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Discrete Neumann problem

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1. Discrete harmonic functions. The lattice points of (x, y) plane have coordinates (mh, nh), where m and n take on the values $0, \pm 1, \pm 2, \ldots$ and h is a positive constant. Let the four points $P_1(x+h, y), P_2(x, y+h), P_3(x-h, y), P_4(x, y-h)$ be called the neighbors of a lattice point $P(x, y)$ and let I be a finite and connected[1] lattice (set of lattice points). For this lattice I we define the lattice Γ as the set of lattice points which do not belong to I themselves, but which possess at least one neighbor belonging to I. Setting $L = I + \Gamma$, any point of I is called an interior point of L and any point of Γ is called a boundary point of L. It is clear that the four neighbors of an interior point belong to L.

Let u be a lattice function defined on L; u is single-valued on I but may be multi-valued on Γ. Each value of u at a boundary point P has a meaning only when it is associated with one interior neighbor P_i and this boundary value may be denoted by $u(\widehat{PP_i})$ or $u(\widehat{P_iP})$.

The operator $\mathfrak{D}$ is defined by

$$\mathfrak{D}u(P) = \frac{1}{h^2}\{\sum_1^4 u(P_i) - 4u(P)\},$$

where P_i $(i=1, 2, 3, 4)$ denote the four neighbors of an interior point P; if P is a neighbor of a boundary point P_i, we have to take $u(P_i) = u(\widehat{PP_i})$. At the points where $\mathfrak{D}u$ vanishes, u is said to be discrete harmonic. If u is defined on L and is discrete harmonic at all points of I, u is said to be discrete harmonic on I. The difference equation $\mathfrak{D}u=0$ can be considered as the direct analogue of the Laplace partial differential equation[2].

Let P_i be an interior neighbor of a boundary point P and let $s(P)$ be the number of its interior neighbors. We put

$$\partial u(\widehat{PP_i}) = \frac{1}{h}\{u(\widehat{PP_i}) - u(P_i)\}.$$

THEOREM 1. *Let u and v be lattice functions defined on L. Then*

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1) A set of lattice points is called connected if any two points of the set can be connected by a chain of neighbor points which belong to the set.
2) The theory of discrete harmonic functions is developped in [1] and [2]. Numbers in brackets refer to the references at the end of the paper.

$$h\sum_{P\in I}\{u(P)\mathfrak{D}v(P)-v(P)\mathfrak{D}u(P)\}$$
$$=\sum_{P\in\Gamma}\sum_{i=1}^{s(P)}\{u(\widehat{PP_i})\partial v(\widehat{PP_i})-v(\widehat{PP_i})\partial u(\widehat{PP_i})\}\,. \tag{1}$$

Setting $v\equiv 1$ in (1), we have

COROLLARY. *If u is discrete harmonic on I, then*

$$\sum_{P\in\Gamma}\sum_{i=1}^{s(P)}\partial u(\widehat{PP_i})=0\,. \tag{2}$$

Let $P(x, y)$ be an interior point of L and put

$$\begin{aligned}
\partial_x u(P) &= \frac{1}{h}\{u(P_1)-u(P)\} && \text{if} \quad P_1(x+h,\ y)\in I\,,\\
&= \partial u(\widehat{PP_1}) && \text{if} \quad P_1(x+h,\ y)\in \Gamma\,,\\
\partial_y u(P) &= \frac{1}{h}\{u(P_2)-u(P)\} && \text{if} \quad P_2(x,\ y+h)\in I\,,\\
&= \partial u(\widehat{PP_2}) && \text{if} \quad P_2(x,\ y+h)\in \Gamma\,.
\end{aligned}$$

Let $P(x, y)$ be a boundary point of L and put

$$\begin{aligned}
\partial_x u(P) &= -\partial u(\widehat{PP_1}) && \text{if} \quad P_1(x+h,\ y)\in I\,,\\
&= 0 && \text{if} \quad P_1(x+h,\ y)\bar{\in} I\,,\\
\partial_y u(P) &= -\partial u(\widehat{PP_2}) && \text{if} \quad P_2(x,\ y+h)\in I\,,\\
&= 0 && \text{if} \quad P_2(x,\ y+h)\bar{\in} I\,.
\end{aligned}$$

