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On Conditions in order that two Uniform Spaces are uniformly homeomorphic

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The topology of a compact space is, as well known, characterized by the lattice of its closed basis.

The purpose of this paper is to establish analogous theories in the case of complete uniform spaces, i.e. we characterize the uniform topology of a uniform space by the lattice of its uniform basis. Obviously, it is impossible to characterize the uniform topology of a complete space by the lattice of an arbitrary uniform basis[1], for every metric space has the same lattice of uniform basis $\{\mathfrak{M}_1, \mathfrak{M}_2, \mathfrak{M}_3, \ldots\}$.

For example, let us consider the reason why the ring of all continuous functions characterizes the topology of a compact space. Then we shall see that it is the reason that for an arbitrary point a and its arbitrary neighbourhood U, there exists a continuous function f such that $f(a)=0$, $f(x)=1$ $(x \in U^c)$. Similarly if we assume the existence of uniform coverings with some local properties in the lattice of uniform basis, then we can characterize the uniform topology of a complete uniform space by such a lattice.

We concern ourselves with the lattice $L(R)$ of uniform basis of a complete uniform space R, satisfying the following conditions,

1) *if* $\mathfrak{M}_x, \mathfrak{M}_y \in L(R)$, *then* $\mathfrak{M}_x \times \mathfrak{M}_y \in L(R)$,

2) *if* $\mathfrak{M} \ni L(R)$, *then for an arbitrary* $M \in \mathfrak{M}$ *there exists* $\mathfrak{M}(M)$ *in* $L(R)$ *such that* i) $M \notin \mathfrak{M}(M)$;
ii) $\mathfrak{M} \ni M' \not\supset M$ *implies* $M' \in \mathfrak{M}(M)$,

3) *for an arbitrary point* $a \in R$ *and its arbitrary neighbourhood* $U(a)$, *there exists* $\mathfrak{M}'(U(a))$ *in* $L(R)$ *such that* $S(a, \mathfrak{M}'(U(a))) \subset U(a)$[2], *for a uniform covering* $\mathfrak{M}'(a)$ *defined for* a, $a \notin M \in \mathfrak{M}'(U(a))$ *implies* $M \notin \mathfrak{M}'(a)$,

or 2′) *if* $\mathfrak{M} \in L(R)$, *then for an arbitrary* $M \in \mathfrak{M}$, *there exists* $\mathfrak{M}(M)$ *in* $L(R)$ *such that* i) $M \notin \mathfrak{M}(M)$;
ii) $M' \in \mathfrak{M}$, $\overline{M'} \not\supset M$ *imply* $M' \in \mathfrak{M}(M)$,

3′) *for an arbitrary point* $a \in R$ *and its arbitrary neighbourhood* $U(a)$,

1) If $L(R)$ is a set of open uniform coverings of R, and if for every uniform covering $\mathfrak{U}$ of R, there exists some element $\mathfrak{M}$ of $L(R)$ such that $\mathfrak{M} < \mathfrak{U}$, then we call $L(R)$ a uniform basis of R.

2) $S(a, \mathfrak{M}) = \cup \{M \mid a \in M \in \mathfrak{M}\}$.

there exist a neighbourhood V *of* a *and a uniform covering* $\mathfrak{M}'(U(a))$ *in* $L(R)$ *such that* $S(V, \mathfrak{M}'(U(a)))\subset U(a)$;
for a uniform covering $\mathfrak{M}'(a)$ *defined for* a, $M\in\mathfrak{M}'(U(a))$, $M\frown V=\phi$ *imply* $M\notin\mathfrak{M}'(a)$.

Remarks 1. The conditions 2) and 2′) show the existence of uniform coverings which do not include a given uniform covering locally.

3) and 3′) show the existence of a uniform covering which is included by a given uniform covering locally.

2. When we regard two coverings $\mathfrak{M}$, $\mathfrak{N}$ such that $\mathfrak{M}<\mathfrak{N}$, $\mathfrak{M}>\mathfrak{N}$ as the same covering, $L(R)$ is a lattice; otherwise $L(R)$ is a directed system.

The following propositions are valid whichever we may choose, but we regard here $L(R)$ as a lattice for the simplicity of notations. Hence, for example $M\notin\mathfrak{M}$ means that for every open set $M'\supset M$, $M'\notin\mathfrak{M}$ holds.

