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On topological groups.

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0. **Introdnction.** In this paper, we shall deal with arbitrary topological groups by means of their *Markoff-extensions*: the definition of an *Markoff-extension* is given in Section 1.

Generally speaking, though the representation theory in matric-algebras plays an important rôle in studying topological groups,[1] it becames occationally meaning-less for some type of groups, which have no usual (non-trivial) representations; as well known, minimally almost periodic groups are those. However, the Markoff-extension seems to be useful for any topological groups.

Section 2 is devoted to an exposition of the relation between the representation of a topological group and those of its Markoff-extension. In Section 3, we shall concern the duality theorem of any topological groups, which we would rather call the *co-duality theorem*. Our theorem coincides with the famous one of Tannaka and Krein[2] in maximally almost periodic cases at all, but even if a group is minimally almost periodic, ours may remain still useful.

This duality theorem is, on the other hand, considered as the representation theorem in B-algebra, and the process from Theorem 4 to Theorem 5 gives one proof for the Tannaka-Krein's duality theorem.

The space of almost periodic functions is considerd as a commutative B*-algebra. This investigation is done in Section 4.

Finally, in Section 5, we shall try the theorem of K. Iwasawa[3] concerning the group-rings as an interesting application of Markoff-extensions.

1. **Preliminary theorem.** We begin with the noted theorem of A. Markoff and S. Kakutani on free topological groups.[4] That is stated as follows: For any completely regular topological space Γ, there exists a free topological group F with the following properties;

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1) Recently, Banach representation theory has been developed as in 13), 19), 20), etc. But in them, groups are restricted in locally compact case.
2) T. Tannaka 24), and M. Krein 10).
3) K. Iwasawa 5) and 6).
4) A. Markoff 11) and S. Kakutani 8).

α) $\Gamma \subset [F]$,

β) Γ generates F algebraically,

γ) any continuous mapping φ of Γ into any topological group $\mathfrak{G}$ is extended up to the continuous homomorphism Φ of F into $\mathfrak{G}$ such that

$$\Phi(x) = \varphi(x) \text{ on } \Gamma,$$

where $[F]$ is the set of all elements of F with the same topology as F.

Now we always assume that G is a topological group, then G is a uniform space with that topology, due to A. Weil,[5] and is completely regular; so is the topological space $[G]$, where the brackets are used in the above sense.

We shall next consider a continuous mapping ϕ of $[G]$ into G such that

$$\phi(x)_{x \in [G]} = x_{x \in G} \quad \text{(identity mapping)},$$

that is,

(1.1) $$\phi([G]) = G.$$

Then according to the Markoff and Kakutani's theorem mentioned above, we can obtain a free topological group F such that;

(1.2) $[G]$ generates F algebraically,

(1.3) $\phi(F) \subset G$,

and

(1.4) $[G] \subset [F]$.

Combining (1.1) with (1.3), we have

$$G = \phi([G]) \subset \phi(F) \subset G,$$

that is,

(1.5) $\phi(F) = G$.

Since F is a free topological group, F is maximally almost periodic (max, a. p.); this fact is due to T. Nakayama.[6] Then we conclude:

Theorem 1.[7] *For any topological group G, there exists at least one max. a. p. topological group G, which has the following properties:*

α) *There exists a continuous homomorphism ϕ of G_0 onto G.*

β) *$[G] \subset G_0$ and $[G]$ is the group-generator of G_0.*

γ) *ϕ is invariant on $[G]$, i.e.*

$$\phi(x) = x, \quad x \in [G].$$

For a given topological group G, we can consider the family of all such G_0 and denote it by Π_G. Then it is certain that Π_G is not empty. For G_1, G_2 of

5) A. Weil 26).

6) T. Nakayama 14).

7) P. Samuel has proved that any topological group is the image of a free topological group; in P. Samuel 17).

Π_G, if G_1 is topologically homomorphic to G_2, one writes $G_2 \geqq G_1$. Thus Π_G forms a partly orderd set by this binary operation $\geqq$, whose greatest extreme is the free topological group.

Each element of Π_G is called an *Markoff-extension* of G, and if $G_0 = G_0'$ for every G_0 with $G_0 \geqq G_0'$, G_0 is called *irreducible*, while the rest *reducible*. A max. a, p. topological group is obviously the irreducible Markoff-extension of itself.

Theorem 2. *The homomorphism ϕ, which is continuous, of an Markoff-extension G_0 of G onto G is further a open mapping.*

Proof. Let H_0 be the kernel of homomorphism ϕ. The natural mapping ϕ_{H_0} from G_0 to G_0/H_0 being topological, *i.e.* continuous and open, an open set $U_0 \subset G_0$ is mapped to an open set $U_{H_0} \subset G_0/H_0$ by ϕ_{H_0} and $V_0 = \phi_{H_0}^{-1}(U_{H_0})$ is also open in G_0.

Putting $V_0 \frown [G] = V_G$, we shall prove that V_G is open in $[G] \subset G_0$. For an arbitrary $x \in V_G$, there exists an open neighborhood $U(x)$ in G and also in $[G]$; while x being an element of V_0, there must be an open set $V(x)$ in V_0.

Since $W(x) = V(x) \frown U(x)$ is not empty, $W(x)$ must be an open set contained in V_G. Thus, V_G is open and hence

$$\phi(U_0) = \phi(V_0) = \phi(V_G)$$

must be open in G.

Corollary. *G_0/H_0 is topologically isomorphic to G, in the symbols*;

(1.6) $$G_0/H_0 \cong G.$$

2. **Representations and $*$-Representations.** Let G_0 be an (irreducible or reducible) Markoff-extension of G and H_0 the kernel of the homomorphism ϕ, $\phi(G_0) = G$, *i.e.*

$$\phi(H_0) = e,$$

where e is the unit of G. We call such H_0 the Markoff-kernel of G_0. Let $H_{(x)}$ be the restclass containg x, while H_x the restclass which corresponds to x, with respect to the factor-group G_0/H_0. Then we have immediately that $H_{(x)} = H_x$.