THEOREM 2. *Let u and v be lattice functions defined on L. Then*

$$h\sum_{P\in I}u(P)\mathfrak{D}v(P)+h\sum_{P\in L}\{\partial_x u(P)\partial_x v(P)+\partial_y u(P)\partial_y v(P)\}$$
$$=\sum_{P\in L}\sum_{i=1}^{s(P)}u(\widehat{PP_i})\partial v(\widehat{PP_i}) \tag{3}$$

Especially

$$h\sum_{P\in I}u(P)\mathfrak{D}u(P)+h\sum_{P\in L}\{[\partial_x u(P)]^2+[\partial_y u(P)]^2\}$$
$$=\sum_{P\in\Gamma}\sum_{i=1}^{s(P)}u(\widehat{PP_i})\partial u(\widehat{PP_i})\,. \tag{4}$$

Theorems 1 and 2[1] are analogues of Green's identities.

2. Discrete Neumann problem. The problem of constructing a discrete harmonic function on I with prescribed partial differences on Γ is called the discrete Neumann problem. In a previous note the author established the following existence theorem[2].

1) (1) and (4) are given in the Book of H. Batemann [1]. (3) results from (1) and (4). See also [3]. Discrete Dirichlet problem and its numerical computations are treated in my previous paper [5].

2) See [5]. In this paper the author proved that the discrete Neumann problem can be reduced to a discrete Dirichlet problem by using the so-called conjugate transformation:

$$u(x+h,\ y)-u(x,\ y)=v(x,\ y+h)-v(x,\ y)\,,$$
$$u(x,\ y)-u(x,\ y-h)=v(x-h,\ y)-v(x,\ y)$$

THEOREM 3. *Let $f(\widehat{PP_i})$ be a given lattice function defined on Γ, such that* $\sum_{P\in\Gamma}\sum_{i=1}^{s(P)} f(\widehat{PP_i})=0$. *Then there exists a lattice function u which satisfies*

$$\mathfrak{D}u(P)=0 \qquad \text{for } P\in I,$$
$$\partial u(\widehat{PP_i})=f(\widehat{PP_i}) \quad \text{for } P\in\Gamma.$$

Such a function is uniquely determined except an additive constant.

A different proof based on the use of Dirichlet Principle will be given here. Let Φ be the familly of lattice functions φ defined on L, which satisfies the condition

$$\sum_{P\in\Gamma}\sum_{i=1}^{s(P)} f(\widehat{PP_i})\,\varphi(\widehat{PP_i})=k,$$

where we suppose $f\not\equiv 0$ without loss of generality and k is a constant ($\neq 0$), and let θ be an arbitrary lattice function defined on L, such that

$$\sum_{P\in\Gamma}\sum_{i=1}^{s(P)} f(\widehat{PP_i})\,\theta(\widehat{PP_i})=0.$$

If we put

$$\inf_{\varphi\in\Phi}\sum_{P\in L}\{[\partial_x\varphi(P)]^2+[\partial_y\varphi(P)]^2\}=K<\infty,$$

there exists a lattice function $u\in\Phi$, such that

$$\sum_{P\in L}\{[\partial_x u(P)]^2+[\partial_y u(P)]^2\}=K.$$

Since $u+\alpha\theta=u_\alpha\in\Phi$, we obtain for any real constant α

$$\begin{aligned}\sum_{P\in L}\{[\partial_x u_\alpha(P)]^2+[\partial_y u_\alpha(P)]^2\}-\sum_{P\in L}\{[\partial_x u(P)]^2+[\partial_y u(P)]^2\}\\ =2\alpha\sum_{P\in L}\{\partial_x u(P)\partial_x\theta(P)+\partial_y u(P)\partial_y\theta(P)\}\\ +\alpha^2\sum_{P\in L}\{[\partial_x\theta(P)]^2+[\partial_y\theta(P)]^2\}\geqq 0.\end{aligned}$$

Hence it follows

$$\sum_{P\in L}\{\partial_x u(P)\partial_x\theta(P)+\partial_y u(P)\partial_y\theta(P)\}=0.$$