3. In this paper German capitals $\mathfrak{M}$, $\mathfrak{N}$, $\mathfrak{U}$, $\mathfrak{P}$, $\mathfrak{Q}$, ... are used for uniform open coverings of $L(R)$ except in Lemma 5 and corollaries.

4. Condition 2′) is weaker than 2), and 3′) is stronger than 3). $\mathfrak{M}(M)$ in 2) and 2′) are strictly denoted by $\mathfrak{M}(M, \mathfrak{M})$, for $\mathfrak{M}(M)$ is defined by M and $\mathfrak{M}$.

Definition. For two uniform coverings $\mathfrak{M}$, $\mathfrak{N}$, we denote by $\mathfrak{M}<\mathfrak{N}$ the fact that $\mathfrak{M}\not<\mathfrak{P}$ and $\mathfrak{M}<\mathfrak{P}\vee\mathfrak{N}$ hold for some uniform covering $\mathfrak{P}$.

Definition. A subset μ of $L(R)$ satisfying the following conditions is called an *n.b.-set*,
there exists a uniform covering $\mathfrak{M}(\mu)$ for μ such that

(A) for an arbitrary uniform covering $\mathfrak{M}_x$: $\mathfrak{M}_x<\mathfrak{M}'(\mu)$,
there exists $\mathfrak{M}$ in μ such that

i) $\mathfrak{M}<\mathfrak{M}_x$,

ii) $\mathfrak{M}\not<\mathfrak{P}_1$, $\mathfrak{P}_2$ and $\mathfrak{M}<\mathfrak{P}_i\vee\mathfrak{M}'(\mu)$ $(i=1, 2)$ imply $\mathfrak{M}_x\not<\mathfrak{P}_1\vee\mathfrak{P}_2$ for arbitrary uniform coverings $\mathfrak{P}_1$, $\mathfrak{P}_2$,

(We shall denote such a relation between $\mathfrak{M}$ and $\mathfrak{M}_x$ by $\mathfrak{M}\leq\mathfrak{M}_x(\mathfrak{M}'(\mu))$ or simply $\mathfrak{M}\leq\mathfrak{M}_x$.)

(B) for arbitrary coverings $\mathfrak{M}_i$ $(i=1, \dots, k)$ of μ and for arbitary uniform covering $\mathfrak{U}$, there exists a uniform covering $\mathfrak{P}$ such that

$$\mathfrak{U}\not<\mathfrak{P},\quad \bigvee_{i=1}^{k}\mathfrak{M}_i<\mathfrak{M}'(\mu)\vee\mathfrak{P}.$$

Lemma 1. *The family* $\mu(a)$ *of all uniform coverings* $\mathfrak{M}'(U(a))\in L(R)$ *which are defined for neighbourhoods* $U(a)$ *of* a *such that* $U(a)\in\mathfrak{M}'(a)$ *by the condition* 3) *or* 3′) *is a n.b.-set if we consider* $\mathfrak{M}'(a)$ *in* 2) *or in* 2′) *as* $\mathfrak{M}'(\mu(a))$ *in the condition of n.b.-set.*

Proof. Firstly we show that $\mu(a)$ satisfies (A) when $L(R)$ satisfies 1), 2), 3). If $\mathfrak{M}_x<\mathfrak{M}'(a)$, then we denote by $\mathfrak{M}$ the uniform covering $\mathfrak{M}'(M_x)$ in condition 3) for M_x such that $a \in M_x \in \mathfrak{M}_x$. And we denote by $\mathfrak{P}$ the uniform covering $\mathfrak{M}(M)$ in condition 2) for $M \in \mathfrak{M}$ such that $a \in M \subset M_x$. Then obviously, $\mathfrak{M} \not< \mathfrak{P}$, $\mathfrak{M} < \mathfrak{P} \vee \mathfrak{M}_x$, i.e. $\mathfrak{M} < \mathfrak{M}_x$.

Now let $\mathfrak{M} \not< \mathfrak{P}_i$, $\mathfrak{M} < \mathfrak{P}_i \vee \mathfrak{M}'(a)$ $(i=1, 2)$, then since $a \notin M \in \mathfrak{M}$ implies $M \notin \mathfrak{M}'(a)$, there exist $M_1, M_2 \in \mathfrak{M}$ such that $a \in M_1 \notin \mathfrak{P}_1$, $a \in M_2 \notin \mathfrak{P}_2$. Hence from $S(a, \mathfrak{M}) \subset M_x$, we get $M_x \notin \mathfrak{P}_1 \vee \mathfrak{P}_2$, i.e. $\mathfrak{M}_x \not< \mathfrak{P}_1 \vee \mathfrak{P}_2$.

