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Generalized Laplacian and Balayage Theory

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§ 1. Introduction

1. 1. This paper has two central themes; the first is to establish a generalized notion of Laplacian operator and the second is the construction of balayage in a general situation by using a well-known theorem of Krein-Milman in the theory of linear topological spaces.

In an earlier period, so-called Laplacian has been considered as a linear operator from functions to functions, that is, f being a function of class C^p $(p \geqq 2)$, the Laplacian (in the classical type, which shall be denoted by $\dot{\Delta}$ hereafter) transforms f into a function of class C^{p-2}, $\dot{\Delta}f = \sum_{i=1}^n \frac{\partial^2 f}{\partial x_i^2}$.

Recently, this concept has been developed, notably in the following two directions: the one is appeared in the theory of Riemannian (or Euclidean) manifolds, in which the generalized operator of Laplace-Beltrami $\Delta_0 = (\delta + d)^2 = d\delta + d\delta$ transforms a form of class C^p, degree r, into another form of class C^{p-2} of the same degree; especially if we consider forms α, of degree 0, which are nothing but usual functions, we have

$$(\beta,\ \Delta_0\alpha) = \int \beta(\Delta_0\alpha)^* = \int \beta(-\dot{\Delta}\alpha)\, e_{1\cdots n}\, dx_1 \cdots dx_n\,,$$

where $e_{1\cdots n}$ means the component of Levi-Civita's tensor $(=\sqrt{g}$, g being the determinant $\| g_{ij} \|$ of the fundamental covariant tensors g_{ij}, under taking the coordinate system to be positive). Thus, setting $d(\Delta\alpha) = (-\dot{\Delta}\alpha)e_{1\ldots n}dx_1 \ldots dx_n$, we see that the application $\alpha \longrightarrow \Delta\alpha$ represents a linear operator from functions to measures and $(\beta, \Delta_0\alpha)$ becomes a usual integral $\int \beta d(\Delta\alpha)$, for which the potential related to the Knaster-Hodge's parametrix $\omega(x, y)$ is represented as

$$(\omega,\ \Delta_0\alpha) = \int \omega(x, y)\, d(\Delta\alpha).$$

Even for forms of general degree, these circumstances are still preserved in considering $\Delta\alpha = (\Delta_0\alpha)^*$ instead of $\Delta_0\alpha$ itself.

1. 2. Another developement has been appeared in the theory of distributions of L. Schwartz [16]; his extended Laplacian $\mathring{\Delta}$ is defined in the space of distributions; however, a superharmonic distribution H is reduced to an almost superharmonic function of M. Brelot's sense, which is further equal to a uniquely determined ordinary superharmonic function almost everywhere. On the other hand, since

$-\mathring{\Delta}H$ is positive, it is nothing but a positive measure. Such being the case, the Laplacian of L. Schwarts represents a linear operator from a certain kind of functions to measures, and it is very remarkable that, as far as we concern ourselves with super- or subharmonic distributions, we shall be satisfied with such an operator $\Delta = (-\mathring{\Delta})$ that transforms functions to measures, without considering any extended sense of Laplacian such as $\mathring{\Delta}$.

Of course, $\mathring{\Delta}$ has an important meaning in some other aspects, for instance, in the relation with the elementary solutions of some partial differential equations of distributions; in fact, the most elegant proof of the decomposition theorem of F. Riesz, owing to L. Schwartz, depends on the use of such elementary solution. We shall establish this theorem in a more general situation in § 2, which also guarantees its classial form by refering to the discussion in *Example 3*, § **4.** Theorem 5 shows a necessary and sufficient condition for the global decomposition of this type; such problem has arrisen in S. Hitotumatu [8], who gives a sufficient condition for this problem.

1. 3. Thus, our first task is to construct a general notion of Laplacian, which is a linear operator from some kinds of functions to measures and general potential as its inverse. We shall find that such a generalized investigation involved not to speak of the classical study of super- or subharmonic functions and potentials, but also that of Fourier transformation (with its inverse transformation) and the Fourier expansion of almost periodic functions, etc.

This study gives otherwise an estimation for how much conditions to be sufficient to support the several important results in the classical potential theory, and from another point of view it is an *abstract potential theory* without any notion of "*metric*".

These are the contents of the first half of this paper.

1. 4. The second half is devoted to the new construction of balayage which plays an important rôle in the theory of potentials, earlier and modern, especially in the problem of Dirichlet. H. Cartan [4], [5], has constructed the balayage theory in using the projection method in a (pre-) Hilbert space which is generated from some measures; our present tool is however the noted Krein-Milman's extreme-points theorem in the theory of linear topological space, as is said before. There are some coveniences peculiar to our method, that is, we can treat the balayage of such measures as are not necessarily of finite energy from the start (contrary to this, H. Cartan's construction is at first done only for the measures of finite energy and afterwards, using the foregoing results, for general measures); moreover, we can obtain some close relation between regular (boundary-) points and extreme points of considering convex set, **6. 5,** which offers a short criterion of sufficiency in order that a boundary-point be regular, Prop. 16.

Finally, we may say that the balayage of the present sense is more akin to the classical one.

1. 5. The last paragraph is occupied by the application of the foregoing discussion to Dirichlet problem and its extension, by a functional-analytic (or Banach space) method; some extension of the same type has been treated by M. Brelot in other way.

I express my hearty thanks to Prof. Dr. M. Inoue for his precious guidance throughout this work.

§ 2. Generalized Laplacian

2. 1. Let E be a locally compact space. For a given domain D of E we shall define:

$\mathfrak{L}^+(D)$ is a convex set of real- (or complex-) valued functions defined on D, satisfying the following conditions;

i) $\mathfrak{L}^+(D)$ forms a positive cône, *i.e.* for positive numbers $\alpha, \beta \geqq 0$ and $f, g \in \mathfrak{L}^+(D)$, $\alpha f+\beta g$ belongs also to $\mathfrak{L}^+(D)$.

ii) for any open set $G \subset D$, the absolute value $|f(x)|$ of every $f \in \mathfrak{L}^+(D)$ cannot be identically infinite on G, *i.e.* $|f(x)| \not\equiv +\infty$ for $x \in G$,

iii) if $f=g$ excepting a non-dense set in D for $f, g \in \mathfrak{L}^+(D)$, then $f=g$ everywhere on D.

According as the functions of $\mathfrak{L}^+(D)$ are real or complex, we call $\mathfrak{L}^+(D)$ itself *real* or *complex* respectively.

$\mathfrak{L}(D)$ is a linear space generated algebraically from the convex set $\mathfrak{L}^+(D)$, over the real or complex field according as $\mathfrak{L}^+(D)$ is real or complex, in which we define $f_1-f_2=g_1-g_2$ if and only if $f_1+g_2=g_1+f_2$ (as functions) for $f_i, g_i \in \mathfrak{L}^+(D)$ $(i=1, 2)$; in other words, $\mathfrak{L}(D)$ is a free linear system with the generator $\mathfrak{L}^+(D)$. We remark that, in an open set $U \subset D$ in which both $f(x)$ and $g(x)$, $f, g \in \mathfrak{L}^+(D)$, are finitely definitive, we can identify $f-g \in \mathfrak{L}(D)$ with the function $f(x)-g(x)$.

Throughout this paper, we shall assume that the functions of $\mathfrak{L}^+(D)$ are at least semi-continuous (sometimes, continuous) unless otherwise specified by adding further terms.

2. 2. We shall next fix our notation for spaces of functions and measures according to N. Bourbaki [1] as follows:

$\mathfrak{M}^+(D)$ is a convex set of positive Radon measures defined on D, and $\mathfrak{M}(D)$ is the linear envelope of $\mathfrak{M}^+(D)$ over the real or complex number field; we denote by $K(X)$ the Banach space (algebra) of real or resp. complex continuous functions of the compact support defined on X, X being any set, then $\mathfrak{M}(D)$ is the dual space of $K(D)$ if $\mathfrak{M}(D)$ is furnished of the topology of simple convergence in $K(D)$, *i.e.* vague topology, and it is a space of Montel, that is, any subset of $\mathfrak{M}(D)$ which is bounded with respect to the norm $\|\mu\|$ of meausre μ is always relatively vague compact.

2. 3. We shall understand by a *local laplacian* Δ_D (related to D) such an operator from $\mathfrak{L}(D)$ into $\mathfrak{M}(D)$ that;

Δ_1) Δ_D is linear, *i.e.* $\Delta_D(\alpha f+g)=\alpha\Delta_D(f)+\Delta_D(g)$ for $f, g\in\mathfrak{L}(D)$,

Δ_2) $\Delta_D(f)$ is in $\mathfrak{M}^+(D)$ if and only if $f\in\mathfrak{L}^+(D)$.

For two domains D_1 and D_2 such that $D_1\subset D_2(\subset E)$, we shall make the assumption:

(A. 1) $\mathfrak{L}^+(D_1)\supset\mathfrak{L}^+(D_2)$, therefore $\mathfrak{L}(D_1)$ may be considered as a linear subspace of $\mathfrak{L}(D_2)$.

Then, Δ_D is assumed to have a further condition;

Δ_3) if $D_2\subset D_1$, then we have for any $f\in\mathfrak{L}(D_1)$

(2・1) $\Delta_{D_2}(f)=(\Delta_{D_1}(f))_{D_2}$ (connective relation),

where $(\cdot)_D$ means the restriction of measure $(\cdot)$ to D. Using merely Δ instead of Δ_E, we see directly that $\Delta_D(f)=(\Delta(f))_D$ for every $f\in\mathfrak{L}(E)\subset\mathfrak{L}(D)$.

THEOREM 1. (*Extension theorem*). *Let* $f\in\mathfrak{L}(D_1\cup D_2)$: *if* $f\in\mathfrak{L}^+(D_1)$ *and simultaneously* $\in\mathfrak{L}^+(D_2)$, *then* $f\in\mathfrak{L}^+(D_1\cup D_2)$.

Proof. Suppose now this were not so, and that $\int\varphi d\Delta_{D_1\cup D_2}(f)<0$ for a $\varphi\in K(D_1\cup D_2)$, $\varphi\geqq 0$. Decomposing φ into $\varphi_1+\varphi_2$, where $\varphi_i\geqq 0$ and $\varphi_i\in K(D_i)$ for $i=1$, 2, we have

$$\int\varphi d\Delta_{D_1\cup D_2}(f)=\int\varphi_1 d\Delta_{D_1\cup D_2}(f)+\int\varphi_2 d\Delta_{D_1\cup D_2}(f)=\int\varphi_1 d\Delta_{D_1}(f)+\int\varphi_2 d\Delta_{D_2}(f)\geqq 0,$$

which is absurd.

2. 4. Next we define the *potential operator* ϕ (inverse operator of Δ_D in a sense for every $D\subset E$) which is a linear mapping of a linear subspace $\hat{\mathfrak{M}}(D)$ of $\mathfrak{M}(D)$ into $\mathfrak{L}(E)$ $(\subset\mathfrak{L}(D))$ and satisfies the following conditions;

ϕ_1) $\phi(\mu)\in\mathfrak{L}^+(E)$ (hence $\in\mathfrak{L}^+(D)$) for every $\mu\in\hat{\mathfrak{M}}^+(D)=\hat{\mathfrak{M}}(D)\frown\mathfrak{M}^+(D)$,

ϕ_2) as far as the integral has a meaning, it holds

(2・2) $$\int\varphi(\mu)d\nu=\int\phi(\nu)d\mu \quad \text{(Fubini's relation)},$$

and moreover

ϕ_3) $\Delta_D\phi(\mu)=\mu$ (reciprocity).

In this paper, for the sake of simplicity and utility we shall assume that, in the case of real $\mathfrak{M}(D)$, $\phi(\mu)$ is well defined for all $\mu\in\mathfrak{M}^+(D)$ either as a function of $\mathfrak{L}^+(E)$ or as the function which is identically infinite, and assume moreover that, if D is relatively compact, $\phi(\mu)$ is in the class $L_\infty(E)$ of all functions vanishing at ∞*). Also in the case of complex $\mathfrak{M}(D)$, always assume $\phi(\mu)$ to be defined for all $\mu\in\mathfrak{M}(D)$ and bounded on E as far as D is relatively compact.

Thus, under these situations if D is relatively compact, we can set

(2・3) $$\hat{\mathfrak{M}}(D)=\mathfrak{M}(D)$$

and we shall see (at § 4) that these assumptions are valid in all of the later

*) Every function f of $L_\infty(E)$ is characterized; for a given $\varepsilon>0$, there corresponds a compact F_ε such that $|f(x)|<\varepsilon$ for every $x\in E-F_\varepsilon$.

cited examples.

On account of this (2·3) and the reciprocity ϕ_3), we see that $\varDelta_D$ is then an *onto-mapping* from $\mathfrak{L}(D)$ to $\mathfrak{M}(D)$ for any relatively compact D, because for every $\mu\in\mathfrak{M}(D)$, $\phi(\mu)$ is well defined and is in $\mathfrak{L}(D)$.

In the later discussion there are some cases where we might need another assumption for $\phi(\mu)$ such that;

$$(2\cdot2)' \qquad \int\phi(\mu)d\mu \geqq 0 \quad (=0, \text{ if and only if } \mu = 0).$$

However, it is certain that there exist some important examples in which this assumption is rejected, so that we adopt it only when we say so definitely.

2. 5. For a given domain $G\subset D$, such an element $f\in\mathfrak{L}(D)$ that $\varDelta_G(f)=0$ is said to be *harmonic in* G; the set of all $f\in\mathfrak{L}(D)$ which are harmonic in G is denoted by $\mathfrak{H}_G(D)$; we write $\mathfrak{H}_D(D)$ for $\mathfrak{H}(D)$. Each $\mathfrak{H}_G(D)$ (or also $\mathfrak{H}(D)$) forms evidently a linear subspace of $\mathfrak{L}(D)$, which is $\subset\mathfrak{L}_+(D)$.

PROPOSITION 1. i) *Let* G *and* G' *be two sub-domains of* D *such that* $G'\subset G$. *If* $f\in\mathfrak{L}(D)$ *is harmonic in* G, *then so is it in* G'. ii) *If the support of* $\mu\in\mathfrak{M}(D)$ *has no intersection with* $G\subset D$, *then* $\phi(\mu)$ *is harmonic in* G. (This guarantees the existence of non-trivial harmonic functions).

In fact, i) $\varDelta_{G'}(f)=(\varDelta_G(f))_{G'}=(0)_{G'}=0$ and ii) $\varDelta_G\phi(\mu)=(\varDelta_D\phi(\mu))_G=\mu_G=0$.

PROPOSITION 2. *Let* G *and* G' *be two domains in* E; *for every* $f\in(G\cup G')$ *we have*

$$(2\cdot4) \qquad \varDelta_G\phi\varDelta_{G'}(f) = \varDelta_{G'}\phi\varDelta_G(f) = \varDelta_{G\cap G'}(f).$$

Proof. By $\varDelta_3$), it follows that $\varDelta_G\phi\varDelta_{G'}=(\varDelta_{G\cup G'}\phi\varDelta_{G'})_G=(\varDelta_{G'})_G=\varDelta_{G\cap G'}$; exchangeing G and G' mutually, we obtain the above equalities.

THEOREM 2. *Let* G *be relatively compact*;

i) $\mathfrak{L}(D)/\mathfrak{H}_G(D)\cong\mathfrak{M}(G)$,*)

ii) *Each residue-class of the quotient* $\mathfrak{L}/\mathfrak{H}_G$ *contains one and only one potential* $\phi(\mu)$ *of* $\mu\in\mathfrak{M}(G)$,

iii) *The difference of any two elements* f *and* g *in the same residue-class is harmonic in* G.

This immediately implies the well-know decomposition of F. Riesz:

$$(2\cdot5) \qquad f=\phi\varDelta_G(f)+f_G \quad \text{in } G, \text{ where } f_G\in\mathfrak{H}_G(D),$$

for every $f\in\mathfrak{L}^+(D)$, and the uniqueness of this decomposition is also evident.

Even if G is not relatively compact, the decomposition (2·4) is valid for such $f\in\mathfrak{L}^+(D)$ that $\varDelta_G(f)\in\hat{\mathfrak{M}}^+(G)$; in fact, we have $\varDelta_G(f-\phi\varDelta_G(f))=\varDelta_G(f)-\varDelta_G\phi\varDelta_G(f)=0$. The uniqueness comes from the fact: suppose now $f=\phi(\nu)+f_G'$ in G, $f_G'\in\mathfrak{H}_G(D)$, then it follows that $\varDelta_G(f)=\varDelta_G\phi(\nu)=\nu$ and hence $f_G=f_G'$ in G.

2. 6. In this section we shall assume that the considering integralation is never meaningless; let f, $g\in\mathfrak{L}^+(D)$ be given and by the above theorem put $f=\phi(\varDelta_G(f))+f_G$ and $g=\phi(\varDelta_G(g))+g_G$, where f_G, $g_G\in\mathfrak{H}_G(D)$, $G\subset D$. By ϕ_2), it holds

*) The sign $\cong$ means an isomorphism between two linear spaces.