Setting $u=\theta$, $v=u$ in (3), we get

$$h\sum_{P\in I}\theta(P)\mathfrak{D}u(P)=\sum_{P\in\Gamma}\sum_{i=1}^{s(P)}\theta(\widehat{PP_i})\partial u(\widehat{PP_i})$$

If we select θ such that $\theta(P_0)=1$ at an arbitrary interior point P_0 and $\theta(P)=0$ elsewhere, we obtain $\mathfrak{D}u(P_0)=0$. This shows that u is discrete harmonic on I. Hence it holds

$$\sum_{P\in\Gamma}\sum_{i=1}^{s(P)}\theta(\widehat{PP_i})\,\partial u(\widehat{PP_i})=0$$

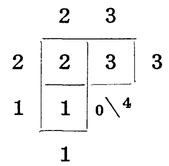
for any θ. From this we see by elementary calculation that there exists a constant $c(\neq 0)$ such that

$$\partial u(\widehat{PP_i})=cf(\widehat{PP_i}).$$

Hence $v=u/c$ is discrete harmonic and satisfies the boundary condition $\partial v(\widehat{PP_i})=f(\widehat{PP_i})$. The existence of a solution is thus proved.

Suppose now two solutions u_1 and u_2. Then $w=u_1-u_2$ is discrete harmonic on I and satisfies $\partial w(\widehat{PP_i})=0$ for every boundary point of L. Since w is discrete harmonic, it attains its maximum or minimum on Γ, but in view of $\partial w(\widehat{PP_i})=0$, this is impossible unless w reduces to a constant. Hence the theorem 3 is completely proved.

REMARK. As we see in the diagram below (the enclosed three points are the interior points), the multiplicity of boundary values can not be removed.



The discrete Neumann function $N(P, Q)$, $P\in L$, $Q\in I$, is defined by

$$\mathfrak{D}_P N(P, Q)=0 \qquad \text{for } P\neq Q,\ P\in I,$$

$$\mathfrak{D}_P N(P, Q)=-\frac{M}{h} \qquad \text{for } P=Q,$$

$$\partial N(\widehat{PP_i}, Q)=-1 \qquad \text{for } P\in\Gamma,$$

$$\sum_{P\in\Gamma}\sum_{i=1}^{s(P)} N(\widehat{PP_i}, Q)=0,$$

where we put $M=\sum_{P\in\Gamma} s(P)$.

We have then the following symmetry property of $N(P, Q)$.

THEOREM 4. *If R and Q are interior points of L, then*

$$N(R, Q)=N(Q, R). \tag{5}$$

PROOF. Put $I'=I-R-Q(R\in I, Q\in I)$ and construct $L'=I'+\Gamma''$ for this I'. Setting $u(P)=N(P, Q)$ and $v(P)=N(P, R)$ in (1), we get

$$\begin{aligned}&\sum_{P\in\Gamma'}\sum_{i=1}^{s(P)}\{N(\widehat{PP_i}, Q)\partial N(\widehat{PP_i}, R)-N(\widehat{PP_i}, R)\partial N(\widehat{PP_i}, Q)\}\\ &=\sum_{P\in\Gamma}\sum_{i=1}^{s(P)}\{N(\widehat{PP_i}, Q)\partial N(\widehat{PP_i}, R)-N(\widehat{PP_i}, R)\partial N(\widehat{PP_i}, Q)\}\\ &\quad+hN(Q, R)[\mathfrak{D}_P N(P, Q)]_{P=Q}-hN(R, Q)[\mathfrak{D}_P N(P, R)]_{P=R}\\ &=M\{N(R, Q)-N(Q, R)\}=0.\end{aligned}$$

This proves (5).

