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On the homotopy groups of Stiefel manifolds

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J. H. C. Whitehead [3] gave a cellular decomposition of the Stiefel manifold $V_{k+m,m}$ of m-frames in Euclidian $(k+m)$-space, and he and Baratt-Peacher [1] determined the homotopy groups $\pi_{k+j}(V_{k+m,m})$ for $j=1, 2, 3$.

In the present paper, by making use of the above J. H. C. Whitehead's result and the Steenrod square, we shall gave a reduced cell complex of P^l_{k-1} and determine $\pi_{k+j}(V_{k+m,m})$ for $j\leqq 5$ $(k\geqq j+2)$.

The basic tools used are following:

i) (*Whitehead's theorem*) [3 theorem 3]. If $r<2k$, then $\pi_r(V_{n,m})=\pi_r(P^l_{k-1})$, $l=\mathrm{Min}(r+1, n-1)$, $k=n-m$, where P^l_{k-1} is a space obtained from the l-dimensionul projective space by shrinking its $(k-1)$-dimensional hyperplane to a point.

ii) (*Squaring formula*). If we denote by u^j the generator of $H^j(P^l_{k-1}; Z_2)$, we have $Sq^iu^j=\binom{i}{j}u^{i+j}$ for $i+j\leqq l$, where $\binom{j}{i}$ is the binomal coefficient with the usual conventions.

I am deeply grateful to Prof. H. Toda for his kind advices during the preparation of this paper.

1. Notations.

We shall use the following notations throughout this paper.

P^n_{k-1}: the space obtained from the l-dimensional projective space by shrinking its $(k-1)$-dimensional hyperplane to a point.

We denote $\pi_{n+r}(P^{n+k}_{n-1})$, $\pi_{n+r}(P^{n+k}_{n-1}, P^{n+k-1}_{n-1})$ by π^k_r, σ^k_r respectively.

Let $K=L \cup_f e^{n+1}$ be a complex such that e^{n+1} is attached to L by a mapping f.

A map $\bar{g}e^{n+1}$: $(E^{p+1}, \dot{E}^{p+1}) \rightarrow (K, L)$ is defined as follows, where g is a map S^p to S^n; $\bar{g}e^{n+1}$ maps E^{p+1} in e^{n+1} by the suspension of g, $\bar{g}e^{n+1}|\dot{E}^{p+1}$ in K by $f\cdot g$.

Now if $f\cdot g$ is a nullhomotopic in L, we denote by ge^{n+1}: $S^{p+1}=E^{p+1}_+ \cup E^{p+1}_- \rightarrow K$ the following map: $ge^{n+1}|E^{p+1}_+$ maps E^{p+1}_+ in e^{n+1} by the suspension of g, and $ge^{n+1}|E^{p+1}_-$ is a null homotopy of $f\cdot g$.

$\{\bar{g}e^{n+1}\}_q$, $\{ge^{n+1}\}_q$ are cyclic subgroups of order q of $\pi_{p+1}(K, L)$, $\pi_{p+1}(K)$ which are generated by $\{\bar{g}e^{n+1}\}$, $\{ge^{n+1}\}$ whose representatives are $\bar{g}e^{n+1}$, ge^{n+1} respectively.

We denote the generators and these representatives of $\pi_{n+1}(S^n)$, $\pi_{n+2}(S^n)$, $\pi_{n+3}(S^n)$, by the same letters η, ε, ν respectively.

We denote the m-dimensional cell of P^l_{n-1} and the generator of $H^m(P^l_{n-1}, Z_2)$ by the same letter e^m.

