Journal of the Institute of Polytechnics,
Osaka City University,
Vol. 4, No. 1, Series A

On relations between lattices of finite uniform coverings of a metric space and the uniform topology of the space

By Jun-iti NAGATA

(Received July 10, 1953)

We characterized a complete uniform space by the lattice of uniform coverings satisfying some two conditions in the previous paper.[1] But for simplicity of the theory it is desirable to use a lattice consist of finite uniform coverings only. In the case of a totally bounded space the possibility of such a restriction is obvious.

In the case of a metric space the totality of finite uniform coverings are not uniform basis generally, but then we can use a lattice of finite uniform coverings for characterizing its uniform topology. In this paper we shall show that a lattice of finite uniform coverings of a complete metric space characterizes the uniform topology and that in the case of a general metric space the lattice characterizes the completion of the space.

We concern ourselves with a lattice $L(R)$ consist of open finite uniform coverings of a complete metric space R satisfying the following conditions,

1) *if* $\mathfrak{U}, \mathfrak{V} \in L(R)$, *then* $\mathfrak{U} \vee \mathfrak{V} \in L(R)$,
2) *if* U, V *are some open sets such that* $U \frown V = \phi$, $V \neq \phi$, *then there exists* $\mathfrak{M} \in L(R)$ *such that* $U \in \mathfrak{M}$, $V \notin \mathfrak{M}$,
3) $L(R)$ *is a basis of the totallity of finite uniform coverings of* R.[2]

Remarks. The order $\mathfrak{U} < \mathfrak{V}$ between elements of $L(R)$ is the relation that $\mathfrak{U}$ is refiner than $\mathfrak{V}$. We denote by $\mathfrak{U} \vee \mathfrak{V}$ the uniform covering $\{W \mid W \in \mathfrak{U}$ or $W \in \mathfrak{V}\}$. In $L(R)$ we regard two equivalent coverings[3] as the same element. Hence the notation $U \in \mathfrak{M}$ means the fact that for some $U' \supset U$, $U' \in \mathfrak{M}$ holds. In condition 2) we assume implicitly that R has no isolated points.

Definition. We denote by $U < V$ the fact that $V \in \mathfrak{M} \in L(R)$ implies $U \in \mathfrak{M}$.

Definition. We mean by a *max. family for* $\mathfrak{U}(\in L(R))$ a subset μ of $L(R)$ having the property that $\mathfrak{P}_i \in \mu$ $(i = 1 \cdots k)$ imply $\mathfrak{U} \not< \bigvee_{i=1}^{k} \mathfrak{P}_i$ and for every $\mu' \supsetneqq \mu$ this condition does not hold.

---

1) On Uniform Homeomorphism between two Uniform Spaces, this journal Vol. 3, No. 1-2, 1952.

2) If for every element $\mathfrak{A}$ of a family A of coverings of R there exists $\mathfrak{U} \in L(R)$ such that $\mathfrak{U} < \mathfrak{A}$, then we call $L(R)$ a basis of A.

3) If $\mathfrak{U} < \mathfrak{V}$, $\mathfrak{V} < \mathfrak{U}$ hold, then we say that $\mathfrak{U}$ and $\mathfrak{V}$ are equivalent.

Lemma 1. *In order that a subset μ of $L(R)$ is a max. family for $\mathfrak{U}$ it is necessary and sufficient that $\mu=\{\mathfrak{M}\mid\cdot U\notin\mathfrak{M}\in L(R)\}$ for some $U\in\mathfrak{U}$ such that $V\in\mathfrak{U}$, $V>U$ imply $U>V$.*

Proof. Let $\mu=\{\mathfrak{M}\mid U\notin\mathfrak{M}\}$, $U\in\mathfrak{U}$, and let $V\in\mathfrak{U}$, $V>U$ imply $U>V$. If $\mathfrak{P}_i\in\mu$ $(i=1,\cdots,k)$, then from $U\notin\bigvee_{i=1}^{k}\mathfrak{P}_i$ we get $\mathfrak{U}\not<\bigvee_{i=1}^{k}\mathfrak{P}_i$.

Next if $\mathfrak{N}\notin\mu$, then there exists $N\in\mathfrak{N}$ such that $N\supset U$. We denote by V_i $(i=1,\cdots.l)$ all the elements of $\mathfrak{U}$. If $V_i\not>U$ $(i=1\cdots l)$, then there exists $\mathfrak{B}_i\in L(R)$ such that $V_i\in\mathfrak{B}_i$, $U\notin\mathfrak{B}_i$; hence $\mathfrak{B}_i\in\mu$ $(i=1\cdots l)$. If $V_i>U$, then from the property of U, $U>V_i$ holds. Since $U\in\mathfrak{N}$, we get $V_i\in\mathfrak{N}$. Therefore we get $\mathfrak{U}<(\bigvee_{i=1}^{l}\mathfrak{B}_i)\vee\mathfrak{N}$, $\mathfrak{B}_i\in\mu$ $(i=1\cdots l)$, *i.e.* μ is a max. family.