Let u be a discrete harmonic function on I and let Q an interior point of L. Put $I'=I-Q$ and construct $L'=I'+\Gamma''$ for this I'. Then by Theorem 1

$$\begin{aligned}&\sum_{P\in\Gamma'}\sum_{i=1}^{s(P)}\{u(\widehat{PP_i})\partial N(\widehat{PP_i}, Q)-N(\widehat{PP_i}, Q)\partial u(\widehat{PP_i})\}\\ &=\sum_{P\in\Gamma}\sum_{i=1}^{s(P)}\{u(\widehat{PP_i})\partial N(\widehat{PP_i}, Q)-N(\widehat{PP_i}, Q)\partial u(\widehat{PP_i})\}\\ &\qquad+\sum_{i=1}^{4}\{u(Q)\partial N(\widehat{QQ_i}, Q)-N(Q, Q)\partial u(\widehat{QQ_i})\}\end{aligned}$$

$$= -\sum_{P\in\Gamma}\sum_{i=1}^{s(P)} u(\widehat{PP_i}) - \sum_{P\in\Gamma}\sum_{i=1}^{s(P)} N(\widehat{PP_i}, Q)\partial u(\widehat{PP_i})$$
$$+hN(Q, Q)\mathfrak{D}u(Q) - hu(Q)[\mathfrak{D}_P N(P, Q)]_{P=Q}$$
$$= -\sum_{P\in\Gamma}\sum_{i=1}^{s(P)} u(\widehat{PP_i}) - \sum_{P\in\Gamma}\sum_{i=1}^{s(P)} N(\widehat{PP_i}, Q)\partial u(\widehat{PP_i}) + Mu(Q) = 0\,.$$

Hence

$$u(Q) = \frac{1}{M}\sum_{P\in\Gamma}\sum_{i=1}^{s(P)} N(\widehat{PP_i}, Q)\partial u(\widehat{PP_i}) + \frac{1}{M}\sum_{P\in\Gamma}\sum_{i=1}^{s(P)} u(\widehat{PP_i})\,.$$

Thus we have proved

THEOREM 5. *Let u be a discrete harmonic function on I. Then*

$$u(Q) = \frac{1}{M}\sum_{P\in\Gamma}\sum_{i=1}^{s(P)} N(\widehat{PP_i}, Q)\partial u(\widehat{PP_i}) + \frac{1}{M}\sum_{P\in\Gamma}\sum_{i=1}^{s(P)} u(\widehat{PP_i})\,. \tag{6}$$

Finally we prove

THEOREM 6. *Let $f(\widehat{PP_i})$ be a given lattice function defined on Γ, such that*

$$\sum_{P\in\Gamma}\sum_{i=1}^{s(P)} f(\widehat{PP_i}) = 0\,. \tag{7}$$

Then the lattice function u defined by

$$u(Q) = \frac{1}{M}\sum_{P\in\Gamma}\sum_{i=1}^{s(P)} N(\widehat{PP_i}, Q)f(\widehat{PP_i}) \qquad for \quad Q\in I\,, \tag{8}$$

$$u(\widehat{QQ_i}) = u(Q_i) + hf(\widehat{QQ_i}) \qquad for \quad Q\in\Gamma\,, \tag{9}$$

is discrete harmonic on I. This shows that u is a solution of the discrete Neumann problem with the prescribed boundary condition $\partial u(\widehat{PP_i}) = f(\widehat{PP_i})$.

PROOF. By (5), (7) and (8), we have for an interior point Q,

$$u(Q) = \frac{1}{M}\sum_{P\in\Gamma}\sum_{i=1}^{s(P)} N(\widehat{PP_i}, Q)f(\widehat{PP_i})$$
$$= \frac{1}{M}\sum_{P\in\Gamma}\sum_{i=1}^{s(P)} N(Q, P_i)f(\widehat{PP_i})\,.$$

If Q is not a neighbor of a boundary point, we have at once

$$\mathfrak{D}u(Q) = \frac{1}{M}\sum_{P\in\Gamma}\sum_{i=1}^{s(P)} \mathfrak{D}_Q N(Q, P_i)f(\widehat{PP_i}) = 0\,.$$

