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On uniform convergence on uniform space

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In the present paper we shall show that the umiform topology of a uniform space can be defined by the uniform convergence of uniform d.s.p. (= directed sets of points), which satisfies analogous conditions to those of ordinary convergence on topological space and is equivalent with the uniform convergence of uniform filters satisfying some conditions. Furthermore we shall discuss the special case of a metric space.

Let us denote by R a T_1-space and by A a directed system,[1] and then we mean by a *uniform d.s.p.* a mapping $\varphi_\alpha(p)(\alpha\in A,\ p\in R)$ from $A\otimes R$ into R. Here we introduce the notion "*uniform convergence of* $\varphi_\alpha(p)$ "and denote by $\varphi_\alpha(p)\to$ the fact that $\varphi_\alpha(p)$ uniformly converges.

About uniform convergence we consider the following four conditions,

D_1) $\varphi_\alpha(p)=p$ *for every* α *and* p *implies* $\varphi_\alpha(p)\to$.

D_2) *If* $\varphi_\alpha(p)\to$ *and* $\varphi_\alpha'(p)\to(\alpha\in A)$, *then for every cofinal subset* B[2] *of* A *and for every* $\{p_\beta|\beta\in B\}$ $(p_\beta\in R)$, *there exists* $\psi_\gamma(p)(\gamma\in C,\ p\in R)$ *such that* $\psi_\gamma(p)\to$ *and* $\{(p_\gamma,\psi_\gamma(p_\gamma))|\gamma\in C\}\subseteqq\{(\varphi_\beta(p_\beta),\varphi_\beta'(p_\beta))|\beta\in B\}$ *for some* $\{p_\gamma|\gamma\in C\}$ $(p_\gamma\in R)$,[3]

D_2) *If for a uniform d.s.p.* $\varphi_\alpha(p)$ $(\alpha\in A)$ *and for every cofinal subset* B *of* A *and every* $\{p_\beta|\beta\in B\}$ $(y_\beta\in R)$, *there exists* $\psi_\gamma(p)$ $(\gamma\in C,\ p\in R)$ *such that* $\psi_\gamma(p)\to$ *and* $\{(p_\gamma,\psi_\gamma(p))|\gamma\in C\}\subseteqq\{(p_\beta,\varphi_\beta(p_\beta)|\beta\in B\}$ *for some* $\{p_\gamma|\gamma\in C\}$ $(p_\gamma\in R)$, *then* $\varphi_\alpha(p)\to$.

D_4) $\varphi_\beta(p)\to$ *implies* $\varphi_\alpha(p_0)\to p_0$ *for each point* p_0 *of* R.

Let us note here the well-known conditions of uniform neighbourhood.[4]

U_1) $\{U_x(p_0)|x\in X\}$ *is a neighbourhood basis of* p_0 *for a fixed point* $p_0\in R$.

U_2) *For every* $x\in X$ *and* $y\in X$, *there exists* $z\in X$ *such that* $U_z(p)\subseteqq U_x(p)\cap U_y(p)$ *hold for all* $p\in R$.

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1) Notions and notations in this paper are due chiefly to J. W. Tukey, Convergence and uniformity in topology (1940) and to A. Weil, Sur les espaces a structure uniform et sur la topologie générale (1938).

2) We denote by φ_α, φ'_α and ψ_γ uniform d. s. p. We mean by a cofinal subset of A a subset B such that for every element α of A, $\beta\geqq\alpha$ holds for some $\beta\in B$.

3) $(p_\gamma,\psi_\gamma(p_\gamma))$ means a point of $R\otimes R$ and hence $\{(p_\gamma\,\psi_\gamma(p_\gamma))|\gamma\in C\}\subseteqq\{(\varphi_\beta(p_\beta)\,\varphi_\beta'(p_\beta))|\beta\in B\}$ means that for every $\gamma\in C$ there exists $\beta\ni B$ such that $p_\gamma=\varphi_\beta(p_\beta)$, $\psi_\gamma(p_\gamma)=\varphi_\beta'(p_\beta)$.

4) A. Weil, loc. cit.

U_3) *For every* $x \in X$, *there exists* $y \in X$ *such that* $p \in U_y(r)$ *and* $q \in U_y(r)$ *imply* $q \in U_x(p)$ *for every* $r \in R$.

Next we give the relation between uniform convergence and uniform neighbourhood.

T_{na}) $\varphi_\alpha(p)$ $(\alpha \in A)$ *uniformly converges, if and only if for every uniform neighbourhood* $\{U_x(p) | p \in R\}$ *there exists* $\alpha(x) \in A$ *such that* $\alpha \geqq \alpha(x)$ *implies* $\varphi_\alpha(p) \in U_x(p)$ *for every* $p \in R$.

T_{an}) *A set* $\{U(p) | p \in R\}$ *of neighbourhoods of* p *is a uniform neighbourhood, if and only if for every uniformly converging* $\varphi_\alpha(p)$ $(\alpha \in A)$, *there exists* $\alpha \in A$ *such that* $\varphi_\alpha(p) \in U_x(p)$ *for every* $p \in R$.

REMARKS. Using $R \otimes R$ after Weil,[5] we get the following conditions $D_1')$—$D_4')$, which are equivalent to $D_1)$—$D_4)$. Let us denote by $\varDelta$ the subset of $R \otimes R$ consisting of all the points (p, p), by uniform d.s.p. $\varphi_\alpha(\alpha \in A)$ a family of subsets $\varphi_\alpha \subseteqq R \otimes R$ such that $\varphi_\alpha = \{(p, \varphi_\alpha(p)) | p \in R\}$ for a mapping $\varphi_\alpha(p)$ from R into R and by $\varphi_\alpha \rightarrow$ the fact that φ_α uniformly converges, then

D_1') $\varphi_\alpha = \varDelta$ *for every* $\alpha \in A$ *implies* $\varphi_\alpha \rightarrow$,

D_2') *if* $\varphi_\alpha \rightarrow$ *and* $\varphi_\alpha' \rightarrow$, *then for every* $G \subseteqq R \otimes R$ *such that* $\{\beta | G \frown \varphi_\beta^{-1} \varphi_\beta \neq \phi\}$ *is cofinal in* A, *there exists* $\psi_\gamma(\gamma \in C)$ *such that* $\psi_\gamma \frown G \neq \phi$ *for all* γ,

D_3') *if for* $\varphi_\alpha(\alpha \in A)$ *and for every* $G \subseteqq R \otimes R$ *such that* $\{\beta | G \frown \varphi_\beta \neq \phi\}$ *is cofinal in* A, *there exists* $\psi_\gamma(\gamma \in C)$ *such that* $\psi_\gamma \rightarrow$ *and* $\psi_\gamma \frown G \neq \phi$ *for all* γ, *then* $\varphi_\alpha \rightarrow$,

D_4') $\varphi_\alpha \rightarrow$ *implies* $\varphi_\alpha(p_0) \rightarrow p_0$.