$\int \phi(\Delta_G(f))d\Delta_G g = \int \phi(\Delta_G(g))d\Delta_G f$, from which it follows

$$(2\cdot6) \qquad \int_G (f-f_G)d\Delta_G g = \int_G (g-g_G)d\Delta_G f$$

or equivalently

$$(2\cdot6)' \qquad \int f d\Delta_G g - \int g d\Delta_G f = \int_G (f_G d\Delta_G g - g_G d\Delta_G f)^{*)}.$$

Putting $[f, g]_G = \int_G (f d\Delta g - g d\Delta f)$, we see that

i) $[f, g]_G = -[g, f]_G$, and hence $[f, f]_G = 0$.

ii) $[f, g]_G$ is bilinear with respect to f and g.

iii) If f is harmonic in G (*i.e.*, $f = f_G$ in G), it holds

$$[f, g]_G = \int_G f d\Delta g,$$

consequently, if both f and g are harmonic in G, $[f, g]_G = 0$.

iv) $[\phi(\mu), \phi(\nu)]_G = 0$ for any μ and $\nu \in \hat{\mathfrak{M}}(E)$.

v) Denoting by ε_x the point measure of total mass $+1$ (*Dirac measure*) placed on a point $x \in E$ and by $\phi(x)$ the potential of ε_x, *i.e.* $\phi(\varepsilon_x)$, we have for $f \in \mathfrak{L}^+(D)$ and $G \subset D$,

$$(2\cdot7) \qquad f_G(x) = [f, \phi(x)]_G \quad \text{in } G \ (=0, \text{ in } D-G).$$

In fact, from iii), iv) and (2·5), it follows $[f, \phi(x)]_G = [\phi\Delta_G(f), \phi(x)]_G + [f_G, \phi(x)]_G = [f_G, \phi(x)]_G = f_G(x)$.

vi) If the unit function 1 ($1(x)=1$ identically) is in $\mathfrak{H}(D)$, we have

$$(2\cdot8) \qquad [f]_G = [1, f]_G = \int_G d\Delta f$$

for $f \in \mathfrak{L}^+(D)$ and $G \subset D$, which shall be called "*f-capacity of G*" (*cf.* **5·5**).

In respecting to (2·8), $[f]_G$ can be linearly prolonged to a linear functional on $\mathfrak{L}(D)$, satisfying the following conditions; α) $[f]_G = 0$ if and only if $f \in \mathfrak{H}_G(D)$, β) $f \in \mathfrak{L}^+(D)$ implies $[f]_G \geqq 0$. We note in passing, in the later mentioned Example 3 (§ **4**), we can easily see by the classical Green's formula in vector analysis that

$$[f]_G = -\frac{1}{N_n}\iint_{\partial G} \frac{\partial f}{\partial n} dS, \qquad N_n = \Gamma(n/2)/2(n-2)\pi^{n/2},$$

where ∂G designates the boundary of G (supposed to be regular now), dS being the surface element on it, and n refers to the outwards normal with respect to ∂G. In the case, $[\phi(\mu)]_G = \int d\mu = -\frac{1}{N_n}\iint_{\partial G} \frac{\partial\phi(\mu)}{\partial n} dS$ for any μ with the support in G, which shows the classical Gauss theorem in potential theory.

vii) On the other hand, if $1 = \phi(e)$, e being the origin of E, we have

$$(2\cdot8)' \qquad [f]_G = [1, f]_G = -\int_{E-G} d\Delta f \ (\text{if } e \in G), \text{ or } = \int_G d\Delta f \ (\text{if } e \bar{\in} G),$$

since in this case $f(e) = \int_E d\Delta f$ on account of (2·12) below. Finally, we remark that $\chi_G(x) = [1, \phi(x)]_G$ is the characteristic function of $G \subset D$.

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*) We remark that $\int d\Delta_G f = \int_G d\Delta_G f = \int_G d\Delta f$

2. 7. Let E be a (locally compact) topogical group having the Haar measure dx and denote $\phi(\varphi)=\phi(\varphi dx)$ for $\varphi \in K(E)$, or $\in \hat{K}(E)$, the space of measurable functions with comapact supports in E. From $(2\cdot 6)'$ it follows that, if the support of φ is contained in D,

$$(2\cdot 9) \qquad \int_E f\varphi dx = \int_D \phi(\varphi) d\Delta_D(f) + \int_E f_D \varphi dx,$$

since it must be that the harmonic part of $\phi(\varphi)$ vanishes on D. In particular, if $D=E$ and $f_D=0$ (such is the case, *e.g.*, in which $f=\phi(\mu)$, or f is arbitrary but $\mathfrak{H}(E)$ itself consists only of zero function as in Example 1, 2 (§4)), then as far as $\phi(\mu)$ is integrable with respact to $\Delta(f)$, we have

$$(2\cdot 10) \qquad \int f\varphi\, dx = \int \phi(\varphi) f \Delta(f) \quad (\textit{Bochner formula})$$

From this point of view, we may call $(2\cdot 9)$ an extended formula of Bochner. More generally, under such condition that $f_D=0$, we have directly from $(2\cdot 2)$ that

$$(2\cdot 11) \qquad \int f d\mu = \int \phi(\varphi) d\Delta(f)$$

and particularly,

$$(2\cdot 12) \qquad f(x) = \int \phi(x) d\Delta(f).$$

Of course, $(2\cdot 11)$ and $(2\cdot 12)$ are valid whether E is a group or not.

2. 8. We shall finally investigate about the integral representation of the operator ϕ and $f\in \mathfrak{L}^+(E)$: To see it, put at first

$$(2\cdot 13) \qquad \Phi(x,y) = (\phi(x))(y)$$

then we can state

THEOREM 3. *If* $\mu \in \hat{\mathfrak{M}}^+(E)$, $\phi(\mu)$ *is represented as follows,*

$$(2\cdot 14) \qquad (\phi(\mu))(x) = \int \Phi(x,y) d\mu(y),$$

where $\Phi(x,y)$ *is symmetrical with respect to* x *and* y, $\Phi(x,y)=\Phi(y,x)$, *and it belongs to* $\mathfrak{L}^+(E)$ *in respect of both* x *and* y, *i.e.* $\Phi\in\mathfrak{L}^+(E\times E)$.

In fact, $\Phi(x,y)=\int \phi(x) d\varepsilon_y = \int \phi(y) d\varepsilon_x = \Phi(y,x)$ and moreover one sees

$$\phi(\mu)(x) = \int \phi(\mu) d\varepsilon_x = \int (\phi(x))(y) d\mu(y) = \int \Phi(x,y) d\mu(y).$$

For $f\in\mathfrak{L}^+(E)$ having $\Delta f\in\hat{\mathfrak{M}}(E)$ and $f_E=0$, it holds

$$(2\cdot 15) \qquad f(x) = \int \Phi(x,y) d\Delta f(y). \qquad (\textit{cf.}\ (2\cdot 12))$$

$\Phi(x,y)$ is called the "*kernel function*" of the potential operator ϕ.

§ 3. The real cases and modulus principle.

3. 1. Throughout this paragraph we restrict ourselves within the real $\mathfrak{L}^+(D)$ and assume always that: given a general domain $D\subset E$, which is however supposed to be a union of a countably infinite number of compact sets at present, the following three conditions are fulfiled;

(R. 1) every constant function (on D) is contained in $\mathfrak{H}(D)$,

(R. 2) $\mathfrak{L}^+(D)\cap\hat{K}(D)=(0)$, where 0 means the zero function on D, and moreover

(R. 3) for any f, $g\in\mathfrak{L}^+(D)$, $f\cap g$ also belongs to $\mathfrak{L}^+(D)$.

PROPOSITION 3. *Let $f\in\mathfrak{L}^+(D)$ and $h\in\mathfrak{H}(D)$; if $f\geqq h$ on $D-G$ for a relatively compact G such that $\bar{G}\subset D$, then $f\geqq h$ everywhere on D.*

Proof. Put $f_0=(f\cap h)-h$, then $f_0\in\mathfrak{L}^+(D)$ by (R. 3) and $f_0=0$ on $D-G$, hence $f_0=0$ everywhere on D by (R. 2), *i. e.* $f\geqq f\cap h=h$ on D.

PROPOSITION 4. *If $f\in\mathfrak{H}(D)\cap L_\infty(D)$, then f is identically zero on D.*

Given a positive number $\varepsilon>0$, let G be a relatively compact domain such that $|f(x)|<\varepsilon$ on $D-G$. By the above Prop. 3, it yields that $f\leqq\varepsilon$ and simultaneously $-\varepsilon\leqq f$ everywhere on D, that is, $|f|\leqq\varepsilon$ on D, from which follows that $f=0$ on D, ε being arbitrary.

3. 2. In the real case, we shall always lay down another assumption concerning with the opeation ϕ as follows:

(A. 2) The application of $\mu\in\mathfrak{M}^+(D)$ to $\phi(\mu)\in\mathfrak{L}^+(D)$ is lower semi-continuous with respect to the vague topology of $\mathfrak{M}^+(D)$.

We begin with a fundamental proposition as follows:

THEOREM 4. *If $f\in\mathfrak{L}^+(D)\cap L_\infty(D)$, then $f\geqq 0$ on D. Threfore, the potential $\phi(\mu)$ of positive measure is everywhere positive on D under the condition $\phi(\mu)\in\mathfrak{L}^+(D)$.*

Proof. Suppose it were not so and, for any sufficiently small positive number $\varepsilon>0$, put $h=(f\cap(-\varepsilon))+\varepsilon$, then h vanishes in $D-F$ for a suitable compact $F\subset D$, so that h must be 0 in D since $h\in\mathfrak{L}^+(D)$. Thus, $f\cap(-\varepsilon)=\varepsilon$ or equivalently $f\geqq-\varepsilon$. ε being arbitrary, one concludes that $f\geqq 0$.

Next, take a series of compact sets K_j such that $\cup_j K_j=D$; denoting the restriction of μ in K_j by μ_j for each j, we have by hypothesis (mentioned in **2. 4**) $\phi(\mu_j)\in\mathfrak{L}^+(D)\cap L_\infty(D)$, since K_j is compact, and so $\phi(\mu_j)\geqq 0$ on account of the avove argument. Therefore, we get $\phi(\mu)=\lim_j\phi(\mu_j)\geqq 0$, as desired: this completes the proof of Theorem.

The similar method shows that, if $f\in\mathfrak{L}^+(D)$ and x_0 is any (inner) point of D, $f(x_0)\geqq\inf_{x\in G}f(x)$ for all $x\in D$, or evuivalently $f(x_0)$ cannot be smaller than the greatest lower bound of $f(x)$ in D. But, in order to state positively that $f(x_0)$ is greater than the greatest lower bound of $f(x)$, $x\in D$, supposing f is not constant in D, one shall need some conition about mean values.

3. 3. Now, under the assumptions (R. 1)$\sim$(R. 3), we research a necssary and sufficient condition for the decomposition of *F.* Riesz's type (2. 5) in the case where D is not necessaly relatively compact. Here is the result which we desire:

THEOREM 5. *A necessary and sufficient cnodition that $f\in\mathfrak{L}_+(D)$ might be decomposed in the from $f=\phi\Delta_D(f)+f_D$, $f_D\in\mathfrak{H}(D)$ (or equivalently that $\Delta_D(f)\in\hat{\mathfrak{M}}^+(D)$),* *is that there exists a* $h \in \mathfrak{H}(D)$ *such as* $f \geqq h$ *on* D.

Proof. 1). Let $\{D_j\}$ be the family of relatively compact domains such that $\bar{D}_j \subset D_{j+1}$ and $D = \cup_{j=1}^{\infty} D_j$; by Theorem 2 we have

$$\begin{cases} f = \phi\Delta_{D_j}(f) + f_{D_j}, & f_{D_j} \in \mathfrak{H}(D_j) \\ f = \phi\Delta_{D_{j+1}}(f) + f_{D_{j+1}}, & f_{D_{j+1}} \in \mathfrak{H}(D_{j+1}). \end{cases}$$

Since $\Delta_{D_{j+1}}(f_{D_j}) = \Delta_{D_{j+1}}(f - \phi\Delta_{D_j}(f)) \geqq 0$ by Prop. 2 $(\Delta_{D_{j+1}}\phi\Delta_{D_j}(f) = \Delta_{D_j}(f))$, we have $f_{D_j} \in \mathfrak{L}^+(D_{j+1})$. On the other hand, $f_{D_j} = f - \phi\Delta_{D_j}(f) \in \mathfrak{L}^+(D - \bar{D}_j)$ because $\phi\Delta_{D_j}(f) \in \mathfrak{H}(D - \bar{D}_j)$ by Prop. 1, ii). Thus, owing to Theorem 1, we conclude that $f_{D_j} \in \mathfrak{L}^+(D_{j+1} \cup (D - \bar{D}_j)) = \mathfrak{L}^+(D)$.

2) Suppose now $f \geqq 0$ on D, then $\varepsilon \leqq -\phi\Delta_{D_j}(f) = -f + f_{D_j} \leqq f_{D_j}$ on $D - F$ for a suitable compact F (in fact, $\phi\Delta_{D_j}(f) \in L_\infty(D)$; see the assumption noted in **2. 4**), hence by Prop. 3 $f_{D_j} \geqq \varepsilon$ everywhere on D. Thereby it comes that

$$\begin{aligned} \phi\Delta_D f &\leqq \varliminf_{j\to\infty} \phi\Delta_{D_j}(f) = \varliminf_{j\to+\infty}(f - f_{D_j}) \\ &= f - \varlimsup_{j\to\infty} f_{D_j} \leqq f - \varepsilon < f, \end{aligned}$$

that is, $\phi\Delta_D f \not\equiv +\infty$ on D or in other words $\Delta_D f \in \hat{\mathfrak{M}}^+(D)$.

For a $f \in \mathfrak{L}^+(D)$ such that $f \geqq h$, $h \in \mathfrak{H}(D)$, one has only to put $g = f - h \geqq 0$. Then $g \geqq 0$ and so $\Delta_D(f) = \Delta_D(g) \in \hat{\mathfrak{M}}^+(D)$, which proves the sufficiency.

3) The necessity seems somewhat evident, since $\phi(\mu) \geqq 0$.

Hereafter in this pargraph, we shall define, for any $\mu \in \hat{\mathfrak{M}}(D)$, the integral $I(\mu)$ called *energy of* μ as follows;

$$\begin{aligned} (3\cdot1) \qquad I(\mu) &= \int \phi(\mu)d\mu \\ &= \int \phi(\mu_1)d\mu_1 + \int \phi(\mu_2)d\mu_2 - 2\int \phi(\mu_1)d\mu_2, \end{aligned}$$

as far as the last term is not meaningless, where $\mu = \mu_1 - \mu_2$, $\mu_i \in \mathfrak{M}^+(D)$ for $i = 1, 2$, and assume always that $I(\mu) \geqq 0$ (finite or infinite) and $=0$ if and only if $\mu = 0$. Therefore, the assumption $(2\cdot2)'$ is held here.

Then we can state:

THEOREM 6. *Let* f *be in* $\mathfrak{L}^+(D)$ *and* $f \geqq 0$; *if* $\mu \in \mathfrak{M}^+(D)$ *satisfies,* $I(\mu) < +\infty$ *and* $f \geqq \phi(\mu)$ *on a kernel of* μ, *then the above inequality takes place in the whole* D.

Proof. Putting $f \frown \phi(\mu) = g$, then $g \in \mathfrak{L}^+(D)$ and by Theorem 4 f, $\phi(\mu) \geqq g \geqq 0$ on D, besides $\phi(\mu) = g$ on a kernel of μ by hypothesis. $g \geqq 0$ implies the possibility of the following decomosition; $g = \phi\Delta_D(g) + g_D$, in which $g_D \in \mathfrak{H}(D)$ but simultaneously $g_D \in L_\infty(D)$, so that by Prop. 4 $g_D = 0$ on D. Consequently, $g = \phi\Delta_D(g)$. Now consider the integral;

$$I = \int_D \phi(\mu - \Delta_D(g))d(\mu - \Delta_D(g)) = -\int (\phi(\mu) - g)d\Delta_D(g) \leqq 0,$$

but I is always non-negative, then I must be $=0$. Thus, $\mu = \Delta_D(g)$ and hence $f \geqq g = \phi\Delta_D(g) = \phi(\mu)$ everywhere on D. It remains to show the integral I being reasonable. In fact,

$$\int g d\Delta_D(g) \leqq \int \phi(\mu) d\Delta_D(g) = \int \phi(\Delta_D g) d\mu = \int g d\mu \leqq \int \phi(\mu) d\mu\,,$$

which is $< +\infty$. This concludes the proof.

In this Theorem, we may take $f \in \mathfrak{L}^+(D) \frown L_\infty(D)$ (instead of $f \in \mathfrak{L}^+(D)$ and $f \geqq 0$); of course by Theorem 4 we get $f \geqq 0$.

When f = constant in D, this Theorem shows the maximum principle of Maria-Frostman's type.

3. 4. Now, we shall exhibit some applications of the results obtained before. The first is;

PROPOSITION 5. *If $f \in \mathfrak{L}^+(D)$ is upwards bounded by $\phi(\mu)$ for a $\mu \in \mathfrak{M}^+(D)$ with compact support, then $f = \phi(\nu)$ for a suitable $\nu \in \mathfrak{M}^+(D)$.*

Proof. By Theorem 1 and Prop. 4, it is clear.

