectional areas. And in divided each areas, stresses are treated as constant.
3. Cover concrete area is considered as separated after the yield of reinforcement. And it is to be ignored for flexural rigidity, but to be considered to calculate the coordinate point \((x, y)\) for obtaining stress-strain relationship.

The flow chart for the present numerical procedure is shown in Fig. 3. In this flow, the strain of each integration point is calculated by using Bernoulli's assumption. In case of the elastic state (Fig. 4a), the coordinate is measured from the centroid of cross section. If considering the partially yielded cross section for the elasto-plastic state (Fig. 4b), the coordinate is measured from the neutral point.

\[ \varepsilon = \varepsilon_0 + \phi_x x - \phi_y y \]  
\[ \varepsilon_k = \varepsilon_0 + \phi_x x_k - \phi_y y_k \]

where \(\varepsilon_0\) is the axial strain; \(\phi_x\) and \(\phi_y\) are the curvatures with respect to \(x\)- and \(y\)-axes respectively; \((x, y)\) is the coordinate of the integration point of each finite element for concrete; and \((x_k, y_k)\) is the coordinate of each reinforcement.

\[ \varepsilon = \varepsilon_0 + \phi_y (x_k + x_n) - \phi_x (y_k + y_n) \]  
\[ \varepsilon_k = \varepsilon_0 + \phi_y (x_k + x_n) - \phi_x (y_k + y_n) \]

where \((x_n, y_n)\) is the coordinate of centroid of cross section for neutral point; and the other characters are the same to the case of elastic state one.

After calculation of strains, the stresses of concrete and reinforcement are calculated according to the Japanese seismic design code [1]. The stress-strain relationship of concrete shown in Fig. 5(a) is given by the
following expression.

\[
\sigma_c = E_c \varepsilon_c \left(1 - \frac{1}{n} \left(\frac{\varepsilon_c}{\varepsilon_{cc}}\right)^{n-1}\right)
\quad \text{when } 0 \leq \varepsilon_c \leq \varepsilon_{cc}
\]

\[
\sigma_c = \sigma_{cc} - E_{des} (\varepsilon_c - \varepsilon_{cc})
\quad \text{when } \varepsilon_{cc} \leq \varepsilon_c \leq \varepsilon_{cu}
\]  

(4a)

where

\[
n = \frac{E_c \varepsilon_{cc}}{E_c \varepsilon_{cc} - \sigma_{cc}}
\]  

(4b)

and where \(\sigma_c\) is the concrete stress; \(\sigma_{cc}\) is the confined concrete stress with lateral reinforcement; \(\varepsilon_c\) is the concrete strain; \(\varepsilon_{cc}\) is the strain at peak stress; \(\varepsilon_{cu}\) is the ultimate strain; \(E_c\) is the modulus of elasticity of concrete, and \(E_{des}\) is the falling inclination after the peak stress.

![Stress-Strain Relations](image)

(a) Concrete  
(b) Reinforcement

Fig. 5. Stress-Strain Relations

The stress-strain relationship of reinforcement shown in Fig. 5 (b) is given by

\[
\sigma_s = E_s \varepsilon_s \quad \text{when } \varepsilon \leq \varepsilon_{sy}
\]

(5a)

\[
\sigma_s = \sigma_{sy} \quad \text{when } \varepsilon > \varepsilon_{sy}
\]

(5b)

\[
\sigma_s = -\sigma_{sy} \quad \text{when } \varepsilon < -\varepsilon_{sy}
\]

(5c)

where \(\sigma_{sy}\) is the yield strength of reinforcement; \(\sigma_s\) is the stress of reinforcement; \(E_s\) is the modulus of elasticity of reinforcement; \(\varepsilon_{sy}\) is the yield strain of reinforcement, and \(\varepsilon_s\) is the strain of reinforcement.

4. Numerical Examples

Several numerical examples are shown to verify the present method. In the numerical computing, the width of the cross section is fixed as constant value, i.e. 2.0 m, and the height has been changed from 2.0 m to 4.0 m. The

![Definition of Positive Directions for Bending Moments and Curvatures](image)

Fig. 6. Definition of Positive Directions for Bending Moments and Curvatures
Fig. 7. Moment-Curvature Relations (h/b = 1.0)
Fig. 8. Moment-Curvature Relations \( h/b = 1.5 \)
Fig. 9. Moment-Curvature Relations (h/b = 2.0)
details of dimension of the cross sections are presented in Table 1. The cross section is first loaded axially by compression force $P$ that is equal to the normal stress $0.98 \text{ N/mm}^2$, and keeping the compression force $P$, and then the curvature $\phi_0$ is increased proportionally in magnitude from zero. Definition of positive directions for bending moments $M_x$ and $M_y$ and curvatures $\phi_x(=\phi_{0,0^\circ})$, $\phi_{0,45^\circ}$, and $\phi_y(=\phi_{0,90^\circ})$ are shown in Fig. 6.

The numerical results for moment-curvature relations subjected to the above mentioned conditions are presented in Figs. 7-9 compared with the evaluation method in the code [1]. In these figures, marked points denote the following events; ● yielding point of reinforcement, ▲ peak strength points, and ■ maximum deformation capacity respectively.

5. Concluding Remarks

The result of compared simplified estimation method with regulated method, the maximum error is occurred with 10% at ultimate deformation capacity in case $h/b=2.0$. These results show that the required accuracy is mostly satisfied irrespective of each bending direction.

References


<table>
<thead>
<tr>
<th>Height $h$ (m)</th>
<th>Width $b$ (m)</th>
<th>$h/b$</th>
<th>Reinforcement Ratio (%)</th>
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<td>1.0</td>
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<td>1.5</td>
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<td>2.0</td>
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<td>0.62</td>
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