The condition D_3) (or D_3')) combining with D_2) for $\varphi_\alpha(p) = p$ (or with D_2')) is analogous to the so-called star-convergence conditions of ordinary convergence: $\varphi(\alpha | \alpha \in A) \rightarrow p$, if and only if for each A confinal in B and some $\psi(\gamma | \gamma \in C)$ converging to p, $\psi(C) \subseteqq \varphi(B)$.[6] D_2) (or D_2')) is analogous to the idem potent condition of ordinary convergence in T-space: If $\varphi_\beta(\alpha | \alpha \in A_\beta) \rightarrow \varphi(\beta)$ and $\varphi(\beta | \beta \in B) \rightarrow p$, then there exists $\psi(\gamma | \gamma \in C)$ such that $\psi(C) \subseteqq \{\varphi_\beta(\alpha) | \alpha \in A_\beta, \beta \in B\}$ and $\psi(\gamma | \gamma \in C) \rightarrow p$,[7] but in uniform space it is a condition having the same content as U_3). D_4) or D_4') shows the agreement of the uniform topology with the topology.

THEOREM 1. *If a uniform topology is defined by uniform coverings satisfying* U_1)—U_3) *in* R, *then defining uniform convergence by* T_{na}), *we get* D_1)—D_4) *about the uniform convergence.*

Conversely, if a uniform topology is defined by uniform convergence satisfying D_1)—D_4) *in* R, *then defining uniform coverings by* T_{an}), *we get* U_1)—U_3) *about the uniform coverings.*

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5) A. Weil, loc. cit.
6) J. W. Tukey, loc. cit.
7) This condition is due to A. Komatu, Theory of topological spaces (1947, in Japanese).

Proof. 1. It is obvious that $U_1)$ and $T_{nd})$ imply $D_1)$.

$U_2)$, $U_3)$ and $T_{nd})$ imply $D_2)$, for let $\varphi_\alpha(p)\rightarrow$, $\varphi'_\alpha(p)\rightarrow$ and let $\{p_\beta \mid \beta\in B\}$ be a subset of R for B confinal in A, then for each $x\in X$ from $U_3)$ and $T_{nd})$ there exists $\beta(x)\in B$ such that that $B\ni\beta\geqq\beta(x)$ implies $\varphi_\beta(p)\in U_x(\varphi'_\beta(p))$, where $\{U_x \mid x\in X\}$ is the system of all the uniform neighbourhoods. Since from $U_2)$, X is a directed system, we can define a uniform d.s.p. $\psi_x(p)$ $(x\in X)$ so that $\psi_x(p)\in U_x(p)$ and $\psi_x(\varphi'_{\beta(x)}(p_{\beta(x)}))=\varphi_{\beta(x)}(p_{\beta(x)})$. Then $\psi_x(p)$ obviously satisfies the condition of $\psi_\gamma(p)$ in $D_2)$.

$T_{nd})$ implies $D_3)$, for if $\varphi_\alpha(p)\nrightarrow$,[8] then there exist a uniform neighbourhood $U_x(p)$, and for some B confinal in A and $p_\beta(\beta\in B)$, $\varphi_\beta(p_\beta)\notin U_x(p_\beta)$ holds for every $\beta\in B$. If $\{(p_\gamma,\psi_\gamma(p_\gamma)) \mid \gamma\in C\}\subseteqq\{(p_\beta,\varphi_\beta(p_\beta)) \mid \beta\in B\}$, then for every $\gamma\in C$ there exists $\beta\in B$ such that $p_\gamma=p_\beta$ and $\psi_\gamma(p_\gamma)=\varphi_\beta(p_\beta)\notin U_x(p_\beta)$, *i.e.* $\psi_\gamma(p)\notin U_x(p_\gamma)$. Therefore $\psi_\gamma(p)\nrightarrow$.

It is obvious that $U_1)$ and $T_{nd})$ imply $D_4)$.

2. Firstly we prove the following two propositions.

If $D_1)$, $D_2)$, $T_{dn})$ and $\varphi_\alpha(p)\rightarrow$ hold, then for every uniform neighbourhood $U_x(p)$ there exists $\alpha(x)$ such that $\alpha\geqq\alpha(x)$ implies $\varphi_\alpha(p)\in U_x(p)$ for all $p\in R$. For assume that $\varphi_\alpha(p)\rightarrow$, and that for B cofinal in A $\varphi_\beta(p_\beta)\notin U_x(p_\beta)$ $(\beta\in B)$, then from $D_1)$, $D_2)$ there would exist a uniform d.s.p. $\psi(p)$ $(\gamma\in C)$ such that $\psi_\gamma\rightarrow$ and $\{(p_\gamma\psi_\gamma(p_\gamma)) \mid \gamma\in C\}\subseteqq\{(p_\beta\,\varphi_\beta(p)) \mid \beta\in B\}$ for some $\{p_\gamma \mid \gamma\in C\}$. But $\psi_\gamma(p_\gamma)=\varphi_\beta(p_\beta)\notin U_x(p_\beta)=U_x(p_\gamma)$ for every $\gamma\in C$, which contradicts $T_{dn})$.