THEOREM 7. *If K is compact, then there exists a measure $\mu \in \mathfrak{M}^+(E-K)$ such that $\phi(\mu)=1$ on K.*

Proof. For each $x \in K$, consider the set $V(x) = \{y\,;\ \phi(\varepsilon_x)(y) > 1\}$ (if $\phi(\varepsilon_x)(x) \leqq 1$, take $\phi(\alpha\varepsilon_x)$ instead of $\phi(\varepsilon_x)$ for a suitable positive number α that $\phi(\alpha\varepsilon_x)$ might be > 1 in x); then, as $\phi(\nu)$ is lower semi-continuous for $\nu \in \mathfrak{M}^+(D)$, such defined $V(x)$ is open and assumed not to be void. Since K is compact, it is covered by a finite oumber of $V(x_i)$, $x_i \in K$, for $i = 1, 2, \cdots, n$; putteing $\mu = \sum_{i=1}^n \varepsilon_{x_i}$ (or $\sum_{i=1}^n \alpha_i \varepsilon_{x_i}$) we see $\phi(\mu) > 1$ on $\cup_{i=1}^n V(x_i) \supset K$. Then $f = \phi(\mu) \frown 1$ is a desired one; we shall now prove this.

Such f just obtained is $= 1$ on a certain open set W containing K, so that $f = 1$ on K. By Prop. 5, f must be $= \phi(\nu)$ for a certain positive meausure ν which is necessarily distributed in $E-K$, since f is harmonic in W. Thus, Theorem 7 is completely proved.

This theorem is available for the later discussion of balayage in § **6**.

§ 4. Examples of Δ and ϕ.

We shall exhibit some improtant and concrete examples, at first, of complex cases and, succeedingly, of real cases.

4. 1. *Example 1.* Let E be the one-dimensional Euclidean space R^1, that is the additive group of real numbers, and $\mathfrak{L}^+(E)$ the collection of all continuous positive definite functions defined on E. Furthermore, denote by $L^1(E)$ the commtative self-adjoint Banach algebra consisting of integrable functions on E, with respect to the involution $\varphi^*(x) = \overline{\varphi(-x)}$ and the multiplication $\varphi \cdot \psi(x) = \int \varphi(x-y)\psi(y)dy$ (convolution). Then, the convex set $\mathfrak{L}_0^+(E)$ of functions $\in \mathfrak{L}^+(E)$ with norms less than 1 constitutes a weakly compact convex set in the unit sphere of the dual space $(L(E^1))^*$ of $L^1(E)$ in regard to the relation

(4·1) $$f(\varphi)=\int \overline{f(x)}\varphi(x)dx \quad \text{for} \quad f\in \mathfrak{L}_0^+(E),\ \varphi\in L^1(E)\ .$$

By the noted theorem of Krein-Milman, $\mathfrak{L}_0^+(E)$ coincides with the weakly closure of linear envelope of extreme points of $\mathfrak{L}_0^+(E)$ in $(L^1(E))^*$. Since $L^1(E)$ is commutative, each extreme point χ is multiplicative, that is, $\chi(\varphi\cdot\psi)=\chi(\varphi)\chi(\psi)$, so that the corresponding function $\chi(x)$ must be a character of E. More concretely, $\chi(x)$ is in the form; $\chi(x)=e^{iyx}$ for a $y\in E$.

Denoting by $\hat{\varphi}$ the Fourier transform of $\varphi\in L^1(E)$, *i.e.* $\hat{\varphi}(t)=\int e^{itx}\varphi(x)\,dx$, it is easily seen by Stone-Gelfand's approximation theorem that the collection $\hat{L}$ of all $\hat{\varphi}$ constitutes a dense sub-space of $L_\infty(E)$ (in fact, we see that $\widehat{\varphi\cdot\psi}=\hat{\varphi}\cdot\hat{\psi}$). Put

(4·2) $$\mu_f(\hat{\varphi})=f(\varphi) \quad \text{for} \quad f\in\mathfrak{L}^+(E),\ \varphi\in L^1(E)\ ;$$

if $f\in\mathfrak{L}_0^+(E)$, there exists, for every $\varepsilon>0$, a finite number of $t_k\in E$ such that $|f(\varphi)-\sum\alpha_k e^{it_kx}(\varphi)|<\varepsilon$, $\sum\alpha_k=1$. Thus we see that $|f(\varphi)|\leqq\sum\alpha_k|\hat{\varphi}(t_k)|+\varepsilon\leqq\|\hat{\varphi}\|_\infty+\varepsilon$. ε being arbitrary, we have for any $f\in\mathfrak{L}^+(E)$ that $|\mu_f(\hat{\varphi})|\leqq\|f\|_\infty\cdot\|\varphi\|$, or in other words, μ_f is a bounded linear functional on $\hat{L}$, so that, on $K(E)$. Thus, μ_f defines a bounded measure on E. We shall denote

(4·3) $$\Delta(f)=\mu_f \quad \text{and} \quad \Delta_D(f)=(\mu_f)_D\ ,\ D\subset E.$$

4. 2. It remains to prove the positiveness of thus obtained measure $\Delta(f)$. Let $\varphi\in K(E)$ and $\varphi\geqq 0$; suppose now $\hat{\psi}\longrightarrow\sqrt{\varphi}$ uniformely on E, then $\widehat{\psi^*\psi}=|\hat{\psi}|^2\longrightarrow\varphi$ uniformely and accordingly $0\leqq f(\psi^*\psi)=\mu_f(|\hat{\psi}|^2)$, since f is positive definite. Thus, we conclude $\mu_f(\varphi)\geqq 0$.

On the other hand, for a bounded measure $\mu\in\mathfrak{M}(E)$, we shall define the operator ϕ as follows;

(4·4) $$\phi(\mu)(x)=\int e^{ixt}d\mu(t),\quad \Phi(x,y)=e^{ixy},$$

for which it is easily seen by simple calculation that $\phi(\mu)\in\mathfrak{L}^+(E)$ as far as $\mu\in\hat{\mathfrak{M}}^+(E)$ and by Fubini's theorem $\Delta\phi(\mu)=\mu$ (reciprocity ϕ_3), § **2**). In passing, we shall make a short remark; for a point measure $+1$ on x, it holds $\phi(\varepsilon_x)=e^{ixt}$ and $\phi(\varepsilon_e)=1$, e being the orginal point of E.

In the present case, $\mathfrak{L}(E)$ and so $\mathfrak{M}(E)$ must be complex, and $\mathfrak{L}(D)$ is considered as a space of functions since every $f\in\mathfrak{L}^+(E)$ is bounded; $|f(x)|\leqq\|f\|_\infty=f(e)$, so that $f-g$ represents a function just as it is. Moreover, $\mathfrak{H}(E)$ consists only of the zero function, and this implies every f_E, harmonic part of f in E, has to vanish, that is, the *Riesz's decomposition* is now in the from;

(4·5) $$f=\phi\Delta(f) \quad \text{(Inversion formula).}$$

Combining this with (4·4), we have $f(e)=\phi\Delta(f)(e)=\int d\Delta(f)$.

4. 3. Let E be n-dimensional Euclidean space R^n for $n\geqq 2$, in which every point x is assigned by coordinates $(x_1,x_2,\cdots,x_n)$. Then replacing e^{ixt} by $e^{(x_1t_1+\cdots+x_nt_n)}$, the preceding investigations remain valid thoroughly in this case.

4. 4. *Example 2.* Again let E be one-dimensional Euclidean space R^1; let $\mathfrak{L}(E)$ be the complex linear space of all continuous almost periodic functions defined on E, while $\mathfrak{L}^+(E)$ the collection of those which have the Fourier expansions in such forms; $\sum a_j e^{ix_jt}$ and $a_j \geqq 0$ for every j. The definition of $\mathfrak{M}(E)$ is, however, somewhat different from the others, that is, the topology of E for $\mathfrak{M}(E)$ is assumed to be *discrete*. For theis reason, every measure of $\mathfrak{M}(E)$ is also discrete and, if it has the compact support, it is nothing but a finite measure. Now we define: if f has the Fourier expansion $\sum a_j e^{ix_jt}$,

(4·6) $$\Delta_D(f) = \sum a_j d\mathcal{E}_{x_j},\ x_j \in D.$$

$\phi(\mu)$ is defined also by (4·4), and $\mathfrak{H}(E)=0$. But the inversion formula (4·5) does not necessarily hold true.

4. 5. We shall investigate slightly on the representation theory of the group E. Given a domain $D \subset E$, let N_D be a linear subspace of $L^1(E)$ consisting of such $f \in L^1(E)$ whose Fourier transformation belongs to $K(D)$, *i.e.* has the support contained in D. Then we have

THEOREM 8. *The translation S_x, $x \in E$,*

$$S_x\varphi = \varphi_x,\ \varphi_x(t) = \varphi(t-x) \ \textit{ for } \ \varphi \in L^1(E),$$

is invariant on every N_D, $D \subset E$. If $f \in \mathfrak{L}^+(E)$ (in the sence of Example 1) is harmonic in D, and never in D' such that $D \subset D'$ properly, then N_D coincides with the collection of such $\varphi \in L^1(D)$ that $\iint f(x-y)\overline{\varphi(x)}\varphi(y)dxdy = 0$. Then the mapping $x \longrightarrow S_x$ is a faithful unitary representation of E by the unitary operators in a Hilbert space H_f completed from the quotient space $L^1(E)/N_D$ having the inner product and norm;

(4·7) $$(\dot{\varphi}, \dot{\psi})_f = \iint f(x-y)\overline{\varphi(x)}\psi(y)dxdy$$
$$= \int f(x)\varphi^*\psi(x)dx, \quad \|\dot{\varphi}\|_f = (\dot{\varphi}, \dot{\varphi})_f^{1/2},$$

where $\dot{\varphi}$ means the corresponding element in $L^1(E)/N_D$ for $\varphi \in L^1(E)$.

Proofs come from the general unitary representation theory and (2·10) above. About the former, see H. Cartain-R. Godement [6], L. H. Loomis [11], S. Matsushita [12] [13], etc. We shall remark that, for a suitably choosen element $\dot{\theta} \in H_f$, f is represented as follows:

$$f(x) = (\dot{\theta}, S_x\dot{\theta})_f.$$

4. 6. *Example 3.* Let E be n-dimensional Euclidean space $R^n (n \geqq 3)$, and $\mathfrak{L}^+(D)$ the convex set of all superharmonic functions defined on $D \subset E$; C^p designates the collection of functions having continuous partial derivatives up to the order p, $1 \leqq p < +\infty$.

For a $\varphi \in K(D) \frown C^2$ whose support is in a relatively compact domain B such that $B \subset D$, we define

*) See *e.g.* T. Radó [17].

(4·8) $$\int \varphi \, d\Delta_D(f) = \int f(-\dot{\Delta}\varphi)\, dx, \text{ for any } f \in \mathfrak{L}^+(D),$$

where $\dot{\Delta}\varphi = \sum_{i=1}^{n} \frac{\partial^2 \varphi}{\partial x_i^2}$. Indeed, we see immediately that, as there exists a sequence of $f_j \in \mathfrak{L}^+(B) \frown C^2$ such that $f_j \nearrow f$ it holds by the classical Green's formula, if $\varphi \geqq 0$, then

$$0 \leqq \int_B \varphi(-\dot{\Delta} f)\, dx = \int_B (-\dot{\Delta}\varphi) f_j dx \longrightarrow \int (-\dot{\Delta}\varphi) f dx\,.$$

Thus, $\Delta_D(f)$ is positively linear on $K(D) \frown C^2$, since this is positively dense in $K(D)$,*) $\Delta_D(f)$ is uniquely prolonged up to a positive measure on D, which is a direct consequence of a Proposition of N. Bourbaki [1]. Conversely, we have:

PROPOSITION 6. *Let f be an element of $\mathfrak{L}(D)$, real linear envelope of $\mathfrak{L}^+(D)$. In order that $\Delta_D(f) \in \mathfrak{M}^+(D)$, it is sufficient that $f \in \mathfrak{L}^+(D)$.*

Proof. Suppose now $f = f_1 - f_2$ for $f_i \in \mathfrak{L}^+(D)$ $(i = 1, 2)$. If $\Delta_D(f) \in \mathfrak{M}^+(D)$ and consequently $\Delta_B(f) \in \mathfrak{M}^+(D)$ for any $B \subset D$, then for any sphere $\Sigma \subset D$ we can choose a relatively compact domain B such that $\Sigma \subset B$, $\bar{B} \subset D$, and a sequence of $f_j^i \in \mathfrak{L}^+(B) \frown C^2$ such that $f_j^i \nearrow f_i (j \longrightarrow +\infty)$ for $i = 1, 2$. Take morenver a sequence of spheres Σ_k of the common center x_0 with that of Σ, which converges to the centre x_0.

Denoting the spherical measure with total mass $+1$ on $\partial\Sigma$ by λ, on $\partial\Sigma_k$ by λ_k, we see that $0 \leqq \int \phi(\lambda_k - \lambda) d\Delta_B(f) = \int \phi(\lambda_k - \lambda) d(\Delta_B(f_1) - \Delta_B(f_2)) = \lim_{j\to\infty} \int \phi(\lambda_k - \lambda)$ $d\mu_j^1 - d\mu_j^2$, where $\phi(\mu)$ is the Newtonian potential of μ,

(4·9) $$\phi(\mu)(x) = N_n \int r^{2-n}(x, y) d\mu(y), \quad N_n = \Gamma(n/2)/2(n-2)\pi^{n/2},$$

$r(x, y)$ being the Euclidean distance between x and y in E, and $d\mu_j^i = (-\Delta f_j^i)\, dx$ for each j and $i = 1, 2$. On the other hand, one sees $\lim_{j\to\infty} \int \phi(\lambda_k - \lambda) d\mu_j^i = \lim_{j\to\infty} \int \phi(\mu_j^i)$ $(d\lambda_k - d\lambda) = \lim_{j\to\infty} \int f_j^i\, (d\lambda_k - d\lambda) = \int f_i (d\lambda_k - d\lambda)$ for $i = 1, 2$. Hence $0 \leqq \int (f_1 - f_2)$ $(d\lambda_k - d\lambda)$ for each k; if $f_2(x_0) \neq +\infty$, the function $f_1 - f_2$ is definitive in x_0 and $(f_1 - f_2)(x_0) - \left(\int f_1 d\lambda - \int f_2 d\lambda \right) = \lim_{k\to\infty} \int f_1 (d\lambda_k - d\lambda) - \lim_{k\to\infty} \int f_2 (d\lambda_k - d\lambda) \geqq 0$. Thus, the function $f_1 - f_2$ is defined almost everywhere on D and satisfies $(f_1 - f_2)(x_0) \geqq \int (f_1 - f_2) d\lambda$ for every $\Sigma \subset D$, which implies that $f_1 - f_2$ is *almost superharmonic* on D in the sense of Szpilrajn,**) so that there exists one and only one $f_0 \in \mathfrak{L}^+(D)$ which coincides with $f_1 - f_2$ almost everywhere on D. Consequently, $f_0 + f_2$ is equal to f_1 almost everywhere, therefore, properly everywhere on D, that is, $f_0 = f_1 - f_2$ $\in \mathfrak{L}^+(D)$. Thus, $f = f_1 - f_2 \in \mathfrak{L}(D)$ represents a function f_0 and is in $\mathfrak{L}^+(D)$, which proves the Proposition.

Thus, every $h \in \mathfrak{H}(D)$ is harmonic in D in the proper sense, that is, $\dot{\Delta} h = \sum_{i=1}^{n} \frac{\partial^2 h}{\partial x_2^i} = 0$ in D.

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*) By Bourbaki's terminology, it is "*positivement riche.*" **) T. Radó; Loc. cit.

Next, we shall show the reciprocity $\phi_3)$ (in §2): Let λ and λ' be spherical measures (of total mass $+1$) placed on the surfaces of spheres Σ and Σ' with the common center respectively, where $\Sigma\subset\Sigma'\subset D$. For $\mu\in\hat{\mathfrak{M}}^+(D)$, choose an open set B (relatively compact) such that $\bar{B}\subset D$, $\Sigma'\subset B$, and a series of $f_j\in\Gamma(B)\frown C^3$ such that $f_j\nearrow\phi(\mu)$ in B. Since $\phi(\lambda-\lambda')\in K(B)$, $\geqq 0$, we have

$$\int\phi(\lambda-\lambda')d\varDelta_D\phi(\mu)=\int\phi(\lambda-\lambda')d\varDelta_B\phi(\mu)$$
$$=\lim_{j\to\infty}\int\phi(\lambda-\lambda')d\mu_j=\lim_{j\to\infty}\int\phi(\mu_j)d\varDelta_B\phi(\mu)$$
$$=\lim_{j\to\infty}\int f_jd(\lambda-\lambda')=\int\phi(\mu)d(\lambda-\lambda'),$$

which is equal to $\int\phi(\lambda-\lambda')d\mu$, where $\mu_j=(-\dot{\varDelta}f_j)dx_B$: $\phi(\lambda-\lambda')$ being positively dense in $K(D)$, we conclude that $\mu=\varDelta_D\phi(\mu)$.

Finally, we remark that the assumptions (R. 1)$\sim$(R. 3) in §3 and (2·2)$'$ are always held here.

4. 7. *Example 4*. Let E be two-dimensional Euclidean space R^2 or the open unit circle $|z|<1$ in the complex number plane Z^2. The operator $\varDelta_D$ is defined in the same way as in the above *Example* 3, but the definition of ϕ is alternated respectively as follows:

$$\phi(\mu)(x)=\iint_{R^2}\frac{1}{2\pi}\log\frac{1}{|x-t|}d\mu(t). \tag{4·10}$$

$$\phi(\mu)(x)=\iint_{|t|<1}\log\left|\frac{1-\bar{x}t}{x-t}\right|d\mu(t). \tag{4·10$'$}$$

In the second case, the same situation as in the above *Example* remains as it is; however, in the first case, the condition (2·2)$'$ is not necessarily true for a general $\mu\in\mathfrak{M}(E)$ and yet $\phi(\mu)\in L_\infty(E)$ is false.

