Harmonic functions with two singular points

By Hirosi Yamasuge

(Received Nov. 9, 1956)

In this paper we assume that \( \mathcal{M} \) is a closed orientable analytic Riemannian manifold with a positive-definite metric \( ds^2 = g_{ik}dx^i dx^k \) where \( g_{ik} \) are holomorphic functions of \( x^1, \ldots, x^n \).

In 1 we shall prove the existence of a harmonic function \( \varphi \) with two singular points such that

\[
\varphi \text{ is harmonic in } \mathcal{M} - \xi_1 - \xi_2,
\]

if \( n > 2 \),

\[
\lim_{x \to \xi_i} (n-2)\omega_n r^{n-2}(x, \xi_i) \varphi(x) = (-1)^{i-1}, \quad i = 1, 2,
\]

and if \( n = 2 \),

\[
\lim_{x \to \xi_i} 2\pi \varphi(x)/\log r(x, \xi_i) = (-1)^i, \quad i = 1, 2,
\]

where \( r(x, \xi_i) \) is the geodesic distance between \( x \) and \( \xi_i \), and \( \omega_n \) is the surface area of the \( n \)-dimensional unit sphere. Hence we may consider that \( \varphi(x) \) is the potential at \( x \) of the pair of masses which has the mass 1 at \( \xi_1 \) and the mass \(-1\) at \( \xi_2 \).

Now we consider the equipotential surface \( U_C \) given by \( \varphi = C \). We shall say that a point is stational if all first partial derivatives of \( \varphi \) are zero at this point, and say that a stational point is non-degenerate if at this point the determinant \( \left| \frac{\partial^2 \varphi}{\partial x^i \partial x^j} \right| \) is not zero. We change \( C \) from \( +\infty \) to \(-\infty \), then \( U_C \) is homeomorphic with a sphere if \( \left| C \right| \) is sufficiently large, and the topological structure of \( U_C \) changes only when \( U_C \) passes stational points. Hence if we assume that all stational points are non-degenerate, there are close relations between the number of all stational points and the topological structure of \( \mathcal{M} \). We shall state about them in 2.

1. Existence of a harmonic function with two singular points

Let \( G \) be a sufficiently small geodesic sphere in \( \mathcal{M} \) and \( \xi \) an arbitrary interior point of \( G \). Then the Laplace's equation \( \Delta \Xi = 0 \) has a solution

\[
\Xi(x, \xi) = \begin{cases} 
-\frac{1}{2\pi} \log r(x, \xi) \cdot u(x, \xi) + v(x, \xi), & (n = 2), \\
\frac{1}{(n-2)\omega_n} r^{2-n}(x, \xi) u(x, \xi) + \log r(x, \xi) \cdot v(x, \xi), & (n > 2), 
\end{cases}
\]

defined for \( x, \xi \) in \( G \), where \( u, v \) are holomorphic with respect to \( x, \xi \), and \( u(\xi, \xi) = 1 \). See [1].

Let \( \xi_1 \) and \( \xi_2 \) be two interior points given in \( G \).
Putting
\[ h = \Xi(x, \xi_1) - \Xi(x, \xi_2), \]
we have
\[ \Delta h = 0 \quad \text{in} \quad G - \xi_1 - \xi_2. \]
We shall say that a form \( \alpha \) is regular harmonic in a domain \( D \) if \( da = 0 \) and \( \bar{a}a = 0 \) in \( D \). Then we have

**Lemma.** There exists 1-form \( e \) possessing the following properties:

- \( e \) is regular harmonic in \( \mathcal{M} - \xi_1 - \xi_2 \)

and
\[ e = dh + f \quad \text{in} \quad G - \xi_1 - \xi_2, \]
where \( f \) is a regular harmonic 1-form in \( G \).

**Proof.** By the general existence theorem shown in [1] it is sufficient to prove that \( dh \times BG = \int_{BG} * dh = 0 \) for the surface \( BG \) of a geodesic sphere \( G \).