$\mu(a)$ satisfies also (B). If $\mathfrak{M}_i \in \mu(a)$ $(i=1, \dots, k)$ and $\mathfrak{U} \in L(R)$, then putting $\mathfrak{P} = \mathfrak{M}(U, \vee \mathfrak{M}_i)$ for U such that $a \in U \in \mathfrak{U}$, we get $\mathfrak{U} \not< \mathfrak{P}$ and $\bigvee_{i=1}^{k} \mathfrak{M}_i < \mathfrak{P} \vee \mathfrak{M}'(a)$.

Next we consider the case that $L(R)$ satisfies 1), 2′), 3′). If $\mathfrak{M}_x < \mathfrak{M}'(a)$, then we denote by $\mathfrak{M}$ the uniform covering $\mathfrak{M}'(M_x)$ in 3′) for M_x such that that $a \in M_x \in \mathfrak{M}_x$, and by $\mathfrak{P}$ $\mathfrak{M}(M)$ in 2′). Then $\mathfrak{M} < M_x$.

If $\mathfrak{M} \not< \mathfrak{P}_i$, $\mathfrak{M} < \mathfrak{P}_i \vee \mathfrak{M}'(a)$ $(i=1, 2)$, then since $M \frown V = \phi$, $M \in \mathfrak{M}$ imply $M_i \notin \mathfrak{P}_i$ $(i=1, 2)$, there exist $M_i \in \mathfrak{M}$ $(i=1, 2)$ such that $M_i \notin \mathfrak{P}_i$, $M_i \frown V \neq \phi$. Since $S(V, \mathfrak{M}) \subset M_x$, we get $M_x \notin \mathfrak{P}_1 \vee \mathfrak{P}_2$, i.e. $\mathfrak{M}_x \not< \mathfrak{P}_1 \vee \mathfrak{P}_2$. $\mu(a)$ satisfies also (B), for $\mathfrak{M}_i \in \mu(a)$ $(i=1, \dots, k)$, $\mathfrak{U} \in L(R)$ imply $\mathfrak{U} \not< \mathfrak{P}$, $\vee \mathfrak{M}_i < \mathfrak{P} \vee \mathfrak{M}'(a)$ for $a \in U \in \mathfrak{U}$, $\mathfrak{P} = \mathfrak{M}(U, \vee \mathfrak{M}_i)$.

Lemma 2. *If $L(R)$ satisfies conditions 1), 2′), and if $\mu \ni \mathfrak{M} \leq \mathfrak{M}_x$ $(\mathfrak{M}'(\mu))$ for an n.b.-set μ, then there exists $M_x \in \mathfrak{M}_x$ such that if $\mathfrak{M} \ni M \notin \mathfrak{P}$, $\mathfrak{M} < \mathfrak{P} \vee \mathfrak{M}'(\mu)$ for any $\mathfrak{P} \in L(R)$, then $M \subset S(S(M_x, \mathfrak{M}_x), \mathfrak{M}_x)$.*

Proof. Let $\mu \ni \mathfrak{M} \leq \mathfrak{M}_x$ $(\mathfrak{M}'(\mu))$, then since $\mathfrak{M} < \mathfrak{M}_x < \mathfrak{M}'(\mu)$, there exists $M_1 \in \mathfrak{M}$ such that $M_1 \subset M_x \in \mathfrak{M}_x$; $M_1 \notin \mathfrak{P}_1'$, $\mathfrak{M} < \mathfrak{P}_1' \vee \mathfrak{M}'(\mu)$ for some $\mathfrak{P}_1'$. Then we can show that $M_2 \notin \mathfrak{P}_2$, $\mathfrak{M} < \mathfrak{P}_2' \vee \mathfrak{M}'(\mu)$ for an arbitrary $\mathfrak{P}_2'$ imply $M_2 \subset S(S(M_x, \mathfrak{M}_x), \mathfrak{M}_x)$.

