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Positively infinite singularities of solutions of linear elliptic partial differential equations

By Masao INOUE

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1. Let Ω be a bounded plane region (connected open set) and let there be given a certain equation in one unknown function $u: \mathfrak{F}[u]=0$. We shall say that u is an $\mathfrak{F}$-function in an open set contained in Ω provided u is there a solution[1] of $\mathfrak{F}[u]=0$.

Let E be a closed set contained in Ω. We shall say that E is of $\mathfrak{F}$-capacity zero and write $C_{\mathfrak{F}}(E)=0$, provided there exist an open set O containing E and an $\mathfrak{F}$-function $u(x, y)$ in O-E, such that

$$\lim_{(x,y)\to(x',y')} u(x, y) = +\infty$$

for all boundary points (x', y') of O-E belonging to E. Such a function u will be called an Evans' function for E with respect to $\mathfrak{F}[u]=0$.

Especially if

$$\Delta u \equiv \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

is taken for $\mathfrak{F}[u]=0$, $C_{\mathfrak{F}}(E)=0$ means that E is of logarithmic capacity zero, which we shall write $C(E)=0$ in the sequel.

The main purpose of this paper is to give some conditions for that $C_{\mathfrak{F}}(E)=0$ implies $C(E)=0$ or conversely $C(E)=0$ implies $C_{\mathfrak{F}}(E)=0$. Applications will be made to solutions of linear partial differential equations of elliptic type. Finally we shall obtain an extension of the theorem on removable singularities for harmonic functions.

2. We define the generalized Laplace operator $\Delta^* u$ by

$$\Delta^* u(x, y) = \lim_{\rho\to 0} \frac{4}{\rho^2}\left\{ \frac{1}{2\pi}\int_0^{2\pi} u(x+\rho\cos\theta, y+\rho\sin\theta)d\theta - u(x, y)\right\}$$

or

$$\Delta^* u(x, y) = \lim_{\rho\to 0} \frac{8}{\rho^2}\left\{ \frac{1}{\pi\rho^2}\int_0^{\rho}\int_0^{2\pi} u(x+r\cos\theta, y+r\sin\theta)\, rdrd\theta - u(x, y)\right\}.$$

It turns out that, if u is twice continuously differentiable, $\Delta^* u$ exists and is identically equal to Δu.

THEOREM 1. *Let O be a bounded open set in the plane, E^* the exterior frontier of a closed set E lying in O, and O^* the portion of O exterior to E^*. Let $a(x, y)$, $b(x, y)$ be continuously differentiable functions in O^* such that a, b, a_x, b_y are uniformly*

1) A solution must be continuous and have continuous derivatives of all orders appeared in $\mathfrak{F}[u]=0$.

bounded there, and let $\varphi(x,y,u)$ *be a continuous function for* $(x,y)\in O^*$ *and* $-\infty< u<+\infty$, satisfying

(1) $$\sup_{(x,y)\in O^*,\,u>1}\frac{|\varphi(x,y,u)|}{u}<+\infty$$

or

(2) $$\inf_{(x,y)\in O^*,\,u>1}\varphi(x,y,u)>-\infty.$$

If there exists in O^* *a solution* $u(x,y)$ *of the equation*

$$\varDelta^*u+au_x+bu_y+\varphi(x,y,u)=0,$$

possessing the boundary behavior

$$\lim_{(x,y)\to(x^*,y^*)}u(x,y)=+\infty$$

for all points (x^*,y^*) *of* E^*, *then the logarithmic capacity of* E *is zero.*

Proof. Without loss of generality we can assume that the boundary of O, say $B(O)$, consists of a finite number of smooth curves and that u is defined continuously in $O^*\cup B(O^*)-E^*$ and $u>1$ there.

Let D be any component (connected open set) of O^* and put

$$B_0=B(D)\cap E^*,\ B_1=B(D)\cap B(O).$$

Denoting by D'_n the set of points (x,y) of D at which $u(x,y)<n$ $(n=1,2\cdots)$, $B(D'_n)$ contains B_1 if n is large enough. Let D^*_n be the component of D'_n which has B_1 as boundary components. Shrinking D^*_n to D_n by changing $B(D^*_n)-B_1$, we construct a sequence of regions D_n bounded by a finite number of smooth curves, such that