4. 8. *Example 5*. Again, let E be n-dimensional, $E=R^n$ for $n\geqq 3$, and D a domain $\subset E$, in which a Green's function $G(x,y)$ is regular; leave the definitions of $\mathfrak{L}^+(D)$ and $\varDelta_G(G\subset D)$ as those in *Example 3* and adopts that of ϕ as follows;

$$\phi(\mu)(x)=\int_G G(x,y)d\mu(y). \tag{4·11}$$

For thus defined $\varDelta_D$ and ϕ, it is easily seen that the whole circumstances as in *Example* 3 remain completely. In fact, we have $\varDelta_D(\phi(\mu))=\varDelta_D(\mathring{\phi}(\mu)-\int_D h(x,y)d\mu(y))=\varDelta_D\mathring{\phi}(\mu)=\mu$, where $\mathring{\phi}$ denotes the potential operator ϕ in *Example 3* and $H(x)=\int_D h(x,y)d\mu(y)$ is harmonic in D in the sense of *Example 3*.

4. 9. *Example 6*. Let $E=R^n(n\geqq 2)$ and define $\hat{\mathfrak{M}}_\alpha(E)$ as a subspace of $\mathfrak{M}(E)$ constituted by such measures μ that

$$\|\mu\|_\alpha^2=\int[\phi_{\frac{\alpha}{2}}(\mu)(x)]^2dx<+\infty \qquad (0<\alpha<n),$$

where

$$(4\cdot12) \qquad \phi_\beta(\mu) = \int H_n(\beta) r^{\beta-n}\, d\mu, \qquad H_n(\beta) = \frac{\left(\frac{n-\beta}{2}\right)\pi^{-n/2}}{2^\beta\cdot\Gamma(n/2)}$$

for a certain fixed β, $0<\beta<n$ (the *potential of β-order* in the sence of M. Riesz). Take as $\mathfrak{L}(E)$ the space of $\phi_\alpha(\mu)$ for all $\mu\in\hat{\mathfrak{M}}_\alpha(E)$, which is linear since $\|\mu_1+\mu_2\|_\alpha \leqq \|\mu_1\|_\alpha + \|\mu_2\|_\alpha$ (see H. Cartan [3])). Then, we can define the operators $\varDelta$ and ϕ as follows;

$$(4\cdot12) \qquad \varDelta f=\mu(\text{for } f=\phi_\alpha(\mu)) \text{ and } \phi(\mu)=\phi_\alpha(\mu),$$

for which $\varDelta\phi(\mu)=\mu$.

§ 5. Spectrum, mean value and capacity.

5. 1. We now observe some important values related to the generalized Laplacian. Let Π_x be a certain directed family of measures $\pi_x^k\in\hat{\mathfrak{M}}(E)$, $x\in E$, such that

$$(5\cdot1) \qquad \lim_{k\to\infty}\phi(\pi_x^k)(t)=\begin{cases}1, & \text{if } t=x,\\ 0 & \text{if } t\neq x.\end{cases}$$

Then we call the value

$$(5\cdot2) \qquad \sigma(f,x)=\lim_{k\to\infty}\int(f-f_E)\,d\pi_x^k,\ f\in\mathfrak{L}^+(E)$$

the *spectrum* (or more precisely *point-spectrum*) of f in x. In virtue of (2·2), when every $\phi(\pi_x^k)$ is integrable with pespect to $\varDelta(f)$, we have

$$(5\cdot3) \qquad \sigma(f,x)=\lim_{k\to\infty}\int\phi(\pi_x^k)d\varDelta(f).$$

Thus, if $\varDelta(f)$ has a continuous density in a neighborhood of x, (5·2) yields that $\sigma(f,x)=0$; and so $\sigma(\phi(\varphi),x)=0$ for all $x\in E$ if $\varphi\in K(E)$, where E is a topological group.

In the *Examples* in § 4, where E is assumed to be n-dimensional Euclidean space for $1\leqq n<+\infty$, one may meet with two distinguished cases; let $\Sigma_{x,k}$ ($k=1,2,\cdots$) be spheres (closed intervals for $n=1$) with common center x, and Λ_x the series of measures $\lambda_x^k=dx_{\Sigma_{x,k}}$, restrictions of dx in $\Sigma_{x,k}$.

A) In the first case (*Example* 1, 2)*), x shall be always fixed upon the original point e of E and the radius of $\Sigma_k=\Sigma_{e,k}$ will tend to $+\infty$ as $k\longrightarrow+\infty$. Since $\phi(\mu)(e)=\int d\mu$ for any $\mu\in\hat{\mathfrak{M}}(E)$, it holds that $\sigma_k\equiv\phi(\lambda_e^k)(e)=\int_{\Sigma_k}dx$ = *total mass of Σ_k*, for the Lebesgue measure dx in usually topologized E. Thus, one may put

$$(5\cdot4) \qquad \pi_x^k=\phi(\mathcal{E}_{-x})\frac{\lambda_e^k}{\sigma_k}=e^{-itx}\frac{\lambda_e^k}{\sigma_k},$$

which statisfies the condition (5·1). Indeed, we see

$$\lim_{k\to\infty}\phi(\pi_x^k)(y)=\lim_{k\to\infty}\frac{1}{\sigma_k}\int e^{it(x-y)}d\lambda_e^k=\lim_{k\to\infty}\frac{1}{\sigma_k}\int e^{it(x-y)}dx, \text{ which is } = 1 \text{ if } x=y,$$

otherwise = 0.

*) In this case, $f_E=0$, so that $\sigma(f,x)=\lim_{k\to\infty}\int f\,d\pi_x^k$.

If E is one-dimensional, for a suitable non-decreasing bounded function $V(x)$*) defined on $(-\infty, \infty)$, $f \in \mathfrak{L}^+(E)$ is represented as $f(x) = \int_{-\infty}^{\infty} e^{ixt} dV(t)$, cf. (4·5), and by plain calculation it is seen that

$$\sigma(f, x) = \lim_{\omega \to \infty} \frac{1}{2\omega} \int_{-\omega}^{\omega} f(t) e^{-ixt} dt = V(t+0) - V(t-0) .$$

The spectrum in e, $\sigma(f) = \sigma(f, e)$, is called the *mean value of* f, which is linear with respect to f, $\sigma(\alpha f + g) = \alpha\sigma(f) + \sigma(g)$, and $\sigma(f) \geqq 0$ for $f \geqq 0$. This plays an important rôle especially in *Example* 2, on account of its invariability for translations, that is, as is easily seen it holds

$$\sigma(f) = \sigma(f_s) \quad \text{for} \quad f_s(x) = f(x-s),$$

whatever $s \in E$ may be. Such is a proper situation to *Example* 2.

B) In the second case (*Example* 3, 4), we can take as the sequence of measures

$$(5\cdot4)' \qquad \pi_x^k = \frac{\lambda_x^k}{\sigma_k},$$

where the radius of $\Sigma_{x,k}$ will tend to 0 as $k \longrightarrow +\infty$, contrarily to the former case. Putting now $\rho_k \equiv \int_{\Sigma_{x,k}} dx$ = *total mass of* $\Sigma_{x,k}$, the value $\rho_k(f, x)$;

$$(5\cdot5) \qquad \rho_k(f, x) = \frac{1}{\rho_k} \int f d\lambda_x^k$$

is called the *mean value of* f in $\Sigma_{x,k}$. We remark; in the first case, it happens that $\sigma_k = \rho_k$ for all k, but in the latter $\sigma_k \neq \rho_k$.

5. 2. In the case where the assumptions (R. 1)～(R. 3) are held and the kernel function of ϕ is a function only of distance we have that $\phi(x) \frown k$ is in $\mathfrak{L}^+(E)$ for every constant $k > 0$ and moreover it is equal to $\phi(\lambda_x(k))$ for a suitable $\lambda_x(k) \in \mathfrak{M}^+(E)$. Denote by $\partial\Sigma_x(k)$ the set of such $t \in E$ that $\phi(\lambda_x(k))(t) = k$, we see by the metric condition for the kernel function of ϕ that $\partial\Sigma_x(k)$ must be a surface of sphere with the center x and $\lambda_k(k)$ is distributed on $\partial\Sigma_x(k)$, since $\phi(\lambda_x(k))$ is harmonic outside of $\partial\Sigma_x(k)$.

We call such $\lambda_x(k)$ the *spherical mean of* ε_x on $\partial\Sigma_x(k)$; conversely, for any sphere Σ_x with center x, we have the spherical mean of ε_x on $\partial\Sigma_x$. To see this, we only put $k = \phi(\varepsilon_x)(t)$ for $t \in \partial\Sigma_x$; we shall write $\lambda_{\Sigma_x} = \lambda_x(\phi(\varepsilon_x)(t))$ for $t \in \partial\Sigma_x$.

For every $f \in \mathfrak{L}^+(D)$ and sphere $\Sigma_x \subset D$, the value

$$(5\cdot6) \qquad \lambda_{\Sigma_x}(f) = \int f d\lambda_{\Sigma_x}$$

is call *spherical mean of* f on Σ_x. Since $\phi(\lambda_{\Sigma_x}) = \phi(x) \frown k \leqq \phi(x)$ for $k = \phi(x)(t)$, $t \in \Sigma_x$, we have $\phi(\nu)(x) - \lambda_{\Sigma_x}(\phi(\nu)) = \int \phi(\nu) d(\varepsilon_x - \lambda_{\Sigma_x}) = \int (\phi(x) - \phi(\lambda_{\Sigma_x})) d\nu \geqq 0$ (or $= 0$ if the support of ν is contained in $E - \Sigma_x$) for any $\nu \in \mathfrak{M}^+(E)$, so that $(\varepsilon_x)_{\Gamma}^{0} = \lambda_{\Sigma_x}$ for $\Sigma_x = \bar{D}$ (since $f(x) = \lambda_{\Sigma_x}(f)$ for all $f \in H(\Sigma_x)$); about the notations $H(\cdot)$, $(\cdot)_{\Gamma}^{0}$, etc. see §**5.** Thus, we state

*) $V(x) = \int_{-\infty}^{x} d\Delta(f)$

PROPOSITION 7. *For $f \in \mathfrak{L}^+(D)$ (or $\in \mathfrak{H}(D)$) and any $\Sigma_x \subset D$,*

(5·7) $$f(x) \geqq \lambda_{\Sigma_x}(f) \ \ (or\ resp. = \lambda_{\Sigma_x}(f)).$$

5. 3. Hereafter, let E be always n-dimensional Euclidean space and put $\Phi_0(x, y) = r^{-1}(x, y)$; $4\pi\Phi_0(x, y)$ is obviously the kernel function of 3-dimensional newtonian potential. For a suitable function $\eta(t)$ defined for $0 < t < +\infty$ (or $-\infty < t < +\infty$), which is monotone increasing, continuous there, providing that $\lim_{t\to\infty} \eta(t) = +\infty$ and $\lim_{t\to 0} \eta(t) = 0$ (or resp. $\lim_{t\to -\infty} \eta(t) = 0$), if we can put

(5·8) $$\Phi_0(x, y) = \eta(\Phi(x, y)) \text{ for every } x, y \in E,$$

then the kernel function $\Phi(x, y)$ of a cartain potential operator ϕ is said to be *of type* η: $\eta(t)$ is usually convex or concave, since so is Φ itself.

The exact forms of such $\eta(t)$ for some concrete examples shall be enumerated as follows:

i) If $n = 3$ and ϕ is newtonian, $\eta(t) = 4\pi\cdot t$.

ii) If $n > 3$ and ϕ is newtonian, $\eta(t) = C_n t^{\frac{1}{n-2}}$,

where

$$C_n = \frac{1}{\sqrt[n-2]{N_n}} = \begin{cases} \dfrac{1}{\pi^{1/2}} \sqrt[2m-2]{\dfrac{(m-2)\,!}{4\pi}} & \text{for } n = 2m, \\ \dfrac{1}{\pi^{1/2}} \sqrt[2m-2]{\dfrac{1\cdot 3\cdot 5 \cdots\cdots (2m-3)}{2^{m+1}\pi}} & \text{for } n = 2m+1. \end{cases}$$

iii) If $n = 2$ and ϕ is logarithmic type, $\eta(t) = exp(t/2\pi)$

iv) If $n \geqq 2$ and ϕ is of order α in the sense of M. Riesz, $0 < \alpha < n$,

$$\eta(t) = t^{\frac{1}{\alpha-1}} / \sqrt[n-\alpha]{H_n(\alpha)}\,.$$

We shall hereafter restrict ourselves to such η that its inverse function $\eta^-(\cdot)$ would exist; η^{-1} is also continuous, monotone increasing and satisfies that $\lim_{t\to\infty} \eta^{-1}(t) = +\infty$. We define the *$\eta$-norm of $f \in \mathfrak{L}^+(E)$* as follows;

(5·9) $$\| f \|_\eta = \sup_{x \in E} \eta(f(x))$$

for $f \in \mathfrak{L}^+(E)$, where η is adopted in accordance with the considering potential ϕ. On the other hand, $(f)_\infty$ means $\sup_{x \in E} f(x)$; $(f)_\infty$ may be negative, nevertheless $\| f \|_\eta$ is always positive, *i.e.*

i) $\| f \|_\eta \geqq 0$,

and morever

ii) $\eta((f)_\infty) = \| f \|_\eta$,

iii) if $f \geqq 0$, then $(f)_\infty = \| f \|_\infty$ $(= \sup_{x \in E} | f(x) |) \geqq 0$.

5. 4. In view of the relation mentioned in **2. 3** (2·2), we can extend the definition of the local laplacian Δ_D for a general open set or compact X in E in such a fashion that

(5·10) $$\Delta_X(f) = \pi_X(\Delta(f)) \ (= (\Delta f)_X),$$

where $\pi_X(\mu)$ means the restriction of μ in X. For a countable union or intersection of compact or open sets X_i, we can define also the local laplacian $\Delta_{\cup X_i}$, $\Delta_{\cap X_i}$, in repeating the followin two transpositions;

(5·11) $$\Delta_{X_1 \cap X_2}(f) = \Delta_{X_1}(\phi \Delta_{X_2}(f)) = \Delta_{X_2}(\phi \Delta_{X_1}(f))$$

and

(5·11)′ $$\Delta_{X_1 \cup X_2}(f) = \Delta_{X_1}(f) + \Delta_{X_2}(f) - \Delta_{X_1 \cap X_2}(f).$$

Matter of course, $f \in \mathfrak{L}^+(D)$ brings $\Delta_X(f) \in \mathfrak{M}^+(D)$ for any such X that is a countable combination, by taking intersection and union, of compact or open sets $X_i \subset D$.

Now we shall define the norm of operator $|||\Delta_X|||$ as as follows: let an open or compact set $X \subset E$ be given, then we define

(5·12) $$|||\Delta_X||| = \sup_{(f)_\infty \leqq 1} \|\Delta_X(f)\|,$$

for $f \in \mathfrak{L}^+(E)$ such that $f = \phi \Delta f$, where $\|\mu\|$ means the norm of measure μ (in the present case, $\|\mu\| = \int d\mu$ since $f \in \mathfrak{L}^+(E)$ implies $\Delta_X(f) \in \mathfrak{M}^+(E)$). Thus defined norm $|||\Delta_X|||$ satisfies the following condions;

i) $|||\Delta_X||| \geqq 0$,

ii) if $X \subset Y$, then $|||\Delta_X||| \leqq |||\Delta_Y|||$,

iii) $|||\Delta_{X \cup Y}||| \leqq |||\Delta_X||| + |||\Delta_Y|||$.

Proof. i) and ii) are trivial. To see iii), it is sufficient to prove for the case where X and Y are mutually disjointed on account of ii); indeed, we see

$$|||\Delta_{X \cup Y}||| = \sup_{(f)_\infty \leqq 1} \|\Delta_{X \cup Y}(f)\| \leqq \sup_{(f)_\infty \leqq 1} (\|\Delta_X(f)\| + \|\Delta_Y(f)\|) \leqq \sup_{(f)_\infty \leqq 1} \|\Delta_X(f)\| + \sup_{(g)_\infty \leqq 1} \|\Delta_Y(g)\| = |||\Delta_X||| + |||\Delta_Y|||.$$

Next, let $\Phi(x, y)$ be of η-type for such a η that $M = \max((f)_\infty, \|f\|_\eta) \neq 0$ as far as $\Delta(f) \neq 0$. Then we have an important inequality:

PROPOSITION 8. *for every $f \in \mathfrak{L}^+(E)$ with $f = \phi \Delta(f)$, it holds*

(5·13) $$\|\Delta_X(f)\| \leqq M \cdot |||\Delta_X|||.$$

Proof. Put $f_0 = f/M$ (assuming $\Delta f \neq 0$), then $(f_0)_\infty = (f)_\infty / M \leqq 1$ and by definition $|||\Delta_X||| \geqq \|\Delta_Y(f_0)\| = \|\Delta_X(f)\|/M$, from which follows (5·13). If $\Delta f = 0$, then the assertion is trivial.

We shall remark that, in the quoted examples in **5. 3**, we find each η satisfies the above condition, so that (5·13) is valid in these examples.