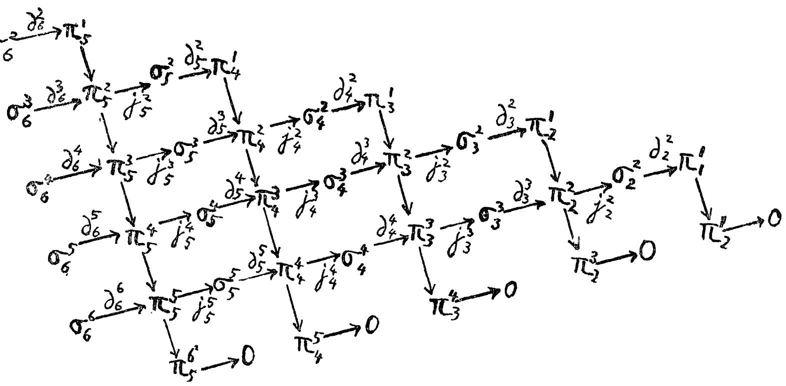
2. Reduced complex of P^{n+i}_{n-1}.

Let n-dimensional CW-complexes K, L be of the same homotopy type. Consider two complexes $K' = K \underset{\alpha}{\cup} e^{n+1}$ and $L' = L \underset{\beta}{\cup} e'^{n+1}$, where α and β are characteristic mappings of e^{n+1} and e'^{n+1} respectively. If $f\cdot\alpha$ is homotopic to β in L, then $g\cdot\beta$ is homotopic to α in K, and L', K' are of the same homotopy type. Therefore if we can construct a reduced complex of P^{n+i}_{n-1}, then $L = K \underset{f_{n+i}}{\cup} e^{n+i+1}$ is a reduced complex of P^{n+i+1}_{n-1} where f_{n+i} is a map representing an element of $\pi_{n+i}(K)$. Now P^n_{n-1} is an n-sphere. Hence by the determination of the homotopy class of the characteristic map f_{n+i} for each i, we can determine the homotopy type of P^{n+i}_{n-1}.

Throughout this paper we use the notation P^{n+i}_{n-1} to denote a reduced complex of P^{n+i}_{n-1}.

3. The homotopy type and homotopy groups of P^l_{n-1}.

We consider the following diagram

(∗) 

In this diagram, the sequence $\cdots \longrightarrow \sigma^k_r \xrightarrow{\partial^k_r} \pi^{k-1}_{r-1} \longrightarrow \pi^k_{r-1} \xrightarrow{j^k_{r-1}} \sigma^k_{r-1} \longrightarrow \cdots$ are exact.

Divide the following 4 cases.

Case 1: $n = 4l$.

In this diagram (∗), we have $\pi^1_r = \pi_{n+r}(S^n)$ and $\sigma^k_r \approx \pi_{n+r-1}(S^{n+k-1})$. By a property of the projective space, e^{n+1} is attached to e^n by a mapping of degree 0. Therefore we have $P^{n+1}_{n-1} = S^n \vee S^{n+1}$, $\pi^1_1 = \{e^{n+1}\}_\infty + \{\eta e^n\}_2$ and $\pi^2_1 = \{\eta e^{n+1}\}_2 + \{\varepsilon e^n\}_2$.

Since $Sq^1 e^{n+1} \neq 0$ and $Sq^2 e^n = 0$, ∂e^{n+1} covers e^{n+1} with a mapping of degree 2 and does not cover e^n. Hence $P^{n+2}_{n-1} = S^n \vee Y^{n+2}$, where Y^{n+2} is the suspended space of the projective plane whose homotopy groups are studied by H. Toda.[1]

1) See [2], p. 79.

We have $\pi_2^1=\{\eta e^{n+1}\}_2+\{\varepsilon e^n\}_2$, Image $j_2^2=0$, $\sigma_3^2=\{\bar{\eta}e^{n+2}\}_2$ and Image $\partial_3^2=0$. These imply that π_2^2 is isomorphic to π_2^1.