To see this, assume that there exist M_2 and $\mathfrak{P}_2'$ such that $M_2 \notin \mathfrak{P}_2'$, $\mathfrak{M} < \mathfrak{P}_2' \vee \mathfrak{M}'(\mu)$ and $M_2 \not\subset S(S(M_x, \mathfrak{M}_x), \mathfrak{M}_x)$. Then putting $\mathfrak{Q}_1 = \mathfrak{M}(M_1, \mathfrak{M}_x)$, $\mathfrak{Q}_2 = \mathfrak{M}(M_2, \mathfrak{M}_x)$ by condition 2′) and $\mathfrak{P}_1 = \mathfrak{P}_1' \vee \mathfrak{Q}_1$, $\mathfrak{P}_2 = \mathfrak{P}_2' \vee \mathfrak{Q}_2$. we get $\mathfrak{M} \not< \mathfrak{P}_i$, $\mathfrak{M} < \mathfrak{P}_i' \vee \mathfrak{M}'(\mu) < \mathfrak{P}_i \vee \mathfrak{M}'(\mu)$ $(i=1, 2)$ from $M_i \notin \mathfrak{P}_i' \vee \mathfrak{Q}_i$ $(i=1, 2)$.

In the other hand let M be an arbitrary element of $\mathfrak{M}_x$, then when $M \frown M_x = \phi$, we get $\overline{M} \frown M_x = \phi$, $\overline{M} \not\supset M_1$, and accordingly $M \in \mathfrak{Q}_1 < \mathfrak{P}_1$. When $M \frown M_x = \phi$, we get $M \subset S(M_x, \mathfrak{M}_x)$, $\overline{M} \subset S(S(M_x, \mathfrak{M}_x), \mathfrak{M}_x) \not\supset M_2$, $\overline{M} \not\supset M_2$, and accordingly $M \in \mathfrak{Q}_2 < \mathfrak{P}_2$. Therefore we get $\mathfrak{M}_x < \mathfrak{P}_1 \vee \mathfrak{P}_2$, which contradicts the condition (A) of μ.

We shall denote by $S(\mathfrak{M}, \mathfrak{M}_x)$ $S(S(M_x, \mathfrak{M}_x), \mathfrak{M}_x)$ in this lemma, which is an element of $\mathfrak{M}_x^{**}$.

Lemma 3. *If $L(R)$ satisfies the conditions 1), 2′) and if μ is a n.b.-set of $L(R)$, then $\bigcap_{i=1}^{k} S(\mathfrak{M}_i, \mathfrak{M}_i') \neq \phi$ for arbitrary $\mathfrak{M}_i, \mathfrak{M}_i'$ such that $\mu \ni \mathfrak{M}_i \leq \mathfrak{M}_i' < \mathfrak{M}'(\mu)$ $(i=1, \dots, k)$.*

Proof. Assume that $\bigcap_{i=1}^{k} S(\mathfrak{M}_i, \mathfrak{M}_i') = \phi$, $\mu \ni \mathfrak{M}_i \leq \mathfrak{M}_i' < \mathfrak{M}'(\mu)$ $(i=1, \dots, k)$, then we see that $\bigvee_{i=1}^{k}\mathfrak{M}_i<\mathfrak{M}'(\mu)\vee\mathfrak{P}$ implies $\mathfrak{U}=\bigwedge_{i=1}^{k}\mathfrak{M}_i<\mathfrak{P}$, which contradicts the condition (B) of μ.

Let $U=\bigcap_{i=1}^{k}M_i$ $(M_i\in\mathfrak{M}_i)$ be an arbitrary element of $\mathfrak{U}=\wedge\mathfrak{M}_i$, then since $\bigcap_{i=1}^{k}S(\mathfrak{M}_i,\mathfrak{M}_i')=\phi$ by the assumption, $M_i\not\subset S(\mathfrak{M}_i,\mathfrak{M}_i')$ for some i. Hence $M_i\notin\mathfrak{P}$ implies $\mathfrak{M}\not<\mathfrak{M}'(\mu)\vee\mathfrak{P}$; hence from $\mathfrak{M}<\mathfrak{M}'(\mu)\vee\mathfrak{P}$ we get $M_i\in\mathfrak{P}$. Therefore we get $U\in\mathfrak{P}$, i.e. $\mathfrak{U}<\mathfrak{P}$.

By this lemma $\{\overline{S(\mathfrak{M},\mathfrak{M}_x)}\,|\,\mu\ni\mathfrak{M}\leq\mathfrak{M}_x<\mathfrak{M}'(\mu)\}$ is a cauchy filter.

Therefore from the completeness of R we get $\bigcap\{\overline{S(\mathfrak{M},\mathfrak{M}_x)}\,|\,\mu\ni\mathfrak{M}\leq\mathfrak{M}_x<\mathfrak{M}'(\mu)\}=a\in R$. Thus a point $a\in R$ corresponds to an n.b.-set μ. We shall denote by $\mu\to a$ this correspondence.