In the contrary, let μ be a max. family for $\mathfrak{U}$, then there exists $U\in\mathfrak{U}$ such that $U\notin\mathfrak{P}_\alpha$ for all $\mathfrak{P}_\alpha\in\mu$. Since $\mathfrak{U}$ is a finite covering, there exists some $V\in\mathfrak{U}$ such that $V>U$; $\mathfrak{U}\ni V'>V$ implies $V>V'$. Since $U\notin\mathfrak{P}_\alpha$ for all α, $V\notin\mathfrak{P}_\alpha$ holds for all α, too. Hence we get $\mu\subset\{\mathfrak{M}\mid V\notin\mathfrak{M}\}$. Therefore from the maximum property of μ we get $\mu=\{\mathfrak{M}\mid V\notin\mathfrak{M}\}$.

Definition. We mean by a *chauchy sequence of* $L(R)$ a sequence $\{\mu_n\mid n=1,2,\cdots\}$ of max. families of $L(R)$ such that $\mu_n\supset\mu_{n+1}$, and for every $\mathfrak{U}\in L(R)$ and for some μ_n, $\mathfrak{U}\notin\mu_n$ holds.

Remarks. By lemma 1 let us assume that $\mu_n=\{\mathfrak{M}\mid U_n\notin\mathfrak{M}\}$ $(n=1,2,\cdots)$. In order that $\mu_n\supset\mu_{n+1}$ it is necessary and sufficient that $U_n>U_{n+1}$. We note that the last formula implies $U_{n+1}\subset\bar{U}_n$. For in the contrary case we get from the condition 2) of $L(R)$ an element $\mathfrak{U}$ of $L(R)$ such that $U_n\in\mathfrak{U}$, $U_{n+1}-\bar{U}_n\notin\mathfrak{U}$, and accordingly $U_{n+1}\notin\mathfrak{U}$. This consequence contradicts the fact that $U_{n+1}<U_n$.

Lemma 2. *If $\mu_n=\{\mathfrak{M}\mid U_n\notin\mathfrak{M}\}$ $(n=1,2,\cdots)$, then in order that $\{\mu_n\mid n=1,2,\cdots\}$ is a chauchy sequence of $L(R)$ it is necessary and sufficient that $\{U_n\mid n=1,2,\cdots\}$ is a chauchy sequence*[4] *of R.*

Proof. Since $U_n\in\mathfrak{U}$ implies $\mathfrak{U}\notin\mu_n$, the sufficiency of the condition is obvious.

Now assume that $\{U_n\mid n=1,2,\cdots\}$ is no chauchy sequence of R, and assume that $U_n\not\subset S_m(x)$ for all n and for all $x\in R$, where $S_m(x)=\{y\mid\rho(x.y)<1/2^m\}$; ρ is the distance between x and y. Then there exist $x_1, y_1\in U_1=U_{n_1}$ such that $y_1\notin S_m(x_1)$. If $S_{m+1}(x_1)\frown U_n\neq\phi$ for all n, then for the uniform covering $\mathfrak{M}=\{\overline{(S_{m+1}(x_1))}^c, S_m(x_1)\}$[5] we can take a refinement $\mathfrak{U}\in L(R)$ of $\mathfrak{M}$ by condition 3) of $L(R)$. Since $U_n\notin\mathfrak{M}$ for all n, $U_n\notin\mathfrak{U}$ hold for all n; hence $\mathfrak{U}\in\mu_n$, and

4) We mean by a chauchy sequence of R a sequence $U_n(n=1,2,\dots)$ of open sets of R such that $U_u>U_{n+1}$, and the diameters of U_n tend to zero.

5) We denote by A^c the complement of A. Since $\{S_{m+2}(x)\mid x\in R\}<\mathfrak{M}$, $\mathfrak{M}$ is a uniform covering of R.

hence $\{\mu_n\}$ is no chauchy sequence of $L(R)$. In the case that $S_{m+1}(y_1)\frown U_n \neq \phi$ for all n, we see analogously that $\{\mu_n | n=1,2,\cdots\}$ is no chauchy sequence of $L(R)$.

