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Multiplicative Linear Functionals on B-algebras

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Introduction

1. In the present paper, we shall make a survey of multiplicative linear functionals defined on not necessarily commutative B-algebras or B*-algebras.[0]

In commutative cases, the notion of these functionals coincides with that of the so-called *Gelfand's functions* on maximal ideals, which have been studied by *V. R. Šmulian* [13], *I. Gelfand* [1], *N. Dunford* and *E. Hille* [4], etc.

As well known, the existence of these functionals is equivalent to that of two-sided maximal ideals which are simultaneously maximal as left or right ideals. However their existence is not always assured in general cases, some results based on them should be yet expected; for instance, the functional radical (whose definition shall be given later), which is always in existence and never meaningless, suggests the algebraic structure to a certain extent.

Multiplicative linear functionals are related to the spectrums closely in such a manner that, for every multiplicative linear φ, each value $\varphi(x)$ belongs to the spectrum $\sigma(x)$ of x; and conversely, if the commutativity holds. Though the spectrum theory plays an important rôle in studying a B-algebra, it concerns itself in each element individually, while the theory of multiplicative linear functionals seems to be somewhat *in the large.*

(I) is devoted to the investigation of multiplicative functionals' situation among usual linear ones. In (II), the relation between algebraic and functional structures is indicated; that is our main intention. In the last (III), we shall concern ourselves in the representation of B-algebras and resume the *almost periodic algebras*, which are discussed in our another paper; *S. Matsushita* [10].

(I)

2. We shall recall some fundamental notions and fix our notations at first. Let $\mathfrak{R}$ or $\mathfrak{K}$ be the field of the reals or the complices respectively.

$\mathfrak{A}(\mathfrak{S})$ is a *Banach algebra* (*B-algebra*) over the scalar field $\mathfrak{S}$, which is $\mathfrak{R}$ or $\mathfrak{K}$; $\mathfrak{A}^*(\mathfrak{S})$ is a B*-algebra over $\mathfrak{S}$.

Here, a *B*-algebra* $\mathfrak{A}^*(\mathfrak{K})$ is a B-algebra which admits such a *-operation that is a conjugate linear, involutely anti-automorphism for every a, b in $\mathfrak{A}^*(\mathfrak{K})$, *i. e.*

0) We shall always assume that a B- or B*-algebra has unit e of norm 1.

$*_1$) $(\alpha_1 a+\alpha_2 b)^* = \bar{\alpha}_1 a^* + \bar{\alpha}_2 b^*$, where $\alpha_i \in \mathfrak{K}$ with conjugate $\bar{\alpha}_i \in \mathfrak{K}$,

$*_2$) $(a^*)^* = a$,

$*_3$) $(ab)^* = b^* a^*$,

and further has the norm condition;

$$\| aa^* \| = \| a \| \cdot \| a^* \| . \tag{2.1}$$

For $\mathfrak{A}^*(\mathfrak{K})$, the first condition $*_1$) must be replaced by;

$*_1)'$ $(\alpha_1 a+\alpha_2 b)^* = \alpha_1 a^* + \alpha_2 b^*$, $\alpha_i \in \mathfrak{R}$.

We shall sometimes abbreviate $\mathfrak{A}(\mathfrak{S})$, $\mathfrak{A}^*(\mathfrak{S})$ to $\mathfrak{A}$, $\mathfrak{A}^*$ only, if there is no confusion, and always assume that $\mathfrak{A}$ or $\mathfrak{A}^*$ has unit e of norm 1.

A B-algebra over $\mathfrak{R}$, $\mathfrak{A}(\mathfrak{R})$, is embeded into that over $\mathfrak{K}$; see *I. Kaplansky* [8], and if $\mathfrak{A}^*(\mathfrak{R})$ is commutative, $\mathfrak{A}^*(\mathfrak{R})$ is evidently a sub-algebra of $\mathfrak{A}^*(\mathfrak{R}) \times \mathfrak{A}^*(\mathfrak{R}) \equiv (\mathfrak{A}^* \times \mathfrak{A}^*)(\mathfrak{K})$ by the correspondence;

$$a \in \mathfrak{A}^*(\mathfrak{R}) \longleftrightarrow \left(\frac{a+a^*}{2}, \frac{a-a^*}{2}\right) \in (\mathfrak{A}^* \times \mathfrak{A}^*)(\mathfrak{K}). \tag{2.2}$$

Conversely, if $\mathfrak{A}^*(\mathfrak{K})$ is commutative, the hermitian kernel of $\mathfrak{A}^*$ is a B-algebra over $\mathfrak{R}$, all of whose elements are *hermitian* (*i. e.* $a=a^*$).