We shall distinguish the group-operation of G_0 from that of G, writing the former by $x_0 \cdot y_0$ in G_0 and the latter by xy in G, while we denote the inverse of $x_0 \in G_0$ by $x_0^{\cdot -1}$ and that of $x \in G$ in usual way, *i.e.* by x^{-1}. Then we have

(2.1) $$H_x \cdot H_y = H_{x \cdot y} = H_{xy} = H_{(xy)}.$$

We next denote the set of such elements, *differences in a sense*, that

(2.2) $$\partial(x_1, x_2, \dots, x_n) = x_n^{\cdot -1} \cdot x_{n-1}^{\cdot -1} \cdots x_1^{\cdot -1} \cdot x_1 x_2 \dots x_n$$

for $x_k \in G$: $n, k = 1, 2, \dots$, by $\hat{H}_0$, calling it the essential Markoff-kernel of G_0. Obviously, $\hat{H}_0 \subset H_0$. Furthermore we hold;

lemma 1. $\hat{H}_0$ is the generator of H_0.

Proof. $[G]$ being the group-generator of G_0, each $x_0 \in H_0$ is represented as

(2.3) $$x_0 = x_1 \cdot x_2 \ldots x_n ;\ x_k \in G ,$$

and

(2.4) $$x_1 x_2 \ldots x_n = e .$$

Then we see that

(2.5) $$x_0 = (x_1 x_2 \ldots x_n) \cdot (x_n^{\cdot -1} \cdot x_{n-1}^{\cdot -1} \cdots x_1^{\cdot -1} x_1 x_1 x_2 \ldots x_n)^{\cdot -1}$$
$$= e \cdot \delta^{\cdot -1}(x_1, x_2, \ldots, x_n)$$

and $\delta(x_1, x_2, \ldots, x_n) \in \hat{H}_0$; that is, for each $x_0 \in \hat{H}_0$, we have

(2.5) $$x_0 = e \cdot \hat{x}_0^{\cdot -1}, \quad \hat{x}_0 \in \hat{H}_0 .$$

$\hat{H}_0$ being a subset of H_0, we have also

(2,6) $$\hat{x}_0 = e \cdot \hat{x}_0'^{\cdot -1}, \quad \hat{x}_0' \in \hat{H}_0 \ ;\ i.e.\ e = \hat{x}_0 \cdot \hat{x}_0' ,$$

and we see consequently $x_0 = \hat{x}_0 \cdot \hat{x}_0' \cdot \hat{x}_0^{\cdot -1}$. This proves the Lemma.

Let $D(x_0)$, $x_0 \in G_0$, be a continuous (irreducible) unitary -equivalent representation of G_0, *i.e.* a continuous normal representation in the sense of J, von Neumann.[8] Such $D(x_0)$ does not necessarily become a representation of G. Then, we shall investigate the necessary and sufficient condition for $D(x_0)$, in order that it might be a continuous normal representation of G. If $D(x_0)$ is an algebraic representation of G, the continuity of it on G is easily proved. Hence, it is sufficient to restrict our treatments to purely algebraic ones.

Theorem 3. *For a normal (irreducible) representation of G_0, an Markoff-extension of G, the following three conditions are mutually equivalent;*

i) *$D(x_0) = E$ for all $x_0 \in H_0$,*

ii) *$D(x_0) = E$ for all $x_0 \in \hat{H}_0$,*

iii) *$D(x)$ is a normal (irreducible) representation of G,*

where E is unit matrix with same dimension as $D(x_0)$.

Proof of ii)⇄iii.). iii)→ii) is clear. We show that, if $D(x_0)$, $x_0 \in G_0$, has the property ii), $D(x)$, $x \in G$, forms a representation of G, We have easily

$$D(\delta((x, y)) = D(y^{\cdot -1} \cdot x^{\cdot -1} \cdot xy)$$
$$= D(y^{\cdot -1}) D(x^{\cdot -1}) D(xy) = D(y)^{-1} D(x)^{-1} D(xy) = E ,$$

that is,

(2.7) $$D(xy) = D(x) D(y)$$

for every $x, y \in G$, and consequently iii) is satisfied, providing $D(e) = E$,

Proof of i)⇄ii). i)→ii) is clear. If ii) is fulfilled for $D(x_0)$, from Lemma 1, we have for every $x_0 \in H_0$

$$x_0 = \hat{x}_1 \cdot \hat{x}_2 \ldots \hat{x}_n ;\ \hat{x}_k \in H_0 .$$

Then we have,

8) J. von Neunann 15).

$$D(\hat{x}_0) = D(\hat{x}_1)D(\hat{x}_2)\dots D(\hat{x}_n)$$
$$=E\cdot E\dots E = E.$$

Thus ii)→i) is proved. This completes the proof of the theorem.

We now come to the desirable condition, under which the (irreducible) normal representation of G_0 becomes that of G, but we shall pursue the study of representations of G further.

If G has a continuous normal (irreducible) representation $D(x)$, $x \in G$, putting

(2.8) $$D_0(x_0) = D(x) \quad \text{for all } x_0 \in H_x,$$

it is easy to see that such $D_0(x_0)$, $x_0 \in G_0$, is a continuous normal (irreducible) representstion of D_0, and the condition (2.8) is characterised by only condition such that,

(2.9) $$D_0(x_0) = E \quad \text{for all } x_0 \in \hat{H}_0.$$

These facts enable us to establish

Corollary. *A necessary and sufficient condition for a topological group G to have a continuous normal (irreducible) representation is that one of its Markoff-extension G_0 has a continuous normal (irreducible) representation $D_0(x_0)$, such that*

$$D(x_0) = E \quad \textit{on } \hat{H}_0 \ (\textit{or equivalenty on } H_0)$$

where $\hat{H}_0$ is the essential Markoff-kernel of G_0.