Definition. For two n.b.-sets μ,ν, we shall denote by $\mu\sim\nu$ the following relation between μ and ν,
for each $\mathfrak{M}_x\in L(R)$, there exists $\mathfrak{M}_0\in L(R)$ such that $\mathfrak{M}_0<\mathfrak{M}'(\mu)\wedge\mathfrak{M}'(\nu)$; if $\mathfrak{M}_1\in\mu$, $\mathfrak{M}_2\in\nu$, $\mathfrak{M}_1\leq\mathfrak{M}_0$, $\mathfrak{M}_2\leq\mathfrak{M}_0$ and if $\mathfrak{M}_1\not<\mathfrak{P}_1$, $\mathfrak{M}_1<\mathfrak{P}_1\vee\mathfrak{M}_0$, $\mathfrak{M}_2\not<\mathfrak{P}_2$, $\mathfrak{M}_2<\mathfrak{P}_2\vee\mathfrak{M}_0$, then $\mathfrak{M}_x\not<\mathfrak{P}_1\vee\mathfrak{P}_2$ holds.

Lemma 4. *In order that $\mu\sim\nu$, it is necessary and sufficient that μ and ν correspond to one and the same point a by the above correspondence. i.e. $\mu\to a$, $\nu\to a$.*

Proof. Firstly we show that $\mu\to a$, $\nu\to a$ imply $\mu\sim\nu$. Let $\mathfrak{M}_x$ be an arbitrary covering of $L(R)$, and let $a\in M_x\in\mathfrak{M}_x$. Then let us take $\mathfrak{M}_0\in L(R)$ such that $S(a,\overline{\mathfrak{M}_0^{**}})\subset M_x$, $\mathfrak{M}_0<\mathfrak{M}'(\mu)\wedge\mathfrak{M}'(\nu)$.[3)]

If $\mathfrak{M}_1\in\mu$, $\mathfrak{M}_2\in\nu$, $\mathfrak{M}_i\leq\mathfrak{M}_0$ $(i=1,2)$, then since $a\in\overline{S(\mathfrak{M}_i,\mathfrak{M}_0)}$, we get $S(\mathfrak{M}_i,\mathfrak{M}_0)\subset M_x$ $(i=1,2)$. Let $\mathfrak{P}_1,\mathfrak{P}_2$ be arbitrary uniform coverings such that $\mathfrak{M}_i\not<\mathfrak{P}_i$, $\mathfrak{M}_i<\mathfrak{P}_i\vee\mathfrak{M}_0$ $(i=1,2)$, then since $\mathfrak{M}_0<\mathfrak{M}'(\mu)\wedge\mathfrak{M}'(\nu)$, there exist $M_i\in\mathfrak{M}_i$ $(i=1,2)$ such that $M_i\notin\mathfrak{P}_i$, $M_i\subset S(\mathfrak{M}_i,\mathfrak{M}_0)$ $(i=1,2)$. Hence $M_i\subset M_x$; hence we get $M_x\notin\mathfrak{P}_1\vee\mathfrak{P}_2$, i.e. $\mathfrak{M}_x\not<\mathfrak{P}_1\vee\mathfrak{P}_2$.