Here, the *hermitian kernel* $H(\mathfrak{A}^*)$ of $\mathfrak{A}^*$ means the set of all hermitian elements, $a=a^*$, in $\mathfrak{A}^*$. An hermitian element is said to be *self-adjoint*, too. It is easy to see $H(\mathfrak{A}^*)$ being a B-algebra, if and only it $\mathfrak{A}^*$ is commutative. However $\mathfrak{A}^*$ is not commutative, $H(\mathfrak{A}^*)$ becomes a complete normed algebra except for the associativity of products, by the *Segal's product*;

$$a \circ b = \frac{1}{2}(ab+ba),^{1)} \tag{2.3}$$

which is commutative, distributive and preserves hermitian elements; see *I. E. Segal* [12].

3. We shall next define tensors, multiplicative linear functionals, etc. According to O. Veblen, we consider a multilinear functional $\tau^n(a_1, \dots, a_n)$ on the n-fold Kronecker products $\mathfrak{A}^{(n)} \equiv \mathfrak{A} \times \cdots \times \mathfrak{A}$ of $\mathfrak{A}(\mathfrak{S})$, whose values are contained in $\mathfrak{S}$, for $n=1, 2, \dots$, and call it a (covariant) *n-tensor*[2]; 1-tensor is nothing but a usual functional on $\mathfrak{A}$. We suppose always that, for a B*-algebra $\mathfrak{A}^*(\mathfrak{K})$, every n-tensor has an added condition;

$$\tau^n(a_1^*, \dots, a_n^*) = \bar{\tau}^n(a_1, \dots, a_n), \tag{3.1}$$

where $\bar{\tau}^n$ means the conjugate value of τ^n in $\mathfrak{K}$; for 1-tensor $\tau(a) \equiv \tau^1(a)$,

$$\tau(a^*) = \bar{\tau}(a). \tag{3.1}$$

1) This product is written in the form;

$$a \circ b = \frac{1}{4}((a+b)^2 - (a-b)^2).$$

2) When $\mathfrak{S}$ is reals, 1-tensors (vectors) forms the conjugate space of $\mathfrak{A}$ as a vector space; the set of all contravariant vectors coincides with $\mathfrak{A}$ itself.

If 1-tensor $\varphi(a)$ satisfies the additional condition

$$\varphi(ab)=\varphi(a)\,\varphi(b) \tag{3.2}$$

for every a, b in $\mathfrak{A}$, we call it a ***multiplicative linear functional*** on $\mathfrak{A}$.

The vector space over $\mathfrak{S}$ of all n-tensors is denoted by $V^n(\mathfrak{A}, \mathfrak{S})$ or briefly $V^n(\mathfrak{A})$, while $V^0(\mathfrak{A})$ is defined as $\mathfrak{S}$ itself. The set of all multiplicative linear functionals is denoted by $\Phi(\mathfrak{A})$, which is clearly a subset of $V^1(\mathfrak{A})$.

By analogy with *G. Hochschild's* methods [5], we define a coboundary operation δ_φ with respect to each $\varphi\in\Phi(\mathfrak{A})$ as follows;

$$\begin{aligned}(\delta_\varphi\cdot\tau^n)&(a_1,\dots,a_{n+1})\\ &=\varphi(a_1)\cdot\tau^n(a_2,\dots,a_{n+1})\\ &\quad+\sum_{i=1}^{n}(-1)^i\cdot\tau^n(a_1,\dots,a_ia_{i+1},\dots,a_{n+1})\\ &\quad+(-1)^{n+1}\cdot\tau^n(a_1,\dots,a_n)\cdot\varphi(a_{n+1}),\end{aligned} \tag{3.3}$$

which defines a homorphism of $V^n(\mathfrak{A})$ into $V^{n-1}(\mathfrak{A})$ and fulfils an impartant relation

$$\delta_\varphi\cdot\delta_\varphi=0\,. \tag{3.4}$$

If $\delta_\varphi\cdot\tau^n=0$, τ^n is called an *(n, φ)-cocycle* and, if there exists a $\tau^{n-1}\in V^{n-1}(\mathfrak{A})$ such that $\tau^n=\delta_\varphi\cdot\tau^{n-1}$, τ^n is called an *(n, φ)-coboundary*; the set of all (n, φ)-cocycles is denoted by $Z^n(\mathfrak{A}, \varphi)$, while that of all (n, φ)-coboundaries by $B^n(\mathfrak{A}, \varphi)$. Since $\theta=0\ (x)\in\Phi(\mathfrak{A})$, $\delta\equiv\delta_\theta$ is in reason and represented as

$$\begin{aligned}(\delta\tau^n)&(a_1,\dots,a_{n+1})\\ &=\sum(-1)^i\,(a_1,\dots,a_ia_{i+1},\dots,a_{n+1})\end{aligned} \tag{3.5}$$

which is independent of the choice of φ, while δ_φ is not.

We abbreviate $Z^n(\mathfrak{A}, \theta)$ or $B^n(\mathfrak{A}, \theta)$ to $Z^n(\mathfrak{A})$ or $B^n(\mathfrak{A})$ respectively.