(3) $$1<u<n \qquad \text{in } D_n\cup B(D_n),$$

$$n-1<u<n \qquad \text{on } B(D_n)-B_1$$

and $$D_n\subset D_{n+1}\subset D_{n+2}\subset\ldots,\ D_n\to D.$$

We now form the harmonic function $h_n(x,y)$ which is continuous in $D_n\cup B(D_n)$, harmonic in D_n and equal to u on $B(D_n)$. Then

$$\varDelta^*(u-h_n)+au_x+bu_y+\varphi(x,y,u)=0 \qquad \text{in } D_n$$

and

$$u-h_n=0 \qquad \text{on } B(D_n).$$

Therefore we may write $u-h_n$ in the form

(4) $$u(x,y)-h_n(x,y)=\frac{1}{2\pi}\iint_{Dn}\left\{au_\xi+bu_\eta+\varphi(\xi,\eta,u)\right\}G_n(\xi,\eta\,;x,y)d\xi d\eta,$$

where $G_n(\xi,\eta\,;x,y)$ is the Green's function for D_n (with respect to the Laplace equation) with pole at (x,y). To prove $C(B_0)=0$, it suffices to show that

$$\lim_{n\to\infty}\frac{h_n(x,y)}{n}=0.$$

Let $G(\xi,\eta\,;x,y)$ be the generalized Green's function for D with pole at (x,y). We then have in view of (1) and (3),

$$\lim_{n\to\infty}\frac{\varphi(\xi,\eta,u)}{n}G_n(\xi,\eta\,;x,y)=0,$$

$$\left| \frac{\varphi(\xi, \eta, u)}{n} G_n(\xi, \eta ; x, y) \right| < AG(\xi, \eta ; x, y)$$

with $A = \sup_{(\xi,\eta) \in O^*_, u>1} \left| \frac{\varphi(\xi, \eta, u)}{u} \right| < +\infty$, except at $(\xi, \eta) = (x, y)$, and

$$0 < \iint_D G(\xi, \eta ; x, y) d\xi d\eta < +\infty.$$

Hence we obtain

$$\lim_{n\to\infty} \frac{1}{n} \iint_{Dn} \varphi(\xi, \eta, u) G_n(\xi, \eta ; x, y) d\xi d\eta = 0. \tag{5}$$

Next consider

$$\iint_{Dn} au_\xi G_n(\xi, \eta ; x, y) d\xi d\eta.$$

In view of vanishing of G_n on $B(D_n)$ and by integration by parts, this integral can be brought in the form

$$-\iint_{Dn} u \frac{\partial}{\partial \xi} \left\{ aG_n(\xi, \eta ; x, y) \right\} d\xi d\eta$$

$$= - \iint_{Dn} a_\xi u G_n(\xi, \eta ; x, y) d\xi\, d\eta - \iint_{Dn} au \frac{\partial}{\partial \xi} G_n(\xi, \eta ; x, y) d\xi d\eta.$$

Since a_ξ is uniformly bounded in D, we have similary as the preceding argument

$$\lim_{n\to\infty} \frac{1}{n} \iint_{Dn} a_\xi u G_n(\xi, \eta ; x, y) d\xi d\eta = 0. \tag{6}$$

Let $H_n(\xi, \eta)$ be the harmonic function in D_n with the boundary values $\log r = \log \sqrt{(\xi - x)^2 + (\eta - y)^2}$ on $B(D_n)$. Then

$$G_n(\xi, \eta ; x, y) = \log \frac{1}{r} + H_n(\xi, \eta),$$

so that

$$\frac{\partial}{\partial \xi} G_n(\xi, \eta ; x, y) = \frac{x - \xi}{r^2} + \frac{\partial}{\partial \xi} H_n(\xi, \eta),$$

and so

$$\iint_{Dn} au \frac{\partial}{\partial \xi} G_n(\xi, \eta ; x, y) d\xi d\eta$$

$$= \iint_{Dn} au \frac{x - \xi}{r^2} d\xi d\eta + \iint_{Dn} au \frac{\partial}{\partial \xi} H_n \, d\xi d\eta.$$

We now obtain by Schwarz inequality

$$\left(\iint_{Dn} au \frac{\partial}{\partial \xi} H_n \, d\xi d\eta \right)^2 \leqq \left(\iint_{Dn} \left(\frac{\partial}{\partial \xi} H_n \right)^2 d\xi d\eta \right) \left(\iint_{Dn} (au)^2 \, d\xi d\eta \right).$$

By Dirichlet's principle[2)] we can find a positive constant K such that

$$\iint_{D_n} \left\{ \left(\frac{\partial}{\partial \xi} H_n \right)^2 + \left(\frac{\partial}{\partial \eta} H_n \right)^2 \right\} d\xi d\eta < K \qquad \text{for all } n.$$

2) R. Courant, Dirichlet's Principle, Conformal Mapping and Minimal Surfaces (1950), p. 11. K may depend on (x, y).