Conversely we show that $\mu\to a$, $\nu\to b$, $a\neq b$ imply $\mu\not\sim\nu$ (the negation of $\mu\sim\nu$). Take a definite $\mathfrak{M}_x$ such that $S(S(a,\mathfrak{M}_x)\mathfrak{M}_x)\frown S(b,\mathfrak{M}_x)=\phi$. Then let $\mathfrak{M}_0$ be an arbitrary uniform covering of $L(R)$ such that $\mathfrak{M}_0<\mathfrak{M}'(\mu)\wedge\mathfrak{M}'(\nu)$. For these $\mathfrak{M}_x$ and $\mathfrak{M}_0$, we take $\mathfrak{N}_i$ $(i=1,2)$ such that $S(a,\overline{\mathfrak{N}_1^{**}})\subset M'\in\mathfrak{M}_x$, $S(b,\overline{\mathfrak{N}_2^{**}})\subset M''\notin\mathfrak{M}_x$ and $\mathfrak{M}_i$ $(i=1,2)$ such that $\mathfrak{M}_1\in\mu$, $\mathfrak{M}_2\in\nu$; $\mathfrak{M}_1\leq\mathfrak{N}_1<\mathfrak{M}_0$, $\mathfrak{M}_2\leq\mathfrak{N}_2<\mathfrak{M}_0$. Then $M_1\subset S(\mathfrak{M}_1,\mathfrak{N}_1)\subset M'$ and $M_2\subset S(\mathfrak{M}_2,\mathfrak{N}_2)\subset M''$ hold for $M_i\in\mathfrak{M}_i$ such that $M_i\notin\mathfrak{P}_2'$, $\mathfrak{M}_i<\mathfrak{P}_i'\vee\mathfrak{M}_0$ $(i=1,2)$. Putting $\mathfrak{Q}_i'=\mathfrak{M}(M_i,\mathfrak{M}_i\vee\mathfrak{M}_x)$, $\mathfrak{P}_i=\mathfrak{P}_i'\vee\mathfrak{Q}_i'$ $(i=1,2)$, we get $\mathfrak{M}_i<\mathfrak{P}_i'\vee\mathfrak{M}_0<\mathfrak{P}_i\vee\mathfrak{M}_0$ and $\mathfrak{M}_i\not<\mathfrak{P}_i$ from $M_i\notin\mathfrak{P}_i$.

Next let M be an arbitrary element of $\mathfrak{M}_x$, then either $M\frown S(a,\mathfrak{M}_x)=\phi$ or $M\frown S(b,\mathfrak{M}_x)=\phi$ holds from the property of $\mathfrak{M}_x$.

If $M\frown S(a,\mathfrak{M}_x)=\phi$, then $M\frown M'=\phi$ and $\overline{M}\not\supset M_1$; hence $M\in\mathfrak{Q}_1'<\mathfrak{P}_1$.

3) $\overline{\mathfrak{M}}=\{\overline{M}\,|\,M\in\mathfrak{M}\}$.

If $M \frown S(b, \mathfrak{M}_x)=\phi$, then $M \in \mathfrak{Q}_2' < \mathfrak{P}_2$ holds in the same way. Therefore we get $\mathfrak{M}_x < \mathfrak{P}_1 \vee \mathfrak{P}_2$, i.e. $\mu \sim \nu$.

Now we classify all n.b.-sets of $L(R)$ by the relation $\sim$, and we denote by by $\mathfrak{L}(R)$ the family of all classes. Then there exists a one-to-one correspondence between R and $\mathfrak{L}(R)$ as we have seen. We denote by $\mathfrak{L}(A)$ the image of a subset A of R in $\mathfrak{L}(R)$ by this correspondence.

We define uniform coverings in this set $\mathfrak{L}(R)$ as follows.

Definition. We call a covering $\{\mathfrak{L}(U_\alpha)\}$ of $\mathfrak{L}(R)$ a uniform covering of $\mathfrak{L}(R)$ when $\{\mathfrak{L}(U_\alpha)\}$ satisfies the following condition:

When we choose an element μ from each class $\{\mu\}$ of $\mathfrak{L}(R)$ in an arbitrary way, there exists $\mathfrak{U} \in \mathfrak{L}(R)$ such that
if $\mu_\alpha \notin \mathfrak{L}(U_\alpha)$ $(\mathfrak{L}(U_\alpha) \in \{\mathfrak{L}(U_\alpha)\}$, then we can choose certain μ_1, μ_2 from $\{\mu_\alpha\}$ so that there exist $\mathfrak{M}_{0i} < \mathfrak{M}'(\mu_i)$ $(i=1, 2)$ for which $\mu_i \ni \mathfrak{M}_i \leq \mathfrak{M}_{0i}$ $(i=1, 2)$ imply $\mathfrak{M}_i < \mathfrak{P}_i \vee \mathfrak{M}_{0i}$, $\mathfrak{M}_i \not< \mathfrak{P}_i \vee \mathfrak{N}_i$ and $\mathfrak{U} < \mathfrak{N}_1 \vee \mathfrak{N}_2$ for certain $\mathfrak{N}_i$, $\mathfrak{P}_i$ $(i=1, 2)$.

Lemma 5. *In order that* $\{\mathfrak{L}(U_\alpha)\}$ *is a uniform covering of* $\mathfrak{L}(R)$, *it is necessary and sufficient that* $\{U_\alpha\}$ *is a uniform covering of* R.