Some topological groups have not any non-trivial representations, even when they are locally bicompact or further Lie-groups; minimally a. p. groups are those.

Now we shall define a new operation of matrices. Let G_0 be an Markoff-extension of G and $D(x_0)$ a continuous normal (irreducible) representation of G_0. Markoff-kernel H_0 and H_x $(=x\cdot H_0)$ are defined as above. Then we define

(2.10) $$D(x_0)*D(y_0) = D(x_0)D(y_0)\Delta(x, y),$$

where $x_0 \in H_x$, $y_0 \in H_y$ and

(2.11) $$\Delta(x_1, x_2, \dots, x_n) = D(\delta(x_1, x_2, \dots, x_n)).$$

For any $x, y \in G$, we have immediately

$$D(x)*D(y) = D(x)D(y)\Delta(x, y)$$
$$= D(x)D(y)D(y^{\cdot -1}\cdot x^{\cdot -1}\cdot xy)$$
$$= D(x\cdot y\cdot y^{\cdot -1}\cdot x^{\cdot -1}\cdot xy) = D(xy).$$

Thus $D(x)$ becomes a kind of representation of G with respect to the operation*, and, it is certain, this representation is continuous. We call it a *-representation of G based on G_0.

Theorem 4. *A *-representation $D(x)$ of G based on G_0 coincides with the usual one, if and only if*

i) $$\Delta(x, y) = E \ \textit{for all } x, y \in G,$$

or equivalently

ii) $D(x_0)=E$ *on the(essential or not) Markoff-kerner of* G_0.

All these facts together with the approximation theorem of Weierstrass-Neumann bring us the considerations about a. p. functions on G.

Again, let G_0 be an markoff-extension of G, and $A(G_0)$ the space of all continuous complex-valued a. p. functions on G_0, which becomes a B*-algebra as we see in the following.

If there exists such $f\in A(G_0)$, *i.e.* a. p. function in G_0, that all for $x_0, x_0'\in H_x$ and $x\in G$,

$$f(x_0)=f(x_0'), \tag{2.12}$$

the collection of all such f is denoted by $\hat{A}(G_0)$. It may consist of only constant functions for some G_0, and if $f\in\hat{A}(G_0)$, the translation of f by G, $fa(x_0)=f(a\cdot x_0)$ for $a\in G$, must be contained in $\hat{A}(G_0)$, too.

Now, we put, for $f\in\hat{A}(G_0)$,

$$\tilde{f}(x)=f(x_0)_{x_0\in H_x}. \tag{2,13}$$

Then we have:

Corollary. *For any* $f\in\hat{A}(G_0)$, $\tilde{f}$ *is a. p. on G. Conversely, if f is a. p. on G*, $f_0(x_0)$, $x_0\in G_0$, *which is defined by* (2.13);

$$f_0(x_0)=\tilde{f}(x) \text{ on each } H_x,$$

is a. p. on G_0.

We complehend, consequently, that if $G_1\geqq G_2$ in Π_G, there are no more a. p. functions on G_2 than on G_1. Especially, a. p. functions on G are contained in those on G_0, $G_0\in\Pi_G$, in the above sense.

The totality of $\tilde{f}$, $f\in A(G_0)$, coincides with $A(G)$, *i.e.*

$$\hat{A}(G_0)\cong A(G), \tag{2.14}$$

and if G is min. a. p., $\hat{A}(G_0)$ consists of only constant functions on G_0. With regard to the mean of an a. p. function, we may suppose and easily prove that

$$M_x[\tilde{f}(x)]_G=M_{x_0}[f(x_0)]_{G_0}, \tag{2.15}$$

for every $f\in\hat{A}(G_0)$.

3. **Duality theorem and B-algebra.**[9] Let G be a topological group and G_0 an Markoff-extension of G respectively. $M(G_0)$ is complex B-space of all bounded functions on G_0 with uniform norm $\|f\|=\sup.|f(x_0)|$; $f\in M(G_0)$, $x_0\in G_0$, while $D(G_0)$ a normed subspace of $M(G_0)$ which is the set of all finite linear aggregates of the elements of all irreducible mutually non-equivalent continuous normal representations $D^{\omega}(x_0)=(D^{\omega}_{ij}(x_0))$, *i.e.* the set of all Fourier polynomials of $D^{\omega}_{ij}(x_0)$, where $\{D^{\omega}\}$ is a complete (mutually non-equivalent) representation system of finite degrees of G_0.

9) About B-algebras, see E. Hille 4), W. Ambrose 1), I. Kaplansky 9), etc.

Now we see immediately that $\boldsymbol{M}(G_0)$ itself is regarded as a B–algebra, while $\boldsymbol{D}(G_0)$ a normed subring. It is obvious that both $\boldsymbol{M}(G_0)$ and $\boldsymbol{D}(G_0)$ have the same algebraic unit $\boldsymbol{1}$ with $\| \boldsymbol{1} \| = 1$.

Further, $\boldsymbol{S}$ is the total operator ring on $\bar{\boldsymbol{D}}(G_0)$ with the norm $|S\| = \sup_{\|f\| \leq 1} \| S \cdot f \|$, that is, the set of all bounded linear operators on $\boldsymbol{D}(G_0)$, then $\mathfrak{S}$ itself is also a B–algebra, where $\bar{\boldsymbol{D}}(G_0)$ means the completion of $\bar{\boldsymbol{D}}(G_0)$ by uniform norm in $\boldsymbol{M}(G_0)$; $\bar{\boldsymbol{D}}(G_0)$ coincides with the set of all continuous a. p. functions on G_0, $A(G_0)$. Next, we consider the set of all regular elements S of $\mathfrak{S}$, which have in addition

$$\text{(3.1)} \qquad \begin{aligned} &\text{i)} \quad S(f \cdot g) = S(f) \cdot S(g), \\ &\text{ii)} \quad S(\bar{f}) = \overline{S(f)}, \text{ (bar means the conjugation)} \end{aligned}$$