Hence

$$\frac{1}{n}\left|\iint_{D_n} au\frac{\partial}{\partial\xi}H_n\,d\xi\,d\eta\right|<\sqrt{K}\left(\iint_{D_n}\left(\frac{au}{n}\right)^2 d\xi d\eta\right)^{\frac{1}{2}}$$

$$\to 0 \text{ as } n\to\infty.$$

Thus

$$\lim_{n\to\infty}\frac{1}{n}\iint_{D_n} au\frac{\partial}{\partial\xi}H_n\,d\xi d\eta=0.$$

On the other hand it is easy to show that

$$\lim_{n\to\infty}\frac{1}{n}\iint_{D_n} au\,\frac{x-\xi}{r^2}\,d\xi d\eta=0.$$

Therefore we get

$$\lim_{n\to\infty}\frac{1}{n}\iint_{D_n} au\frac{\partial}{\partial\xi}G_n(\xi,\eta\,;x,y)\,d\xi d\eta=0. \tag{7}$$

(6) and (7) give us

$$\lim_{n\to\infty}\frac{1}{n}\iint_{D_n} au_\xi\,G_n(\xi,\eta\,;x,y)\,d\xi d\eta=0. \tag{8}$$

Similarly we have

$$\lim_{n\to\infty}\frac{1}{n}\iint_{D_n} bu_\eta\,G_n(\xi,\eta\,;x,y)\,d\xi d\eta=0. \tag{9}$$

Finally it follows from (4), (5), (8) and (9), that

$$\lim_{n\to\infty}\frac{h_n(x,y)}{n}$$

$$=\lim_{n\to\infty}\left[\frac{u(x,y)}{n}-\frac{1}{2\pi n}\iint_{D_n}\left\{au_\xi+bu_\eta+\varphi(\xi,\eta,u)\right\}G_n(\xi,\eta\,;x,y)\,d\xi d\eta\right]=0$$

for any (x,y) of D. This proves $C(B_0)=0$.

Let us now assume (2) instead of (1). Then we have

$$\liminf_{n\to\infty}\frac{1}{n}\iint_{D_n}\varphi(\xi,\eta\,;u)G_n(\xi,\eta\,;x,y)\,d\xi d\eta\geqq 0$$

instead of (5), so that we get

$$\limsup_{n\to\infty}\frac{h_n(x,y)}{n}\leqq 0.$$

However $h_n(x,y)>0$ for all n. Consequently

$$\lim_{n\to\infty}\frac{h_n(x,y)}{n}=0.$$

This proves $C(B_0)=0$.

O^* has at most a finite number of components $D^{(k)}\,(k=1,2,\cdots,m)$ of the type considered above. Putting $B_0^{(k)}=B(D^{(k)})\cup E^*$, we have $C(B_0^{(k)})=0$ for $k=1,2,\cdots,m$. Therefore

$$C(E^*)\leqq\sum_{k=1}^{m}C(B_0^{(k)})=0.$$

From this it follows that $C(E^*)=0$, consequently E^* is identical to E and $C(E)=0$. The theorem is thus proved completely.

As an application of this theorem we can state:

Let $a(x,y)$, $b(x,y)$ be continuously differentiable in Ω, and let $\varphi(x,y,u)$ be a continuous function for $(x,y)\in\Omega$ and $-\infty<u<+\infty$, satisfying

$$(1') \qquad \sup_{(x,y)\in F,\, u>1} \frac{|\varphi(x,y,u)|}{u} < +\infty$$

or

$$(2') \qquad \inf_{(x,y)\in F,\, u>1} \varphi(x,y,u) > -\infty$$

for each closed subset F *of* Ω.

Let the equation

$$\Delta^* u + a u_x + b u_y + \varphi(x, y, u) = 0$$

be taken for $\mathfrak{F}[u]=0$, *then* $C_{\mathfrak{F}}(E)=0$ *implies* $C(E)=0$.