5. 5. We shall now define the capacity by using the *η-norm*: as well known, there have been several ways to define the capacity, for instance, that of N. Wiener, de La Vallée Poussin, O. Fastman [7], or H. Cartan [4], etc., and unfortunately they are not necessarily conciliatory each other. Our present methode of definition is akin to that of O. Frostman, but includes any other ones mentioned above.

For a compact or open $X \subset E$, we define the *capacity* $c(X)$ as follows; for $f \in \mathfrak{L}^+(E)$ with $f = \phi \Delta_X(f)$ and $\|\Delta_X(f)\| = 1$,

(5·14) $$c(X) = K_\eta \sup(\|f\|_\eta^{-1}), \quad K_\eta = \eta(1).$$

For a general set $X \subset E$, we define further the inner capacity $c^i(X)$ of X by $\sup(c(F))$ for all compact F such that $F \subset X$. In a dual fashion, we define the outer capacity $c^e(X)$ of X by $\inf(c(U))$ for all open U such that $X \subset U$. Therefore, if X is compact, $c(X) = c^i(X)$ and if X is open, $c(X) = c^e(X)$.

As immediate consequences of definition, we have:

c_1) $\quad c(X) \geqq 0$, and $c^i(X), c^e(X) \geqq 0$.

c_2) If $X \subset Y$, then $c(X) \leqq c(Y)$ (X, Y being compact or open) and $c^i(X) \leqq c^i(Y)$, $c^e(X) \leqq c^e(Y)$,

c_3) $\quad c^i(X) \leqq c^e(X)$ for every X.

Moreover, we have an important inequality;

c_4) $\quad \eta^{-1}(K_\eta c(X)^{-1}) \leqq (f)_\infty / \| \Delta_X(f) \|$.

From this, we get a characteristic property of capacity zero, which plays an important rôle in the later discussions. That is:

Proposition 9. *If $c^i(X)$ is zero, then there exists no measure μ having the support in X and whose potential $\phi(\mu)$ is finite.*

Proof. Let F be a compact set contained in X and suppose that $\pi_F(\mu) \neq 0$; then $0 \leqq c(F) \leqq c^i(X) = 0$ and hence by c_4) $+\infty \leqq \eta^{-1}(0^{-1}) \leqq (\phi(\mu))_\infty / \| \mu \| < +\infty$, which is absured.

We shall next extend the notion of "*f-capacity*" defined in **2. 6.** for a general open or compact $X \subset E$ as follows; putting $[f, g]_X = \int_X (f d\Delta g - g d\Delta f)$ for $f, g \in \mathfrak{L}^+(E)$ and

$$(2\cdot6)' \qquad [f]_X = [1, f]_X = \int_X d\Delta f,$$

we call $\eta([f]_X)/K_\eta$ the "*f-(η-) capacity of X.*" Then we assert:

c_5) Let X be compact and η is separating type (see **5. 6.**) ; if there exists such $f_0 \in \mathfrak{L}^+(E)$ with $f_0 = \phi\Delta_X(f_0)$ that $f_0 = 1$ on X except a set of capacity 0, then

$$(5\cdot16) \qquad c(X) = \frac{1}{K_\eta}\eta([f_0]_X).$$

In fact, suppose now there were such $f \in \mathfrak{L}^+(E)$ that $f = \phi\Delta_X(f)$, $\|\Delta_X(f)\| = 1$ and $(f_\infty) < 1/v$, $v = [f_0]_X = \|\Delta_X(f_0)\|$, then we have $1 = \int f_0 d\Delta_X(f) = \int f d\Delta_X(f_0) < \frac{1}{v}\int d\Delta_X(f_0) = 1$ (refer to Prop. 10 above), which is absurd; so that, it must be $\|f\|_\eta^{-1} \leqq \eta(v)/K_\eta^2$ for every $f \in \mathfrak{L}^+(E)$ with $\|\Delta_X(f)\| = 1$ and $f = \phi\Delta_X(f)$, from which (5·16). Such f_0 is called "*equilibrum potential of X.*" In regard to **2. 6.**, for Newtnian η in $E = R^3$, we have $c(X) = N_3[f_0]_X = N_3[f_0]_G = -4\pi\iint_{\partial G}\frac{\partial f_0}{\partial n}dS$ (Wiener's definition) for a regular domain $G \supset X$.

5. 6. We shall proceed to study about the capacity and consider the relation between the norm of Δ_X and the capacity. Therefore, we should restrict ourselves to the to the case where $\Phi(x, y)$ is of η-type. For the present discussion, it is convenient to name two distinguished type of η; the first is *separating type*, that is $\eta(ts) = \frac{1}{K_\eta}\eta(t)\eta(s)$ for every $t, s \geqq 0$. Such is the one appeared in the example i), ii), or

iii) in **5. 3**, while in the example iv) there, η has the following property; $\eta(t+s) = \eta(t)\eta(s)$. This type is called *character type.*

THEOREM 9. *If η is of separating type, then it holds*

$$c(X) = \frac{1}{K_\eta}\eta(|||\varDelta_X|||). \tag{5・15}$$

Proof. 1) We see at first that, if η is separating type, then every $f \in \mathfrak{L}^+(E)$ with $f = \phi\varDelta(f)$ is non-negative. In fact, $\varPhi(x, y) = \eta^{-1}(\varPhi_0(x, y))$ is always non-negative whatever x, y may be, hence so is $\phi\varDelta(f) = \int \varPhi(x, y)d\varDelta(f)$ as far as $f \in \mathfrak{L}^+(E)$.

By this reason, the condition $f = \phi\varDelta_X(f)$ in the *definition of the capacity* may be weakened as follows;

$$c(X) = K_\eta \sup_{\|\varDelta_X(f)\|=1} \|f\|_\eta^{-1}, \text{ for } f = \phi\varDelta(f) \in \mathfrak{L}^+(E). \tag{5・12)$'$}$$

2) Let $\|\varDelta_X(f)\| = 1$ and $(f)_\infty = k > 0$, then putting $f_0 = f/k$, we see $f_0 = \phi\varDelta(f_0) \in \mathfrak{L}^+(E)$ and $(f_0)_\infty = 1$, so that $|||\varDelta_X||| \geqq \|\varDelta_X(f_0)\| = \|\varDelta_X(f)\|/k = 1/(f)_\infty$. Varying f under the restriction $\|\varDelta_X(f)\| = 1$ and $f = \phi\varDelta_X(f)$, we have

$$\eta(|||\varDelta_X|||) \geqq \sup \eta\left(\frac{1}{(f)_\infty}\right) = \sup \frac{K_\eta^2}{\eta((f)_\infty)} = K_\eta^2 \sup \|f\|_\eta^{-1} = K_\eta c(X).$$

3) Conversely, let $(f)_\infty \leqq 1$, $f = \phi\varDelta(f)$ and $\|\varDelta_X(f)\| = m > 0$, then putting $f_0 = f/m$, we see that $f_0 = \phi\varDelta(f_0)$, $(f_0)_\infty \leqq 1/m$ and $\|\varDelta_X(f_0)\| = 1$, so that by definition $c(X) \geqq K_\eta \|f_0\|_\eta^{-1} = K_\eta/\eta((f)_\infty) \geqq \frac{1}{K_\eta}\eta(m)$. Varying $f = \phi\varDelta(f)$ under the the condition $(f)_\infty \leqq 1$, we have $c(X) \geqq \frac{1}{K_\mu} \sup \eta(\|\varDelta_X(f)\|) = \frac{1}{K_\mu}\eta(|||\varDelta_X|||)$. Combining this with 2), we conclude (5・13). Thus, the proof is completed.

COROLLARY 1. *If η is of separating type, we have*

$$\eta^{-1}(K_\eta c(\sum_{i=1}^n X_i)) \leqq \sum_{i=1}^n \eta^{-1}(K_\eta c(X_i)). \tag{5・17}$$

By (5・15) and iii) in **5. 4**, this is evident.

Partictlarly, when E is n-dimentional ($n \geqq 3$) and $\varPhi$ is newtonian or otherwise of order α for $n-1 \geqq \alpha > 0$, then it holds

$$c(\sum_{i=1}^n X_i) \leqq \sum_{i=1}^n c(X_i). \tag{5・17)$'$}$$

Indeed, let $k = n-2$ and $N = 1/\sqrt[n-2]{N_n}$ in the first cace, or let $k = n-\alpha$ and $N = 1/\sqrt[n-\alpha]{H_n(\alpha)}$ in the second case (see §4), then we have $k \geqq 1$ and by the above Corollary $\frac{1}{N^k}(Nc(\sum_{i=1}^n X_i))^k \leqq \sum_{i=1}^n \frac{1}{N^k}(Nc(K_i))^k$, from which it comes by using Minkowski's inequality that $c(\sum_{i=1}^n X_i) \leqq (\sum_{i=1}^n c(X_i)^k)^{1/k} \leqq \sum_{i=1}^n c(X_i)$.

COROLLARY 2. $\eta(\|\varDelta_X(f)\|) \leqq \|f\|_\eta \cdot c(X)$ (precision of (5・13)).

This is an immediate consequence of c_4).

In newtoniain or α-order potential case, independing upon the assigned density to $\varPhi$, the definition of capacity here is just equivalent to that of de La Vallée Poussin or of Frostman as is easily seen. For example, if E is 3-dinemsional, one has

$$c(X) = |||\,\Delta_X\,|||\,.$$

5. 7. Let us now consider about η of *chracter type*. Since η is continuous, η should be in the from $\eta(t) = e^{Kt}$ for a certain constant K. Here, we shall restrict ourselves to a positive K(*e.g.*, in the example iii) **5. 3**, we have $K = 1/2\pi$), and suppose further that there exists a $f \in \mathfrak{L}^+(E)$ with $f = \phi\Delta_X(f)$, such that $c(X) = K_\eta \|f\|_\eta^{-1}$ and $\|\Delta_X(f)\| = 1$ (such is the equilibrum potention of X, if its existenc were proved). Then, it is possible to write $c(X) = K_\eta e^{-KV}$, where $V = (f)_\infty$.

If $V \leqq 1$, then $1 = \|\Delta_X(f)\| \leqq |||\Delta_X|||$, so that $KV \leqq K \leqq K|||\Delta_X|||$ and hence $c(X) = K_\eta e^{-KV} \geqq K_\eta e^{-K|||\Delta_X|||}$.

If $V \geqq 1$, then $f_0 = f/V \in \mathfrak{L}^+(E)$, $f_0 = \phi\Delta(f_0)$ and $(f_0)_\infty = 1$, so that $1/V = \|\Delta_X(f)\|/V = \|\Delta_X(f_0)\| \leqq |||\Delta_X|||$. Therefore, $c(X) = K_\eta e^{-KV} \leqq K_\eta e^{-K/|||\Delta_X|||}$. Summarizing these, we assert:

Theorem 10. *If η is of character type and X admits such a measure μ that* $c(X) = \|\phi(\mu)\|_\eta^{-1}$, then

α) $c(X) \geqq K_\eta/\eta(|||\Delta_X|||)$, when $c(X) \geqq 1$,

β) $c(X) \leqq K_\eta/\eta(|||\Delta_X|||^{-1})$, when $c(X) \leqq 1$.

§ 6. General construction of balayage.

6. 1. Let E be a general locally compact space. Throughout this paragraph and the next, we shall denote by letters F, ∂F, $int F$ and $ext F$, a compact set in E, its boundary, interior and exterior respectively. Further, $C(\cdot)$ denotes a Banach space (simultaneously, Banach algebra with respect to the usual product and the usual norm) of all continuous functions defined in $(\cdot)$.

Now we shall constitute a normed linear space $H(D)$, $D = int F$ of a given compact F, which is generated of all such bounded $f \in \mathfrak{L}^+(E)$ that $\Delta(f) \in \mathfrak{M}(E-D)$, with respect to the norm;

(6·1) $\|f\|_D = \sup_{x \in D} |f(x)|$ $(= \sup_{x \in \Gamma} |f(x)|$ for a certain $\Gamma \subset F)$.

Remark: Matter of course, since such f is bounded in $\bar{D}$, so that we might as well consider $H(D)$ as to be constituted of the collection $\mathfrak{L}_D$ of all such functions $f \in \mathfrak{L}(E) \frown B(\bar{D})$ that $\Delta(f) \in \mathfrak{M}(E-D)$; then denoting by $\mathfrak{N}_D$ the linear subspace of $\mathfrak{L}_D$ consisting of all such f that $\|f\|_D = 0$, we have $H(D) \cong \mathfrak{L}_D/\mathfrak{N}_D$. So far as there occurs no confusion, we shall use the same letter f for an element of $H(D)$ as well for the corresponding function of $\mathfrak{L}_D$.

We shall assume that:

*) Some subspace $B_0(E)$ of $B(E)$, the space of all bounded potentials in E, is dense in $C(F)$ on every $F \subset E$.

Next, $\mathfrak{M}_0(D)$ denotes a convex set of all such $\mu \in \mathfrak{M}^+(\bar{D}) = \mathfrak{M}^+(F)$ that $\|\mu\| = \int d\mu = 1$. Then, every $\mu \in \mathfrak{M}_0(D)$ defines a linear continuous functional $\mu^\wedge$ on the linear normed space $H(D)$ in the following manner:

$$(6\cdot2)\qquad \mu^{\wedge}(f) = \int f\,d\mu.$$

The collection, $\mathfrak{M}_0^{\wedge}(D)$, of such $\mu^{\wedge}$ forms a weakly relatively compact convex set in the dual space $(H(D))^*$ of $H(D)$, with respect to the weak topology as funtionals, which shall be called w^*-topology hereafter; that is,

PROPOSITION 10. *$\overline{\mathfrak{M}}_0^{\wedge}(D)$ is w^*-compact convex in $(H(D))^*$, $=\mathfrak{M}_0(D)$ vaguely.*

The convexity is obvious, since so is $\mathfrak{M}_0(D)$. Thus we need only to prove that $\mathfrak{M}_0^{\wedge}(D)$ is w^*-compact in $(H(D))^*$, and it is achieved in the following manner.

In fact, we see at first that $\mathfrak{M}_0(D)$ is vaguely compact in $\mathfrak{M}(E)$, since the set of such $\mu\in\mathfrak{M}(E)$ that $\|\mu\|=1$ is relatively compact (see N. Bourbaki [1]) and $\mathfrak{M}_0(D)$ is closed for the vague topology. On the other hand, $H(D)$ is considered as a linear subspace of $B(D)=B(\overline{D})/\mathfrak{N}_D$; $\mathfrak{M}_0(D)$ is contained in the unit sphere Ξ of $(B(D))^*$, whose w^*-topology is stronger than that of $(C(F))^*=\mathfrak{M}(F)$; Ξ is w^*-compact in $(B(D))^*$, so that $\overline{\mathfrak{M}}_0(D)$ is w^*-compact, too. Since the application $\mu\longrightarrow\mu^{\wedge}$, defined by $(6\cdot2)$, from the dual space of $B(D)$ into $(H(D))^*$ is continuous, so that $\overline{\mathfrak{M}}_0^{\wedge}(D)$ is also compact and $=\mathfrak{M}_0(D)$ *vaguely* in $(H(D))^*$, which completes the proof of Proposition.

6. 2. Our chief tool to construct the balayage is the noted theorem of M. Krein-D. Milman [10] on extreme points of regular convex set; see also N. Bourbaki [2]. By that theorem, we can state:

THEOREM 11. *$\mathfrak{M}_0^{\sim}(D)=\overline{\mathfrak{M}}_0^{\wedge}(D)$ possesses the extreme points, whose closed convexly linear envelope coincides with $\mathfrak{M}_0^{\sim}(D)$ itself; in other words, denoting the set of all extreme points of $\mathfrak{M}_0^{\sim}(D)$ by Ext. $\mathfrak{M}_0^{\sim}(D)$, for any $\mu^{\sim}\in\mathfrak{M}_0^{\sim}(D)$, $f\in H(D)$ and $\varepsilon>0$, there exist a finite number of $\mu_i^{\sim}\in$ Ext. $\mathfrak{M}_0^{\sim}(D)$ such that*

$$(6\cdot3)\qquad |\mu^{\sim}(f)-\textstyle\sum\alpha_i\mu_i^{\sim}(f)|<\varepsilon,\ \text{where}\ \sum\alpha_i=1,\ \alpha_i>0.^{*)}$$

Throughout the following discussions, we shall set a natural assumption that: for any two points x_1 and x_2, $x_1\neq x_2$ of F, there exists at least one point $z\in E-F$ such that

$$(6\cdot4)\qquad \phi(x_1)(z)\neq\phi(x_2)(z).$$

This assumption is valid in all the *Examples* quoted in §4: for instance, in *Example 1, 2*, it is clear since $\phi(x)=e^{i\lambda x}$, and in *Example 3, 4*, we owe it to the fact that $\phi(x)$ is defined by metric condition.

Then under this assumption we have

THEOREM 12. *If $\mu^{\wedge}\in$ Ext. $\mathfrak{M}_0^{\sim}(D)$, μ is a point-measure of total mass $+1$, ε_x, placed in a certain point $x\in\overline{D}=F$.*

Proof. Suppose now the support K_μ of μ contains at least two points x_1 and x_2, $x_1\neq x_2$, and take neighborhoods $U(x_i)$ of x_i $(i=1,2)$ such that $U(x_1)\cap U(x_2)$ is empty. Denote further the restrictions of μ in $U(x_1)$, $U(x_2)$ and $K_\mu-(U(x_1)\cup U(x_2))$ by μ_1, μ_2 and μ_3 respectively. Setting $\mu_i^*=\mu_i/\alpha_i$ for $\alpha_i=\|\mu_i\|\geqq0$ $(i=1,$

*) Since $\mathfrak{M}_0(D)\subset\overline{\mathfrak{M}}_0^{\wedge}(\Gamma)$ (in $(H(D))^*$), we can take $\mu_i\in\mathfrak{M}_0^{\wedge}(\Gamma)$ as such $\mu_i^{\sim}$; in fact, $\overline{\mathfrak{M}}^{\wedge}(\Gamma)$ is also regularly convex and any $\mu^{\sim}\in\mathfrak{M}_0^{\sim}(D)$ cannot be separated from $\overline{\mathfrak{M}}_0^{\wedge}(\Gamma)$.