We consider the characteristic map of e^{n+3}. Since $Sq^2e^{n+1}=0$, ∂e^{n+3} does not cover e^{n+1}. And we may suppose that ∂e^{n+3} does not cover e^n. This fact is proved as follows: Since the fibre bundle $V_{n+4,4}/V_{n+3,3}=S^{n+3}$ has a cross-section for $n=4l$,[2] we have $\pi_{n+2}(P_{n-1}^{n+3})=\pi_{n+2}(V_{n+4,4})\approx\pi_{n+2}(V_{n+3,3})=\pi_{n+2}(P_{n-1}^{n+2})=\{\eta e^{n+1}\}_2+\{\varepsilon e^n\}_2$. If ∂e^{n+3} covers e_n then $\{\varepsilon e^n\}$ is a nullhomotopic map in π_2^3. This is a contradiction. Then we have $P_{n-1}^{n+3}=S^n\vee Y^{n+2}\vee S^{n+3}$.

We have $\pi_2^3=\{\eta e^{n+1}\}_2+\{\varepsilon e^n\}_2$, $\pi_3^1=\{\varepsilon e^{n+1}\}_2+\{\nu e^n\}_{24}$, $\pi_3^2=\{\eta e^{n+2}\}_4+\{\nu e^n\}_{24}$,[1] Image $\partial_4^3=0$ and Kernel $\partial_3^3=\{e^{n+3}\}_\infty$. These implies that $\pi_3^3=\{e^{n+3}\}_\infty+\{\eta e^{n+2}\}_4+\{\nu e^n\}_{24}$.

Since $Sq^1e^{n+3}\neq 0$ and $Sq^2e^{n+2}\neq 0$, ∂e^{n+4} covers e^{n+3} with a mapping of degree 2 and covers e^{n+2} by η.

Consider the following two cases.

i) $l\equiv 0$ *mod* 2. Since $Sq^4e^n=0$, ∂e^{n+4} covers e^n by $k\nu$ with even k. However the homotopy type of P_{n-1}^{n+4} depends only on k mod 2. This is proved as follows: Let $K=(S^n\vee Y^{n-2}\vee S^{n+3})\smile e^{n+4}$, where ∂e^{n+4} covers S^{n+3} with a mapping of degree 2, e^{n+2} by η and S^n by $k\nu$. Let $K'=(S'^n\vee Y'^{n+2}\vee S'^{n+3})\smile e'^{n+4}$, where $\partial e'^{n+4}$ covers S'^{n+3} by a mapping of degree 2, e'^{n+2} by η and S'^n by $k'\nu$ with $k'=k+2a$. Consider a map f: $(S^n\vee Y^{n+2}\vee S^{n+3})\to(S'^n\vee Y'^{n+2}\vee S'^{n+3})$ such that i) $f|S^n$ maps S^n in S'^n with degree 1, ii) $f|Y^{n+2}$ maps e^{n+2} in e'^{n+2} with degree 1, iii) f maps the upper hemi-sphere E_+^{n+3} of S^{n+3} in S'^{n+3} with degree 1 and maps the lower hemi-sphere E_-^{n+3} in S'^n by $a\nu$. Consider also a map f': $(S'^n\vee Y'^{n+2}\vee S'^{n+3})\to(S^n\vee Y^{n+2}\vee S^{n+3})$ such that f maps S'^n, e'^{n+2}, $E_+'^{n+3}$ are mapped in S^n, e^{n+2}, S^{n+3} with degree 1 respectively and $E_-'^{n+3}$ is mapped in S^n by $-a\nu$. Then $f\cdot f'$, $f'\cdot f$ are homotopic to identity maps of P_{n-1}^{n+3}, $P_{n-1}'^{n+3}$ respectively. On the other hand, f and f' are extended to mappings g: $K\to K'$, g': $K'\to K$ such that $g\cdot g'$, $g'\cdot g$ are homotopic to identity mappings of K, K' respectively. Thus we may suppose that ∂e^{n+4} does not cover e^n.