If $S_{m+1}(x_1)\frown U_{n'}=\phi$, $S_{m+1}(y_1)\frown U_{n''}=\phi$, then for $n \geqq max(n', n'')=n_2$ from $U_n \subset \bar{U}_{n'}$, $U_n \subset \bar{U}_{n''}$ we get $S_{m+1}(x_1)\frown U_n=\phi$ and $S_{m+1}(y_1)\frown U_n=\phi$. Then we can take x_2, $y_2 \in U_{n_2}$ such that $S_m(x_2) \not\ni y_2$. If $S_{m+1}(x_2)\frown U_n \neq \phi$ for all n or $S_{m+1}(y_2)\frown U_n \neq \phi$ for all n, then we can conclude that $\{\mu_n | n=1,2,\cdots\}$ is no chauchy sequence of $L(R)$ as in the previous manner. In the contrary case $S_{m+1}(x_2)\frown U_n=\phi$, $S_{m+1}(y_2)\frown U_n=\phi$ hold for some n_3 and for all $n \geqq n_3$. Then we take x_3, $y_3 \in U_{n_3}$ such that $y_3 \notin S_m(x_3)$. By an inductive consideration we get the conclusion that $\{\mu_n | n=1,2,\cdots\}$ is no chauchy sequence of $L(R)$ or the conclusion that there exists a sequence x_i, y_i $(i=1,2,\cdots)$ of points of R such that x_i, $y_i \in U_{n_i}$; $y_i \notin S_m(x_i)$, $S_{m+1}(x_i)\frown U_{n_j}=\phi$, $S_{m+1}(y_i)\frown U_{n_j}=\phi$ $(j \geqq i+1)$.

In the last case we get a finite uniform covering $\mathfrak{M}=\{\bigcup_{i=1}^{\infty} S_{m+1}(x_i),\ R-\bigcup_{i=1}^{\infty} x_i\}$, for which $U_{n_i} \notin \mathfrak{M}$ hold for all i. For $x_i \in U_{n_i}$ implies $U_{n_i} \not\subset R-\bigcup_{i=1}^{\infty} x_i$, and $y_i \notin \bigcup S_{m+1}(x_i)$ combining with $y_i \in U_{n_i}$ implies $U_{n_i} \not\subset \bigcup_{i=1}^{\infty} S_{m+1}(x_i)$. By the condition 3) of $L(R)$, we take $\mathfrak{U}$ such that $\mathfrak{M} > \mathfrak{U} \in L(R)$. Then for an arbitrary U_n, $n_i \geqq n$ implies $U_{n_i} < U_n$; hence from $U_{n_i} \notin \mathfrak{U}$ we conclude that $U_n \notin \mathfrak{U}$ Therefore $\mathfrak{U} \in \mu_n$ for all n, *i.e.* $\{\mu_n | n=1,2,\cdots\}$ is no chauchy sequence of $L(R)$ also in this case.

Definition. We denote by $\{\mu_n | n=1,2,\cdots\} \sim \{\nu_n | n=1,2,\cdots\}$ the relation between two chauchy sequences of $L(R)$ such that for every $\mathfrak{U} \in L(R)$ there exist two elements μ_n, ν_n of the sequence and some max. family λ such that $\lambda \supset \mu \cup \nu$, $\mathfrak{U} \notin \lambda$.

Lemma 3. *In order that* $\{\mu_n | q=1,2,\cdots\} \sim \{\nu_n | n=1,2,\cdots\}$ *it is necessary and sufficient that* $\{U_n | n=1,2,\cdots\}$ *and* $\{V_n | n=1,2,\cdots\}$ *are equivalent chauchy sequences of* R, *where* $\mu_n=\{\mathfrak{M} | U_n \notin \mathfrak{M}\}$, $\nu_n=\{\mathfrak{M} | V_n \notin \mathfrak{M}\}$.

Proof. The sufficiency of the condition is obvious.

If $\{U_n\}$ and $\{V_n\}$ are not equivalent in R, then for some m $U_n \cup V_n \not\subset S_m(x)$ hold for all n and for all $x \in R$. Hence in the same way as in the previous proof we get $\mathfrak{U} \in L(R)$ such that $U_n \cup V_n \notin \mathfrak{U}$ for all n. Take $\mathfrak{V} \in L(R)$ such that $\bar{\mathfrak{V}}=\{\bar{V} | V \in \mathfrak{V}\} < \mathfrak{U}$. If $\mathfrak{V} \notin \lambda$ for some max. family $\lambda=\{\mathfrak{M} | W \notin \mathfrak{M}\}$, and if $\lambda \supset \mu_n \cup \nu_n$, then $U_n < W$, $V_n < W$; hence from $W \in \mathfrak{V}$, $U_n \cup V_n \subset \bar{W} \subset U \in \mathfrak{U}$, but this is impossible. Therefore the negation of $\{\mu_n | n=1,2,\cdots\} \sim \{\nu_n | n=1,2,\cdots\}$ holds.