By (3.4), it is clear that

$$B^n(\mathfrak{A}, \theta)\subset Z^n(\mathfrak{A}, \varphi) \tag{3.6}$$

and their residue group (difference module)

$$H^n(\mathfrak{A}, \varphi)\equiv Z^n(\mathfrak{A}, \varphi)/B^n(\mathfrak{A}, \varphi)$$

is called the *n-dimensional φ-cohomology group* of $\mathfrak{A}$ over $\mathfrak{S}$.

It would be intersting to investigate the characters of thus defined cohomology groups, but we restrict ourselves to only some remarks, here.

4. In the following $\mathfrak{S}$ will be the reals $\mathfrak{R}$. Denote by $N^n(\mathfrak{A})$ the subspace of $V^n(\mathfrak{A})$, which consists of all τ^n such that $\tau^n(e,\dots,e)=0$, and put

$$\hat{V}^n(\mathfrak{A})=V^n(\mathfrak{A})/N^n(\mathfrak{A}), \tag{4.1}$$

which brings about a homomorphic mapping

$$\tau^n\in V^n(\mathfrak{A})\longrightarrow\hat{\tau}^n\subset\hat{V}^n(\mathfrak{A}), \tag{4.2}$$

that is,

$$\dot{\tau}^n(a_1, \dots, a_n) \equiv \tau^n(a_1, \dots, a_n) \quad (modulo\ N^n(\mathfrak{A})),$$

then we hold

$$\hat{\tau}^n(e, \dots, e) = \tau^n(e, \dots, e). \tag{4.3}$$

According to the mapping (4.1), we get

$$Z^n(\mathfrak{A}) \longrightarrow \hat{Z}^n(\mathfrak{A}) \subset \hat{V}^n(\mathfrak{A}).$$

If $\tau^n(e, \dots, e)$ (or $\hat{\tau}^n(e, \dots, e)$)$=1$, τ^n(or $\hat{\tau}^n$) is said to be *normalized*. Then, for evety τ^n such that $\tau^n \not\equiv 0$ (*mod.* $N^n(\mathfrak{A})$), we can normalized it by setting

$$\tau^n_0(a_1, \dots, a_n) = \tau^n(a_1, \dots, a_n)/\tau^n(e, \dots, e), \tag{4.4}$$

and we see immediately that

$$\hat{\tau}^n_0 = (\hat{\tau}^n)_0 = \hat{\tau}^n(a_1, \dots, a_n)/\tau^n(e, \dots, e)$$

by (4.2), and again

$$\hat{\tau}^n_0 \equiv (\dot{\tau}^n)_0 \quad (mod.\ N^n(\mathfrak{A})). \tag{4.5}$$

Then we assert:

Theorem 1. i) *If n is even,*

a) *all $\hat{Z}^n(\mathfrak{A}, \varphi)$, $\varphi \in \Phi$, are mutually disjointed,*

b) *every $\hat{Z}^n(\mathfrak{A}, \varphi)$ is linearly independent of other $\hat{Z}^n(\mathfrak{A}, \psi)$, $\varphi \neq \psi$, with respect to finite convex combinations; i. e., $\hat{\tau}^n_0 = \sum_{j=1}^{n} \alpha_j \hat{\tau}^n_j$, where $\tau^n_0 \in \hat{Z}^n(\mathfrak{A}, \varphi_0)$, $\tau^n_j \in \hat{Z}^n(\mathfrak{A}, \varphi_j)$, $\alpha_j \geqq 0$ and $\sum_{j=1}^{n} \alpha_j = 1$, yields $\hat{Z}^n(\mathfrak{A}, \varphi_0) = \hat{Z}^n(\mathfrak{A}, \varphi_j)$ for suitable j, say j_0, and $\alpha_{j_0} = 1$, $\alpha_j = 0$ for $j \neq j_0$.*

ii) *If n is odd, every $\hat{Z}^n(\mathfrak{A}, \varphi) \equiv 0$ (mod. $N^n(\mathfrak{A})$).*

Lemma 1. *All normalized f in $V^1(\mathfrak{A})$, with $f(a^2) \geqq 0$ on every squares $a^2 \in \mathfrak{A}$, forms a (regular) convex set $C^1(\mathfrak{A}) \subset V^1(\mathfrak{A})$, Φ^0 is contained in the set of extremes of $C^1(\mathfrak{A})$, wher Φ^0 means $\Phi - \theta$.*

R. V. Kadison [7] has proved, for an ordered (not necessarily associative) algebra, that the convex set of normalized positive (*i. e.* positive on positive elements) linear functionals has the extremes which coincide Φ^0 itself. To prove the above Lemma, we go somewhere analogously; by the assumption for f in $C^1(\mathfrak{A})$, we hold the *Cauchy-Schwarz's Lemma* in a certain general situation such that

$$(f(a))^2 \leqq f(a^2). \tag{4.6}$$

Using (3.3), we can prove easily that multiplicative f cannot be a midpoint of no segment of g_1, $g_2 \in C^1(\mathfrak{A})$.