Since Image $\partial_3^4=\{2e^{n+1}\}+\{\eta e^{n+2}\}$, we have $\pi_3^4=\{e^{n+3}\}_8+\{\nu e^n\}_{24}$.

ii) $l\equiv 1$ *mod* 2. Since $Sq^4e^n\neq 0$, ∂e^{n+4} covers e^n by $k\nu$ with odd k. By the same reason as in the case $l\equiv 0$ (2), the homotopy type of P_{n-1}^{n+4} depends only on k mod 2. Thus we may suppose $k=1$. Hence we have Image $\partial_3^4=\{2e^{n+3}\}+\{\eta e^{n+2}\}+\{\nu e^n\}$ and $\pi_3^4=\{e^{n+3}\}_{48}+\{\eta e^{n+2}\}_4$.

We have $\pi_4^1=\{\nu e^{n+1}\}_{24}$, $\sigma_5^2=\{\bar{\nu}e^{n+2}\}$, Image $\partial_5^2=2\nu e^{n+1}$, $\sigma_4^2=\{\bar{\varepsilon}e^{n+2}\}$ and j_4^2 is an onto-homomorphism. Thus we have $\pi_4^2=\{\varepsilon e^{n+2}\}_2+\{\nu e^{n+1}\}_2$.

Since Image $\partial_5^3=0$, $\sigma_4^3=\{\bar{\eta}e^{n+3}\}$, j_4^3 is an onto-homomorphism and $2\{\eta e^{n+3}\}=0$ in π_4^3, we have $\pi_4^3=\{\eta e^{n+3}\}_2+\{\varepsilon e^{n+2}\}_2+\{\nu e^{n+1}\}_2$.

We have $\sigma_5^4=\{\bar{\eta}e^{n+4}\}$, Image $\partial_5^4=\{\varepsilon e^{n+2}\}$ and Image $j_4^4=0$. These imply that $\pi_4^4=\{\eta e^{n+3}\}_2+\{\nu e^{n+1}\}_2$.

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2) For example, see Steenrod's book: The topology of bibre bundles, p. 142.

Since $Sq^2 e^{n+3} \neq 0$ and $Sq^4 e^{n+1} = 0$ for $l \equiv 0$, ∂e^{n+5} covers e^{n+3} by η and e^{n+1} by $k\nu$ with even k. However $2\{\nu e^{n+1}\}$ is nullhomotopic in π^4_4, and so we may suppose that ∂e^{n+5} does not cover e^{n+1}.

Since $Sq^2 e^{n+3} \neq 0$ and $Sq^4 e^{n+1} \neq 0$ for $l \equiv 1$, we may suppose that ∂e^{n+5} covers e^{n+3} by η and e^{n+1} by ν.

Therefore we have $\pi^5_4 = \{\nu e^{n+1}\}_2$ for both cases.

Since $\pi^1_5 = 0$ and Kernel $j^3_5 = \{12\nu e^{n+2}\}$, we have $\pi^2_5 = \{12\nu e^{n+2}\}_2$.

We have also Image $\partial^3_5 = 0$, $\sigma^3_5 = \{\bar{\varepsilon} e^{n+3}\}$ and j^3_5 is an onto-homomorphism. These imply that $\pi^3_5 = \{\varepsilon e^{n+3}\}_2 + \{12\nu e^{n+2}\}_2$.

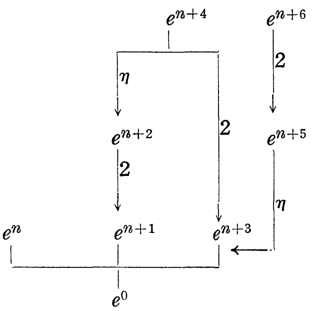
Since $\sigma^4_6 = \{\bar{\varepsilon} e^{n+4}\}_2$, Image $\partial^4_6 = \{12\nu e^{n+2}\}$ and Image $j^4_5 = 0$, we have $\pi^4_5 = \{\varepsilon e^{n+3}\}_2$.