From lemma 3 we can classify all the chauchy sequences of $L(R)$ by the relation $\sim$. We denote by $\mathfrak{L}(R)$ the set of all such classes. From this lemma and the completeness of R we get a one-to-one correspondence between R and

$\mathfrak{L}(R)$; hence we denote by $\mathfrak{L}(A)$ the image of a subset A of R in $\mathfrak{L}(R)$ by this correspondence.

Definition. We mean by a *uniform covering of* $\mathfrak{L}(R)$ a covering $\{\mathfrak{L}(U'_\alpha)|\alpha\in A\}$ of $\mathfrak{L}(R)$ such that there exists a definite covering $\{\mathfrak{L}(U_\alpha)\}$: $\{\mathfrak{L}(U_\alpha)\}^{\Delta *}<\{\mathfrak{L}(U'_\alpha)\}$[6] and for an arbitrary binary covering $\{\mathfrak{L}(U), \mathfrak{L}(V)\}>\{\mathfrak{L}(U_\alpha)\}$, there exists $\mathfrak{U}\in L(R)$ such that $\mu_n\in\{\mu_n|n=1,2,\cdots\}\notin\mathfrak{L}(U)$, $\nu_m\in\{\nu_n|n=1,2,\cdots\}\notin\mathfrak{L}(V)$ imply $\mathfrak{U}<\mathfrak{U}'{}^\vee V'$ for some $\mathfrak{U}'\in\mu_n$ and $\mathfrak{V}'\in\nu_m$.

Lemma 4. *In order that* $\{\mathfrak{L}(U_\alpha')\}$ *is a uniform covering of* $\mathfrak{L}(R)$ *it is necessary and sufficient that* $\{U'_\alpha\}$ *is a uniform covering of R.*

Proof. *Sufficiency.* Let $\{U'_\alpha\}$ be a uniform covering of R, then there exists a uniform covering $\{U_\alpha\}$ of R such that $\{U_\alpha\}^{\Delta *}<\{U_\alpha'\}$, *i.e.* $\{\mathfrak{L}(U_\alpha)\}^{\Delta *}<\{\mathfrak{L}(U_\alpha')\}$. If $\{\mathfrak{L}(U), \mathfrak{L}(V)\}$ is an arbitrary binary covering of $\mathfrak{L}(R)$ such that $\{\mathfrak{L}(U), \mathfrak{L}(V)\}>\{\mathfrak{L}(U_\alpha)\}$, then since $\{U_\alpha\}<\{U,V\}$ in R, $\{U,V\}$ is a binary uniform covering of R. Hence from condition 3) of $L(R)$ there exists $\mathfrak{U}\in L(R)$ such that $\bar{\mathfrak{U}}<\{U,V\}$. If $\mu_n\in\{\mu_n\}\notin\mathfrak{L}(U)$, $\nu_m\in\{\nu_n\}\notin\mathfrak{L}(V)$ and if $\mu_n=\{\mathfrak{M}|U_n\notin\mathfrak{M}\}$, $\nu_m=\{\mathfrak{M}|V_m\notin\mathfrak{M}\}$, then $\{U_n\}$ converges to $a\notin U$ and $\{V_n\}$ converges to $b\notin V$. Let $U'\in\mathfrak{U}$, then from $\bar{\mathfrak{U}}<\{U,V\}$, $\bar{U}'\subset U$ or $\bar{U}'\subset V$ holds. If $\bar{U}'\subset U$, then from $a\notin U$ and from $a\in\bar{U}_n$ we get $\bar{U}'\not\supset U_n$. Hence from condition 2) of $L(R)$ there exists $\mathfrak{U}(U')\in L(R)$ such that $U'\in\mathfrak{U}(U')$, $U_n\notin\mathfrak{U}(U')$. If $\bar{U}'\subset V$, then analogously there exists $\mathfrak{U}(U')$ such that $U'\in\mathfrak{U}(U')$, $V_m\notin\mathfrak{U}(U')$. Hence ${}^\vee\{\mathfrak{U}(U')|\bar{U}'\subset U\}=\mathfrak{U}'\in\mu_n$, ${}^\vee\{\mathfrak{U}(U')|\bar{U}'\subset V\}=\mathfrak{V}'\in\nu_m$ and $\mathfrak{U}<\mathfrak{U}'{}^\vee\mathfrak{V}'$. Therefore $\{\mathfrak{L}(U_\alpha')\}$ is a uniform covering of $\mathfrak{L}(R)$ by the above definition.