Next $\varphi_x(p)\in U_x(p)$ for every uniform neighbourhood $U_x(p)$ $(x\in X)$ and for every $p\in R$ implies $\varphi_x(p)\rightarrow$, when $D_3)$ and $T_{dn})$ hold. For assume that $\varphi_x(p)\nrightarrow$, then from $D_3)$ there exist B cofinal in X and $\{p_\beta \mid \beta\in B\}\subseteqq R$ such that $\{(p_\gamma\,\psi_\gamma(p)) \mid \gamma\in C\}\subseteqq\{(p_\beta\,\varphi_\beta(p_\beta)) \mid \beta\in \beta\}$ implies $\psi_\gamma(p)\nrightarrow$. Hence defining $U(p)=R-\{\varphi_\beta(p_\beta) \mid p_\beta=p\}$, we get a uniform covering $U(p)$ satisfying $T_{an})$. Hence from the former proposition there exists $\alpha(x)$ such that $\alpha\geqq\alpha(x)$ implies $\varphi_\alpha(p)\in U(p)$ for all $p\in R$, but for $\beta\geqq\alpha(x)$, $\beta\in B$ $\varphi_\beta(p_\beta)\notin U(p_\beta)$ would also hold, which is a contradiction.

$D_3)$, $T_{dn})$ and $D_4)$ imply $U_1)$. For if $U_x(p_0)\subsetneqq U(p_0)$ for some neighbourhood $U(p_0)$ of p_0 and for every uniform neighbourhood $U_x(p)$, then defining $\varphi_x(p)$ so that $\varphi_x(p_0)\in U_x(p_0)-U(p_0)$, $\varphi_x(p)=p(p\neq p_0)$, we get a uniformly converging $\varphi_x(p)$ by the latter of the above propositions. Hence from $D_4)$ $\varphi_x(p_0)\rightarrow p_0$ would hold, which is impossible.

$D_1)$, $D_2)$ and $T_{dn})$ imply $U_2)$. For if $\{U_x(p) \mid p\in R\}$ and $\{U_y(p) \mid p\in R\}$ are uniform neighbourhood, then from the former of the above propositions there exist $\alpha(x)$, $\alpha(y)$ such that $\alpha\geqq\alpha(x)$ implies $\varphi_\alpha(p)\in U_x(p)$ for every $p\in R$, and $\alpha\geqq\alpha(y)$ implies $\varphi_\alpha(p)\in U_y(p)$ for every $p\in R$. Hence $\alpha\geqq\alpha(x)$ and $\alpha\geqq\alpha(y)$ imply $\varphi_\alpha(p)$

8) $\varphi_\alpha(p)\nrightarrow$ denotes the negation of $\varphi_\alpha(p)\rightarrow$.

$\in U_x(p)\frown U_y(p)$, and hence $\{U_x(p)\frown U_y(p)\mid p\in R\}$ is a uniform neighbourhood satisfying the condition of $U_z(p)$ in U_2).

Lastly we prove that D_2), D_3) and T_{dn}) imply U_3). Assume that for some uniform neighbourhood $U_x(p)$ and for every uniform neighbourhood $U_y(p)$ $(y\in X)$, there exist p_y, q_y and r_y such that $p_y\in U_y(r_y)$, $q_y\in U_y(r_y)$ and $q_y\notin U_x(p_y)$. Hence defining $\varphi_y(r_y)=p_y, \varphi_y(p)\in U_y(p)$; $\varphi_y'(r_y)=q_y$, $\varphi_y'(p)\in U_y(p)$, we get uniform converging $\varphi_y(p)$ and $\varphi_y'(p)$ by the latter-mentioned proposition. Therefore there would be a uniform converging d.s.p. $\psi_\gamma(p)$ satisfying the condition in D_2) for φ_y, φ_y' and $\{r_y\mid y\in X\}$, *i.e.* for every γ there would be y such that $\psi_\gamma(\varphi_y(r_y))=\varphi_y'(r_y)\notin U_x(p_y)=U_x(\varphi_y(r_y))$. This contradicts T_{dn}) and completes the proof of this theorem.

THEOREM 2. *If we derive from uniform convergence,* $\rightarrow$ *satisfying* D_1)—D_4) *uniform neighbourhoods by* T_{dn}) *and from the uniform neighbourhoods uniform convergence,* $\rightsquigarrow$ *by* T_{nd}), *then the convergence* $\rightsquigarrow$ *is equivalent to* $\rightarrow$. *Conversely, if we derive from uniform neighbourhoods* $\mathfrak{U}=\{U_x(p)\mid x\in X\}$ *satisfying* U_1)— U_3) *uniform convergence by* T_{nd}) *and from the uniform convergence uniform neighbourhoods* $\mathfrak{U}'=\{U'_{x'}(p)\mid x'\in X'\}$ *by* T_{dn}), *then* $\mathfrak{U}$ *and* $\mathfrak{U}'$ *are equivalent.*[9]

Proof. 1. If $\varphi_\alpha(p)\rightarrow$, then from the former proposition in 2 of the proof of Theorem 1, $\varphi_\alpha(p)\rightsquigarrow$ is obvious.

Conversely, if $\varphi_\alpha(p)\not\rightarrow$, then from D_3) there exists B confinal in A and $p_\beta(\beta\in B)$ such that $\{(p_\gamma\,\psi_\gamma(p_\gamma))\mid\gamma\in C\}\subseteqq\{(p_\beta\,\varphi_\beta(p_\beta))\mid\beta\in B\}$ implies $\psi_\gamma(p)\not\rightarrow$. Hence defining $U(p)=R-\{\varphi_\beta(p_\beta)\mid p_\beta=p\}$, we get a uniform neighbourhood $\{U(p)\mid p\in R\}$. Therefore $\varphi_\beta(p_\beta)\notin U(p_\beta)$ for all $\beta\in B$, and hence $\varphi_\alpha(p)\not\rightsquigarrow$ from T_{nd}).

2. We prove the latter-half of the theorem. If $U_x(p)\in\mathfrak{U}$, then since $\varphi_\alpha(p)\rightarrow$ implies $\varphi_\alpha(p)\in U_x(p)$ for some $\alpha(x)$ and every $\alpha\geqq\alpha(x)$, $U_x(p)\in\mathfrak{U}'$ from T_{dn}).