Proof. Firstly we prove that if $\{U_\alpha\}$ is a uniform covering of R, then $\{\mathfrak{L}(U_\alpha)\}$ is a uniform covering of $\mathfrak{L}(R)$. We take $\mathfrak{U} \in L(R)$ such that $\mathfrak{U}^{*\Delta} < \{U_\alpha\}$.

Let $\mu_\alpha \notin \mathfrak{L}(U_\alpha)$, $\mu_\alpha \rightarrow a_\alpha$, then $a_\alpha \notin U_\alpha$. Now we see that for an arbitrary point a_1 of $\{a_\alpha\}$ and some a_2 of $\{a_\alpha\}$, $a_2 \notin S(a_1, \mathfrak{U}^*)$. For $S(a_1, \mathfrak{U}^*) \subset U_\alpha$ for some α from $\mathfrak{U}^{*\Delta} < \{U_\alpha\}$; hence $a_\alpha \notin S(a_1, \mathfrak{U}^*)$ from $a_\alpha \notin U_\alpha$. We denote by a_2 this a_α.

We take μ_i such that $\mu_i \rightarrow a_i$ $(i=1. 2)$ as the ones in the above definition. And we take $\mathfrak{M}_{0i} \in L(R)$ $(i=1, 2)$ such that $\mathfrak{M}_{0i} < \mathfrak{M}'(\mu_i)$; $S(a_i, \overline{\mathfrak{M}_{0i}^{**}}) \subset U_i'$, $a_i \in U_i' \in \mathfrak{U}$. If M_i $(i=1, 2)$ is arbitrary coverings such that $\mu_i \ni \mathfrak{M}_i \leq \mathfrak{M}_{0i}$, then there exist $\mathfrak{P}_i$ and $M_i \in \mathfrak{M}_i$ $(i=1, 2)$ such that $M_i \notin \mathfrak{P}_i$, $\mathfrak{M}_i < \mathfrak{P}_i \vee \mathfrak{M}_{0i}$ and then $a_i \in \overline{S(\mathfrak{M}_i, \mathfrak{M}_{0i})} \supset M_i$ $(i=1, 2)$; hence $M_i \subset V_i'$ $(i=1, 2)$. Putting $\mathfrak{N}_i = \mathfrak{M}(M_i, \mathfrak{M}_i \vee \mathfrak{U})$ by conditions 2'), we get $\mathfrak{M}_i \not< \mathfrak{P}_i \vee \mathfrak{N}_i$. Then each $U \in \mathfrak{U}$ satisfies either $a_1 \notin S(U, \mathfrak{U})$ or $a_2 \notin (U, \mathfrak{U})$. If $a_1 \notin S(U, \mathfrak{U})$, then $U_1' \frown U=\phi$, and $\bar{U} \frown M_1=\phi$; hence $\bar{U} \not\supset M_1$ and accordingly $U \in \mathfrak{N}_1$. If $a_2 \notin S(U, \mathfrak{U})$, then $U \in \mathfrak{N}_2$ in the same way. Therefore $\mathfrak{U} < \mathfrak{N}_1 \vee \mathfrak{N}_2$, i.e. $\{\mathfrak{L}(U_\alpha)\}$ is a uniform covering of $\mathfrak{L}(R)$.

Conversely we show that if $\{U_\alpha\}$ is not a uniform covering of R, then $\{\mathfrak{L}(U_\alpha)\}$ is not a uniform covering of $\mathfrak{L}(R)$. Let $\mathfrak{U}$ be a certain uniform covering of $L(R)$, then since $\{U_\alpha\}$ is not uniform covering, $\mathfrak{U} \not< \{U_\alpha\}$ holds; hence there exists $U \in \mathfrak{U}$ such that $U \frown U_\alpha^c \neq \phi$ for all $U_\alpha \in \{U_\alpha\}$. Take $a_\alpha \in U \frown U_\alpha^c$ for each U_α, then $\mu_\alpha \notin \mathfrak{L}(U_\alpha)$ for μ_α such that $\mu_\alpha \rightarrow a_\alpha$. We chose μ_1, μ_2 from $\{\mu_\alpha\}$ in an arbitrary way, and consider arbitrary uniform coverings $\mathfrak{M}_{0i}$ $(i=1, 2)$ such that $\mathfrak{M}_{0i} < \mathfrak{M}'(\mu_i)$. For these μ_i and $\mathfrak{M}_{0i}$, we choose $\mathfrak{N}_{0i}$ and $\mathfrak{M}_i \in \mu_i$ $(i=1, 2)$ such that $S(a_i, \overline{\mathfrak{N}_{0i}^{**}}) \subset U$, $\mathfrak{M}_i \leq \mathfrak{N}_{0i} < \mathfrak{M}_{0i}$ $(i=1, 2)$. Then $a_i \in \overline{S(\mathfrak{M}_i, \mathfrak{N}_{0i})} \subset U$.