and denote that by $\mathfrak{S}$. $\mathfrak{S}$ is obviously a subset of $\boldsymbol{S}$, but not a subalgebra. However, $\mathfrak{S}$ forms a group contained in $\boldsymbol{S}$. In fact we have

$$\text{(3.2)} \quad \begin{aligned} S_1 \cdot S_2^{-1}(f \cdot g) &= S_1 \cdot S_2^{-1}(S_2 \cdot S_2^{-1}(f) \cdot S_2 \cdot S_2^{-1}(g)) = S_1 \cdot S_2^{-1} \cdot S_2(S_2^{-1}(f) \cdot S_2^{-1}(g)) \\ &= S_1(S_2^{-1}(f) \cdot S_2^{-1}(g)) = S_1 S_2^{-1}(f) \cdot S_1 S_2^{-1}(g), \end{aligned}$$

$$\text{(3.3)} \quad \begin{aligned} S_1 \cdot S_2^{-1}(\bar{f}) &= S_1 \cdot S_2^{-1}(\overline{S_2 S_2^{-1}(f)}) = S_1 \cdot S_2^{-1} \cdot S_2(\overline{S_2^{-1}(f)}) = \overline{S_1(S_2^{-1} f)} \\ &= \overline{S_1 \cdot S_2^{-1}(f)} \end{aligned}$$

Lemma 2. For each $S \in \mathfrak{S}$, $\| S \| = 1$.

Proof. From (3.1), i), we have for unit function $\boldsymbol{1} \in A(G_0)$

$$\text{(3.4)} \qquad S(\boldsymbol{1}) = \boldsymbol{1},$$

and $1 = \| \boldsymbol{1} \| = \| S(\boldsymbol{1}) \| \leq \| S \|$. On the other hand, for every $f \in A(G_0)$ with $\| f \| \leq 1$, we have

$$\begin{aligned} \| S(f) \| &= \| S(f) \cdot \overline{S(f)} \| = \| S(f) S(\bar{f}) \| \\ &= \| S(f \cdot \bar{f}) \| \leq \| S \| . \end{aligned}$$

and hence $\| S \|^2 \leq \| S \|$, that is $\| S \| \leq 1$. This completes the proof of Lemma. $\mathfrak{S}$ is not void, since every S_{a_0} or ${}_{a_0}S$ is contained in $\mathfrak{S}$, where S_a or ${}_{a}S$ is a translation operator such that

$$\text{(3.5)} \qquad S_{a_0}(f) = = f(x_0 \cdot a_0) \quad \text{resp.} \quad {}_{a_0}S(f) = f(a_0^{-1} \cdot x_0)$$

for a_0, $x_0 \in G_0$. Denoting the totality of such S_{a_0} or ${}_{a_0}S$ by $\mathfrak{S}_{G_0}$ or ${}_{G_0}\mathfrak{S}$, we see that $\mathfrak{S}_{G_0}$ or ${}_{G_0}\mathfrak{S}$ is a group which is algebraically isomorphic to G_0. The isomorphism is directly ontained from the maximal almost-periodicity of G_0. Though $\mathfrak{S}$ has norm-topology, we introduce another topology, *i.e.* a weak topology in $\mathfrak{S}$ such that a neighborhood $U_{s_0}(f_2, \ldots, f_n; \varepsilon)$ is defined as

$$\text{(3.6)} \qquad \begin{aligned} &U_{s_0}(f_1, f_2, \ldots f_n; \varepsilon) \\ &= \{S \mid \| S(f_j) - S_0(f_j) \| < \varepsilon\}, \end{aligned}$$

for $j = 1, 2, \ldots, n$.

From Lemma 2, we have

(3.7) $$\| S(f) \| \leqq 1,$$

for every $S \in \mathfrak{S}$ and f with $\| f \| \leqq 1$. If $S \neq S'$, there exists f_0 with $\| f_0 \| \leqq 1$ such that $S(f_0) \neq S'(f_0)$. Then, due to A. Tychonoff,[10] $\mathfrak{S}$ turns out to a bicompact group with the weak topology, and $\mathfrak{S}_{G_0}$ is algebraically isomorphic and continuous image of G_0. According to the normal subgroup (the Markoff-kernel) H_0 of G_0, there exists a normal subgroup $\mathfrak{H}_{G_0}$ of $\mathfrak{S}_{G_0}$ such that $\mathfrak{S}_{G_0}/\mathfrak{H}_{G_0}$ is the algebraically isomorphic and continuous image of G_0/H_0 ; *i.e.*

(3.8) $$G \cong G_0/H_0 \quad \gtrsim \mathfrak{S}_{G_0}/\mathfrak{H}_{G_0}.$$

(al. isomorph.) (al. isomorph.)
homeomorph. continuous.

Denoting the commutor of ${}_{G_0}\mathfrak{S}$ in S by $\mathfrak{S}_0$, we conclude:

Theorem 5. (*Generalized Duality and Representation Theorem*) *With the definition above, we hold; for any topological group G,*

i) *$\mathfrak{S}_0$ is a bicompact group in a B-algebra (operaor-algebra),*

ii) *there exists an algebraically isomorphic and continuous mapping ϕ of G onto $\mathfrak{S}_G$ such that $\mathfrak{H}*\mathfrak{S}_G$ is a dense sub-group of $\mathfrak{S}_0$ for a suitable normal sub-group $\mathfrak{H}$ of $\mathfrak{H}*\mathfrak{S}_G$ (i.e. $=\mathfrak{S}_{G_0}$).*

iii) *if G is max. a. p., ϕ is a continuous isomorphism of G onto a dense sub-group $\mathfrak{S}_G$ of $\mathfrak{S}_0$, and if G is bicompact, G is continuously isomorphic to $\mathfrak{S}_0$ itself.*

Here, $\mathfrak{A}*\mathfrak{B}$ means a group-extension $\mathfrak{G}$ of $\mathfrak{B}$ by $\mathfrak{A}$, such that $\mathfrak{G}/\mathfrak{A}=\mathfrak{B}$.