For an arbitrary function \( g \)
\[ d * dg = \ast d_{\partial} g = \ast \Delta g. \]
Hence if \( g \) is harmonic in a domain \( D \) and its first derivatives are continuous in \( \overline{D} - D + BD \), then
\[ \int_{BD} * dg = \int_D d * dg = \int_D - \ast \Delta g = 0. \]
Applying the above to \( dh, d\Xi(x, \xi_1) \) and \( d\Xi(x, \xi_2) \), we have
\[ \int_{BG} * dh = \int_{BG} * d\Xi(x, \xi_1) - \int_{BG} * d\Xi(x, \xi_2) \]
where \( G_\xi, G_\eta \) are geodesic spheres of the center \( \xi \) and of the radius \( \eta \) respectively. Moreover we can verify that
\[ \lim_{\delta \to 0} \int_{BG_\delta} * d\Xi(x, \xi_i) = -1, \quad i = 1, 2. \]
Hence \( \int_{BG} * dh = 0 \), q.e.d. [See the proof of Theorem 2 in [4]].

Now we consider the periods of \( e \) along loops which do not pass through the points \( \xi_1 \) and \( \xi_2 \). Since \( e \) is closed, the periods of \( e \) depend only on homology classes of loops in \( \mathcal{M} \). Let \( \gamma_i(i = 1, \ldots, R_1) \) be a base for 1-cycles of \( \mathcal{M} \). Then by the theorem of de Rham there exists harmonic 1-form \( e' \) such that
\[ \int_{\gamma_i} e = \int_{\gamma_i} e', \quad (i = 1, \ldots, R_1). \]
Let \( P_0 \) be a fixed point in \( \mathcal{M} \) and \( P \) an arbitrary point in \( \mathcal{M} \).

Put
\[ \varphi(P) = \int_{P_0}^P (e - e'), \]
then the periods of \( \varphi(P) \) on every loops are always zero. Hence \( \varphi(P) \) is an one valued function defined in \( \mathcal{M} \). From the construction of \( \varphi \) we have

**Theorem 1.** For arbitrary two points \( \xi_1 \) and \( \xi_2 \) given in a small subdomain
Harmonic functions with two singular points

There exists a harmonic function \( \varphi \) such that

\[ \Delta \varphi = 0 \quad \text{in} \quad \mathbb{R} - \xi_1 - \xi_2, \]

and in a neighbourhood of \( \xi_i \) (\( i = 1, 2 \))

\[ \varphi(x) = \begin{cases} \frac{1}{(n-2)\omega_n} r^{2-n}(x, \xi_i)u_i + \log r(x, \xi_i) \cdot v_i + w_i, & (n > 2), \\ -\frac{1}{2\pi} \log r(x, \xi_i) \cdot u_i + v_i, & (n = 2), \end{cases} \]

where \( u_i, v_i \) and \( w_i \) are holomorphic functions of \( x \), and \( u_i(\xi_i) = 1 \).

2. Relations between the number of stational points and the topological structure of \( \mathbb{R} \).

From now on let us assume that all stational points are non-degenerate. Then in a suitable coordinate system, the Taylor's expansion of \( \varphi \) at every stational point becomes

\[ \varphi(x) = C + (-x_1^2 - \cdots - x_n^2 + x_{n+1}^2 + \cdots + x_n^2) + \chi(x). \]

Hence the stational point \( x = 0 \) is isolated, and since \( \mathbb{R} \) is closed, the number of stational points is finite.

Suppose \( \nu = 0 \) and take a sufficiently small positive number \( \delta \). Then the closed subdomain \( G_\delta \) given by the inequality \( \varphi(x) \leq C + \delta \) is homeomorphic with a sphere. Using maximum principle for \( \varphi \), we see that \( \varphi \) is the constant \( C + \delta \) in \( G_\delta \). By [1], \( \varphi \) is holomorphic in \( \mathbb{R} - \xi_1 - \xi_2 \). Therefore \( \varphi \) would be identically the constant \( C + \delta \), contrary to (2) of Theorem 1. Hence \( \nu \geq 1 \).