3. We shall denote by $\{\mathfrak{F}\}$ a family of all $\mathfrak{F}$-functions defined in any open subset of Ω and impose the following conditions to $\{\mathfrak{F}\}$:

(C_1) If $f \in \{\mathfrak{F}\}$, $g \in \{\mathfrak{F}\}$ in some open set O, then $\lambda f + \mu g \in \{\mathfrak{F}\}$ in O for any real numbers λ and μ.

(C_2) If $f_n \in \{\mathfrak{F}\}$ $(n=1,2,\cdots)$ in some open set O and f_n converge uniformly toward f in any closed subset of O, then $f \in \{\mathfrak{F}\}$ in O.

(C_3) Let E be a closed set of measure zero in Ω. Then it is possible to find a region D containing E such that, for each $z \in E$, there is a function

$$G(\zeta, z) = w(\zeta, z) \log \frac{1}{|\zeta - z|} + v(\zeta, z)$$

which belongs to $\{\mathfrak{F}\}$ in $D-E$ as a function of ζ, where $w(\zeta, z)$ and $v(\zeta, z)$ satisfy

$$M > w(\zeta, z) > m, \quad |v(\zeta, z)| < N \quad \text{for } \zeta \in D-E,\ z \in E,$$

M, m and N denoting positive constants not depending on ζ, z.

THEOREM 2. *Let* $\{\mathfrak{F}\}$ *satisfy the conditions* (C_1), (C_2) *and* (C_3). *If* $C(E)=0$, *then* $C_{\mathfrak{F}}(E)=0$.

Proof.[3] Take n points $z_1, z_2, \cdots, z_n$ on E and put

$$M_n = \inf_{z_1, z_2, \cdots z_n} \{\max_{z\in E} |(z-z_1)(z-z_2)\cdots(z-z_n)|\}.$$

Then $C(E)=0$ implies $m_n = -\log \sqrt[n]{M_n} \to +\infty$ as $n \to \infty$. Hence we can choose n_j for $j=1,2,\cdots$ so that $m_{n_j} \geqq 2^j$, and find a sequence of sets of points $\{z^0_{1,j}, z^0_{2,j}, \cdots, z^0_{n_j,j}\}$ such that

$$\max_{z\in E} |(z-z^0_{1,j})(z-z^0_{2,j})\cdots(z-z^0_{n_j,j})| = M_{n_j}.$$

Then it is known that

$$(10) \qquad \sum_{j=1}^{\infty} \left(\frac{1}{2^j n_j} \sum_{k=1}^{n_j} \log \frac{1}{|\zeta - z^0_{k,j}|} \right) \to +\infty$$

as $\zeta \bar{\in} E$ approaches to any point of E.

We shall now define a corresponding function for $\{\mathfrak{F}\}$. Since E is of measure zero, there is a region $D-E$ for which exist $\mathfrak{F}$-functions $G(\zeta, z)$ mentioned in (C_3). Putting

$$u_j(\zeta) = \frac{1}{n^j} \sum_{k=1}^{n_j} G(\zeta, z^0_{k,j}),$$

3) Our method for the proof is analogous to that of the Evans' theorem for harmonic functions. G. C. Evans, Potentials and positively infinite singularities of harmonic functions, Monatsh. für Math. u. Phys. 43(1936), pp.419–424.

we define

$$U(\zeta)=\sum_{j=1}^{\infty}2^{-j}u_j(\zeta). \tag{11}$$

We shall show at first that the series (11) converges uniformly in $D-E$. For any closed subregion F of $D-E$, we find a positive constant L such that

$$|G(\zeta,z)|<L \qquad \text{for } \zeta\in F,\ z\in E.$$

Then $|u_j(\zeta)|<L$, so that the series (11) converges uniformly on F. Hence it follows from (C_1) $u_j\in\{\mathfrak{F}\}$, and so from (C_2) $U\in\{\mathfrak{F}\}$ in $D-E$.