To completes the proof of this theorem, it remains for us to prove the denseness of $\mathfrak{H}*\mathfrak{S}_G$ in $\mathfrak{S}_0$. iii) is a direct result of it. However, we shall remark it soon after.

For each $S \in \mathfrak{S}_0$, we put

(3.9) $\tilde{S}(f)=(S\cdot f)(e)$; for $f \in A(G_0)$, $e=$unit of G_0,

and get the set $\tilde{\mathfrak{S}}_0$ of such linear functionals $\tilde{S}$.

Lemma 3. *$\tilde{\mathfrak{S}}_0$ is algebraically isomorphic to $\mathfrak{S}_0$.*

If $S_1 \neq S_2$ in $\mathfrak{S}_0$, there exist $f \in A(G_0)$ and $x_0 \in G_0$ such that

$$(S_1\cdot f)(x_0) \neq (S_2\cdot f)(x_0),$$

and putting $S_{x_0}\cdot f = g$,

$$\tilde{S}_i(g) = (S_i\cdot g)(e) = (S_i\cdot S_{x_0} f)(e)$$
$$= S_i\cdot f(x_0\cdot e) = S_i\cdot f(x_0), \text{ for } i = 1, 2.$$

It implies that $\tilde{S}_1(g) \neq \tilde{S}_2(g)$ from (3.8), thus $\tilde{\mathfrak{S}}_0$ and $\mathfrak{S}_0$ are one-to-one corresponding.

Now we define the product in $\tilde{\mathfrak{S}}_0$ by

10) An excellent proof has been obtained by C. Chevalley and O. Frink, Bull. A. S. S. 47 (1941).

(3.10) $$\tilde{S}_1\cdot\tilde{S}_2=\widetilde{S_1\cdot S_2},$$

and get the algebraic isomorphism between $\tilde{\mathfrak{S}}_0$ and $\mathfrak{S}_0$, *i.e.* $\tilde{\mathfrak{S}}_0$ is a group which is isomorphic to $\mathfrak{S}_0$.

Particularly, (3.9) is realized for $D(x_0)$ as the form such that

(3.11) $$\tilde{S}_1\cdot\tilde{S}_2(D_{ij})=\sum_k\tilde{S}_1(D_{ik})\tilde{S}_2(D_{kj}),$$

where $D(x_0)=(D_{ij}(x_0))$.

We next introduce a weak topology in $\mathfrak{S}_0$ by such the way that ; a neighborhood $\tilde{U}_{\tilde{S}_0}(f_1, f_2, \dots, f_n;\varepsilon)$ is defined as

(3.12) $$\tilde{U}_{\tilde{S}_0}=\{S\,|\,|\tilde{S}(f_j)-\tilde{S}_0(f_j)|<\varepsilon\},$$

for $j=1, 2, \dots, n$. Then the correspondence of $\mathfrak{S}_0$ onto $\tilde{\mathfrak{S}}_0$ is continuous and $\tilde{\mathfrak{S}}_0$ is bicompact. Of course, the inverse correspondence of $\tilde{\mathfrak{S}}_0$ onto $\mathfrak{S}_0$ is also continuous *i.e.* $\mathfrak{S}_0$ and $\tilde{\mathfrak{S}}_0$ are isomorphic.

Then we can modify the Theorem 5 as follows:

Theorem. 6. *Usual Duality Theorem) For any topological G, there exists an algebraically isomorphic and continuous mapping $\tilde{\phi}$ of G onto $\tilde{\mathfrak{S}}_G$, for which $\tilde{\mathfrak{H}}_G*\tilde{\mathfrak{S}}_G$ is a dense subgroup of a bicompact group $\tilde{\mathfrak{S}}_0$ for suitable $\tilde{\mathfrak{H}}_G$. If G is max. a. p., $\tilde{\phi}(G)=\tilde{\mathfrak{S}}_G$ and if G is bicompact, $\tilde{\phi}(G)=\tilde{\mathfrak{S}}_0$.*

Proof of the denseness of $\tilde{\mathfrak{S}}_{G_0}=\tilde{\mathfrak{H}}_G*\tilde{\mathfrak{S}}_G$ in $\tilde{\mathfrak{S}}_0$: As G_0 is *max. a. p.*, from Theorem 7 later, we have

(3.13) $$A(G_0)\simeq \boldsymbol{C}(\tilde{\mathfrak{S}}_0) \quad \text{(in norm-preserving fashion).}$$

If $\tilde{\mathfrak{S}}_{G_0}$ is not dense in $\tilde{\mathfrak{S}}_0$, there exists a point φ_0 in $\tilde{\mathfrak{S}}_0-\overline{\mathfrak{S}}_{G0}$ ($\overline{\mathfrak{S}}_{G_0}$ being the closure of $\tilde{\mathfrak{S}}_{G_0}$). By Urysohn's theorem, there exists a function $f\in\boldsymbol{C}(\tilde{\mathfrak{S}}_0)$ such that

(3.14) $$f(\varphi)=\begin{cases}1, \text{ if } \varphi=\varphi_0,\\ 0, \text{ if } \varphi \text{ is in } \overline{\mathfrak{S}}_{G_0}.\end{cases}$$

But this is contradictory with (3.13). Thus, the denseness is proved, and moreover that in Theorem 5 is also complectely verified.

Remark 1: This duality theorem is of the Tannaka-Krein's type and if G max. a. p., it is exactly the Tannaka-Krein's one.[11]

But in general cases, the homomorphism are just in the opposite directions one another; in Tanaka-Krein's theorem the direction of homomorphic mapping is of G to G^0 (=a certain group of functionals on $A(G)$), while that of ours is of $G^0(=\tilde{\mathfrak{S}}_{G_0})$ to G.

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11) T. Tannaka 25) and M. Krein 10), loc. cit. An excellent and plain proof is shown by K. Yosida 27). Also see I. E. Segal 20).