We have $\sigma^5_6 = \{\bar{\eta} e^{n+5}\}$, Image $\partial^5_6 = \{\varepsilon e^{n+3}\}$ and Kernel $\partial^5_5 = \{2e^{n+5}\}$. These imply that $\pi^5_5 = \{2e^{n+5}\}_\infty$ for $l \equiv 0$, $= \{\bar{2} e^{n+5} + \bar{\nu} e^{n+2}\}_\infty$ for $l \equiv 1$, where $f = (\bar{2} e^{n+5} + \bar{\nu} e^{n+2})$: $S^{n+5} \to P^{n+5}_{n-1}$ is a mapping such that the upper hemi-sphere E^{n+5}_+ of S^{n+5} is mapped in e^{n+5} with degree 2 and the lower hemi-shere E^{n+5}_- of S^{n+5} in e^{n+2} by the suspension of ν.

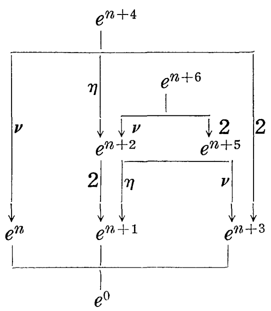
Since $Sq^1 e^{n+5} \neq 0$, e^{n+6} is attached to P^{n+5}_{n-1} by a generator of π^5_5. Thus we have $\pi^6_5 = 0$ for both cases.

Summing up the above, the homotopy type of P^{n+6}_{n-1} is described as follows:

$l \equiv 0$



$l \equiv 1$

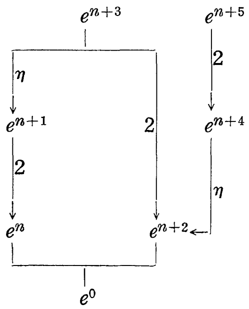


Case 2: $n = 4l + 1$.

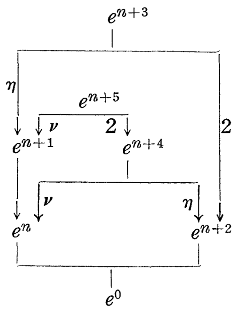
In this case P^{n+5}_{n-1} is of the same homotopy type as the complex K obtained by shrinking e^{n-1} to a point in P^{n+5}_{n-2}.

Therefore the homotopy type of P^{n+5}_{n-1} is described as follows:

$l \equiv 0$



$l \equiv 1$



Then we have:

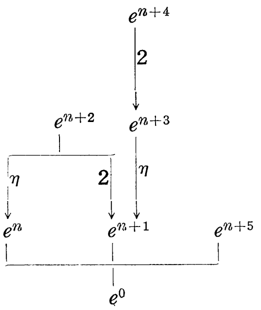
$\pi_1^1=\{\eta e^n\}_2$, $\pi_1^2=\{\eta e^n\}_2$, $\pi_2^1=\{\eta e^{n+1}\}_4$,[1] $\pi_2^2=\{e^{n+2}\}_\infty+\{\eta e^{n+1}\}_4$, $\pi_2^3=\{e^{n+2}\}_8$, $\pi_3^1=\{\varepsilon e^{n+1}\}_2+\{\nu e^n\}_2$,[1] $\pi_3^2=\{\eta e^{n+2}\}_2+\{\varepsilon e^{n+1}\}_2+\{\nu e^n\}_2$, $\pi_3^3=\{\eta e^{n+2}\}_2+\{\nu e^n\}_2$, $\pi_3^4=\{\nu e^n\}_2$, $\pi_4^1=\{12\nu e^{n+1}\}_2$,[1] $\pi_4^2=\{\varepsilon e^{n+2}\}_2+\{12\nu e^{n+1}\}_2$, $\pi_4^3=\{\varepsilon e^{n+2}\}_2$, $\pi_4^4=\{2e^{n+4}\}_\infty$ for $l\equiv 0$, $=\{\bar{2}e^{n+4}+\bar{\nu}e^{n+1}\}_\infty$ for $l\equiv 1$, $\pi_4^5=0$, $\pi_5^1=0$, $\pi_5^2=\{\nu e^{n+2}\}_{24}$, $\pi_5^3=\{\nu e^{n+2}\}_2$, $\pi_5^4\approx\pi_5^5\approx\pi_5^3$.