Necessity. Assume that $\{U_\alpha'\}$ is no uniform convering of R and that $\{\mathfrak{L}(U_\alpha)\}^{\Delta *}<\{\mathfrak{L}(U_\alpha')\}$, then $\{U_\alpha\}^{\Delta *}<\{U_\alpha'\}$. We denote by $\mathfrak{S}_n$ the uniform covering $\{S_n(x)|x\in R\}$ of R. Putting $\mathfrak{A}=\{U_\alpha'\}$, for every n we get $S_n\in\mathfrak{S}_n$ $(n=1,2,\cdots)$ such that $S_n\notin\mathfrak{A}^{\Delta *}$. For this S_1 we take $x_1, y_1\in S_1$ such that $y_1\notin S^2(x_1, \mathfrak{A})$[7]. If $S(x_1, \mathfrak{A})\frown S_{n_i}\neq\phi$ hold for an infinite number of n_i $(i=1,2,\cdots)$ then for $x_{n_i}\in S(x_1, \mathfrak{A})\frown S_{n_i}$ $(i=1,2,\cdots)$, $\mathfrak{A}'=\{S^2(x_1, \mathfrak{A}), R-\bigcup_{i=1}^{\infty}x_{n_i}\}$ is a binary covering of R such that $\mathfrak{A}<\mathfrak{A}'$. Since $\mathfrak{S}_{n_i}\ni S_{n_i}\notin\mathfrak{A}'$ $(i=1,2,\cdots)$, $\mathfrak{A}'$ is no uniform covering of R.[8] If $S(y_1, \mathfrak{A})\frown S_{n_i}\neq\phi$ hold for an infinite number of n_i, then analogously there exsists a binary non-uniform covering $\mathfrak{A}'$ of R such that $\mathfrak{A}<\mathfrak{A}'$.

If $n\geqq n_2$ implies $S(x_1, \mathfrak{A})\frown S_n=\phi$ and $S(y_1, \mathfrak{A})\frown S_n=\phi$ for some n_2, then

6) This notation is due to J. W. Tukey, Convergence and Uniformity in topology, 1940.

7) $S(x_1, \mathfrak{A})={}^\cup\{A|x_1\in A\in\mathfrak{A}\}$, $S^2(x_1, \mathfrak{A})=S(S(x_1, \mathfrak{A}), \mathfrak{A})={}^\cup\{A|A\frown S(x_1, \mathfrak{A})\neq\phi, A\in\mathfrak{A}\}$. See J. W. Tukey, loc. cit.

8) For $S_n\notin\mathfrak{A}^{\Delta *}$ implies $S_{n_i}\not\subset S^2(x_1, \mathfrak{A})$, and $x_{n_i}\in S_{n_i}$ implies $S_{n_i}\not\subset R-\bigcup_{i=1}^{\infty}x_{n_i}$.

we take $x_2, y_2 \in S_{n_2}$ such that $y_2 \notin S^2(x_2, \mathfrak{A})$. For these $x_2, y_2; S_{n_2}$ in the same way as for $x_1, y_1, S_{n_2} := S_1$, we get a binary non-uniform covering $\mathfrak{A}$ of R such that $\mathfrak{A} < \mathfrak{A}'$ or $x_3, y_3; S_{n_3}$ $(n_3 > n_2)$ such that $x_3, y_3 \in S_{n_3}$; $S(x_2, \mathfrak{A}) \cap S_n = \phi$, $S(y_2, \mathfrak{A}) \cap S_n = \phi$ $(n > n_3)$, $y_3 \notin S^2(x_3, \mathfrak{A})$. By such an argument we get a binary non-uniform covering $\mathfrak{A}$ of R such that $\mathfrak{A} < \mathfrak{A}'$ or points x_i, y_i $(i = 1, 2, \cdots)$ of R such that $x_i, y_i \in S_{n_i}$; $x_i \notin S(y_j, \mathfrak{A})$, $y_i \notin S(x_j, \mathfrak{A})$. In the latter case, we get a binary covering $\mathfrak{A}' = \{\bigcup_{i=1}^{\infty} S(x_i\ \mathfrak{A}), R - \bigcup_{i=1}^{\infty} x_i\}$. For this $\mathfrak{A}'$ $\mathfrak{A} < \mathfrak{A}'$ is obvious. Since $x_i \in S_{n_i}$, $S_{n_i} \not\subset R - \bigcup_{i=1}^{\infty} x_i$. From $y_i \in S_{n_i}$ and from $y_i \notin S(x_j, \mathfrak{A})$ for all j, $S_{n_i} \not\subset \bigcup_{i=1}^{\infty} S(x_i\ \mathfrak{A})$ holds. Hence $\mathfrak{S}_{n_i} \not< \mathfrak{A}'$. Since this formula holds for every i, $\mathfrak{A}'$ is no uniform covering of R. Therefore in every case we get a binary non-uniform covering $\mathfrak{A}'$ such that $\mathfrak{L}(\mathfrak{A}) < \mathfrak{L}(\mathfrak{A}')$.