For this reason, Tannaka-Krein's theorem concerns with the very case that G has a representation, but even when G is minimally a. p., our theorem has still a meaning.

Remark 2: The bicompact group $\mathfrak{S}_0$ is directly characterized by the set of all linear multiplicative bounded functionals on $\boldsymbol{A}$ (G_0), which is denoted by $\mathfrak{R}$. Using a neighbourhood $U^0_{\varphi_0}(f^0_1 f^0_2, \ldots, f^0_n; \varepsilon)$ for $\varphi^0_j \in \mathfrak{R}, f^0_j \in \boldsymbol{A}(G_0)$ with $\|f^0_j\| \leqq 1, j = 1, 2, \ldots, n$, such that

$$U^0\varphi_0 = \{\varphi \in \mathfrak{R} \,\big|\, |\varphi(f^0_j) - \varphi_0(f^0_j)| < \varepsilon\}, \tag{3.15}$$

$\mathfrak{R}$ turns to be a weakly bicompact Handdroff space. We can easily verify that such neighbourhood system $\{U_{\varphi_0}\}$ is equivalent to that of $\{U_{\varphi_0}\}$, which is definedby

$$\begin{aligned} &U_{\varphi_0}(f_1, f_2, \ldots, f_n; \varepsilon) \\ &= \{\varphi \in \mathfrak{R} \,\big|\, |\varphi(f_j) - \varphi_0(f_j)| < \varepsilon\}, \end{aligned} \tag{3.16}$$

where $f_j \in \boldsymbol{A}(G_0)$ for $j = 1, 2, \ldots, n$, whose norm is not necessarily $\leqq 1$. This implies that $f(\varphi) = \varphi(f)$, $f \in \boldsymbol{A}(G_0)$, $\varphi \in \mathfrak{R}$, is (uniformly) continuous on $\mathfrak{R}$.

From Lemma 4 stated later, we have $\boldsymbol{A}(G_0) \equiv \boldsymbol{C}(\mathfrak{R})$ where $\boldsymbol{C}(\mathfrak{R})$ is the B-algebra of all continuous functions on $\mathfrak{R}$. This fact together with Theorem 7 later implies

$$\boldsymbol{C}(\tilde{\mathfrak{S}}_0) \simeq \boldsymbol{C}(\mathfrak{R}) \tag{3.17}$$

in an algebraic and norm preserving fashion. From (3.15), it follows that $\mathfrak{R}$ is homeomorphic to $\tilde{\mathfrak{S}}_0$.

The linear multiplicative bounded functionals are studied by V. Šmulian, I. Gelfand, E. Hille, etc.[12] Our further investigations of them will appear in another paper.

Remark 3. For a locally compact group, its irreducible representation theorem is given by Gelfand-Raikov, I. E. Segal, G. Mautner, and H. Yosizawa,[13] But our representation (Theorem 5) is complete only if the group is max a. p. The gap between these two representatious has been filled up in any case.

4. **B*-algebra of a. p. functions.** The space of all continuous a. p. functions on G, $\boldsymbol{A}(G)$, is not only a B-algebra, but also a commutative B*-algebra with the norm conditiond;

$$\|f \cdot f^*\| = \|f\| \cdot \|f^*\|, \tag{4.1}$$

$$\|f\| = \|f^*\| \tag{4.2}$$

that is, a B*-algebra in the sense of C. E. Rickart and I. Kaplansky.[14] As *-operation, we have only to put $f^* = \bar{f}$ (conjugate)

12) V. Šmulian 23), and E. Hille, Proc. Nat. Acad. Sci., 30 (1944) and 4).

13) I. Gelfand and D. Raikov, Math Sbornik, 13 (1944).
I. E. Segal 20), G. Mautner 12), and H. Yoshizawa 28).

14) C. E. Rickart 16) and I. Kaplansky 9).

Suppose that G_0 is max. a. p. and $\tilde{\mathfrak{S}}_{G_0}$, $\tilde{\mathfrak{S}}_0$ have the same meaning as in the preceding. Then, for $S \in \mathfrak{S}_0$, the function

$$(4.3) \qquad f(\tilde{s}) = \tilde{s}(f), \qquad f \in A(G_0)$$

is a (uniformly) continuous function on $\tilde{\mathfrak{S}}_0$. Thus, $A(G_0)$ is a subring of $C(\mathfrak{S}_0)$, the B*-algebra of all continuous complex-valued functions on $\tilde{\mathfrak{S}}_0$.

Lemma 4. (*Stone-Rickart*) *If for every pair of points S, S' in a bicompact space $\mathfrak{S}$, there exists an element f of the subring $\mathfrak{A}_0$ of $C(\mathfrak{S})$ such that $f(s) \neq f(s')$, then $\mathfrak{A}_0 \equiv C(\mathfrak{S})$.*

Originally, G. Šilov[15] proved this Lemma under the condition that $\mathfrak{S}$ is bicompact and metric and later M. H. Stone[16] proved for real $C(\mathfrak{S})$.

Theorem 7. *For max. a. p. G_0, we have*

$$(4.4) \qquad A(G_0) \simeq C(\tilde{\mathfrak{S}}_0).$$

Denoting the set of all maximal ideals of $A(G_0)$ by $\mathfrak{M}_{G_0}$, Gelfand-Neumark[18] proved that

$$(4.5) \qquad C(\mathfrak{M}_{G_0}) \simeq A(G_0),$$

Then, we have

Corollary. *we hold for max. a. p. G_0,*

$$(4.6) \qquad C(\tilde{\mathfrak{S}}_0) \simeq C(\mathfrak{M}_{G_0}).$$

Again, let G_0 be an Markoff-extension of G. As in the preceding mentioned, we hold

$$(4.7) \qquad A(G) \simeq \hat{A}(G_0) \subset A(G_0) = C(\tilde{\mathfrak{S}}_0).$$