Conversely, if for some $U(p)\in\mathfrak{U}'$ and for every $U_x(p)\in\mathfrak{U}$, there exists $p_x\in R$ such that $U_x(p_x)\not\subseteqq U(p_x)$, then defining $\varphi_x(p)$ $(x\in X)$ so that $\varphi_x(p)\in U_x(p)$, $\varphi_x(p_x)\in U_x(p_x)-U(p_x)$, we get a uniformly converging $\varphi_x(p)$. Since $\varphi_x(p_x)\notin U(p_x)$ for every $x\in X$ contradicts T_{dn}), we can conclude the equivalence of $\mathfrak{U}$ and $\mathfrak{U}'$.

Now let us concern ourselves with filters. Generally, let $F_\alpha(p)$ $(\alpha\in A,\ p\in R)$, $G_\beta(p)$ $(\beta\in B,\ p\in R)$ be mappings from $A\otimes R$ into 2^R and from $B\otimes R$ into 2^R respectively,[10] where A, B are certain sets, and R is a T_1-space. We denote by $G_\beta(p)\leqq F_\alpha(p)$ the fact that for every α and some β, $G_\beta(p)\subseteqq F_\alpha(p)$ holds for all $p\in R$.

We mean by a *uniform filter* a mapping $F_\alpha(p)$ $(\alpha\in A,\ p\in R)$ from $A\otimes R$ into 2^R satisfying the following conditions,

9) If for every $U_x(p)\in\mathfrak{U}$ and some $U'_{x'}(p)\in\mathfrak{U}$, $U'_{x'}(p)\subseteqq U_x(p)$ as well as the converse hold, we call $\mathfrak{U}$ and $\mathfrak{U}'$ equivalent.

10) We denote by 2^R the set of all the subsets of R.

i) $\phi \neq F_\alpha(p)$ for every α and p,

ii) for every α, β and for some γ, $F_\gamma(p) \subseteqq F_\alpha(p) \frown F_\beta(p)$ holds for all $p \in R$[11].

Any set $\{F_\alpha^\lambda(p) | \lambda \in L\}$ of uniform filters $F_\alpha^\lambda(p)$ $(\alpha \in A_\lambda)$ has the lowest upper bound $G_\beta(p)$ about the order $\leqq$: $\{G_\beta(p) | \beta \in B\} = \{G | G \supseteqq F_\alpha^\lambda(p)$ for every λ, for some α and for all $p\}$, which is denoted by $\bigvee_{\lambda \in L} F_\alpha^\lambda(p)$. We denote by $F_\alpha(p) \rightarrow$ the statement "the uniform filter $F_\alpha(p)$ *uniformly converges.* We consider the following conditions about uniform convergence of uniform filters,

F_1) *$F_\alpha(p) = p$ for every α and p implies $F_\alpha(p) \rightarrow$,*

F_1) *$F_\alpha(p) \rightarrow$ implies $G_\beta(p) \rightarrow$ for some $G_\beta(p)$ such that $G_\beta(p) \geqq S_\alpha(p)$ $= S(p, \{F_\alpha(p) | p \in R\})$,*[12]

F_3) *for any set $\{F_\alpha^\lambda(p) | \lambda \in L\}$ of uniform filters, $\bigvee_{\lambda \in L} F_\alpha^\lambda(p) = G_\beta(p) \rightarrow$, if and only if every $F_\alpha^\lambda(p) \rightarrow$,*

F_4) *$F_\alpha(p) \rightarrow$ imples the filter $\{F_\alpha(p_0) | \alpha \in A\} \rightarrow p_0$ for each $p_0 \in R$,*

where $F_\alpha(p)$, $F_\alpha^\lambda(p)$, $G_\beta(p)$ denote uniform filters.

REMARK. Using $R \otimes R$, we get the following conditions, which are equivalent to F_1)—F_4). If we mean by a filter $\mathfrak{F}$ on $R \otimes R$ a family of subsets of $R \otimes R$ such that $F_\alpha, F_\beta \in \mathfrak{F}$ implies $F_\gamma \subseteqq F_\alpha \frown F_\beta$ for some $F_\gamma \in \mathfrak{F}$ and $F(p_0) \neq \phi$ for every $F \in \mathfrak{F}$ and $p_0 \in R$, and if we denote by $\mathfrak{F} \rightarrow$ the statement "$\mathfrak{F}$ uniformly converges", then

F_1') $\mathfrak{F} = \{\Delta\} \rightarrow$,

F_2') *$\mathfrak{F} \rightarrow$ implies $\mathfrak{G} \rightarrow$ for some $\mathfrak{G}$ such that $\mathfrak{G} \geqq \mathfrak{S} = \{FF^{-1} | F \in \mathfrak{F}\}$,*

F_3') *$\bigvee_{\lambda \in L} \mathfrak{F}^\lambda \rightarrow$, if and only if $\mathfrak{F}^\lambda \rightarrow$ for every $\lambda \in L$,*

F_4') *$\{F_\alpha | \alpha \in A\} = \mathfrak{F} \rightarrow$ implies $\{F_\alpha(p_0) | \alpha \in A\} \rightarrow p_0$ for each $p_0 \in R$,*

where $\mathfrak{F}$, $\mathfrak{F}^\lambda$, $\mathfrak{G}$ denote filters on $R \otimes R$, and $\mathfrak{G} \geqq \mathfrak{S}$ denotes the fact that for every $G \in \mathfrak{G}$ and for some $S \in \mathfrak{S}$, $G \supseteqq S$.

The following axioms show the relation between uniform filters and uniform d.s.p.

T_{fd}) *$\varphi_\alpha(p) \rightarrow$, if and only if $F_\alpha(p) \rightarrow$, where $F_\alpha(p) = \{\varphi_{\alpha'}(p) | \alpha' \geqq \alpha\}$.*

T_{df}) *$F_\alpha(p) \rightarrow$, if and only if $\varphi_\alpha(p) \rightarrow$ for every $\varphi_\alpha(p)$ such that $\varphi_\alpha(p) \in F_\alpha(p)$.*

THEOREM 3. *If we define uniform convergence of uuiform filters by T_{df}) from uniform convergence of uniform d.s.p. satisfying D_1)—D_4), then it satisfies F_1)—F_4).*

Conversely, if we define uniform convergence of uniform d.s.p. by T_{fd}) from uniform convergence of uniform filters satisfying F_1)—F_4), then it satisfies D_1)—D_4).