Let $\mathfrak{U}$ be an arbitrary uniform covering in $L(R)$, then $\mathfrak{U} \not< \mathfrak{A}'$ holds for this $\mathfrak{A}'$, *i.e.* there exists $U \in \mathfrak{U}$ such that $U \not\subset A, B$ for both elements A, B of $\mathfrak{A}'$. Take x, y so that $x \in U \cap A^c$, $y \in U \cap B^c$, and let $L(x) = \{\mu_n | n = 1, 2, \cdots\}$, $\mathfrak{L}(y) = \{\nu_m | m = 1, 2, \cdots\}$; $\mu_n = \{\mathfrak{M} | U_n \notin \mathfrak{M}\}$, $\nu_m = \{\mathfrak{M} | V_m \notin \mathfrak{M}\}$, then since $\{U_n\}$, $\{V_n\}$ converge to x, y respectively in R, there exist U_n, V_n such that $U_n \subset U$ $V_n \subset U$. For every $\mathfrak{U}' \in \mu_n$, $\mathfrak{B}' \in \nu_n$ we get $U_n \notin \mathfrak{U}'$, $V_n \notin \mathfrak{B}'$. Combining these formulas with the above $U_n \cup V_n \subset U$, we get $U \notin \mathfrak{U}'$, $\mathfrak{B}'$. Hence $\mathfrak{U} \not< \mathfrak{U}' \vee \mathfrak{B}'$ for such $\mathfrak{U}'$ $\mathfrak{B}'$. Therefore $\{\mathfrak{L}(U_\alpha')\}$ is no uniform covering of $\mathfrak{L}(R)$ by the above definition.

By this lemma R and $\mathfrak{L}(R)$, the uniform space having the above defined uniform coverings are uniformly homeomorphic. Since points and uniform coverings of $\mathfrak{L}(R)$ are defined by elements of $L(R)$ and by relations $<$ between elements of $L(R)$, we get the following theorem.

Theorem 1. *In order that two complete metric spaces R_1 and R_2 are uniformly homeomorphic it is necessary and sufficient that $L(R_1)$ and $L(R_2)$ are lattice-isomorphic, where $L(R_1)$, $L(R_2)$ are lattices of finite uniform coverings of R_1, R_2 respectively and satisfy conditions 1), 2), 3),*

Next we concern ourselves with a metric space having no completeness property. We denote by $L_f(R)$ the lattice of all finite uniform coverings of R. We define max. family of $L_f(R)$ as in the above proof of Theorem 1, and we mean by chauchy sequence of $L_f(R)$ a sequence of max. families of $L_f(R)$, $\{\mu_n | n = 1, 2, \cdots\}$ satisfying besides the above conditions the condition that there exists a max. family μ such that $\mu \not\supset \mu_n$ for all n, and $\nu \supsetneqq \mu$ is not valid but $\nu = \{\mathfrak{M} | R \notin \mathfrak{M}\}$. Thus we can characterize a converging chauchy sequence of R by such a chauchy sequence of $L(R)$ and by an analogous argument to the case of complete metric space we get the following,

Corollary. *In order that two metric spaces R_1, R_2 are uniformly homeomorphic it is necessary and sufficient that lattices $L_f(R_1)$, $L_f(R_2)$ of all finite* *uniform coverings of* R_1, R_2 *respectively are lattice-isomorphic.*

This corollary is obvious for totally bounded uniform spaces R_1, R_2, too.

Next let us consider relations between $L(R)$ and the completion $\widetilde{R}$ of R.

Theorem 2. *If* R_1, R_2 *are metric spaces and if* $\widetilde{R_1}$, $\widetilde{R_2}$ *are the completions of* R_1, R_2 *respectively, then in order that* $\widetilde{R_1}$ *and* $\widetilde{R_2}$ *are uniformly homeomorphic it is necessary and sufficient that lattices of finite uniform coverings,* $L(R_1)$, $L(R_2)$ *satisfying conditions 1), 2), 3) are lattice-isomorphic.*[9]