If Q is a neighbor of boundary points Q^j ($1\leq j\leq 3$ except in the trivial case in which I consists of Q only) and if we denote by $s(Q)$ the number of interior neighbors Q_λ of Q, we have

$$\mathfrak{D}u(Q) = \frac{1}{h^2 M}\sum_{P\in\Gamma}\sum_{i=1}^{s(P)}\Big[\sum_{\lambda=1}^{s(Q)} N(Q_\lambda, P_i) - s(Q)N(Q, P_i)\Big]f(\widehat{PP_i})$$
$$+\frac{1}{h}\sum_{j=1}^{4-s(Q)} f(\widehat{QQ^j})$$

$$= \frac{1}{h^2 M} \sum_{P \in \Gamma} \sum_{i=1}^{s(P)} \Big[\sum_{\lambda=1}^{s(Q)} N(Q_\lambda, P_i) - s(Q)N(Q, P_i) - (4-s(Q))h \Big] f(\widehat{PP_i})$$

$$+ \frac{1}{h} \sum_{j=1}^{4-s(Q)} f(\widehat{QQ^j}) + \frac{4-s(Q)}{h} \sum_{P \in \Gamma} \sum_{i=1}^{s(P)} f(\widehat{PP_i})$$

$$= \frac{1}{M} \sum_{P \in \Gamma} \sum_{i=1}^{s(P)} \mathfrak{D}_Q N(Q, P_i) f(\widehat{PP_i}) + \frac{1}{h} \sum_{j=1}^{4-s(Q)} f(\widehat{QQ^j})$$

$$= -\frac{1}{h} \sum_{j=1}^{4-s(Q)} f(\widehat{QQ^j}) + \frac{1}{h} \sum_{j=1}^{4-s(Q)} f(\widehat{QQ^j}) = 0 .$$

Thus the theorem is proved.

3. Numerical computations. The practical values of a solution u of the discrete Neumann problem are computed by the method of successive approximation of Wolf [7] or by solving linear algebraic equations. But it is very easy and rapid to compute u by the formula (8) if we know the values of $N(P, Q)$. For this purpose we have computed $N(P, Q)$ for a square lattice with 25 interior points. The values obtained are given in Table 1, where we take $h=1$ and $\min N(P, Q)=0$ instead of $\sum_{P \in \Gamma} \sum_{i=1}^{s(P)} N(\widehat{PP_i}, Q)=0$. The values are all exact if two final figures are repeated, for example, 0.745 is taken as 0.7454545......

We show also a numerical example in Table 2.

$$v(m, n) = \frac{1}{100} \{u(m, n) - u(3, 3)\} \quad (m, n = 1, 2, \dots, 5)$$

are considered as approximate values of $V(m/10, n/10)$, where $V(x, y)$ is the solution of the Neumann problem:

$$\begin{aligned}
&\Delta V(x, y) = 0 && (0<x<0.6,\ 0<y<0.6),\\
&V(0.3, 0.3) = 0 && \\
&\frac{\partial V}{\partial n} = 2x+y && (x = 0.6,\ 0<y<0.6),\\
&\quad = x-2y && (0<x<0.6,\ y = 0.6),\\
&\quad = -2x-y && (x = 0,\ 0<y<0.6),\\
&\quad = -x+2y && (0<x<0.6,\ y = 0),
\end{aligned}$$

where n denotes the exterior normal of the square domain: $0<x<0.6$, $0<y<0.6$. We see easily $V(x, y)=x^2+xy-y^2-0.09$.

	(1, 1)	(3, 2)	(5, 4)
v	−0.080	0.018	0.206
V	−0.080	0.020	0.200

Table 1 (a)