Next we define a *projection operator* P_i from $V^n(\mathfrak{A})$ to $V^1(\mathfrak{A})$, $1 \leqq i \leqq n$, such that

$$(P_i \tau^n)^1(a) = \tau^n(e, \dots, e, \underset{(i)}{a}, e, \dots, e). \tag{4.7}$$

Lemma 2. *If n is even, for every τ^n and normalized $\tau_0^n \in \hat{Z}^n(\mathfrak{A}, \varphi)$,*

i) $P_1\tau_0^n = P_n\tau_0^n = \varphi$,

ii) $P_i\tau^n = P_j\tau^n$ *for* $2 \leqq i, j \leqq n-1$.

These are immediate results of (3.3).

Proof of Theorem 1: a) of i) and ii) are clear. b) of i) is obtained by the above establised two Lemmas without difficulty; if $\tau_0^n = \sum_{j=1}^{r} \alpha_j \tau_j^n$, $\tau_0^n \in \hat{Z}^n(\mathfrak{A}, \varphi_0)$, $\tau_j^n \in \hat{Z}^n(\mathfrak{A}, \varphi_j)$, $\alpha_j \geqq 0$ and $\sum_{j=1}^{r} \alpha_j = 1$, it must be

$$\varphi_0 = P_1\tau_0^n = \sum_{j=1}^{r} \alpha_j(P_1\tau_j^n) = \sum_{j=1}^{r} \alpha_j\varphi_j ,$$

and according to Lemma 1, it is absurd for $\varphi_0 \neq \varphi_j$, $j=1, \dots, r$.

Remark: The fact $f \in C^1(\mathfrak{A})$ being normalized and the property (3.6) imply that $\| f \| = 1$ for every f in $C^1(\mathfrak{A})$; so to say, $C^1(\mathfrak{A})$ is on the surface of the unit shere in $V^1(\mathfrak{A})$, which is weakly compact. Since it is evident that $C^1(\mathfrak{A})$ is weakly closed, $C^1(\mathfrak{A})$ is *regulary convex* in the sense of M. Krein-V. Šmulyan. Due to *M. Krein and D. Milman* [9], the existence of extreme points of $C^1(\mathfrak{A})$ is directly obtained.

(II)

5. In the following, $\mathfrak{S}$ is either $\mathfrak{R}$ or $\mathfrak{K}$ and $\mathfrak{A}(\mathfrak{S})$ may be occasionally specialized to $\mathfrak{A}^*(\mathfrak{S})$. Let $\mathfrak{N}$ be the *radical* of $\mathfrak{A}$, *i. e.* the set of all x (quasi-nilpotents) such that $e+axb$ has an inverse for every a, b in $\mathfrak{A}$. $\mathfrak{N}$ forms a two-sided ideal of $\mathfrak{A}$ and coincides with the intersection of all left (right) maximal ideals of $\mathfrak{A}$; see *S. Perlis* [11] and *N. Jacobson* [6].

Let $\mathfrak{N}^0$ be the *functional-radical* which is the set of all x such that $\varphi(x)=0$ for all $\varphi \in \Phi$. It is easy to see $\mathfrak{N}^0$ being also a two-sided ideal of $\mathfrak{A}_0$ Moreover we hold:

Lemma 3. $\mathfrak{N} \subset \mathfrak{N}^0$.

In fact, if $\varphi(x) \neq 0$ for a $\varphi \in \Phi^0 \equiv \Phi - \theta$ and $x \in \mathfrak{N}$, putting $a = e/\varphi(x)$, $b = -e$, we have

$$\begin{aligned} \varphi(e+axb) &= \varphi(e-(x/\varphi(x))) = \varphi(e)-(\varphi(x)/\varphi(x)) \\ &= 1-1 = 0 . \end{aligned}$$

On the other hand, $e+axb$ has an inverse and hence it must be $\varphi(e+axb) \neq 0$; these are contradictory. If $\Phi \equiv (\theta)$ only, it follows that $\mathfrak{N}^0 = \mathfrak{A}$ and clearly $\mathfrak{N} \subset \mathfrak{N}^0$.

Definition 1: Φ is said to be *full, semi-full* or *null*, if $\mathfrak{N}^0 = (0)$, $= \mathfrak{N}$ or $= \mathfrak{A}$ respectively.

Φ being null means that Φ consists of θ alone and simultaneously $\Phi^0 \equiv \mathfrak{D} - \theta$ is empty.