Proof. For each $\mathfrak{U} = \{U_\alpha\} \in L(R_1)$ we denote by $\widetilde{\mathfrak{U}}$ the uniform covering $\{(\widetilde{U_\alpha^c})^k | U_\alpha \in \mathfrak{U}\}$ of $\widetilde{R_1}$, where U^c, U^k, $\widetilde{U}$ mean complement in R_1, complement in $\widetilde{R_1}$, closure in $\widetilde{R_1}$ respectively. Putting $\widetilde{L(R_1)} = \{\widetilde{\mathfrak{U}} | \mathfrak{U} \in L(R_1)\}$, we see easily that $L(R_1)$ and $\widetilde{L(R_1)}$ are isomorphic. For $\widetilde{\mathfrak{U}} \ni U \subset V \in \mathfrak{V}$ implies $(\widetilde{U^c})^k \subset (\widetilde{V^c})^k$; hence $\mathfrak{U} < \mathfrak{V}$ implies $\widetilde{\mathfrak{U}} < \widetilde{\mathfrak{V}}$. If $\widetilde{\mathfrak{U}} < \widetilde{\mathfrak{V}}$, then for all $U \in \mathfrak{U}$ there exists $V \in \mathfrak{V}$ such that $(\widetilde{U^c})^k \subset (\widetilde{V^c})^k$; hence $(\widetilde{U^c})^k \cap R_1 = U \subset V = (\widetilde{V^c})^k \cap R_1$. Therefore $\mathfrak{U} < \mathfrak{V}$. Since $\widetilde{\mathfrak{U}} \vee \widetilde{\mathfrak{V}} = \widetilde{\mathfrak{U} \vee \mathfrak{V}}$ is obvious and since $L(R_1)$ satisfies condition 1), $\widetilde{L(R_1)}$ satisfies condition 1). If U', V' are open sets in $\widetilde{R_1}$ such that $V' \neq \phi$, $U' \cap V' = \phi$, then denoting $U' \cap R_1 = U$, $V' \cap R_1 = V$, we get $U \cup V = \phi$, $V \neq \phi$. Hence there exists $\mathfrak{U} \in L(R_1)$, for which $U \subset U_0$ for some $U_0 \in \mathfrak{U}$ and $V \not\subset U_\alpha$ for every $U_\alpha \in \mathfrak{U}$. Then from $U_0^c \subset U^c \subset U'^k$ we get $\widetilde{U_0^c} \subset U'^k$, and hence $U' \subset (\widetilde{U_0^c})^k \in \widetilde{\mathfrak{U}}$. $V' \not\subset \widetilde{U_\alpha}$ for every $\widetilde{U_\alpha} \in \widetilde{\mathfrak{U}}$ is obvious from $V \not\subset U_\alpha$. Thus $\widetilde{L(R_1)}$ satisfies condition 2) in R_1.

Next we shall show that $\widetilde{L(R_1)}$ satisfies condition 3) in $\widetilde{R_1}$. Let $\{U_i | i = 1, \cdots, k\}$ be an arbitrary finite uniform covering of $\widetilde{R_1}$, then taking a uniform covering $\mathfrak{S}$ of $\widetilde{R_1}$ such that $\mathfrak{S}^{**} < \{U_i\}$, we get open sets $G_i = \cup\{S | S' \cap R_1 = S \supset F \in \mathfrak{F}$ for some $S' \in \mathfrak{S}$ and for some $\mathfrak{F}, F$ such that $F \in \mathfrak{F} \in U_i^k\}$ of R_1, where $\mathfrak{F}$ is a maximum chauchy filter of closed sets of R_1, and $\mathfrak{F}$ is also a point of $\widetilde{R_1}$. For example, $F \in \mathfrak{F}$ means that the filter $\mathfrak{F}$ of R_1 contains the subset F of R_1, and $\mathfrak{F} \in U$ means that the point $\mathfrak{F}$ of $\widetilde{R_1}$ is contained in the subset U of $\widetilde{R_1}$. Now we show that $\mathfrak{U} = \{G_i^c\} > \mathfrak{S}$ in R_1, where $\mathfrak{U}$ is not an open covering generally. Assume the contrary and assume that $S \in \mathfrak{S}$, $S' = S \cap R_1$, $S' \cap G_i \neq \phi$ $(i = 1, \cdots, k)$, then there exist open sets S_i of $\widetilde{R_1}$ and maximum chauchy filters $\mathfrak{F}_i$ of R_1 such that $S \cap (S_i \cap R_1) \neq \phi$, $\mathfrak{S} \ni S_i \supset S_i \cap R_1 \supset F_i \in \mathfrak{F}_i \in U_i^k$. For these $\mathfrak{F}_i$ taking $S_i' \in \mathfrak{S}$ such that $\mathfrak{F}_i \in S_i'$, we see easily that $S_i \cap S_i' \neq \phi$. For $S_i \subset S_i'^k$ combining with $S_i \supset F_i \in \mathfrak{F}_i$ implies $\mathfrak{F}_i \in \overline{S_i} \subset S_i'^k$, which contradicts the fact that $\mathfrak{F}_i \in S_i'$. Therefore $S_i' \subset S^2(S, \mathfrak{S})$ from