Proof. Firstly, we prove that $F_\alpha(p) \rightarrow (\alpha \in A)$, if and only if $F_\alpha(p) \leqq U_x(p)$ for the uniform filter $U_x(p)$ of all the uniform neighbourhoods, which are defined by T_{dn}) from uniform d.s.p. Let $F_\alpha(p) \nleqq U_x(p)$, then there exist $U_x(p)$ and p_α for

11) By the order $\alpha \leqq \beta$: $F_\alpha(p) \subseteqq F_\beta(p)$ for all $p \in R$, A is a directed system.

12) Generally, $S(p, \mathfrak{F})$ denotes the set $\cup\{F | p \in F \in \mathfrak{F}\}$.

every $\alpha \in A$ such that $F_\alpha(p_\alpha) \not\subseteq U_x(p_\alpha)$. Defining $\varphi_\alpha(p)$ so that $\varphi_\alpha(p) \in F_\alpha(p)$, $\varphi_\alpha(p_\alpha) \in F_\alpha(p_\alpha) - U_x(p_\alpha)$, we get a uniform d.s.p., $\varphi_\alpha(p) \to$. Hence $F_\alpha(p) \not\to$ by $T_{d f}$).

Conversely, let $F_\alpha(p) \not\to$, then there exists $\varphi_\alpha(p)$ such that $F_\alpha(p) \ni \varphi_\alpha(p) \not\to$. Hence defining $U(p)$ such that $U(p) = R - \{\varphi_\beta(p_\beta) \mid p_\beta = p\}$ for some $\{p_\beta \mid \beta \in B\}$, B confinal in A, we get a uniform neighbourhood. Since for every α, there exists $\beta \in B$ such that $\beta \geqq \alpha$, $\varphi_\beta(p_\beta) \in F_\beta(p_\beta) \subseteqq F_\alpha(p_\beta)$, $\varphi_\beta \notin U(p_\beta)$. Therefore $F_\alpha(p_\beta) \not\subseteqq U(p_\beta)$, *i.e.* $F_\alpha(p) \not\leqq U(p)$.

1. Now we prove the former-half of the theorem. From D_1), F_1) is obviously satisfied.

Next we prove F_2). If $F_\alpha(p) \to$, then $F_\alpha(p) \leqq U_x(p)$, where $U_x(p)$ $(x \in X)$ is the uniform filter of all the uniform neighbourhoods. For an arbitrary $x \in X$, there exists $y \in X$ such that $p, p' \in U_y(q)$ implies $p \in U_x(p')$. Hence for $F_\alpha(p) \subseteqq U_y(p)$ $(p \in R)$ $U_x(p) \supseteqq \cup\{U_y(q) \mid p \in U_y(q)\} \supseteqq \cup\{F_\alpha(q) \mid p \in F_\alpha(q)\} = S_\alpha(p)$. Therefore $U_x(p) \geqq S_\alpha(p)$ and $U_x(p) \to$.

The proof of F_3) is as follows. $F_\alpha^\lambda(p) \leqq G_\beta(p) \to$ implies $G_\beta(p) \leqq U_x(p)$ and accordingly $F_\alpha^\lambda(p) \leqq U_x(p)$, *i.e.* $F_\alpha^\gamma(p) \to$. Conversely, $F_\alpha^\lambda(p) \to (\lambda \in L)$ imply $U_x(p) \supseteqq F_{\alpha(\lambda)}^\lambda(p)$ $(p \in R)$ for every $x \in X$, $\lambda \in L$ and for some $\alpha(\lambda) \in A_\lambda$. Hence $U_x(p) \geqq G_\beta(p)$, *i.e.* $G_\beta(p) \to$.

Finally, $F_\alpha(p) \to$ implies $\varphi_\alpha(p_0) \to p_0$, for $\varphi_\alpha(p) \in F_\alpha(p)$ from D_4) and $T_{d f}$); hence $F_\alpha(p_0) \to p_0$.

2. From F_1), D_1) is obvious.

If $\varphi_\alpha(p) \to$, $\varphi_\alpha'(p) \to$, then $\{\varphi_{\alpha'}(p) \mid \alpha' \geqq \alpha\} = F_\alpha(p) \to$, $\{\varphi'_{\alpha'}(p) \mid \alpha' \geqq \alpha\} = F_\alpha'(p) \to$. Hence from F_3) $F_\alpha(p) \vee F_\alpha'(p) = G_\delta(p) \to$, and from F_2), $S(p, \{G_\delta(q) \mid q \in R\}) = S_\delta(p) \leqq U_\gamma(p) \to$ for some uniform filter $U_\gamma(p)$. Therefore for each γ, there exists $\delta = \delta(\gamma)$ such that $S(p, \{G_\delta(q) \mid q \in R\}) \subseteqq U_\gamma(p)$ $(p \in R)$. Let $\{p_\beta \mid \beta \in B\}$ be an arbitrary subset of R for B confinal in A, then for each δ, there exists $\beta = \beta(\delta)$ such that $F_\beta(p) \cup F'_\beta(p) \subseteqq G_\delta(p)$ $(p \in R)$. Hence for these $\delta = \delta(\gamma)$ and $\beta = \beta(\delta(\gamma))$, $\varphi_\beta(p_\beta)$, $\varphi'_\beta(p_\beta) \in G_\delta(p_\beta)$ and $\varphi'_\beta(p_\beta) \in S_\delta(\varphi_\beta(p_\beta))$ hold; hence we can define $\psi_\gamma(p)$ so that $\psi_\gamma(p) \in S_\delta(p) \subseteqq U_\gamma(p)$, $\psi_\gamma(\varphi_\beta(p_\beta)) = \varphi'_\beta(p_\beta)$. Since $\{\psi_{\gamma'}(p) \mid \gamma' \geqq \gamma\} = H_\gamma(p) \subseteqq U_\gamma(p) \to$ from F_3) and $T_{f d}$), we get $H_\gamma(p) \to$ and $\psi_\gamma(p) \to$, *i.e.* D_2) is concluded.