Since $ab-ba \in \mathfrak{N}^0$ for every pair a, b in $\mathfrak{A}$, the residue-algebra $\mathfrak{A}/\mathfrak{N}^0$ is commutative. Particularly if Φ is full, $\mathfrak{A}$ itself becomes commutative.

Another definition shall be given as follows:

Definition 2: The residue-class of $\mathfrak{A}/\mathfrak{N}^0$ which contains a is called a-class, denoting by C_a. Each element of C_e is called a *quasi-unit* or a *functional-unit*, which is characterized as such x that $\varphi(x)=1$ for all $\varphi\in\Phi^0$.

It is manifest that, for every pair x, y in the same class, it holds $\varphi(x)=\varphi(y)$ for all $\varphi\in\Phi$, and if Φ is full, there exists a φ in Φ (exactly in Φ^0) such that $\varphi(x)\neq\varphi(y)$ for every $x, y\in\mathfrak{A}$. Moreover, the fullness of Φ implies that there is no quasi-unit in $\mathfrak{A}$ except the usual unit e.

Lemma 4. *A quasi-unit a is written in the form $a=e-u$, $u\in\mathfrak{N}^0$, and if a is regular a^{-1} is also represented as the form; $a=e-v$, $v\in\mathfrak{N}^0$, where u and v are reversible one another.*

In fact, C_e is the unit of $\mathfrak{A}/\mathfrak{N}^0$ and hence $a\in C_e$ means $a^{-1}\in C_e$. It follows that $v+u-vu=u+v-uv=0$ from $(e-v)(e-u)=(e-u)(e-v)=e$.

Definition 3: A left (right) ideal $\mathfrak{J}$ is said to be *quasi-proper*, if $\mathfrak{J}$ has no quasi-unit.

Furthermore, we define another product, *the cross product*;

$$a\times b=a+b-ab\,, \tag{5.1}$$

which is associative, $(a\times b)\times c=a\times(b\times c)$, and $a\times 0=0\times a=0$, but not always commutative.

Now we shall announce our principal theorems:

Theorem 2. *The following three conditions are mutually equivalent;*

i) *Φ is semi-full in $\mathfrak{A}$,*

ii) *Every left or right maximal ideal is quasi-proper,*

iii) *$\mathfrak{N}^0$ forms a group with the product (5.1), or equivalently, every quasi-unit is regular.*[3]

Theorem 3. *The following two conditions are equivalent one another for any B^*-algebra, which is always semi-simple;*[4]

i) *$\Phi(\mathfrak{A}^*)$ is full and so semi-full in $\mathfrak{A}^*(\mathfrak{K})$.*

ii) *$\mathfrak{A}^*(\mathfrak{K})$ is commutative.*

6. To prove the Theorems, we shall begin with some preliminary Lemmas. Let J be a left maximal ideal of $\mathfrak{A}(\mathfrak{S})$.

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3) Under the cross product (5.1), $\mathfrak{N}$ forms a group; thus i)→iii) is somewhat trivial owing to this fact.

4) The set of quasi-nilpotent elements a, *i.e.* $\lim\limits_{n\to\infty}\sqrt[n]{\|a^n\|}=0$ or equivalently $e-\mu u$ being regular for every $\mu\in\mathfrak{S}$, contains the radical $\mathfrak{N}$ evidently for any $\mathfrak{A}(\mathfrak{S})$.

From this fact, it follows immediately that $\mathfrak{A}(\mathfrak{S})$ with $\|a^2\|=\|a\|^2$ for every $a\in\mathfrak{A}$ is semi-simple and so is every B*-algebra.

This last result, the semi-simplicity of B*-algebras, is otherwise mentioned by K. Iseki before us; K. Iseki [15].

Lemma 5. *If* $\mathfrak{N}^0_J = \mathfrak{N}^0 - J$, *the intersection of* $\mathfrak{N}^0$ *and the complement of* J, *is not empty, there exists a* u^0 *in* $\mathfrak{N}^0_J$ *such that* $e - u^0 \in J$.

Proof: Since $\mathfrak{N}^0_J$ is not empty, it is possible that one might pick out an element $u \in \mathfrak{N}^0_J$ arbitrarily. Then $\mathfrak{A}\cdot u + J$ is a left ideal which contains J properly, and since J is maximal, $\mathfrak{A}\cdot u + J$ coincides with $\mathfrak{A}$ itself. Hence e is represented as

$$e = a\cdot u + b, \quad a \in \mathfrak{A} \quad \text{and} \quad b \ni J,$$

that is,

$$b = e - a\cdot u,$$

where $a\cdot u \in \mathfrak{N}^0$ and further $\in \mathfrak{N}^0_J$. Putting $u^0 = a\cdot u$, such u^0 fulfils our statement.

The next Lemma is a direct result of the preceding.