(m, n)	(1, 1)	(2, 1)	(3, 1)	(4, 1)	(5, 1)	(1, 2)	(2, 2)	(3, 2)	(4, 2)	(5, 2)
(1, 0)	21.363	11.363	5.454	3.257	2.348	11.363	7.272	3.181	1.969	1.439
(2, 0)	12.363	16.342	8.390	5.433	4.257	8.384	8.272	4.836	3.651	3.081
(3, 0)	7.342	9.278	14 545	9.278	7.342	5.406	5.948	6.518	5.948	5.406
(4, 0)	4.257	5.433	8.390	16.342	12.363	3.081	3.651	4.836	8.272	8.384
(5, 0)	2.348	3.257	5.454	11.363	21.363	1.439	1.969	3.181	7.272	11.363
(6, 1)	2.348	3.257	5.454	11.363	21.363	1.439	1.969	3.181	7.272	11.363
(6, 2)	2.439	3.081	4.518	8.384	12.363	1.796	2.287	3.527	8.272	16.342
(6, 3)	1.887	2.224	2.927	5.406	7.342	1.551	1.857	2.590	5.948	9.278
(6, 4)	1.000	1.112	1.336	3.081	4.257	0.887	1.000	1.254	3.651	5.433
(6, 5)	0.000	0.000	0.000	1.439	2.348	0.000	0.000	0.000	1.969	3.257
(5, 6)	0.000	0.000	0.000	1.439	2.348	0.000	0.000	0.000	1.969	3.257
(4, 6)	1.000	0.887	0.663	1.796	2.439	1.112	1.000	0.745	2.287	3.081
(3, 6)	1.887	1.551	0.909	1.551	1.887	2.224	1.857	1.063	1.857	2.224
(2, 6)	2.439	1.796	0.663	0.887	1.000	3.081	2.287	0.745	1.000	1.112
(1, 6)	2.348	1.439	0.000	0.000	0.000	3.257	1.969	0.000	0.000	0.000
(0, 5)	2.348	1.439	0.000	0.000	0.000	3.257	1.969	0.000	0.000	0.000
(0, 4)	4.257	3.081	1.336	1.112	1.000	5.433	3.651	1.254	1.000	0.887
(0, 3)	7.342	5.406	2.927	2.224	1.887	9.278	5.948	2.590	1.857	1.551
(0, 2)	12.363	8.384	4.518	3.081	2.439	16.342	8.272	3.527	2.287	1.796
(0, 1)	21.363	11.363	5.454	3.257	2.348	11.363	7.272	3.181	1.969	1.439

Table 1 (b)

(m, n)	(1, 3)	(2, 3)	(3, 3)	(4, 3)	(5, 3)	(1, 4)	(2, 4)	(3, 4)	(4, 4)	(5, 4)
(1, 0)	5.454	3.181	0.000	0.000	0.000	3.257	1.969	0.000	0.000	0.000
(2, 0)	4.518	3.527	1.000	1.254	1.336	3.081	2.287	0.745	1.000	1.112
(3, 0)	2.927	2.590	1.600	2.590	2.927	2.224	1.857	1.063	1.857	2.224
(4, 0)	1.336	1.254	1.000	3.527	4.518	1.112	1.000	0.745	2.287	3.081
(5, 0)	0.000	0.000	0.000	3.181	5.454	0.000	0.000	0.000	1.969	3.257
(6, 1)	0.000	0.000	0.000	3.181	5.454	0.000	0.000	0.000	1.969	3.257
(6, 2)	0.663	0.745	1.000	4.836	8.390	0.887	1.000	1.254	3.651	5.433
(6, 3)	0.909	1.063	1.600	6.518	14 545	1.551	1.857	2.590	5.948	9.278
(6, 4)	0.663	0.745	1.000	4.836	8.390	1.796	2.287	3.527	8.272	16.342
(6, 5)	0.000	0.000	0.000	3.181	5.454	1.439	1.969	3.181	7.272	11.363
(5, 6)	0.000	0.000	0.000	3.181	5.454	1.439	1.969	3.181	7.272	11.363
(4, 6)	1.336	1.254	1.000	3.527	4.518	3.081	3.651	4.836	8.272	8.384
(3, 6)	2.927	2.590	1.600	2.590	2.927	5.406	5.948	6.518	5.948	5.406
(2, 6)	4 518	3.527	1.000	1.254	1.336	8.384	8.272	4.836	3.651	3.081
(1, 6)	5.454	3.181	0.000	0.000	0.000	11.363	7.272	3.181	1.969	1.439
(0, 5)	5.454	3.181	0.000	0.000	0.000	11.363	7.272	3.181	1.969	1.439
(0, 4)	8.390	4.836	1.000	0.745	0.663	16.342	8.272	3.527	2.287	1.796
(0, 3)	14.545	6.518	1.600	1.063	0.909	9.278	5.948	2.590	1.857	1.551
(0, 2)	8.390	4.836	1.000	0.745	0.663	5.433	3.651	1.254	1.000	0.887
(0, 1)	5.454	3.181	0.000	0.000	0.000	3.257	1.969	0.000	0.000	0.000