2 and 3), but if $\mu_3 = 0$, μ_3^* should be set $= 0$, we have $(\mu_i^*)^\wedge \in \mathfrak{M}_0^\wedge(D)$ and

$$\mu = \sum_{i=1}^{3} \alpha_i \mu_i^*, \qquad \sum_{i=1}^{3} \alpha_i = 1. \tag{6·5}$$

If it has been proved that $(\mu_1^*)^\wedge \neq (\mu_2^*)^\wedge$, we have $\mu_0^* = (\alpha_1/(1-\alpha_3))\mu_1^{*\wedge} + (\alpha_2/(1-\alpha_3))\mu_2^{*\wedge}$ is an inner point of the segment combining $\mu_1^{*\wedge}$ with $\mu_2^{*\wedge}$, so that $\mu^\wedge = (1-\alpha_3)\mu_0^{*\wedge} + \alpha_3 \mu_3^{*\wedge}$ could not be extreme. Thus, we need only to prove $(\mu_1^*)^\wedge \neq (\mu_2^*)^\wedge$; indeed, by assumption there sxists a point y outside of F such that $\phi(y)(x_1) = \phi(x_1)(y) \neq \phi(y)(x_2)$. Since $\phi(y)$ is continous in D, we can take such neighborhoods $V(x_1)$ and $V(x_2)$ of x_1 and x_2 respectively that $\sup|\phi(y)(x_1') - \phi(y)(x_2')| > \varepsilon$ for a given sufficiently small ε (*e.g.* $< \frac{1}{2}|\phi(y)(x_1) - \phi(y)(x_2)|$), where $x_i' \in V(x_i)$ for $i=1, 2$.

If we have adopted $U(x_i)$ as these $V(x_i)$ $(i = 1, 2)$ from the first, we have certainly that $\int \phi_0(y)\, d\mu_1^* \neq \int \phi_0(y)\, d\mu_2^*$ for the restricted $\phi_0(y)$ of $\phi(y)$ in F ($\phi_0(y)$ is clearly considered as in $H(D)$), whence it follows $\mu_1^{*\wedge} \neq \mu_2^{*\wedge}$ as desired. This completes the proof of Theorem 12.

6. 3. The set of such x that $\varepsilon_x^\wedge$ belongs to $Ext.\mathfrak{M}_0^\sim(D)$ is denoted by Γ_0, then $\Gamma_0 \subset \bar{\Gamma}$ and $\bar{\Gamma}$ is compact. Therefore, $\mathfrak{M}_0(\bar{\Gamma})$ is vaguely compact.

For a given $\mu \in \mathfrak{M}_0(D)$, the collection of measures $\mu_{(f,\varepsilon)} = \sum \alpha_i \mu_i \in \mathfrak{M}_0(\bar{\Gamma})$ which satisfy (6·3) constitutes a base of filter $\mathfrak{F}_\mu$ in $\mathfrak{M}_0(\bar{\Gamma})$ and an ultrafilter (maximal filter) $\mathfrak{F}_\mu^0$ containing $\mathfrak{F}_\mu$ converges to a measure μ_Γ in $\mathfrak{M}_0(\bar{\Gamma})$. Such μ_Γ has the following fundamental properties;

B_1) $\mu_\Gamma^\wedge(f) = \mu^\wedge(f)$ for every $f \in H(D)$, or equivalently

$$\int f\, d\mu = \int f\, d\mu_\Gamma \qquad \text{for every } f \in \mathfrak{L}_D, \tag{6·6}$$

B_2) $\phi(\mu) = \phi(\mu_\Gamma)$ outside of F.

B_3) $(\alpha\mu_1 + \beta\mu_2)_\Gamma = \alpha\mu_{1_\Gamma} + \beta\mu_{2_\Gamma}$ for $\mu_1, \mu_2 \in \mathfrak{M}_0(D)$, $\alpha + \beta = 1$, if these are uniquely determined.

In fact, B_1) is a direct consequence of the definition; by the same reasoning as in the latter half of proof of Theorem 12, for any $y \in E-F$, we have $\phi_0(y) \in H(D)$ and $\phi(\mu)(y) = \int \phi_0(\varepsilon_y)\, d\mu = \int \phi_0(\varepsilon_y)\, d\mu_\Gamma = \phi(\mu_\Gamma)(y)$. Finally, B_3) comes from the following inequality; for every $\varepsilon > 0$, if $|\mu_1^\wedge(f) - \sum \alpha_i \mu_i^{1\wedge}(f)| < \varepsilon$ and $|\mu_2^\wedge(f) - \sum \beta_i \mu_i^{2\wedge}(f)| < \varepsilon$, then

$$\begin{aligned} &|(\alpha\mu_1 + \beta\mu_2)^\wedge(f) - [\alpha(\sum_i \alpha_i \mu_i^{1\wedge}(f)) + \beta(\sum_j \beta_j \mu_j^{2\wedge}(f))]| \\ &\leqq \alpha\, |\mu_1^\wedge(f) - \sum_i \alpha_i \mu_i^{1\wedge}(f)| + \beta\, |\mu_2^\wedge(f) - \sum_j \beta_j \mu_j^{2\wedge}(f)| \\ &< (\alpha + \beta)\varepsilon = \varepsilon, \end{aligned}$$

where $\alpha \sum_i \alpha_i + \beta \sum_j \beta_j = \alpha + \beta = 1$.

6. 4. Now, we define for any positive number α and for $\mu \in \mathfrak{M}_0(D)$ with unique μ_Γ

$$(\alpha\mu)_\Gamma = \alpha\mu_\Gamma, \text{ and } (-\mu)_\Gamma = -\mu_\Gamma; \tag{6·7}$$

then these are well defined and satisfy the above conditions $B_1) \sim B_3)$ as is easily verified. Moreover if $\alpha, \beta > 0$, then $(\alpha\mu+\beta\nu)_\Gamma = (\alpha+\beta)\left(\frac{\alpha}{\alpha+\beta}\mu+\frac{\beta}{\alpha+\beta}\nu\right)$ $= (\alpha+\beta)\left(\frac{\alpha}{\alpha+\beta}\mu_\Gamma+\frac{\beta}{\alpha+\beta}\nu_\Gamma\right)$(by B_3)) $= \alpha\mu_\Gamma+\beta\nu_\Gamma$ for $\mu, \nu \in \mathfrak{M}(\bar{D})$, whence it follows the linearity of the operation $(\cdot)$ for measures with unique μ_Γ.

We call such a measure μ_Γ of $\mathfrak{M}(\bar{\Gamma})$ which satisfies the conditions $B_1 \sim B_2)$ for a given $\mu \in \mathfrak{M}(F)$ a *balayaged mearure* of μ related to Γ.

Of course, a balayaged measure is not necessarily uniquely determined since it depends upon the selection of ultrafilter $\mathfrak{F}_\mu^0$ which contains $\mathfrak{F}_\mu$. However, we have especially:

PROPOSITION 11. *If* $x \in \Gamma_0$, $(\varepsilon_x)_\Gamma$ *is unique and coincides with* ε_x *itself, i.e.* $(\varepsilon_x)_\Gamma = \varepsilon_x$.

In fact, suppose now $\varepsilon_x \neq (\varepsilon_x)_\Gamma$ and put $\mu = \frac{1}{2}(\varepsilon_x+(\varepsilon_x)_\Gamma)$, then $\mu^\wedge = \frac{1}{2}(\varepsilon_x^\wedge + (\varepsilon_x)_\Gamma^\wedge) = (\varepsilon_x)_\Gamma^\wedge = \varepsilon_x^\wedge \in Ext.\mathfrak{M}_0^\sim(D)$, since $(\varepsilon_x)_\Gamma^\wedge = \varepsilon_x^\wedge$ by B_1). But we see, this contradicts with Theorem 12 because $\mu^\wedge$ is not a point measure by hypothesis, from which it follows the Proposition.

This property characterizes the set Γ_0 (whose point shall be called *regular*); the further characteristic properties of Γ_0 will be mentioned later.

PROPOSITION 12. $\phi(\mu) = \phi(\mu_\Gamma)$ *on* $\partial\bar{D}$ *excepting a set of capacity zero, and* $\int d\mu = \int d\mu_\Gamma$.

In fact, suppose now $\phi(\mu) > \phi(\mu_\Gamma)$ in $S \subset \partial\bar{D}$ with $c(S) > 0$, then we can find a compact $K \subset S$ on which a measure ν with $\phi(\nu) \in H(D)$ is placed. For such ν, we have $\int\phi(\nu)\,d\mu = \int\phi(\mu)d\nu > \int\phi(\mu_\Gamma)d\nu = \int\phi(\nu)\,d\mu_\Gamma$, which is absurd; thus $c(S) = 0$. By the same fashion, the set $\subset \partial\bar{D}$ in which $\phi(\mu_\Gamma) > \phi(\mu)$ is of capacity 0. About the last, refer to Theorem 7.

6. 5. We now consider the most important and interesting case, that is, real case where always $\bar{\Gamma} \subset \partial F$ and the assumptions (R. 1) $\sim$ (R. 3), (3. 1) in § 3 (consequently the maximum principle; Theorem 6) are held.

Owing to Prop. 7, if $(R.\ 1) \sim (R.\ 3)$ are held and if Φ is a function only of metric, then the condition $\bar{\Gamma}_0 \subset \partial\bar{D}$ is fulfiled since $\varepsilon_x^\wedge = \frac{1}{2}(\varepsilon_x^\wedge + \lambda_{\Sigma_x}^\wedge)$ and λ_{Σ_x} is not a point measure.

PROPOSITION 13 *If* $\phi(\mu)$ *is bounded, so is* $\phi(\mu_\Gamma)$ *and moreover* $\phi(\mu) \geqq \phi(\mu_\Gamma)$ *everywhere. Such* μ_Γ *is uniquely determined.*

Let $x \in \partial\bar{D}$, then we have for $y \in E-\bar{D}$

$$\phi(\mu)(x) = \lim_{y\to x}\phi(\mu)(y) = \varliminf_{y\to x}\phi(\mu_\Gamma)(y) \geqq \phi(\mu_\Gamma)(x),$$

Owing to Proy. 12, such μ_Γ is not distributed in any set on which $\phi(\mu) \neq \phi(\mu_\Gamma)$, that is, $\phi(\mu) = \phi(\mu_\Gamma)$ on a certain kernel of μ_Γ, so that $\phi(\mu) \geqq \phi(\mu_\Gamma)$ in

$E^{*)}$. Next, let μ'_Γ be another balayaged measure of μ, then we can consider μ_Γ or μ'_Γ being a balayaged measure of μ'_Γ or resp. of μ_Γ, so that $\phi(\mu_\Gamma)=\phi(\mu'_\Gamma)$ and hence $\mu_\Gamma = \Delta\phi(\mu_\Gamma) = \Delta\phi(\eta'_\Gamma) = \mu'_\Gamma$, which proves Prop. 13.

For every $\nu \in \hat{\mathfrak{M}}(E)$ with bounded $\phi(\nu)$, naturally $\phi(\nu_{\bar{D}})$ being bounded, put now

$$\nu_\Gamma = (\nu_{\bar{D}})_\nu + \nu_{E-\bar{D}},$$

then such ν_Γ satisfies also the conditions $B_1)\sim B_3)$ and Prop. 12~13. For every $\mu \in \mathfrak{M}^+(\bar{D})$, put further

$$L_\mu(\phi(\nu)) = \int \phi(\nu_\Gamma)\,d\mu \text{ for all } \phi(\nu) \in B(E).$$

Then, in virtue of the assumption *), such linear bounded functional L_μ (in fact, $|L_\mu(\phi(\nu))| = \left|\int\phi(\nu_\Gamma)d\mu\right| = \left|\int\phi(\mu)d\nu_\Gamma\right| \leqq \|\phi(\mu)\|_\infty \int |d\nu|$) defines a positive measure μ^0_Γ which satisfies

$$(6\cdot8) \qquad \int\phi(\nu)\,d\mu^0_\Gamma = \int\phi(\nu_\Gamma)\,d\mu.$$

On the other hand, for every $\phi(\mu) \geqq 0$ there exists a sequence of $\phi(\mu_j) \in B(E)$ such that $\phi(\mu_j) \nearrow \phi(\mu)$. Then, for any $\phi(\lambda) \in B(E)$

$$\int\phi(\lambda)\,d\mu^0_\Gamma = \int\phi(\lambda_\Gamma)\,d\mu = \int\phi(\mu)\,d\lambda_\Gamma = \lim\int\phi(\mu_j)\,d\lambda_\Gamma$$

$= \lim\limits_j \int\phi(\lambda)d(\mu_j)^0_\Gamma$, that is $(\mu_j)^0_\Gamma \longrightarrow \mu^0_\Gamma$ vaguely; so that, for $\nu \in \mathfrak{M}^+(\bar{D})$,

$$\int\phi(\nu)\,d\mu^0_\Gamma \leqq \varliminf_j \int\phi(\nu)\,d(\mu_j)_\Gamma = \varliminf_j \int\phi(\mu_j)\,d\nu^0_\Gamma = \int\phi(\mu)\,d\nu^0_\Gamma,$$

that is, $\int\phi(\nu)d\mu^0_\Gamma \leqq \int\phi(\mu)d\nu^0_\Gamma$. Replacing μ and ν, we have $\int\phi(\mu)d\nu^0_\Gamma \leqq \int\phi(\nu)d\mu^0_\Gamma$. Comming these, we conclude:

$$(6\cdot8)' \qquad \int\phi(\mu)\,d\nu^0_\Gamma = \int\phi(\nu)\,d\mu^0_\Gamma \text{ for } \mu,\ \nu \in \mathfrak{M}_0(D)$$

THEOREM 13. *μ^0_Γ is a balayaged measure of $\mu \in \mathfrak{M}_0(D)$, and if $\phi(\nu)$ is bounded, $\mu_\Gamma = \mu^0_\Gamma$.*

As an immediate consequence of Prop. 12 and Theorem 13, we have a further characterization of Γ_0 as follows:

PROPOSITION 14. *For any $x \in \Gamma_0$, it holds $(\varepsilon_x)^0_\Gamma = \varepsilon_x$, and hence*

$$(6\cdot9) \qquad \phi(\mu) = \phi(\mu^0_\Gamma) \quad \text{on} \quad \Gamma_0,$$

whatever $\mu \in \mathfrak{M}_0(D)$.

It is possible of course to define μ^0_Γ for a general positive μ or a compound $\mu = \nu_1 - \nu_2$ placed on $F = \bar{D}$, in accordance with extended μ_Γ defined by $(6\cdot7)$, for

---

*) Since $\int\phi(\mu)\,d\mu^0_\Gamma = \int\phi(\mu^0_\Gamma)\,d\mu^0_\Gamma$, we have

$(6\cdot8)''$ $\phi(\mu) = \phi(\mu^0_\Gamma)$ in certain kernel of μ^0_Γ;

we call here such X that $\int_{E-X} |d\nu| = 0$ a *kernel of* ν.

which we state:

PROPOSITION 15. *The application* $\mu \longrightarrow \mu_\Gamma^0$ *brings on a homomorphism of* $\mathfrak{M}(F)$ *into* $\mathfrak{M}(\partial F)$. *Moreover* $\mathfrak{M}(\Gamma_0)$ *is invariant for this application; that is,*

(6·10) $\mu = \mu_\Gamma^0$ *for every* $\mu \in \mathfrak{M}(\Gamma_0)$.

The latter half of the proposition comes from the fact that $\phi(\mu)(x) = \int \phi(\mathcal{E}_x)d\mu = \int \phi((\mathcal{E}_x)_\Gamma^0)d\mu$ (by (6·9)) $= \int \phi(\mu)d(\mathcal{E}_x)_\Gamma^0 = \int \phi(\mu_\Gamma^0)d\mathcal{E}_x$ (by (6·8)′) $= \phi(\mu_\Gamma^0)(x)$ for any $x \in F$; in $E-F$, $\phi(\mu) = \phi(\mu_\Gamma^0)$ is evident, so that this equality holds everywhere in E. Thus, we obtain $\mu = \varDelta\phi(\mu) = \varDelta\phi(\mu_\Gamma^0) = \mu_\Gamma^0$, which concludes the proof.

We call thus obtained μ_Γ^0 *properly balayaged measure* of μ on ∂F, while μ_Γ a *general balayaged measure* of it: hereafter, if we call merely *balayaged* measure of μ with no indefinite article, we shall always mean μ_Γ^0, and the operation $\mu \longrightarrow \mu_\Gamma^0$ is called the *balayage.*

We now want to characterize the balayaged measure μ_Γ^0: this is actually answered as follows;

THEOREM 14. *The balayaged measure* μ_Γ^0 *of* $\mu \in \mathfrak{M}^+(\overline{D})$ *satisfies the following two conditions,*

α) $\phi(\mu) = \phi(\mu_\Gamma^0)$ *on* $(E-\overline{D}) \cup \Gamma_0$, *and on* $\partial\overline{D}$ *except a set of capacity* 0.