Since $\mathrm{Sq}^4 e^{n+2}=0$ for $l\equiv 0$ and $\neq 0$ for $l\equiv 1$, e^{n+6} is attached to e^{n+2} by $k\nu$, where k is even for $l\equiv 0$ and k is odd for $l\equiv 1$. However $2\{\nu e^{n+2}\}=0$ in π_5^5, thus we have the following, the homotopy type depends only on k mod 2, $\pi_5^6=\{\nu e^{n+2}\}_2$ for $l\equiv 0$ and $\pi_5^6=0$ for $l\equiv 1$.

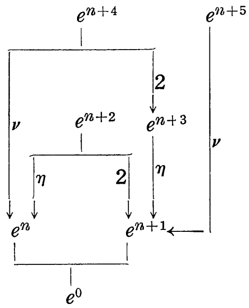
Case 3: $n=4l+2$.

We obtain P_{n-1}^{n+5} from P_{n-2}^{n+5} by the same manner in the preceding section. The homotopy tope P_{n-1}^{n+5} is described as follows:

$l\equiv 0$



$l\equiv 1$



Then we have:

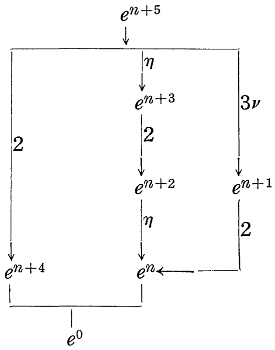
$\pi_1^1=\{e^{n+1}\}_\infty+\{\eta e^n\}_2$, $\pi_1^2=\{e^{n+1}\}_4$, $\pi_2^1=\{\eta e^{n+1}\}_2+\{\varepsilon e^n\}_2$, $\pi_2^2=\{\eta e^{n+1}\}_2$, $\pi_2^3=0$, $\pi_3^1=\{\varepsilon e^{n+1}\}_2+\{\nu e^n\}_{24}$, $\pi_3^2=\{\varepsilon e^{n+1}\}_2+\{\nu e^n\}_{12}$, $\pi_3^3=\{2e^{n+2}\}_\infty+\{\nu e^n\}_{12}$, $\pi_3^4=\{\nu e^n\}_{12}$, $\pi_4^1=\{\nu e^{n+1}\}_{24}$, $\pi_4^2=\{\nu e^{n+1}\}_2$, $\pi_4^3\approx\pi_4^4\approx\pi_4^2$, $\pi_4^5=\{\nu e^{n+1}\}_2$ for $l\equiv 0$, $\pi_4^5=0$ for $l\equiv 1$, $\pi_5^1=0$, $\pi_5^2=\{12\nu e^{n+2}\}_2$, $\pi_5^3=\{\bar{\varepsilon}e^{n+3}+6\bar{\nu}e^{n+4}\}_4$, $\pi_5^4=\{\bar{\eta}e^{n+4}+3\bar{\nu}e^{n+2}\}_8$ where $g=(\bar{\eta}e^{n+4}+3\bar{\nu}e^{n+2})$: $E_+^{n+5}\smile E_-^{n+5}\to P_{n-1}^{n+4}$ is a mapping such that $g|E_+^{n+5}=\bar{\eta}e^{n+4}$ and $g|E_-^{n+5}=3\bar{\nu}e^{n+2}$. We have $g|E_+^{n+5}=\{\eta,2,\eta\}\,e^{n+1}$, where $\{\eta,2,\eta\}$ is a Toda's construction and is known $\{\eta,2,\eta\}=\pm 6\nu$ (See [2, Chap. 5]), and so g represents an element of $\pi_{n+5}(P_{n-1}^{n+5})$, $\pi_5^5=\{e^{n+5}\}_\infty+\{g\}_8$ for $l\equiv 0$, $\pi_5^5=\{\bar{2}e^{n+5}+\bar{\nu}e^{n+2}\}_\infty+\{g\}_8$ for $l\equiv 1$.