Similarly we have \( \nu \leq n - 1 \).

Now let us consider the equipotential surface \( U = C \) and denote it by \( U_C \). If \( U_C \) has no stational point, \( U_C \) is an orientable \((n-1)\)-dimensional manifold.

Change \( C \) from \( +\infty \) to \( -\infty \). Then \( U_C \) moves in \( \mathbb{R} \) but the topological structure of \( U_C \) changes only when \( U_C \) passes the stational points of \( \varphi \).

The case of \( n = 2 \). In this case (3) becomes

\[ \varphi(x) = C - x_1^2 + x_2^2 + \chi(x). \]

Let \( \delta \) be a sufficiently small positive number. Then in a neighbourhood \( V \) of the stational point \( x = 0 \), \( U_{C+\delta} \) are hyperbolas and \( U_C \) is two straight lines. If \( U_C \) passes no stational point, \( U_C \) consists of some loops, and the number of the loops increases by 1 or decreases by 1 whenever \( U_C \) passes a stational point. Suppose \( \{ P_i, Q_j ; i = 1, \ldots, g, j = 1, \ldots, g' \} \) is the complete set of all stational points such that the number of the loops increases by 1 when \( U_C \) passes \( P_i \) and decreases by 1 when \( U_C \) passes \( Q_j \). And if \( C \) is sufficiently large, by (2) \( U_{C+\delta} \) consists of one loops. Hence we have \( g = g' \). Moreover we can assume without loss of generality that \( \varphi(P_i) > \varphi(Q_j) \) for \( i, j = 1, \ldots, g \). Take \( C \) so that \( \varphi(P_i) > C > \varphi(Q_j) \) for \( i, j = 1, \ldots, g \).
Put
\[ (5) \quad \mathcal{M}(a, b) = \{ P \mid a \leq \varphi(P) \leq b \} \]
then
\[ \mathcal{M} = \mathcal{M}(C, \infty) + \mathcal{M}(-\infty, C) \]
where \( \mathcal{M}(C, \infty) \) and \( \mathcal{M}(-\infty, C) \) are homeomorphic to a sphere with \( g \) holes.
Hence we have

**Theorem 2.** If all stational points of \( \varphi \) are non-degenerate, then the number of these points is equal to twice the genus of \( \mathcal{M} \).

The case of \( n=3 \). In this case \( \nu \) of (3) is 2 or 1. Let \( P_1, \ldots, P_g \) be all stational points at which \( \nu=2 \) and \( Q_1, \ldots, Q_g \) all stational points at which \( \nu=1 \). Then in a neighbourhood of \( P_i, U_C \) changes from a hyperboloid of two sheets to a hyperboloid of one sheet, and in a neighbourhood of \( Q_j, U_C \) changes from a hyperboloid of one sheet to a hyperboloid of two sheets. Thus we see easily that the genus of a connected component of \( U_C \) increases by 1 or decreases by 1 according as the component passes point \( P_i \) or \( Q_j \). Moreover if \( C \) is sufficiently large, then by (2), \( U_{\pm C} \) is homeomorphic with a sphere, and hence \( g = g' \).

We may assume without loss of generality that \( \varphi(P_i) > \varphi(Q_j), i, j=1, \ldots, g \). Then similarly to (5) we have
\[ (6) \quad \mathcal{M} = \mathcal{M}(C, \infty) + \mathcal{M}(-\infty, C) \]
where \( \mathcal{M}(C, \infty) \) and \( \mathcal{M}(-\infty, C) \) are homeomorphic with a closed subdomain bounded by a surface of genus \( g \) in \( E^3 \). Thus we have

**Theorem 3.** If all stational points of \( \varphi \) are non-degenerate, then the number \( g \) of these points is even. Take two closed domains bounded by a surface of genus \( g/2 \) in \( E^3 \). Then \( \mathcal{M} \) is obtained from these two domains by identifying their boundaries by a homeomorphism.

**Reference**