It remains to show that

$$\lim_{\zeta\to\zeta'}U(\zeta)=+\infty$$

for all $\zeta'\in E$. To prove this, we may assume that E lies in a square with sides less than 1/2. Otherwise, we devide E into a finite number of closed subsets E_k each of which lies in such a square, and have only to add all $U_k(\zeta)$ defined above for E_k (where D may be the same for all k). If we make this assumption, we can find a small neighborhood V of ζ' in which, except at points $\zeta\in E$,

$$G(\zeta,z)>m\log\frac{1}{|\zeta-z|}-N \quad \text{for any } z\in E.$$

Then

$$u_j(\zeta)>\frac{m}{n^j}\sum_{k=1}^{n_j}\log\frac{1}{|\zeta-z^0_{k,j}|}-N.$$

Hence from (10),

$$U(\zeta)>m\sum_{j=1}^{\infty}\left(\frac{1}{2^jn^j}\sum_{k=1}^{n_j}\log\frac{1}{|\zeta-z^0_{k,j}|}\right)-N\to+\infty \text{ as } \zeta\to\zeta'.$$

Thus $U(\zeta)$ is an Evans' function for E with respect to $\{\mathfrak{F}\}$, and so $C_{\mathfrak{F}}(E)=0$.

As an application of Theorems 1 and 2, we prove :

THEOREM 3. *Let $a(x,y)$, $b(x,y)$, $c(x,y)$ and $f(x,y)$ be continuously differentiable in Ω. Take for $\mathfrak{F}[u]=0$ the linear partial differential equation of elliptic type*

$$\Delta u+au_x+bu_y+cu+f=0. \tag{12}$$

Then $C_{\mathfrak{F}}(E)=0$ is equivalent to $C(E)=0$ [4].

Proof. It is obvious from Theorem 1 that $C(E)=0$ if $C_{\mathfrak{F}}(E)=0$. Let us now suppose $C(E)=0$. Recognizing that the family of all solutions of the equation

$$\Delta u+au_x+bu_y+cu=0 \tag{13}$$

satisfies the conditions (C_1), (C_2) and (C_3), there is, by Theorem 2, a region $D-E$ in which exists an Evans' function u for E with respect to (13). On the other hand, we can find a solution v of the equation

$$\Delta v+av_x+bv_y+cv+f=0$$

4) Theorem 3 is an extension of the Evans' theorem. Recently I. Hong extended this theorem for solutions of the equation $\Delta u+k^2u=0$ with a constant $k>0$. I. Hong, On positively infinite singularities of a solution of the equation $\Delta u+k^2u=0$, Kōdai Math. Sem. Rep. v. 8, n. 1 (1956) pp.9-12.

in D, if the area of D is sufficiently small. Then $w=u+v$ becomes an Evans' function for E with respect to (12). Hence $C_{\mathfrak{F}}(E)=0$ and the theorem is established.

The fact that the family of all solutions of (13) satisfies the conditions $(C_1)\sim(C_3)$ is proved as follows:

(C_1) is trivial. (C_2) is easily verified by applying the fact that the Dirichlet problem with respect to (13) has a unique solution for circles of small radius. To assure (C_3), let us take a small region[5] D in which there exists, for each $z \in E$, a fundamental solution having a logarithmic infinity at z and it is written in the form[6]

$$G(\zeta, z) = \log \frac{1}{|\zeta - z|} + \iint_D \frac{\psi(w, z)}{|w-z|} \log \frac{1}{|\zeta - w|}\, d\tau_w,$$

where $d\tau_w$ denotes the area element with respect to w and $\psi(w,z)$ is uniformly bounded with respect to $w \in D$ and $z \in E$. From this expression, the property required for $G(\zeta, z)$ in (C_3) follows.

4. We now set the following conditions:

(C_4) Let D be any region bounded by a finite number of smooth curves in Ω and let f be any $\mathfrak{F}$-function on $B(D)$.[7] Then the Dirichlet problem for D and f, relative to $\{\mathfrak{F}\}$, has a solution, namely, there is a function which belongs to $\{\mathfrak{F}\}$ in D, is continuous in $D \cup B(D)$ and equal to f on $B(D)$.

(C_5) Let D be any region in Ω. If u is continuous in $D \cup B(D)$, belongs to $\{\mathfrak{F}\}$ in D and vanishes on $B(D)$, then u vanishes identically in D.

THEOREM 4. *Let $\{\mathfrak{F}\}$ satisfy the conditions (C_1), (C_4) and (C_5). Let O be an open set in Ω and E a closed set of $\mathfrak{F}$-capacity zero contained in O. If U is a bounded $\mathfrak{F}$-function in $O-E$, it is possible to define U on E, so that U becomes an $\mathfrak{F}$-function in O.*

Proof. Without loss of generality we can assume that $B(O)$ consists of a finite number of smooth curves and $U \in \{\mathfrak{F}\}$ in $\bar{O}-E$ where $\bar{O}$ denotes $O \cup B(O)$, and moreover that there is in $\bar{O}-E$ an Evans' function u for E with respect to $\{\mathfrak{F}\}$.