Table 1 (c)

(m, n)	(1, 5)	(2, 5)	(3, 5)	(4, 5)	(5, 5)	(m, n)	(1, 5)	(2, 5)	(3, 5)	(4, 5)	(5, 5)
(1, 0)	2.348	1.439	0.000	0.000	0.000	(5, 6)	2.348	3.257	5.454	11.363	21.363
(2, 0)	2.439	1.796	0.663	0.887	1.000	(4, 6)	4.257	5.433	8.390	16.342	12.363
(3, 0)	1.887	1.551	0.909	1.551	1.887	(3, 6)	7.342	9.278	14.545	9.278	7.342
(4, 0)	1.000	0.887	0.663	1.796	2.439	(2, 6)	12.363	16.342	8.390	5.433	4.257
(5, 0)	0.000	0.000	0.000	1.439	2.348	(1, 6)	21.363	11.363	5.454	3.257	2.348
(6, 1)	0.000	0.000	0.000	1.439	2.348	(0, 5)	21.363	11.363	5.454	3.257	2.348
(6, 2)	1.000	1.112	1.336	3.081	4.257	(0, 4)	12.363	8.384	4.518	3.081	2.439
(6, 3)	1.887	2.224	2.927	5.406	7.342	(0, 3)	7.342	5.406	2.927	2.224	1.887
(6, 4)	2.439	3.081	4.518	8.384	12.363	(0, 2)	4.257	3.081	1.336	1.112	1.000
(6, 5)	2.348	3.257	5.454	11.363	21.363	(0, 1)	2.348	1.439	0.000	0.000	0.000

Table 2.

		(1, 1)	(3, 2)	(3, 3)	(5, 4)
(m, n)	f	fN	fN	fN	fN
(1, 0)	−1	−21.3636⋯	− 3.1818⋯	0.000⋯	0.0000⋯
(2, 0)	−2	−24.7272⋯	− 9.6727⋯	−2.000⋯	− 2.2242⋯
(3, 0)	−3	−22.0272⋯	−19.5545⋯	−4.800⋯	− 6.6727⋯
(4, 0)	−4	−17.0303⋯	−19.3454⋯	−4.000⋯	−12.3272⋯
(5, 0)	−5	−11.7424⋯	−15.9090⋯	0.000⋯	−16.2878⋯
(6, 1)	13	30.5303⋯	41.3636⋯	0.000⋯	42.3484⋯
(6, 2)	14	34.1515⋯	49.3818⋯	14.000⋯	76.0666⋯
(6, 3)	15	28.3181⋯	38.8636⋯	24.000⋯	139.1818⋯
(6, 4)	16	16.0000⋯	20.0727⋯	16.000⋯	261.4787⋯
(6, 5)	17	0.0000⋯	0.0000⋯	0.000⋯	193.1818⋯
(5, 6)	−7	0.0000⋯	0.0000⋯	0.000⋯	−79.5454⋯
(4, 6)	−8	− 8.0000⋯	−5.9636⋯	− 8.000⋯	−67.0787⋯
(3, 6)	−9	−16.9909⋯	−9.5727⋯	−14.400⋯	−48.6545⋯
(2, 6)	−10	−24.3939⋯	−7.4545⋯	−10.000⋯	−30.8181⋯
(1, 6)	−11	−25.8333⋯	0.0000⋯	0.000⋯	−15.8333⋯
(0, 5)	−5	−11.7424⋯	0.0000⋯	0.000⋯	− 7.1969⋯
(0, 4)	−4	−17.0303⋯	−5.0181⋯	−4.000⋯	− 7.1878⋯
(0, 3)	−3	−22.0272⋯	−7.7727⋯	−4.800⋯	− 4.6545⋯
(0, 2)	−2	−24.7272⋯	−7.0545⋯	−2.000⋯	− 1.7757⋯
(0, 1)	−1	−21.3636⋯	−3.1818⋯	0.000⋯	0.0000⋯
Σ		−159.9999⋯	35.9999⋯	0.000⋯	412.0000⋯
$u=\Sigma/20$		−7.999 ⋯	1.799 ⋯	0.000⋯	20.600 ⋯

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