Next, let us consider $\varphi_\alpha(p)$ satisfying the star-condition in D_3), then we see $\{\varphi_{\alpha'}(p) \mid \alpha' \geqq \alpha\} = F_\alpha(p) \leqq A_\gamma(p)$ for the uniform filter $A_\gamma(p) = \vee\{F_\alpha^\lambda(p) \mid F_\alpha^\lambda(p) \to\}$. For if we assume the contrary, there exists γ, B cofinal in A and $\{p_\beta \mid \beta \in B\}$ such that $\varphi_\beta(p_\beta) \notin A_\gamma(p_\beta)$ $(\beta \in B)$. Hence $\{(p_\delta, \psi_\delta(p_\delta))\} \subseteqq \{(p_\beta, \varphi_\beta(p_\beta))\}$ implies $\psi_\delta(p_\delta) \notin A_\gamma(p_\delta)$ and $\psi_\delta(p) \not\to$ from the definition of $A_\gamma(p)$, which contradicts the star-condition of $\varphi_\alpha(p)$. Therefore it must be $A_\gamma(p) \geqq F_\alpha(p)$. Since $A_\gamma(p) \to$ from

F_3), we get $F_\alpha(p) \dashrightarrow$ and accordingly $\varphi_\alpha(p) \dashrightarrow$. Thus D_3) is established. D_4) is obvious from F_4).

THEOREM 4. *If from a uniform convergence of uniform filters $F_\alpha(p) \rightarrow$ satisfying F_1)—F_4), by T_{fd}) and T_{df}) the uniform convergence $F_\alpha(p) \rightsquigarrow$ is definde, then $F_\alpha(p) \rightarrow$ and $F_\alpha(p) \rightsquigarrow$ are equivalent.*

Conversely, if from a uniform convrgence of d.s.p. $\varphi_\alpha(p) \rightarrow$ satisfying D_1)—D_4), by T_{df}) and T_{fd}) the uniform convergence $\varphi_\alpha(p) \rightsquigarrow$ is defined, then $\varphi_\alpha(p) \rightarrow$ and $\varphi_\alpha(p) \rightsquigarrow$ are equivalent.

Proof. Firstly, we prove the former-half. If $F_\alpha(p) \rightarrow$, then since $\{\varphi_{\alpha'}(p) | \alpha' \geqq \alpha\} \subseteqq F_\alpha(p)$ for $\varphi_\alpha(p) \in F_\alpha(p)$, from F_3) we get $\{\varphi_{\alpha'}(p) | \alpha' \geqq \alpha\} \rightarrow$, *i.e.* $\varphi_\alpha(p) \rightarrow$. Therefore $F_\alpha(p) \rightsquigarrow$.

Conversely, let $F_\alpha(p) \rightsquigarrow$ and $\mathfrak{G}$ be the set of all the uniform filters $G_\alpha(p)$ such that $G_\alpha(p) = \{\varphi_{\alpha'}(p) | \alpha' \geqq \alpha\}$, $\varphi_\alpha(p) \in F_\alpha(p)$, then since $\varphi_\alpha(p) \rightarrow$ from T_{df}) and $G_\alpha(p) \rightarrow$ from T_{fd}), we get $\vee \{G_\alpha(p) | G_\alpha(p) \in \mathfrak{G}\} = H_\gamma(p) \rightarrow$ from F_3). If we assume $H_\gamma(p) \not\geqq F_\alpha(p)$, then there exist γ, $\{p_\alpha | \alpha \in A\}$ such that $H_\gamma(p_\alpha) \not\supseteqq F_\alpha(p_\alpha)$ $(\alpha \in A)$. Hence defining $\psi_\alpha(p)$ so that $\psi_\alpha(p) \in F_\alpha(p)$, $\psi_\alpha(p_\alpha) \in F_\alpha(p_\alpha) - H_\gamma(p_\alpha)$, we get a uniformly converging $\psi_\alpha(p)$, but $\{\psi_{\alpha'}(p_\alpha) | \alpha' \geqq \alpha\} \not\subseteqq H_\gamma(p_\alpha)$ $(\alpha \in A)$, which contradicts the definition of $H_\gamma(p)$. Therefore it must be $H_\gamma(p) \geqq F_\alpha(p)$, and hence $F_\alpha(p) \rightarrow$ from F_3).

Next we remove to the proof of the remainder. $\varphi_\alpha(p) \not\rightsquigarrow$ implies $\{\varphi_{\alpha'}(p) | \alpha' \geqq \alpha\} = F_\alpha(p) \not\rightarrow$ and accordingly $\psi_\alpha(p) \not\rightarrow$ for some $\psi_\alpha(p) \in F_\alpha(p)$. We denote by $\alpha'(\alpha)$ the element of A such that $\psi_\alpha(p) = \varphi_{\alpha'(\alpha)}(p)$; hence $\alpha'(\alpha) \geqq \alpha$. From D_3) we get B cofinal in A and $p_\beta (\beta \in B)$, for which star-condition about $\psi_\alpha(p)$ does not hold. Let $B' = \{\alpha'(\beta) | \beta \in B\}$ and $p_{\beta'} = p_\beta$ for a definite β such that $\beta' = \alpha'(\beta)$, then for an arbitrary ψ'_γ and $p_\gamma (\gamma \in C)$, $\{(p_\gamma \psi'_\gamma(p_\gamma)) | \gamma \in C\} \subseteqq \{(p_{\beta'} \varphi_{\beta'}(p_{\beta'})) | \beta' \in B'\}$ implies $\{(p_\gamma \psi'_\gamma(p_\gamma)) | \gamma \in C\} \subseteqq \{(p_\beta \psi_\beta(p_\beta)) | \beta \in B\}$. Hence $\psi'_\gamma(p) \not\rightarrow$. Since B is cofinal in A, we get $\varphi_\alpha(p) \not\rightarrow$ from D_1) and D_2).

Conversely, $\varphi_\alpha(p) \rightsquigarrow$ implies $\{\varphi_{\alpha'}(p) | \alpha' \geqq \alpha\} = F_\alpha(p) \rightarrow$ by T_{fd}), and from $\varphi_\alpha \in F_\alpha(p)$ by T_{df}), $\varphi_\alpha(p) \rightarrow$.