By (C_4) there are two $\mathfrak{F}$-functions U^*, u^* in O, such that U^*, u^* are continuous in $\bar{O}$ and $U=U^*$, $u=u^*$ on $B(O)$. Put $w=U-U^*$, $h=u-u^*$. Then h is again an Evans' function for E, and, by assumption, we can find a constant M such that $|w|<M$ in $\bar{O}-E$.

Denoting by O_n the set of points at which $h<n$ $(n=1, 2, \cdots)$, $B(O)$ becomes boundary components of O_n, because $h=0$ on $B(O)$. According to (C_1), $\pm Mh/n$ are $\mathfrak{F}$-functions in O_n with the boundary values $\pm M$ on $B(O_n)-B(O)$ and zero on $B(O)$.

Next we shall show using the condition (C_5) that

$$\frac{Mh}{n} \geqq w \geqq -\frac{Mh}{n} \quad \text{throughout } O_n. \tag{14}$$

5) The smallness of D is required for its area and the magnitude of the diameter of D is irrelevant.

6) See R. Courant-D. Hilbert, Methoden der Mathematischen Physik, II, pp. 279–281.

7) f is said to be an $\mathfrak{F}$-function on a closed set S if f is so in a certain open set containing S.

Suppose that the statement $Mh/n \geqq w$ is false. Then there are a point of O_n at which $w > Mh/n$, and a region $D_n \subset O_n$ containing this point, on whose boundary $w = Mh/n$. On the other hand, by virtue of (C_1), $w - Mh/n$ is continuous in $\bar{D}_n$ and belongs to $\{\mathfrak{F}\}$ in D_n. Hence it follows from (C_5) that

$$w - \frac{Mh}{n} \equiv 0 \quad \text{in } D_n,$$

which is impossible. Thus we have $Mh/n \geqq w$ throughout O_n. The inequality $w \geqq -Mh/n$ is shown similarly.

Letting $n \to \infty$ in (14), we see

$$w = U - U^* = 0$$

at all points of $O - E$, because $O_n \to O - E$ as $n \to \infty$. This proves our theorem.

As an application of these results we can state:

Let E be a bounded closed set of logarithmic capacity zero in a plane region D, and let u be a bounded solution of the equation

$$\Delta u + au_x + bu_y + cu + f = 0 \tag{15}$$

in $D - E$, where a, b, c and f are continuously differentiable in D, and $c \leqq 0$. Then it is possible to define u on E, so that u satisfies (15) *on E.*

To prove this, consider without loss of generality any bounded solution v of (15) in D, then $w = u - v$ is bounded in D and satisfies

$$\Delta w + aw_x + bw_y + cw = 0 \tag{16}$$

in $D - E$. Theorems 3 and 4 now apply to w. In fact, taking the equation (16) for $\mathfrak{F}[w] = 0$, $\{\mathfrak{F}\}$ satisfies the conditions $(C_1) \sim (C_5)$. Hence w, consequently, u is prolongable continuously over E, so that u satisfies again (15) at all points of E.

In the above statement the restriction on the sign of c was imposed to insure the conditions (C_4) and (C_5). But we notice that a set of logarithmic capacity zero is of measure zero, and that Theorem 4 remains true for any E of $\mathfrak{F}$-capacity zero and of measure zero provided the conditions (C_4) and (C_5) are assumed for sufficiently small regions.

On the other hand, the Dirichlet problem for a region D and any continuous boundary value function, with respect to (15), has a unique solution regardless of the sign of c, if the boundary of D is smooth and its area is sufficiently small[8]. So the condition (C_4) is fulfilled, while the condition (C_5) is also assured for sufficiently small regions. Hence the above statement holds with no restriction on the sign of c. Thus we obtain

THEOREM 5. *Let E be a bounded closed set of logarithmic capacity zero in a plane region D, and let u be a bounded solution of the equation*

$$\Delta u + au_x + bu_y + cu + f = 0$$

in $D - E$, where a, b, c and f are continuously differentiable in D. Then it is possible to define u on E, so that u satisfies the above equation on E.

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8) I. G. Petrowsky, Lectures on Partial Differential Equations, (1954), p.232.