Since $\mathrm{Sq}^1 e^{n+5}\neq 0$ and $\mathrm{Sq}^2 e^{n+4}\neq 0$, e^{n+6} is attached to e^{n+5} by a mapping of degree 2 and to e^{n+4} by η. Thus we have $\pi_5^6=\{e^{n+5}\}_{16}$ for $l\equiv 0$ and $\pi_5^6=\{\bar{2}e^{n+5}+\bar{\nu}e^{n+2}\}_8$ for $l\equiv 1$.

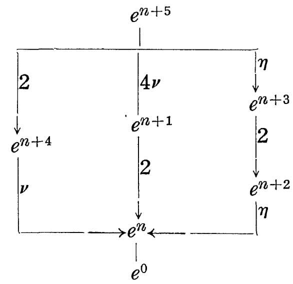
Case 4: $n=4l+3$.

The homotopy type of P_{n-1}^{n+5} is described as follows:

$l \equiv 0$



$l \equiv 1$



The we have:

$\pi_1^1=\{\eta e^n\}_2$, $\pi_1^2=0$, $\pi_2^1=\{\eta e^{n+1}\}_4$,[1] $\pi_2^2=\{2e^{n+2}\}_\infty+\{\eta e^{n+1}\}_2$, $\pi_2^3=\{\eta e^{n+1}\}_2$, $\pi_3^1=\{\nu e^n\}_2+\{\varepsilon e^{n+1}\}_2$,[1] $\pi_3^2\approx\pi_3^3\approx\pi_3^1$, $\pi_3^4=\{\varepsilon e^{n+1}\}_2$ for $l\equiv 1$, $\pi_3^4=\{\nu e^n\}_2+\{\varepsilon e^{n+1}\}_2$ for $l\equiv 0$, $\pi_4^1=\{12\nu e^{n+1}\}_2$,[1] $\pi_4^2=\{\bar{\varepsilon}e^{n+2}+6\bar{\nu}e^{n+1}\}_4$, $\pi_4^3=\{\bar{\eta}e^{n+3}+3\bar{\nu}e^{n+1}\}_8$, $\pi_4^4=\{e^{n+4}\}_\infty+\{\bar{\eta}e^{n+3}+3\bar{\nu}e^{n+1}\}_8$ for $l\equiv 0$, $\pi_4^4=\{\bar{2}e^{n+4}+\bar{\nu}e^{n+1}\}_\infty+\{\bar{\eta}e^{n+3}+3\bar{\nu}e^{n+1}\}_8$ for $l\equiv 1$, $\pi_4^5=\{e^{n+4}\}_{16}$ for $l\equiv 0$, $\pi_4^5=\{\bar{2}e^{n+4}+\bar{\nu}e^{n+1}\}_8$ for $l\equiv 1$, $\pi_5^1=0$, $\pi_5^2=\{\nu e^{n+2}\}_{24}$, $\pi_5^3=\{\nu e^{n+2}\}_2+\{\varepsilon e^{n+3}\}_2$, $\pi_5^4=\{\nu e^{n+2}\}_2+\{\varepsilon e^{n+2}\}_2+\{\eta e^{n+4}\}_2$, $\pi_5^5=\{\nu e^{n+2}\}_2+\{\eta e^{n+4}\}_2$.

Since $Sq^2e^{n+4}\neq 0$ and $Sq^4e^{n+2}=0$ for $l\equiv 0$, e^{n+6} is attached to P_{n-1}^{n+5} by $\{\eta e^{n+4}\}$. Since $Sq^4e^{n+2}\neq 0$ for $l\equiv 1$, e^{n+6} is attached to P_{n-1}^{n+5} by $\{\nu e^{n+2}\}+\{\eta e^{n+4}\}$. Thus we have $\pi_5^6=\{\nu e^{n+2}\}_2$ for both cases.

4. The homotopy groups of the Stiefel manifolds.

From the Whitehead's theorem, we have the tables of the homotopy groups of the Stiefel manifolds.