Lemma 6. $J \supset \mathfrak{N}^0$ *if and only if* J *is quasi-proper, and true is this for a right maximal ideal.*

These facts together with the properties of the radical establish the equivalency of i) and ii) in Theorem 2. The fact that all left or right maximal ideals is quasi-proper implies all quasi-units are regular and conversely, and only the regularity of all quasi-units makes them into a group owing to Lemma 4. This is equivalent to the fact that $\mathfrak{N}^0$ becomes a group by the product (5.1), 0 being its identity. Thus Theorem 2 is completely proved.

Remark: If $\mathfrak{A}$ is a B*-algebra $\mathfrak{A}^*(\mathfrak{R})$, the product (5.1) has the property;

$$(a \times b)^* = b^* \times a^* . \tag{6.1}$$

The proof of i)→ii) in Theorem 3 is already mentioned and somewhat trivial. What remains for us to prove is that the commutativity of $\mathfrak{A}^*(\mathfrak{R})$ implies $\mathfrak{N}^0$ being empty, which comes from the following Theorem.

Theorem 4. (*R. V. Kadison*) *A commutative B*-algebra* $\mathfrak{A}^*(\mathfrak{R})$ *is represented as the ring of continuous functions on* $\mathfrak{M}$, *isomorphically and isometrically, where* $\mathfrak{M}$ *means the bicompact set of all maximal ideals of* $\mathfrak{A}(\mathfrak{R})$.

Remark 1: *R. V. Kadison* [7] has proved this Theorem as an application of a representation theory of strictly real algebras with Hewitt's condition, not analitically but ordered-algebraically, using the spectrum theory.

But in the following proof, we avail ourselves of I. Gelfand's results without reserve.

Remark 2: We leave the topologization and bicompactification of $\mathfrak{M}$ to *I. Gelfand and G. Šilov* [2].

Proof: i) $H(\mathfrak{A}^*)$ is dense in a commutative B-algebra $\overline{H(\mathfrak{A}^*)}$ over $\mathfrak{R}$ with $\| a^2 \| = \| a \|^2$ for every $a \in H(\mathfrak{A}^*)$, because $\| a^2 \| = \| a\cdot a^* \| = \| a \| \cdot \| a^* \|$

$= \| a \|^2$. Owing to *I. Gelfand* [1], $\overline{H(\mathfrak{A}^*)}$ is isometrically isomorphic to $C(\mathfrak{M}_H)$, which is the rings of all continuous (real) functions on the (weakly) bicompact set $\mathfrak{M}_H$ of all maximal ideal of $\overline{H(\mathfrak{A}^*)}$, since $\mathfrak{N}^0(\overline{H(\mathfrak{A}^*)}) = \mathfrak{N}^0(H(\mathfrak{A}^*)) = 0$.

Consequently, $\mathfrak{N}^0 \cap H(\mathfrak{A}^*)$ is empty, and further $\mathfrak{N}^0$ itself must venish, since $u \neq 0 \in \mathfrak{N}^0$ yields $u + u^* \neq 0$ in $H(\mathfrak{A}^*)$. Then Φ^0 is non-void, and $\| a \| = \sup_{\varphi \in \Phi^0} |\varphi(a)|$. By the way, to prove Theorem 3 only all these are sufficient.

ii) Every $a \in \mathfrak{A}^*(\mathfrak{K})$ is decomposed uniquely as follows;

$$a = a_1 + ia_2, \tag{6.2}$$

$$a^* = a_1 - ia_2, \tag{6.3}$$

where $a_1 = \dfrac{a+a^*}{2}$ and $a_2 = \dfrac{a-a^*}{2i}$, both of which belong to $H(\mathfrak{A}^*)$. Then we have, for $\varphi \in \Phi^0$

$$\begin{aligned} \| a \|^2 = \| a_1 + ia_2 \|^2 &\geqq |\varphi(a_1 + ia_2)|^2 \\ &= |\varphi(a_1) + i\varphi(a_2)|^2 \\ &= \varphi^2(a_1) + \varphi^2(a_2) \\ &= \varphi(a_1^2) + \varphi(a_2^2) = \varphi(a_1^2 + a_2^2). \end{aligned}$$

Since $a_1^2 + a_2^2 \in H(\mathfrak{A}^*)$, we see

$$\| a_1^2 + a_2^2 \| = \sup_{\varphi \in \Phi^0} |\varphi(a_1^2 + a_2^2)|.$$

Consequently, $\| a \| \geqq \sqrt{\| a_1^2 + a_2^2 \|}$ and similarly $\| a^* \| \geqq \sqrt{\| a_1^2 + a_2^2 \|}$, which brings us

$$\| a \| = \| a^* \| = \sqrt{\| a_1^2 + a_2^2 \|}, \tag{6.4}$$

for $\| a \| \cdot \| a^* \| = \| a \cdot a^* \| = \| a_1^2 + a_2^2 \|$,

Then we hold

$$\begin{aligned} \| a^2 \| &= \| a_1^2 - a_2^2 + 2i \cdot a_1 a_2 \| \\ &= \| (a_1^2 - a_2^2)^2 + 4a_1 a_2 \|^{\frac{1}{2}} \\ &= \| (a_1^2 + a_2^2)^2 \|^{\frac{1}{2}} = \| a_1^2 + a_2^2 \| = \| a \|^2. \end{aligned}$$

Due to I. Gefand again, we establish the isomertrically isomorphism; $\mathfrak{A}^*(\mathfrak{K}) \cong C(\mathfrak{M})$.