---

9) The completion $\widetilde{R}$ of R consists of all the maximum chauchy filters of closed sets of R. The topology of $\widetilde{R}$ is defined by the closed basis $\{\widetilde{F} | \widetilde{F} = \{\mathfrak{F} | \mathfrak{F} \ni F\}$, F is closed subset of $R\}$. The uniform topology of $\widetilde{R}$ is defined by the uniform coverings $\widetilde{\mathfrak{U}} = \{(\widetilde{U^c})^k | U \in \mathfrak{U}\}$ for uniform coverings $\mathfrak{U}$ of R.

$S \cap S_i \neq \phi$. Therefore $\mathfrak{F}_i \notin U_i$, $\mathfrak{F}_i \in S_i' \subset S^2(S, \mathfrak{S})$; hence $S^2(S, \mathfrak{S}) \not\subset U_i$ $(i = 1, \cdots, k)$, but this contradicts the fact that $\mathfrak{S}^{**} < \{U_i\}$. This contradiction proofs the validity of $\mathfrak{S} < \mathfrak{U}$ in R_1. Hence $\mathfrak{U}$ is a finite uniform covering of R_1 and hence we can take $\mathfrak{V} \in L(R_1)$ such that $\mathfrak{V} < \mathfrak{U}$. Let $V \in \mathfrak{V}$ and let $V \subset G_i^c \in \mathfrak{U}$. If $\mathfrak{F}$ is a maximum chauchy filter of R_1 or a point of $\widetilde{R_1}$ such that $\mathfrak{F} \in U_i^k$ in $\widetilde{R_1}$, then taking $S \in \mathfrak{S}$ such that $S \supset F \in \mathfrak{F}$ for some F, from the definition of G_i we get $R_1 \cap S \subset G_i \subset V^c$. Hence $\mathfrak{F} \in \widetilde{V^c}$, and hence $U_i^k \subset \widetilde{V^c}$, *i.e.* $U_i \supset (\widetilde{V^c})^k$. Since V is an arbitrary element of $\mathfrak{V}$, $\widetilde{\mathfrak{V}} < \{U_i\}$ for $\widetilde{\mathfrak{V}} \in \widetilde{L(R_1)}$. Thus we see that $\widetilde{L(R_1)}$ is a basis of all the finite uniform coverings of $\widetilde{R_1}$, *i.e.* $\widetilde{L(R_1)}$ satisfies 3), too.

If $L(R_1)$ and $L(R_2)$ are isomorphic, then $\widetilde{L(R_1)}$ and $\widetilde{L(R_2)}$ are isomorphic; hence from the above conclusion and from Theorem 1 we get Theorem 2.

For completions of non-metric spaces we get the following propositions by the theorem of my previous paper[10] and by analogous arguements.

Corollary. *If we denote by $\widetilde{R_1}$, $\widetilde{R_2}$ the completions of totally bounded uniform spaces R_1, R_2 respectively, then in order that $\widetilde{R_1}$ and $\widetilde{R_2}$ are uniformly homeomorphic it is necessary and sufficient that lattices $L(R_1)$ and $L(R_2)$ of finite uniform coverings satisfying conditions 1), 2), 3) are lattice-isomorphic.*

Corollary. *If we denote by $\widetilde{R_1}$, $\widetilde{R_2}$ the completions of uniform spaces R_1, R_2 respectively, then in order that $\widetilde{R_1}$ and $\widetilde{R_2}$ are uniformly homeomorphic it is necessary and sufficient that $L(R_1)$ and $L(R_2)$ are lattice-isomorphic, where $L(R_1)$ and $L(R_2)$ are lattices of uniform coverings of R_1 and R_2 respectively and satisfy the following conditions.*

1′) $\mathfrak{U} \in L(R_i)$, $\mathfrak{V} \in L(R_i)$ *imply* $\mathfrak{U} \vee \mathfrak{V} \in L(R_i)$,

2′) *if* $\mathfrak{U} \in L(R_i)$ *and if* $U \neq \phi$ *is an open set of* R_i, *then there exists* $\mathfrak{M} \in L(R_i)$ *such that* $U \notin \mathfrak{M}$; $U^c \supset U' \in \mathfrak{U}$ *implies* $U' \in \mathfrak{M}$,

3′) $L(R_i)$ *is a basis of the totality of uniform coverings of* R_i.

10) Loc. cit.