The uniform topology of a metric space is deciced by uniform convergence of uniform d.s.p. on $N = \{1, 2, \cdots\}$[13].

THEOREM 5. *Metric spaces are characterized as uniform spaces, in which $\varphi_\alpha(p) \rightarrow$ $(\alpha \in A)$, if and only if there exist $\alpha_1, \alpha_2, \cdots \in A$ such that $A \ni \beta_i \geqq \alpha_i$ $(i = 1, 2, \cdots)$ imply $\varphi_{\beta_i}(p) \rightarrow$.*

Proof. If R is a metric space, then there exists a basis of uniform neighbour-

13) The topology of a topological space satisfying first countability axiom is decided by a sequencial convergence by K. Sakakihara, The structure of neighbourhood systems and types of convergences, this journal, Vol. 4, No. 1 (1953).

hoods consisting of a countably infinite uniform neighbourhoods $U_1(p)$, $U_2(p), \cdots$ $(p \in R)$. Hence $\varphi_\alpha(p) \to$ implies $\varphi_\alpha(p) \in U_n(p)$ for some $\alpha_n \in A$ and for every $\alpha \geqq \alpha_n$. Therefore if $\beta_i \geqq \alpha_i$ $(i=1,2,\cdots)$, then for every i, $k \geqq i$ implies $\varphi_{\beta_k}(p) \in U_k(p) \subseteq U_i(p)$ $(p \in R)$, and hence $\varphi_{\beta_i}(p) \to$. Furthermore $\varphi_\alpha(p) \nrightarrow$ implies $\varphi_\beta(p_\beta) \notin U_n(p_\beta)$ $(\beta \in B)$ for some n and B cofinal in A. Hence for an arbitrary sequence $\alpha_1, \alpha_2, \cdots$ of elements of A, there exists $\beta_1, \beta_2, \cdots$ such that $\alpha_i \leqq \beta_i \in B$ and accordingly $\varphi_{\beta_i}(p_{\beta_i}) \notin U_n(p_{\beta_i})$ $(i=1,2,\cdots)$, *i.e.* $\varphi_{\beta_i}(p) \nrightarrow$.

Conversely, if $\varphi_\alpha(p) \to$ satisfies the condition of the theorem, then for the set of all the uniform neighbourhoods $\{U_x(p) \mid x \in X\}$, we define a uniform d.s.p. $\varphi_{(xq)}(p)$ on $X \otimes R$ such that $\varphi_{(xq)}(p) = p (q \notin U_x(p))$, $= q (q \in U_x(p))$, where $X \otimes R$ is a directed system by the order $(xq) \leqq (yr)$ meaning $x \leqq y$. For an arbitrary $U_x(p)$, $(xq_0) \leqq (yq)$ implies $y \geqq x$ and $\varphi_{(yq)}(p) \in U_y(p) \subseteq U_x(p)$, and hence $\varphi_{(xq)}(p) \to$. Hence there exist $(x_i q_i)$ $(i=1,2,\cdots)$ such that $(x_i q_i) \leqq (y_i r_i)$ $(i=1,2,\cdots)$ imply $\varphi_{(y_i r_i)}(p) \to$. Now we shall see that the uniform neighbourhoods $U_i(p) = \{\varphi_{(x_i q)}(p) \mid q \in R\} = U_{x_i}(p)$ form a uniform basis of R. For if we assume that there exist $x \in X$ and p_i $(i=1,2,\cdots)$ such that $U_i(p_i) \nsubseteq U_x(p_i)$, then for $r_i \in U_i(p_i) - U_x(p_i)$, $\varphi_{(x_i r_i)}(p_i) = r_i \notin U_x(p_i)$ $(i=1,2,\cdots)$ hold. Hence $\varphi_{(x_i r_i)}(p) \nrightarrow$, but this contradicts $(x_i r_i) \geqq (x_i q_i)$. Thus R has a uniform basis of countably infinite number of uniform neighbourhoods and is a metric space.

About uniform filters, in the same way we can show the following.

THEOREM 6. *Metric spaces are characterized as uniform spaces satisfying that* $F_\alpha(p) \to$ $(\alpha \in A)$, *if and only if for some* $\alpha_i \in A$ $(i=1,2,\cdots)$, $F_{\alpha_i}(p) \to$.

In a metric space the totality of the uniform coverings consisting of two sets determines the uniform topology, and this important fact leads to theorems holding only in metric spaces by the author[14]; for example, the uniform topology of a complete metric space R is defined by the lattice $U(R)$ of uniformly continuous bounded functions on the space and it is also defined by a lattice $L(R)$ of finite open uniform coverings satisfying 1) $L(R)$ is a basis of the totality of finite uniform coverings of R, 2) $\mathfrak{U} \in L(R)$ and $\mathfrak{V} \in L(R)$ imply $\mathfrak{U} \vee \mathfrak{V} \in L(R)$, 3) for any open sets U, V such that $U \frown V = \phi$, $V \neq \phi$ there exists $\mathfrak{M} \in L(R)$ such that $U \in \mathfrak{M}$. $V \notin \mathfrak{M}$. We get this fact readily from Theorem 5 and from the fact that in a metric space $\varphi_n(p) \to$ $(n=1,2,\cdots)$, if and only if for every binary uniform covering $\mathfrak{U} = \{U_1, U_2\}$ there exists n_0 such that $n \geqq n_0$ implies $\varphi_n(p) \in U_i$ for each p and for some $U_i \in \mathfrak{U}$. We close this note with the following

COROLLARY. *If* S *is a* T_1*-space and if* R *is a uniform space such that* $\bar{R} = S$,

14) J. Nagata, On lattices of functions on topological spaces and of functions on uniform spaces, Osaka Math. Journ. Vol. 1, No. 2 (1949).
J. Nagata, On relations between latties of finite uniform coverings of a metric space and the uniform topology of the space, this journal Vol. 4, No. 1 (1953).

then in order that the uniform topology of R can be extended to S, it is necessary and sufficient that

i) *every d.s.p. in R converging to a point of S satisfies chauchy condition in R,*

ii) *every equivalent d.s.p. in R with a d.s.p. converging to a point of S in R converges to the same point.*