Corollary 4.1. For $\mathfrak{A}(\mathfrak{R})$ with $\| a^2 \| = \| a \|^2$, $\Phi(\mathfrak{A})$ is semi-full if and only if $\mathfrak{A}$ is commutative.

Corollary 4.2. If a B*-algebra $\mathfrak{A}^*(\mathfrak{K})$ is non-commutative, so is $\mathfrak{N}^0$.[5)]

(III)

7. As already mentioned, $C^1(\mathfrak{A})$ is weakly bicompact and Φ^0 is weakly closed in $C^1(\mathfrak{A})$, Φ^0 is also bicompact. Furthermore, $a(\varphi) \equiv \varphi(a)$, $a \in \mathfrak{A}$, is considered as a (uniformly) continuous function on Φ^0 with respect to the weak topology of Φ^0; $u(\varphi)$, $u \in \mathfrak{N}^0$, is a zero-function. Thus we get

5) If B*-algebra $\mathfrak{A}^*$ is non-commutative, $\Phi(\mathfrak{A}^*)$ is never semi-full from the semi-simplicity of $\mathfrak{A}^*$.

Theorem 5. *$\mathfrak{A}/\mathfrak{N}^0$ (or $\mathfrak{A}$ itself, if $\Phi(\mathfrak{A})$ is full), is isomorphically mapped into $C(\Phi^0)$, which is the ring of all continuous functions on bicompact Φ^0.*

The totality of such x that $\varphi(x)=0$ for a $\varphi \in \Phi^0$ formes clearly a two-sided maximal ideal of $\mathfrak{A}$, denoting it by $M(\varphi) \in \mathfrak{M}$.

Conversely, if $\mathfrak{A}$ is commutative, for every maximal ideal M of $\mathfrak{A}$, the Gelfand's function $M(a) \equiv a(M)$, $M \in \mathfrak{M}$, is multiplicative on $\mathfrak{A}$, denothing it by $\varphi(M) \in \Phi^0$. Thus, $\mathfrak{M}$ and Φ^0 is one-to-one corresponded;

$$(7.1) \qquad \begin{cases} M \in \mathfrak{M} \longrightarrow \varphi(M) \in \Phi^0, \\ \varphi \in \Phi^0 \longrightarrow M(\varphi) \in \mathfrak{M}, \end{cases}$$

and $M(\varphi(M))=M$, $\varphi(M(\varphi))=\varphi$. Since, Φ^0 and $\mathfrak{M}$ has the same topology, Φ^0 is homeomorphic to $\mathfrak{M}$;

$$(7.2) \qquad \Phi^0 \simeq \mathfrak{M}.$$

Corollary 5.1. *If $\mathfrak{A}$ is commutative, $\Phi(\mathfrak{A})$ is always semi-full, i.e. $\mathfrak{N}^0=\mathfrak{N}$.*

In fact, from (7.2) and Theorem 5,

$$\mathfrak{A}/\mathfrak{N}^0 \cong C(\Phi^0) \cong C(\mathfrak{M}) \cong \mathfrak{A}/\mathfrak{N},$$

and $\mathfrak{N}^0 \supset \mathfrak{N}$, which imply $\mathfrak{N}=\mathfrak{N}^0$; the last isomorphism is due to the representation theorem of normed rings, see *I. Gelfand* [1]. Thus Theorem 5 is regarded as a certain extension of this.

Thereby, for a commutative $\mathfrak{A}(\mathfrak{S})$, we hold

$$(7.3) \qquad \| a \| = \sup_{M \in \mathfrak{M}} |a(M)| = \sup_{\varphi \in \Phi^0} |\varphi(a)|,$$

which is already used in the proof of Theorem 4 above.

If $e-a^2$ has no inverse for every a of norm 1 in $\mathfrak{A}(\mathfrak{S})$, we say that $\mathfrak{A}(\mathfrak{S})$ has the Hewitt's property; see *E. Hewitt* [7].

Lemma 7. *If $\mathfrak{A}(\mathfrak{S})$ has the Hewitt's property,* $\| a^2 \| = \| a \|^2$.

Proof: Suppose that $\| a^2 \| < \| a \|^2$, then $a^2/\| a \|^2$ is of norm less than 1, that is,

$$\| e-(e-(a/\| a \|)^2)\| = \|(a^2/\| a \|^2)\| < 1.$$

By Hilb's theorem, $e-(a^2/\| a \|^2)$ has an inverse; this is contradictory with the Hewitt's property. Thus, it must be $\| a^2 \| = \| a \|^2$.