Theorem 1. *The table of the homotopy groups* $\pi_{k+2}(V_{k+m,m})$ $(k\geqq 4)$ *is the following*:

k \ m	1	2	3	$4\leqq m$
$4l$	Z_2	Z_2+Z_2	Z_2+Z_2	Z_2+Z_2
$4l+1$	Z_2	Z_4	$Z_\infty+Z_4$	Z_8
$4l+2$	Z_2	Z_2+Z_2	Z_2	0
$4l+3$	Z_2	Z_4	$Z_\infty+Z_2$	Z_2

Theorem 2. *The table of the homotopy groups* $\pi_{k+3}(V_{k+m,m})$ $(k\geqq 5)$ *is the following*:

k \ m	1	2	3	4	$5\leqq m$
$4l$ $\begin{cases} l\equiv 0 \\ l\equiv 1\end{cases}$	Z_{24}	Z_2+Z_{24}	Z_4+Z_{24}	$Z_\infty+Z_4+Z_{24}$	$\begin{cases} Z_8+Z_{24} \\ Z_4+Z_{48}\end{cases}$
$4l+1$	Z_{24}	Z_2+Z_2	$Z_2+Z_2+Z_2$	Z_2+Z_2	Z_2
$4l+2$	Z_{24}	Z_2+Z_{24}	Z_2+Z_{12}	$Z_\infty+Z_{12}$	Z_{12}
$4l+3$ $\begin{cases} l\equiv 0 \\ l\equiv 1\end{cases}$	Z_{24}	Z_2+Z_2	Z_2+Z_2	Z_2+Z_2	$\begin{cases} Z_2+Z_2 \\ Z_2\end{cases}$

Theorem 3. *The table of the homotopy groups* $\pi_{k+4}(V_{k+m,m})$ $(k \geqq 6)$ *is the following*:

k \ m	1	2	3	4	5	$6 \leqq m$
$4l$	0	Z_{24}	Z_2+Z_2	$Z_2+Z_2+Z_2$	Z_2+Z_2	Z_2
$4l+1$	0	Z_2	Z_2+Z_2	Z_2	Z_∞	0
$4l+2$ $\begin{cases} l \equiv 0 \\ l \equiv 1 \end{cases}$	0	Z_{24}	Z_2	Z_2	Z_2	$\begin{cases} Z_2 \\ 0 \end{cases}$
$4l+3$ $\begin{cases} l \equiv 0 \\ l \equiv 1 \end{cases}$	0	Z_2	Z_4	Z_8	$Z_\infty+Z_8$	$\begin{cases} Z_{16} \\ Z_8 \end{cases}$

Theorem 4. *The table of the homotopy groups* $\pi_{k+5}(V_{k+m,m})$ $(k \geqq 7)$ *is the following*:

k \ m	1	2	3	4	5	6	$m \leqq 7$
$4l$	0	0	Z_2	Z_2+Z_2	Z_2	Z_∞	0
$4l+1$ $\begin{cases} l \equiv 0 \\ l \equiv 1 \end{cases}$	0	0	Z_{24}	Z_2	Z_2	Z_2	$\begin{cases} Z_2 \\ 0 \end{cases}$
$4l+2$	0	0	Z_2	Z_4	Z_8	$Z_\infty+Z_8$	Z_{16}
$4l+3$	0	0	Z_{24}	Z_2+Z_2	$Z_2+Z_2+Z_2$	Z_2+Z_2	Z_2

References

[1] M. G. Baratt and G. F. Peacher, *A note on* $\pi_r(V_{n,m})$, Proc. Nat. Acad. Sci, U. S. A. **38** (1952) 119–121.

[2] H. Toda, *Generalized Whitehead products and homotopy groups of spheres*, Jour. of the Inst. of Polytechnics, Osaka City Univ., **3** (1952).

[3] J. H. C. Whitehead, *On the group* $\pi_r(V_{n,m})$ *and shere bundles*, Proc. London Math. Soc., **48** (1944).