Now, according to Šilov and Rickart, we decompose $\mathfrak{S}_0$ to the direct sets (the continuous decomposition in the sense of P. Alexandroff)

$$(4.8) \qquad \tilde{\mathfrak{S}}_0 = \sum \oplus L(s),$$

where

$$(4.9) \qquad L(s) = \{s' \mid f \in \hat{A}(G_0) \text{ impiles } f(s) = f(s')\}.$$

Denoting the unit of $\tilde{\mathfrak{S}}_0$ by s_e, we see immediately that $L(s_e) = L_0$ is a closed normal subgroup of $\tilde{\mathfrak{S}}_0$ such that

$$(4.10) \qquad \mathfrak{L} \simeq \tilde{\mathfrak{S}}_0 / L_0$$

where $\mathfrak{L} \equiv \{L(s)\}$, and moreover $L_0 \subset \bar{\mathfrak{H}}_G$ (the closure of $\mathfrak{H}_G$ in $\mathfrak{S}_0$). Owing to Rickart, we have

$$(4.11) \qquad C(\mathfrak{L}) \simeq \hat{A}(G_0)$$

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15) G. Šilov 22).
16) M. H. Stone 24).
17) Gelfiand and Šilov, Rec. Math., N. S. 9 (1941) and C. Rickart 16), loc. cit.
18) I. Gelfand-M. Neumark 3).

and hence

(4.12) $$C(\tilde{\mathfrak{S}}_0/L_0) = C(\mathfrak{L}) \simeq \hat{A}(G_0) \simeq A(G).$$

Consequently, we come to the extended formula of Theorem 7 as follows;

Theorem 8. *For any topological group G, there exists a bicompact topological group $\mathfrak{G}_G$ such that*

$$A(G) \simeq C(\mathfrak{G}_G)$$

in the norm preserving fashion: With the same definitions of $\mathfrak{S}_0$ and L_0 as in the preceding, $\mathfrak{G}_G$ is written in the form; $\mathfrak{G}_G = \mathfrak{S}_0/L_0$.

This theorem together with the preceding theorem has the same meaning as the representation theorem of a commutative B*-algebra in the sense of I. E. Segal[19], I. Kaplansky[20], and R. V. Kadison[21], which is written in the form; $A(G) \simeq C(X)$ for a suitable bicompact Hausdorff space X.

Next we consider a Lebesgue integral on $\mathfrak{G}_G$ with respect to the Haar's measure m on it;

(4.13) $$\mu(f) = \int f^0(a)\, dm(a),$$

for every $f \in A(G)$ with the corresponding f^0 in C $(\mathfrak{G}_G)$ and $\int dm(a) = 1$.

G itself being considered as the group of measure-preserving automorphisms o $[\mathfrak{G}_G]$, for each $a_0 \in G$, we have

(4.14) $$\mu(S_{a_0} \cdot f) = \int f^0(a_0 a)\, dm(a)$$

$$= \int f_0(a)\, dm(a_0^{-1} a) = \int f^0(a) dm(a) = \mu(f),$$

that is, $\mu(S_{a_0} f) = \mu(f)$. It is clear that, for $f^-(x) = f(x^{-1})$, we hold $\mu(f^-) = \mu(f)$. These implies that $\mu(f)$ satiffies the all properties of a mean value in $A(G)$, and from the uniqueness of mean values[22], we get

Corellary. *The mean value $\mu(f)$ of an element (a. p. function) of $A(G)$ is represented in the form;*

$$\mu(f) = \int_{\mathfrak{G}_G} f(a)\, dm(a).$$

Theorem 8, with the Corollary, has been otherwise prouved in I. E. Segal 19).

5. **Application to locally compact cases. Group-algebras.** In this section, we shall restrict ourselves in the case that G is locally bicompact. We begin by defining the group-algebra $L(G)$ of G in the sense of I. E. Segal[23] with respect to the right-invariant Harr's measure on G; the multiplication and the norm are respectively defined as follows:

19) I. E. Segal 20), about C*-algebras.
20) I. Kaplansky 9), loc. cit.
21) R. v. Kadison 7).
22) J. von Neumann 15), S. Bochner and J. von Neumann 2), and further W Maak 12).
23) I. E. Segal 18), 19).

$$f\times g=\int_G f(xy^{-1})\, g(y)\, dy \quad\text{and}\quad \| f\| =\int_G |f(x)|dx\,.$$

Let G_0 be an Markoff extension of G, and $D(x_0)$ a complex irreducible continuous normal representation of G_0. Then $\Delta(x_1, x_2, \dots, x_n)$; $x_1, x_2, \dots, x_n) \in G$, is a continuous function on $G\times G\cdots\times G$ (n times); cf. (2.11). Now we define that

$$D(f_1)\times D(f_2)\times\cdots\times D(f_n)$$

(5.1)
$$=\underbrace{\iint\cdots\int}_{\substack{G\times G\times\cdots\times G\\ n}} f_1(x_1)\cdot f_2(x_2)\dots f_n(x_n)\cdot D(x_1)*D(x_2)*\cdots*D(x_n)\, dx_1dx_2\dots dx_n,$$

$$=\underbrace{\iint\cdots\int}_{\substack{G\times G\times\cdots\times G\\ n}} f_1D(x_1)\cdot f_2D(x_2)\dots f_nD(x)\Delta(x_1, x_2, \dots, x_n)\, dx_1dx\dots dx_n,$$

where $f_k\in \boldsymbol{L}(G)$, $f_k\cdot D(x_k)=f_k(x_k)\cdot D(x_k)$.

Since $\Delta(x)=1$ for $x\in G$, we see immediately that, when $n=1$,

(5.2)
$$D(f)=\int_G f(x)D(x)\, dx.$$

and for complex numbers α, β,

(5.3)
$$D(\alpha f+\beta g)=\alpha D(f)+\beta D(g)$$

(5.4)
$$\| D(f)\| \leqq M^D\cdot \| f\|,$$

where $\| D\|$ means the usual norm of matrices, i.e. $\| D\| =(\sum_{i,j}|D_{ij}|^2)^{1/2}$, and M^D depends upon D only. (5.4) comes from the fact that each $|D_{ij}|$ is bounded.