β) $\phi(\mu) \geqq \phi(\mu_\Gamma^0)$ *everywhere in E.*

Conversely, μ_Γ^0 *is characterized as a measure* of $\mathfrak{M}^+(\partial\overline{D})$, *whose potential* $\phi(\mu_\Gamma^0)$ *is the minimum among all of* $\phi(\nu)$ *for such* ν *as fulfil the condition* α) *above.*

Proof. We have only to prove the latter half, but it is obtained by a very simple course; $\phi(\mu_\Gamma^0)(x) = \int \phi(\mathcal{E}_x)d\mu_\Gamma^0 = \int \phi(\mathcal{E}_x)_\Gamma^0 d\mu$ (by (6·8)) $= \int \phi(\mathcal{E}_x)_\Gamma^0 d\nu$ (by α)) $\leqq \int \phi(\mathcal{E}_x)d\nu = \phi(\nu)(x)$, for $x \in \overline{D}$, and so in E and μ itself satisfies the condition α).

In the case where $\Gamma_0 \subset \partial\overline{D}$, we call every point of Γ_0 (or of $\partial\overline{D}-\Gamma_0$) a *regular* (resp. *irregular*) *boundary-point* of D.

In passing, we shall show, in the case where $\varPhi$ is of η-type, a short criterion of sufficiency in order that a boundary-point be regular.

PROPOSITION 17. *If we can draw outside of D an osculation sphere* $\sum$ *to* $\partial\overline{D}$ *at a point x, then x is a regular boundary-point of D.*

In fact, let x_0 be the center of $\sum$ and take an inner point z on the segment combing x and x_0; suppose now $\mathcal{E}_x^\wedge = \alpha\mu^\wedge + \beta\nu^\wedge$, $\mu \neq \nu$ and $\alpha, \beta > 0$ with $\alpha+\beta = 1$, then we have $\mu^\wedge(\phi(\mathcal{E}_z))$, $\nu^\wedge(\phi(\mathcal{E}_z)) < \eta^{-1}(r^{-1}(x, z)) = \mathcal{E}_x^\wedge(\phi(\mathcal{E}_z)) = \phi(\mathcal{E}_z)(x)$, which is absurd. Thus, the Proposition is established.

6. 6. According to H. Cartan [5], we shall define the balayage of functions of $\mathfrak{L}^+(E)$. Let G and $F = \overline{G}$ be taken as they were (in the preceding section). For a general $f \in \mathfrak{L}^+(E)$ with $f = \phi(\varDelta_D f) + f_D$, $f_D \in \mathfrak{H}(D)$, $\overline{G} \subset D$, put

(6·11) $f_\Gamma^0 = \phi((\varDelta_D f)_\Gamma^0) + f_D$;

then we see immediately that f_Γ^0 is harmonic in G and satisfies

$$(6\cdot 12)\qquad \begin{array}{ll} \text{i)} & f \geqq f_\Gamma^0 \quad \text{everywhere in } D, \\ \text{ii)} & f = f_\Gamma^0 \quad \text{in } \Gamma_0 \cup (D-\bar{G}). \end{array}$$

Moreover, let f, $g \in \mathfrak{L}^+(D)$ and assume that f_D and g_D are continuous on G, then we have

$$(6\cdot 13)\qquad \int f d(\Delta_D g)_\Gamma^0 = \int g_\Gamma^0 d(\Delta_D f).$$

6. 7. Next, we shall deal with the general case where F is closed but non compact: instead of $\mathfrak{M}_0(D)$ for $D = intF$, we now take $\mathfrak{M}_1(D)$ of measures $\mu \in \mathfrak{M}^+(F)$ with norms less than 1, $\|\mu\| \leqq 1$, and define $\mathfrak{M}_1^\wedge(D)$ as a convex set in $(H(D))^*$, the dual of $H(D)$ whose definition shall be leaved as it was in **6. 1**, consisting of linear functionals $\mu^\wedge$ defined by $(6\cdot 2)$ for all $\mu \in \mathfrak{M}_1(D)$. Since $\mathfrak{M}(E)$ is a Montel space, $\mathfrak{M}_1(D)$ is vaguely compact and, by the same reason as in **6. 1.** $\mathfrak{M}_1^\sim(D)$ is also compact (and naturally convex) in $(H(D))^*$ with respect to the w^*-topology in it, so the theorem of Krein-Milman is again applicable to $\mathfrak{M}_1^\sim(D)$ and Theorem 11 remains valid.

Under the assumption $(6\cdot 2)$ in **6. 2**, which is guaranteed *e. g.* in the newtonian potential case, see S. Matsushita [14], pp. 125~126, we can settle

THEOREM 12^{bis}. *The set of extreme points of $\mathfrak{M}_1^\sim(D)$, $\subset \mathfrak{M}_1^\wedge(D)$ consists of $\varepsilon_x^\wedge$, ε_x being a point measure of total mass $+1$ on $x \in \bar{D}$, and of the zero functional $0^\wedge$.*

By the same fashion as in the proof of Theorem 12, we see that, if $\mu \neq 0$ is not a point measure, then $\mu^\wedge$ cannot be extreme; next assume $\|\mu\| < 1, > 0$, then $\mu^\wedge = \alpha\mu_0^\wedge + (1-\alpha)\, 0^\wedge$, for $\alpha = \|\mu\|$ and $\mu_0 = \mu/\alpha$, which shows $\mu^\wedge$ not being extreme. Thus, Theorem 12^{bis} is proved.

Define Γ_0 as the set of such $x \in F$ that $\varepsilon_x^\wedge \in Ext.\ \mathfrak{M}_1^\sim(D)$, then $\mathfrak{M}_0(\bar{\Gamma})$ is also vaguely compact, and analogically as in **6. 2**, we can define μ_Γ which satisfies $B_1) \sim B_3)$ and, in the case where $\Gamma \subset \partial\bar{D}$, define also μ_Γ^0 for every $\mu \in \mathfrak{M}_1(D)$ and hence $\mu \in \mathfrak{M}^+(\bar{D})$.

For such μ_Γ^0, $(6\cdot 8)$, Theorem 13, Prop. 14, 15 and Theorem 14 remain valid.

6. 8. Let now D be an open set with the boundary ∂D. Take a series of closed sets F_n such that i) $F_n \subset D$, ii) $\overline{D-F_{n+1}} \subset \bar{D} - F_n$, and iii) $\partial D = \cap_{i=1}^\infty (D-F_n)$. For a general $\mu \in \hat{\mathfrak{M}}^+(D)$, denote by μ_n the restriction of μ in F_n and by μ_n^0 the balayaged measure of μ_n in ∂F_n for each $n (n=1, 2, \cdots)$; then we see that μ_n converges vaguely to μ and there exists a sub-sequence $\{\mu_{n_i}^0\}$ of $\{\mu_n^0\}$ such that $\mu_{n_i}^0$ converges to a measure μ_Γ^0 defined in ∂D, with respect to the vague-topology in $\mathfrak{M}(E)$ since $\mathfrak{M}_1^+(D)$ is vague-compact. Such μ_Γ^0 is called the balayaged measure of $\mu \in \mathfrak{M}^+(D)$ in ∂D.

PROPOSITION 17. *$\phi(\mu) \geqq \phi(\mu_\Gamma^0)$ everywhere in E for $\mu \in \mathfrak{M}^+(D)$.*

In fact $\phi(\mu)(x) = \lim_i \phi(\mu_{n_i})(x) \geqq \varliminf_i \phi(\mu^0_{n_i})(x) \geqq \phi(\mu^0_\Gamma)(x)$.

PROPOSITION 18. *For any $\mu,\ \nu \in \hat{\mathfrak{M}}^+(D)$, we have*

$$\int \phi(\mu)\, d\nu^0_\Gamma = \int \phi(\nu)\, d\mu^0_\Gamma. \tag{6·16}$$

Proof. At first, let μ be in $\hat{\mathfrak{M}}^+(F_{n_0})$ for a certain n_0, then we see $\mu_n = \mu$ for $n \geqq n_0$ and $\phi(\mu)$ is contained in $B(E)$. We have

$$\int \phi(\mu)\, d\nu^0_\Gamma = \lim_j \int \phi(\nu^0_j)\, d\mu = \lim_j \int \phi(\mu^0)\, d\nu_j = \int \phi(\nu)\, d\mu^0,$$

whence for general $\mu = \lim \mu_i$ it comes $\int \phi(\mu)\, d\nu^0_\Gamma = \lim_i \int \phi(\nu^0_\Gamma)\, d\mu_i \geqq \varliminf_i \int \phi(\nu)\, d\mu^0_i \geqq \int \phi(\nu)\, d\mu^0_\Gamma$ and, replacing μ and ν one another, $\int \phi(\mu)\, d\nu^0_\Gamma = \int \phi(\nu)\, d\mu^0_\Gamma$.

Thus, Proposition 18 is completely proved. By definition, the following is claimed;

PROPOSITION 19. *If $\nu \in \mathfrak{M}^+(\partial D)$ has a bounded potential, then we have for any $\mu \in \mathfrak{M}^+(D)$,*

$$\int \phi(\mu)\, d\nu = \int \phi(\mu^0_\Gamma)\, d\nu. \tag{6·17}$$

THEOREM 15. *μ^0_Γ satisfies the following conditions:*

α) $\phi(\mu) \geqq \phi(\mu^0_\Gamma)$ *everywhere in E,*

β) $\phi(\mu) = \phi(\mu^0_\Gamma)$ *in $E - \bar{D}$ and, excepting a set of capacity zero, on ∂D,*

γ) $\int \phi(\mu)\, d\nu^0_\Gamma = \int \phi(\nu)\, d\mu^0_\Gamma$ *for any $\nu \in \mathfrak{M}^+(\bar{D})$.*

§ 7. Application to Dirichlet Problem

7. 1. In this paragraph, we shall concern ourselves with some applications of the foregoing discussions, especially with Dirichlet problem. For this sake, we shall be content with the following restricted conditions, all of which are however fulfiled in the case of newtonian potentials in $E = R^n$ for $n \geqq 3$, as noted later; let $D = \operatorname{int} \bar{D}$ be a considered domain, then

1°) Γ_0 is included in the boundary $\partial \bar{D}$, which is always assumed compact.

2°) D is approximated from within by a sequence of D_k such that ∂D_k consists of the regular boundary-points of D_k and $D_k = \mathit{int}\, \bar{D}_k$.

3°) $\partial D - \Gamma_0$ is of inner capacity zero, *i. e.* $c^i(\partial D - \Gamma_0) = 0$. Moreover, the assumptions (R. 1)~(R. 3) are always assumed.

Remark: Condition 2°) is satisfied when the kernel function Φ is of η-type for a certain η quoted in **5. 3**, in the following manner; take spherse $S_\rho(x)$ of radius ρ and center x for every $x \in \partial D - \Gamma_0$ and put $D_k = D - \cup S_{\rho_k}(x)$ for $\rho_k = 1/2^k$. If D_k is not empty, ∂D_k satisfies 2°) owing to Prop. 16, and we have clearly $D_k \longrightarrow D$.

In the case of newtonian potential in $E = R^n (n \geqq 3)$, H. Cartan [5] has proved that there exists a positive measure α such that $\phi(\alpha)$ is continuous in E and

$\int\phi(\mu_1)d\alpha=\int\phi(\mu_2)d\alpha$ implies that $\mu_1=\mu_2$ for $\mu_1, \mu_2\in\hat{\mathfrak{M}}^+(E)$. Using such α, we can prove some important facts; at first, every point $x\in\Gamma_0$ is characterized as such a point that $\alpha_{\bar{D}}(x)=(\alpha_{\bar{D}})^0_\Gamma(x)$ for the restriction $\alpha_{\bar{D}}$ of α on $\bar{D}$. In fact, if the above equality is held, then we have $\int\phi(x)d\alpha_{\bar{D}}=\int\phi(x)d\alpha_{\bar{D}}{}^0_\Gamma=\int\phi((\varepsilon_x)^0_\Gamma)d\alpha_{\bar{D}}$ and hence $\int\phi(x)d\alpha=\int\phi((\varepsilon_x)^0_\Gamma)d\alpha$, so that $\varepsilon_x=\varepsilon_x{}^0_\Gamma$. The inverse is trivial.

Owing to this fact, 3°) is shortly verfied: that is, for every $\nu\in\mathfrak{M}^+(\partial D)$ such that $\phi(\nu)$ is bounded, it holds

$$\int\phi(\alpha_{\bar{D}})d\nu=\int\phi(\alpha_{\bar{D}})d\nu^0_\Gamma=\int\phi((\alpha_{\bar{D}})^0_\Gamma)d\nu,$$

hence the set of $x\in\partial D$ for which $\phi(\alpha_{\bar{D}})(x)>\phi((\alpha_{\bar{D}})^0_\Gamma)(x)$ is of inner capacity zero, then so is $\partial D-\Gamma_0$.

7·2. We start with the following Lemma.

LEMMA. *Let $D=int\ \bar{D}$, $\partial D=\Gamma_0$, then the collection $H^0(D)$ of all the restricted f^0 of $f\in H(D)\cap C(\partial\bar{D})$ on ∂D is dense in the Banach space $C(\partial D)$ of all continuous functions in ∂D.*

Proof. Suppose it were not so, then there must be a continuous linear functional $\xi(\cdot)\neq 0$ such that $\xi(f)=0$ for every $f\in H^0(D)$; on the other hand, such ξ defines a measur μ_ξ on ∂D, generally in a composed type $\mu_\xi=\mu^1_\xi-\mu^2_\xi$ for $\mu^i_\xi\in\mathfrak{M}^+(\partial D)$ $(i=1,2)$. By hypothesis, we have $\int fd\mu^1_\xi=\lim_j\int f_jd\mu^1_\xi=\lim_j\int f_jd\mu^2_\xi=\int fd\mu^2_\xi$ for every $f\in H(D)$, and $f_j\nearrow f$, $f_j\in B(E)\cap C(E)$ for which $f_j=(f_j)^0_\Gamma$ on $\partial\bar{D}$, so that μ^2_ξ is considered as a balayaged measure of μ^1_ξ, but as $\partial D=\Gamma_0$ it must be $\mu^1_\xi=\mu^2_\xi$ by Prop.. 15. Consequently, $\mu_\xi=0$ and this contradicts with the assumption, which guarantees the Lemma.

If $\mathfrak{H}(D)$ is complete for the uniform convergence in D, we get easily the solution of Dirichlet problem for such D that $\partial D=\Gamma_0$ as follows; for every $f\in C(\partial D)$ there exists a sequence of $f^0_j\in H^0(D)$ such that $f^0_j\longrightarrow f$ uniformly on ∂D. Then, we have

$$f_j(x)=\int f^0_jd(\varepsilon_x)^0_\Gamma\longrightarrow\int fd(\varepsilon_x)^0_\Gamma$$

uniformly for all $x\in D$, since $\int d\varepsilon_x=1$. Putting $\tilde{f}(x)=\int fd(\varepsilon_x)^0_\Gamma$ for $x\in\bar{D}$, such $\tilde{f}$ belongs to $\mathfrak{H}(D)$*). Since $f_j(x)$ converges uniformly to $\tilde{f}(x)$ on $\bar{D}$, $\tilde{f}(x)$ is also uniformly continuous on $\bar{D}$, and $\tilde{f}\in\tilde{H}(D)$, the completion of $H(D)\cap C(\partial D)$. On the other hand, $|f(x)|\leqq\|f\|_{\partial D}$ for every $x\in\bar{D}$, where $\|f\|_{\partial D}$ means the norm in $C(\partial D)$. Summing up these, we establish:

PROPOSITION 20. *If $\mathfrak{H}(D)$ is complete for the uniform convergence in D and*

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*) Such f_j is continuos on $\bar{D}$, see Prop. 22.

$\partial D=\Gamma_0$, *then* $\tilde{H}(D)$ *is isometrically isomorphic to* $C(\partial D)$ *by the relation*;

(7・1) $$f(x)=\int f^0 d(\varepsilon_x)^0_\Gamma,\ f\in\tilde{H}(D), f^0\in C(\partial D).$$

More sharply we get:

PROPOSITION 21. *Under the same condion as above,* $C(\partial D)$ *is isometrically isomorphic to* $H_\infty(D)\equiv L_\infty(E\cap\bar{D})\cap\mathfrak{H}(D)$ *so that every* $f\in H_\infty(D)$ *is uniquely represented in the form* (7・1) *for the restriction* f^0 *of* f *in* ∂D.

This Proposition offers the solution of Dirichlet problem for a domain D such that $\partial D=\Gamma_0$ (Dirichlet problem in a classical type).

To prove Prop. 21, we need only to show that the correspondence between f and f^0 is unique: let $f\in H_\infty(D)$ be given, then putting $\tilde{f}(x)=\int_{\partial D} f\, d(\varepsilon_x)^0_\Gamma$, we see by Prop. 20 that $\tilde{f}\in\tilde{H}(D)\subset H_\infty(D)$ (see **3・5**, about $\tilde{f}\in L_\infty(E\cap\bar{D})$)*) and $f-\tilde{f}$ vanishes on ∂D, hence $f-\tilde{f}\in L_\infty(D)\cap\mathfrak{H}(D)=(0)$ by Prop. 4, that is, $f=\tilde{f}$ in D, which completes the proof of Prop. 21.