Proof. Since the necessity of the condition is ovbious, we shall prove only the sufficiency. From the condition i) of the theorem, for every point $p \in S = \bar{R}$ there exists a chauchy d.s.p. $p^{\alpha}(\alpha \in A_p)$ converging to p such that $p^{\alpha} \in R$. For every $x \in X$ there exists $\alpha(x) \in A_p$ such that $A_p \ni \alpha \geqq \alpha(x)$ implies $U_x(p^{\alpha(x)}) \ni p^{\alpha}$, where $\{U_x(p) \mid x \in X, p \in R\}$ is the system of the uniform neighbourhoods defining the uniform topology of R. For $p \in R$ we take $p^{\alpha} = p$. Now for $U \subseteqq R$, we denote by $\widetilde{U}$ the subset $(\overline{U^c})^c$ of S.[15] Furthermore we denote $\cup\{U_y(q) \mid q \in U_x(p)\}$ by $U_y(U_x(p))$, $U_x(U_x(p))$ by $U_x^2(p)$, $U_x(U_x^2(p))$ by $U_x^3(p)$ and so far. We define uniform convergence of uniform d.s.p. in S as follows: $\varphi_{\gamma}(p) \rightarrow (\gamma \in C)$ if and only if for every $x \in X$ there exists $\gamma(x) \in C$ such that $\gamma \geqq \gamma(x)$ implies $\widetilde{U_x^2(p^{\alpha(x)})} \ni \varphi_{\gamma}(p)$ for each $p \in S$. It is easy to verify that this uniform convergence satisfies the conditions $D_1)$—$D_4)$.

Firstly, $\varphi_{\delta}(p) = p$ implies $\varphi_{\delta}(p) \rightarrow$. For every $x \in X$, there exist $y, z \in X$ such that $U_z^2(q) \subseteqq U_y(q) \subseteqq U_y^2(q) \subseteqq U_x(q)$ $(q \in R)$. If $p \notin \widetilde{U_y(p^{\alpha(z)})}$ then there exists $r_{\gamma} \rightarrow p$, where $r_{\gamma} \in V_{\gamma}(p)$, and $\{V_{\gamma}(p) \mid \gamma \in C\}$ is a neighbourhood basis of p in S such that $r_{\gamma} \in R - U_z^2(p^{\alpha(z)})$. Since $U_z(p^{\alpha(z)}) \ni p^{\alpha} \rightarrow p$ $(\alpha \geqq \alpha(y))$, defining $\psi_{(\gamma 1)} = r_{\gamma}$, $\psi_{(\gamma 2)} = p^{\alpha(\gamma)} \in V_{\gamma}(p)$ $(\alpha(\gamma) \geqq \alpha(y))$ for the directed system $\{(\gamma i) \mid \gamma \in C\}$ with the order $(\gamma i) \geqq (\gamma' i') : \gamma \geqq \gamma'$, we get a non chauchy d.s.p. $\psi_{(\gamma i)}$ converging to p, which contradicts i). Hence we get $p \in \widetilde{U_y(p^{\alpha(z)})} \subseteqq \widetilde{U_y U_z U_x(p^{\alpha(x)})} \subseteqq \widetilde{U_x^2(p^{\alpha(x)})}$ from $p^{\alpha(z)} \in U_z(U_x(p^{\alpha(x)}))$, *i.e.* $\varphi_{\delta}(p) \rightarrow$.

Next let us prove $p, \varphi_x(p) \in \widetilde{U_x(p_x)}$ $(x \in X, p_x \in R)$ implies $\varphi_x(p) \rightarrow$. For any $y \in X$, $y' \geqq y$ implies $p_{y'} \in U_y^2(p_y)$ from $U_y(p_y) \frown U_{y'}(p_{y'}) \neq \phi$. Hence $\varphi_{y'}(p) \in \widetilde{U_{y'}(p_{y'})} \subseteqq \widetilde{U_y^3(p_y)} \subseteqq (\widetilde{U_y^4(U_x(p^{\alpha(x)})}))$ for every $x \in X$ from the fact that $p \in \widetilde{U_y(p_y)}$, *i.e.* $\widetilde{U_y(p_y)}$ is a neighbourhood of p in S and accordingly $U_x(p^{\alpha(x)}) \frown U_y(p_y) \neq \phi$. (For $\alpha \geqq \alpha'$ implies $p^{\alpha} \in U_x(p^{\alpha(x)}) \frown U_y(p_y)$ for some $\alpha' \in A_p$.) Therefore taking $y \in X$ for an arbitrary $x \in X$ such that $U_y^4(q) \subseteqq U_x(q)$ $(q \in R)$, we get $\varphi_{y'}(p) \in \widetilde{U_x^2(p^{\alpha(x)})}$ for $y' \geqq y$; hence $\varphi_x(p) \rightarrow$. If $\varphi_{\alpha}(p) \rightarrow$, $\varphi_{\alpha}'(p) \rightarrow (\alpha \in A)$ and if $p_{\beta} \in S$ $(\beta \in B)$ for a cofinal B in A, then for every $x \in X$ there exists $\beta(x) \in B$ such that $\beta \geqq \beta(x)$ implies $\varphi_{\beta}(p_{\beta}), \varphi'_{\beta}(p_{\beta}) \in \widetilde{U_x^2(p_{\beta}^{\alpha(x)})}$. Hence from the above fact we get the condition $D_2)$.

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15) U^c denotes the complement of U in R, and $(\overline{U^c})^c$ denotes the complement of $\overline{U^c}$ in S. $\overline{U^c}$ denotes the closure of U^c in S.

Next if $\varphi_\gamma(p) \nrightarrow$, then there exist $x \in X$, D cofinal in C and $p_\delta \in S$ such that $\varphi_\delta(p_\delta) \notin \widetilde{U_x^2(p_\delta^{\alpha(x)})}$ $(\delta \in D)$. Hence for this D and p_δ $(\delta \in D)$ the star condition of D_3) does not hold. Lastly, $\varphi_\gamma(p) \dashrightarrow$ implies the equivalence of $\varphi_\gamma(p_0)$ with p_0^α for each $p_0 \in S$, and hence from ii) we get $\varphi_\gamma(p_0) \rightarrow p_0$. It is obvious that this uniform topology is an extension of that of R.