Corollary 5.2. *If $\mathfrak{A}(\mathfrak{R})$ with the Hewitt's property is commutative, $\mathfrak{A}(\mathfrak{R})$ is represented as real $C(\chi)$, χ being bicompact.*

This is proved by Corollary 4.1, and Theorem 5.

It is indicated by R. V. Kadison ingeniously that $\mathfrak{A}(\mathfrak{S})$ with the dense subalgebra which is strictly real, *i.e.* $e+a^2$ has always an inverse, and has the Hewitt's property, is commutative and hence represented as real $C(\chi)$, *R. V. Kadison* [7].

8. Finally, we shall consider the case that Φ^0 becomes a topological group. Let G be a maximally almost periodic group and $\mathfrak{A}_G{}^*$ the commutative B*-algebra of all continuous complex-valued almost periodic functions in G; such $\mathfrak{A}_G{}^*$ may be called an *almost periodic algebra* on G.

Now we shall consider $\Phi^0(\mathfrak{A}_G{}^*)$, which has ever been considered by *K. Yosida* [14] to prove the Tannaka-Krein's duality theorem. The following discussions have the same outlines as in [14] but some distinctions in the methods; these are rather in conjunction with our previous note [10].

Lemma 8. *G is dense in* $\Phi_G^0 \equiv \Phi^0(\mathfrak{A}_G{}^*)$ *with respect to the weak topology of* Φ_G^0.

Proof: We denote the elements of $\mathfrak{A}_G{}^*$ by a, b, etc. and those of G by x, y, etc. Then putting

$$\varphi_x(a) = a(x), \tag{8.1}$$

$\varphi_x \in \Phi_G^0$, and it is obvious that the mapping

$$x \in G \longrightarrow \varphi_x \in \Phi_G^0 \tag{8.2}$$

translates continuously and uniquely G into Φ_G^0.

As $\mathfrak{A}_G{}^*$ is full, $\mathfrak{A}_G{}^* \simeq C(\Phi_G^0)$ isomertrically from Theorem 5. If the closure $\overline{G}$ of the image of G by the mapping (8.2) is not equal to the whole Φ_G^0, then there exists a point φ_0 in $\Phi_G^0 - \overline{G}$ and a function f in $C(\Phi_G^0)$ such that

$$f(\varphi) = \begin{cases} 1 \text{ for } \varphi = \varphi_0, \\ 0 \text{ for } \varphi \text{ in } \overline{G}, \end{cases} \tag{8.3}$$

by Urysohn's theorem. But it is contradictory with $\mathfrak{A}_G{}^* \simeq C(\Phi_G^0)$.

Thus, the assumption must be taken away, and the proof is complete.

Now we recall the bicompact group $\tilde{\mathfrak{G}}_0$ in [10], which is proved to contain G as a dense subgroup; Φ_G^0 must be in agreement with $\tilde{\mathfrak{G}}_0$ as a topological space. According to the product of $\tilde{\mathfrak{G}}_0$, Φ_G^0 turns out a bicompact group.

Conversely, if $\Phi(\mathfrak{A}^*)$ is a group for a commutative B*-algebra $\mathfrak{A}^*(\mathfrak{R})$, it holds that $\mathfrak{A}^*(\mathfrak{R}) \simeq C(\Phi^0)$ by Theorem 5, and hence $\mathfrak{A}^*$ is considered as an almost periodic algebra on Φ^0. Thus, we establish:

Theorem 6. *A necessary and sufficient condition that* $\Phi^0(\mathfrak{A}^*)$ *be a compact group for a commutative* $\mathfrak{A}^*(\mathfrak{R})$ *is that* $\mathfrak{A}^*$ *is isometrically isomorphic to an almost periodic algebra on a certain maximally almost periodic group G. In the case,* Φ^0 *contains G as a dense subgroup. If G is bicompact,* $G \simeq \Phi^0$.

Summerizing the above obtained results, we shall denote the family of all bicompact Hausdorff spaces by Θ, and that of all commutative B*-algebras by Ω and consider the mapping $\sim$ between Θ and Ω;

$$\left\{\begin{array}{lll} \sim X = C(X) & \text{for} & X \in \Theta;\ \Theta \longrightarrow \Omega, \\ \sim A = \Phi(A) & \text{for} & A \in \Omega;\ \Psi \longrightarrow \Theta, \end{array}\right. \tag{8.4}$$

then we hold;

$$\sim\sim X = X \quad \text{and} \quad \sim\sim A = A, \tag{8.5}$$

and if X is replaced by a bicompact group G,

$$\sim\sim G = G, \tag{8.6}$$

which is just the *duality theorem* of T. Tannaka.

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