7. 3. Let D be now a general domain which however satisfies the conditions 1°)～3°) above, and assume that $\mathfrak{H}(D)$ forms a Montel space with respect to the topology of compact convergence in D: In view of the discussion in **6. 9**, for a given $x\in D$ there exists a number j_0 for which $x\in\bar{D}_{j_0}$ and we get a sequence of $(\varepsilon_x)^0_j$ ($j\geqq j_0$), balayaged measures of ε_x on ∂D_j, which converges vaguely to $(\varepsilon_x)^0_\Gamma$. Let f_0 be an arbitrary prolonged continuous function in E (with the compact support) of $f\in C(\partial D)$: Since ∂D is assumed to be compect, the existence of such an f_0 is quite doubtless.

Denote further for each j the restriction of f_0 on ∂D_j by f_j, then by (7・1) it holds for $j\geqq j_0$

$$f_j(x)=\int_{\partial D_j} f_j d(\varepsilon_x)^0_j=\int_{\partial D_j} f_0 d(\varepsilon_x)^0_j\longrightarrow\int_{\partial D} f_0 d(\varepsilon_x)^0_\Gamma,$$

which is $=\int_{\partial D} f d(\varepsilon_x)^0_\Gamma$. Set now

(7・2) $$\tilde{f}(x)=\int_{\partial D} f d(\varepsilon_x)^0_\Gamma \qquad \text{for every } x\in D;$$

then $f_j(x)$ converges to $\tilde{f}(x)$ as $j\longrightarrow\infty$ ($j\geqq j_0$) for $x\in D_{j_0}$. On the other hand, denoting the uniform norm of f_0 in E by $\|f_0\|$, which could be $\leqq\|f\|$ (the norm of f in $C(\partial D)$), we see clearly $|f_j(x)|\leqq\|f_0\|$ for every $x\in D_{j_0}$ and all $j\geqq j_0$, so that $\{f_j\}$ are uniformly boubded on D_{j_0}; as $\mathfrak{H}(D_{j_0})$ is a Montel space, $\{f_j\}$ from a relativelry compact set in $\mathfrak{H}(D_{j_0})$ and hence there exists a sub-series of $\{f_j\}$ which converges to a function of $\mathfrak{H}(D_{j_0})$ which must be equal to f for every $x\in D_j$, or equivalently $\tilde{f}$ is harmonic in D_{j_0}. Increasing $j_0(\longrightarrow+\infty)$, we see that $\tilde{f}$

*) In fact, by Thr. 7, there exists a $\phi(\nu)$ such that $\phi(\nu)\geqq\|f\|_{\partial D}$ on ∂D, and $\phi(\nu)\longrightarrow 0$ as $x\longrightarrow\infty$.

is harmonic in every D_j, hence D, *i.e.* $f\in\mathfrak{H}(D)$.

We can state then:

THEOREM 16. *For every $f\in C(\partial D)$, ∂D being compact, there exists the unique solution $\tilde{f}$ of Dirichlet problem with respect to D, such that*

i) *$\tilde{f}$ is harmonic in D and bounded in $\bar{D}$,*

ii) $\lim_{x\to x_0}\tilde{f}(x)=f(x_0)$ for $x\in D$ and $x_0\in\Gamma_0$,

iii) $\tilde{f}\in L_\infty(E\cap\bar{D})$ *(if D is compact, this* iii) *is unnecessary). Moreover, such* f and $\tilde{f}$ *are related by* (7·2).

Proof. It remains us to prove only ii), iii) and the uniqueness of the solution but ii) is an immediate consequence of the following Proposion (mentioned in a general situation) and iii) is somewhat clear for the above obtained $\tilde{f}$; thus, we need only to prove the uniqueness. Now, suppose that we have from the first adopted as $\{D_j\}$ the sequence of domains which were mentioned just in **7. 1.** (that is, $D_j=D-\cup S_{\rho_j}(x)$, $x\in\partial D-\Gamma_0$ and $\rho_j=1/2^j$). Denote the restrictions of $(\varepsilon_x)^0_j$ on $\partial\bar{D}\cap\partial\bar{D}_j$ by $(\varepsilon_x)^1_j$ and on $\partial\bar{D}_j-\partial\bar{D}$ by $(\varepsilon_x)^2_j$ for an arbitrary fixed $x\in D$; a sub-sequence of $\{(\varepsilon_x)^2_j\}$ converges vaguely to a measure ν, necessaily distributed on ∂D. As $\phi(\varepsilon_x)\geqq\underline{\lim}_j\,\phi((\varepsilon_x)^2_j)\geqq\phi(\nu)$, $\phi(\nu)$ is bounded on ∂D and hence in E, while $\lim_j\int d(\varepsilon_x)^2_j=\int d\nu<+\infty$. If $(\nu)_{\partial D-\Gamma_0}\neq 0$, these are contradictory with $c^i(\partial D-\Gamma_0)=0$, hence it must be $(\nu)_{\partial D-\Gamma_0}=0$. $\varepsilon>0$ being given, there is a number j such that, for any $h\in\mathfrak{H}(D)$ which is bounded on $\bar{D}$ and satisfies the conditions i)~iii) above, $\left|\int|\tilde{f}-h|\,d(\varepsilon_x)^2_j\right|<\varepsilon$ and so

$$|\tilde{f}(x)-h(x)|\leqq\int_{\partial D}|\tilde{f}-h|\,d(\varepsilon_x)^0_j=\int_{\partial D_j-\partial D}|\tilde{f}-h|\,d(\varepsilon_x)^2_j<\varepsilon,$$

whence it concludes that $\tilde{f}=h$ in D.

PROPOSITION 22. *In the case where $\Gamma_0\subset\partial D$, for a point $x_0\in\Gamma_0$ and every sequence of points $x_j\longrightarrow x_0$, we have*

$$\lim_{j\to\infty}(\varepsilon_{x_j})^0_\Gamma=\varepsilon_{x_0}\ \textit{for the vague topology.}\tag{7·3}$$

Proof. Since any sub-sequence of $\{(\varepsilon_{x_j})^0_\Gamma\}$ forms a relatively compact set in $\mathfrak{M}^+(\partial D)$, we can choose a sub-sequence of it, say $\{(\varepsilon_{x_k})^0_\Gamma\}$, which converges vaguely to a certain measure ν on ∂D. For every $f\in B(E)\cap C(E)$, we see;

$$f(x_0)=f^0_\Gamma(x_0)\leqq\lim_k f^0_\Gamma(x_k)=\lim_k\int fd(\varepsilon_{x_k})^0_\Gamma=\int fd\nu\leqq f(x_0),$$

so that, $\nu=(\varepsilon_{x_0})_\Gamma=\varepsilon_{x_0}$ since $x_0\in\Gamma_0$; thus, we can conclude that $\{(\varepsilon_{x_j})^0_\Gamma\}$ itself converges ε_{x_0}. This completes the proof.

7. 4. Now we shall restrict ourselves in the case where D is relatively compact; the case where D is not so but ∂D is compact may be treated in an analogous manner.

If $\partial D \neq \Gamma_0$, the isomorphism in Prop. 20 or Prop. 21 must be in fault, because if it were not so, then for any $f \in C(\partial D)$ there would exist a sequence of $g_i \in H(D) \cap C(\bar{D})$ which converges uniformly to f on ∂D, so that

$$f(x_0) = \lim_i g_i(x_0) = \lim_i \int g_i d(\varepsilon_{x_0})^0_\Gamma = \int f d(\varepsilon_{x_0})^0_\Gamma$$

for every $x_0 \in \partial D$, still for $x \in \partial D - \Gamma_0$; on the other hand, the above equality yields that $\varepsilon_{x_0} = (\varepsilon_{x_0})^0_\Gamma$ since f is arbitrary in $C(\partial D)$, which conducts us to an inconsistency.

Then, what relation would take the place of that?

Next will answer for it.

THEOREM 17. *If $\partial D \neq \Gamma_0$, then $H^0(D)$ forms a dense subspace of $C^0(\Gamma_0)$ with respect to the topology of compact convergence in Γ_0. In order that this topology would be replaced by the uniform topology in $C^0(\Gamma_0)$, it is necessary and sufficient that Γ_0 is closed, where $C^0(\cdot)$ means the space of uniform continuous functions.*

Proof. Let K be any compact set in Γ_0. If the restrictions f_K of $f \in H^0(D)$ on K does not form a dense subspace in $C(K)$, there exists at least one linear continuous functional $\xi(\cdot) \neq 0$ on $C(K)$ whicn satisfies $\xi(f_K) = 0$ for every $f \in H^0(D)$; such ξ defines a measure $\mu_\xi = \mu_\xi^1 - \mu_\xi^2$ for $\mu_\xi^i \in \mathfrak{M}^+(K)$ $(i=1, 2)$. In an analog as in the proof of Lemma, it follows easily that $\mu_\xi = 0$, which is absurd. Thus, we see that every $f \in C^0(\Gamma_0)$ is uniformly approximated by functions of $H^0(D)$ on every K. Next, if Γ_0 is not closed and if $f \in C^0(\Gamma_0)$ is uniformly approximated by $g_i \in H^0(D)$ on Γ_0, hence on $\bar{\Gamma}_0$ (since f and g_i are all uniformly continuous in Γ_0), then for a point $x_0 \in \bar{\Gamma}_0 - \Gamma_0$ an inconsistency would occur similarily as just before (since $f(x_0) = \lim_i g_i(x_0)$).

This completes the proof of Theorem 17.

Remark: This theorem will suggest something about the functional determination of the solution for Dirichlet problem, which has been pursued by M. Keldych, M. Inoue [9], etc. We shall not go into details on that interesting subject here, but some further investigation about it may be appeared elsewhere in the future.

7. 5. Extension of Dirichlet problem: In the foregoing sections we have been exclusively occupied in the case where the boundary functions are continuous. We shall now investigate the boundary-value problem of the same type for some more general boundary functions which may permit of the solutions.

We treat here only n-dimensional Euclidean space $E = R^n (n \geqq 3)$ and newtonian potentials. Let D be a relatively compact domain in E, whose boundary ∂D is assumed as a measure space with respect to a certain measure m such that i) $c^i(X) = 0$ implies $m(X) = 0$, and ii) every bounded potential is m-measurable on ∂D.

THEOREM 18. *For every essentially bounded m-measurable function g on ∂D, there corresponds a harmonic function $\tilde{g}$ in D such that; if $x_0 \in \Gamma_0$ has a neigh-*

borhood $U(x_0)$ such that g is continuons in $U(x_0)\cap\partial D$, then $\tilde{g}(x)\longrightarrow g(x_0)$ as $x\in D\longrightarrow x_0$.

Proof. Denote *by* $M(\partial D)$ the Banach space of all m-measurable, essentially bounded functions difined on ∂D, with respect to the norm (of essential maximum) $\|f\|_M=\text{ess.}\max_{x\in\partial D}|f(x)|$. The restricted f_0 of all $f\in H(D)$ on ∂D forms a subspace $B_0(\partial D)$ of $M(\partial D)$.

We shall prove first $B_0(\partial D)$ is dense in $M(\partial D)$; suppose if it were not so, then there exists a non-trivial functional in the conjugated space $(M(\partial D))^*$ of $M(\partial D)$, say ξ, which vanishes on $B_0(\partial D)$. Such ξ difines a Radon measure μ_ξ of bounded variation in ∂D, which satisfies $\mu_\xi(X)=0$ for every X with $m(X)=0$, hence by hypothesis with $c^i(X)=0$; since $\phi(\mu)=\phi(\mu_\Gamma^0)$ excepting a set X with $c^i(X)=0$ for every $\mu\in\mathfrak{M}^+(\bar{D})$, we have then $\int\phi(\mu)d\mu_\xi=\int\phi(\mu_\Gamma^0)d\mu_\xi$.

Applying this for the Cartan's measure α (cited in (7. 1)) with its restrictions $\alpha_{\bar{D}}$ in $\bar{D}$ and α_* in $E-\bar{D}$, we have

$$0=\int_{\partial D}\phi((\alpha_{\bar{D}})_\Gamma^0)d\mu_\xi+\int_{\partial D}\phi(\alpha_*)d\mu_\xi=\int_{\partial D}\phi(\alpha)d\mu_\xi,$$

so that $\mu_\xi=0$, which is inconsistent.

Next, let x be in D and λ_x the measure of spherical mean of ε_x in a sphere $\Sigma_x\subset D$ (of center x), then the functionals φ_x defined by $\varphi_x(f_0)=f(x)=\int_{\partial D}f_0 d(\varepsilon_x)_\Gamma^0$ and ψ_x by $\psi_x(f_0)=\int f d\lambda_x=\int f_0 d(\lambda_x)_\Gamma^0$ are obviously linear continuous on $B_0(\partial D)$ ($\|\varphi_x\|$, $\|\psi_x\|\leqq 1$) and have respectively the unique extensions $\tilde{\varphi}_x$ and $\tilde{\psi}_x$ in $(M(\partial D))^*$ with the same norms. For every $g\in M(\partial D)$ put $g(x)=\varphi_x(g)$, $x\in D$, then for such $f_j\in B_0(\partial D)$ as $\longrightarrow g$(in $M(\partial D)$), it holds $\tilde{\psi}_x(g)=\lim_j\tilde{\psi}_x(f_j)=\lim_j\int f_j d(\lambda_x)_\Gamma^0$ $=\lim_j\int\varphi_y(f_j)\,d\lambda_x(y)=\int\tilde{\varphi}_y(g)d\lambda_x(y)=\int\tilde{g}(y)d\lambda_x(y)$.

On the other hand, $\psi_x=\varphi_x$ implies $\tilde{\psi}_x=\tilde{\varphi}_x$, so that we see that $\tilde{g}$ is harmonic in D.

Next, suppose x_0 to satisfy the condition in Theorem 18 and take a neighborhood $V(x_0)$ of x_0 such that $\bar{V}(x_0)\subset U(x_0)$; putting $V=\bar{V}(x_0)\cap\partial D$, V is compact and, since the restriction g_V of g in V is continuous there, we have a continuous extension $g_V^0\in C(\partial D)$ with $\|g_V^0\|_{\partial D}\leqq\|g_V\|_V\leqq\|g\|_M$. x_0 being in Γ_0, $(\varepsilon_x)_\Gamma^0$ converges vaguely to ε_{x_0}(as $x\in D\longrightarrow x_0$), so that

$$\int_{\partial D-V}d(\varepsilon_x)_\Gamma^0\longrightarrow 0.$$

Then we have

$$g(x_0)=g_V^0(x_0)=\lim_{x\to x_0}\int_{\partial D}g_V^0\,d(\varepsilon_x)_\Gamma^0=\lim_{x\to x_0}\int_V g_V\,d(\varepsilon_x)_\Gamma^0$$
$$=\lim_{x\to x_0}\int_V g\,d(\varepsilon_x)_\Gamma^0=\lim_{x\to x_0}\int_{\partial D}g d(\varepsilon_x)_\Gamma^0=\lim_{x\to x_0}\tilde{g}(x),$$

which proves Theorem 18 completely.

References

N. Bourbaki [1], Intégration, Livre 6, Paris Hermann (1952).

N. Bourbaki [2], Espaces vectoriels topoligiques, Livre 5, Paris, Hermann (1953).

H. Cartan [3], Sur les fondaments de la théorie du potentiels, bull. Sci. Math. Franee, **69** (1941), 71-96.

H. Cartan [4], Théorie du potentiel newtonien : énergie, capacité, suites de potentiels, ibid., **73** (1954), 74-106.

H. Cartan [5], Théorie générale du balayage en potentiel newtonien, Ann. Univ. Grenoble, **22** (1946), 221-280.

H. Cartan-R. Godement [6], Théorie de la dualité et analyse harmonique dans les groupes abeliens localement compacts, Ann. Sci. Ecole Norm, Sup., **64** (1947), 77-99.

O. Frostman [7], Potentiel d'équilibre et capacité des ensembles, Thése, Lund (1935).

S. Hitotumatu [8] Note on the Riesz docomposition of a subharmonic function, Comm. Math. Univ. Sancti Pauli, **3** (1955), 69-94.

M. Inoue [9], Sur la détermination fonctionnelle de la solution du problème de Dirichlet, Mem. Fac. Sci. Kyushu Univ., **5** (1950), 69-74.

M. Krein-D. Milman [10], On extreme points of regular convex sets, Studia Math., **9** (1940), 133-138.

L. Loomis [11], An introduction to abstract harmonic integral, New York Van Nostrand (1952).

S. Matsushita [12], Positive functionals and representation theory on Banach algebras, I. Jour. Inst. Polytech. Osaka City Univ., **6** (1955), 1-18.

S. Matsushita [13], Sur le théorème de Plancherel, Proc. Japan Acad., **30** (1954), 557-561.

S. Matsushita [14], Théorème de Krein-Milman et le balayage de mesures dans la théorie du potentiel, 1-3, ibid. **31** (1955), 643-647, and **32** (1956), 29-34, 125-530.

S. Matsushita [15], Sur la décomposition de F. Riesz, I et II, C. R. Acad. Soi., Paris, **241** (1955), 1251-1254, 1373-1375.

S. Matsushita [16], Lapacien local et la décomposition de F. Riesz, Proc Japan Acad., **32** (1956), 435-440.

T. Radó [17], Subharmonic functions, Chelsea (1949).

L. Schwartz [18], Théorie des distributions, I et II, Paris, Herman (1950).