We generate a normed ring $\boldsymbol{R}_D(\boldsymbol{L}(G))$, or briefly $\boldsymbol{R}_D(\boldsymbol{L})$, from $D(f)$, $f\in \boldsymbol{L}(G)$, by the usual addition of matrices and the multiplication (5.3).

Then we assert:

Theorem 9. *$\boldsymbol{R}_D(\boldsymbol{L})$ is continuously homomorphic to $\boldsymbol{L}(G)$.*

To prove the theorem, we have only to show thar $D(f)\times D(g)=D(f\times g)$. In fact, we hold;

$$\begin{aligned}D(f)\times D(g) &=\iint_{G\times G} f(x)\cdot g(x)D(x)D(y)\Delta(x, y)\, dxdy\\ &=\iint_{G\times G} f(xy^{-1})g(y)D(xy^{-1})D(y)\Delta(xy^{-1}, y)dxdy\\ &=\iint_{G\times G} f(xy^{-1})g(y)D(xy^{-1}\cdot y)D(y^{-1}\cdot(xy^{-1})^{-1}x)dxdy\\ &=\int_G D(x)\int_G f(xy^{-1})g(y)dydx\\ &=\int_G f\times g(x)D(x)dx=D(f\times g).\end{aligned}$$

Let $\psi(x_1, x_2)$ be a complex continuous function on $G\times G$, then we have

Lemma 5. *If we hold*

$$\int\int_{G\times G} f_1(x_1)f_2(x_2)\psi(x_1, x_2)dx_1dx_2 = 0 \tag{5.5}$$

for arbitrary $f_k \in (G)$, $k=1, 2$, *it must be* $\psi(x_1, x_2)$ *for any* $x_k \in G$.

Remark. This lemma is easily extended to the case that $\psi(x_1, x_2, \dots, x_n)$ is continuous on $\underbrace{G\times G\times \dots G}$, and

$$\int\int\cdots\int_{G\times G\times\cdots\times G} f_1(x_1)\ f_2(x_2)\cdots f_n(x_n)\psi(x_1, x_2, \dots, x_n) \cdot dx_1\, dx_2 \dots dx_n = 0 \tag{5.6}$$

Proof of Lemma. ψ being real at first, the set P_ψ of such points in $G\times G$ that $\psi<o$ is open. For any points $p(x_1, x_2)\in P_\psi$, we can select a neighborhood of x_k, $U_k = U(x_k)$, in G, such that the open rectangle $U_1\times U_2 \subset P_\psi$.

For each U_k, we can find a neighbourhood $V_k=V(x_k)\subset U_k$, whose closure $\overline{V}_k$ is bicompact, then $\overline{V}_1\times\overline{V}_2\subset\overline{P}_\psi$.

Now we define a characteristic function f_0 of (the compact carrier) $\overline{V}_k$, as f_k in (5.7), such that

$$f_k^0 = \begin{cases} 1 \text{ on } \overline{V}_k \\ 0 \text{ on } G\text{--}\overline{V}_k \end{cases} \tag{5.7}$$

then it is necessarily that each f_0 belongs to $L(G)$ and

$$\int\int_{G\times G} f_1^0(x_1)f_2^0(x_2)\psi(x_1, x_2,)dx_1dx_2$$

$$= \int\int_{V_1\times V_2} \psi(x_1, x_2)\, dx_1, dx_2 > 0$$

This is contradictory with (5.7); that is, P_ψ is necessarily of measure 0 on $G\times G$. With respect to N_ψ which is the set of such points that $\psi<0$, we go analogously as above, adopting a negative characteristic function as f_k^0, and at last hold that N_ψ is also of measure 0. Thus $\psi=0$ identically: If ψ is complex, decomposing it to $\psi_1+i\psi_2$ (ψ_1 and ψ_2 are real), we can leasily obtain the Lemma.

Now, we put

$$\Phi_D(x_1, x_2) = D(x_1)\times D(x_2) - D(x_1)D(x_2), \tag{5,8}$$

for $x_1, x_2\in G$, and $\Phi_D=(\psi_{ij}^D)$. If $D(f_1)\times D(f_2)=D(f_1)D(f_2)$ for any $f_k\in L(G)$, $k=1, 2$, we have

$$\int\int_{G\times G} f_1(x_1)f_2(x_2)\Phi_D(x_1, x_2)\ dx_1dx_2 = 0,$$

and, from Lemma 5, we assert that $\Phi_D=0$ (0-matrix with same degree as D), *i.e.*

$$D(x_1)D(x_2)\Delta(x_1, x_2) = D(x_1)D(x_2),$$

namely

$$\Delta(x_1, x_2) = E. \tag{5.9}$$

Owing to Section 2, (5.9) shows that such $D(x)$ is nothing but the representation of G.

Then we get easily the equality $D(f_1)\times D(f_2)\times\cdots\times D(f_n)=D(f_1)D(f_2)\cdots D(f_n)$ for arbitrary $f_k\in(G)$ and n; $k=1, 2, \dots, n$.

Theorem of Iwasawa.[24] *The continuous normal representations of G and the continuous representations of $L(G)$ are one-to-one corresponding by the relation*

$$D(f) = \int_G f(x)D(x)dx.$$[25]

All these circumstances convince us of some analogy between the $*$-operation in the representation of G and the $\times$-operation in that of $L(G)$ and corresponding to the corollary in Section 2, we hold

Theorem 10. *A $\times$-representation of $R_D(L(G))$ coincides with the usual one, that is, the representation by the matric-algebra, if and only if $D(x)$ is a continuous normal representation of G. Every finite rep. of $L(G)$ is a special $\times$-rep. as above.*

Remark: A shorter proof of the Iwasawa's theorem will soon appear elsewhere.

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25) Thees results may be extended to $L^{(1,p)}(G)$ without difficulty, for $p\geqq 2$.

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