Now assume that $\mathfrak{M}_i < \mathfrak{P}_i \vee \mathfrak{M}_{0i}$ and $\mathfrak{U} < \mathfrak{N}_1 \vee \mathfrak{N}_2$. If $U \in \mathfrak{N}_1$, then for each element M of $\mathfrak{M}_1$, we see that $M \in \mathfrak{N}_1$ when $M \subset S(\mathfrak{M}_1, \mathfrak{N}_{01}) \subset U$ and that $M \in \mathfrak{P}_1$ from $\mathfrak{M}_1 < \mathfrak{P}_1 \vee \mathfrak{M}_{01}$ when $M \not\subset S(\mathfrak{M}_1, \mathfrak{N}_{01})$. Hence $\mathfrak{M}_1 < \mathfrak{P}_1 \vee \mathfrak{N}_1$. If $U \in \mathfrak{N}_2$, we see that $\mathfrak{M}_2 < \mathfrak{P}_2 \vee \mathfrak{N}_2$ in the same way. Hence $\{\mathfrak{L}(U_\alpha)\}$ is not a uniform covering of $\mathfrak{L}(R)$ by the definition of uniform covering of $\mathfrak{L}(R)$.

By this lemma we see that $\mathfrak{L}(R)$ with the uniform coverings defined above is a uniform space uniformly homeomorphic with R. Since the uniform space $\mathfrak{L}(R)$ is defined only by the lattice-order $<$ from $L(R)$, if R_1 and R_2 are complete uniform spaces, a lattice isomorphism between $L(R_1)$ and $L(R_2)$ generates a uniform homeomorphism between $\mathfrak{L}(R_1)$ and $\mathfrak{L}(R_2)$. Hence we get the following theorem.

Theorem. *In order that two complete uniform spaces R_1 and R_2 are uniformly homeomorphic, it is necessary and sufficient that $L(R_1)$ and $L(R_2)$ are lattice-isomorphic, where $L(R_1)$ and $L(R_2)$ are lattices of uniform bases satisfying conditions* 1), 2), 3) *or conditions* 1), 2′), 3′).

Corollary 1. *If R is a complete uniform space without isolated point, then the uniform topology of R is characterized by any lattice $L(R)$ of uniform basis of R satisfying the following three conditions,*

(1) $\mathfrak{M}_x, \mathfrak{M}_y \in L(R)$ *implies* $\mathfrak{M}_x \wedge \mathfrak{M}_y \in L(R)$,
$\mathfrak{M}_\alpha \in L(R)$ *imply* $\bigvee_\alpha \mathfrak{M}_\alpha \in L(R)$,

(2) *there exist a certain uniform covering* $\mathfrak{M} \in L(R)$ *and* $\mathfrak{M}'$ *such that*
$$M \in \mathfrak{M} \text{ implies } M \notin \mathfrak{M}'.$$

(3) *if* $\mathfrak{M} \in L(R)$, $a \in R$, *and* $U(a)$ *is a neighbourhood of* a, *then for some uniform covering* $\mathfrak{U} = \{U_0, U\}$ *such that* $U_0 \subset U(a)$, $a \notin \bar{U}$.
$$\mathfrak{U} \circ \mathfrak{M} \in L(R) \text{ holds.}^{4)}$$

Proof. If $a \in M \in \mathfrak{M}$, then since R has no isolated point, there exist points b, c such that $b, c \in M$; $b \neq c$; $b, c \neq a$, Take $\mathfrak{M}(a)$ such that $c \notin S(b, \mathfrak{M}(a))$, then for $\mathfrak{M}'(a) = \mathfrak{M}' \wedge \mathfrak{M}(a)$, $L(R)$ satisfies conditions 1), 2′), 3′).

Corollary 2 *If R is a connected complete space, then the uniform topology of R is characterized by any lattice $L(R)$ of uniform basis satisfying the following three conditions.*

(1) $\mathfrak{M}_x, \mathfrak{M}_y \in L(R)$ *implies* $\mathfrak{M}_x \wedge \mathfrak{M}_y \in L(R)$,
$\mathfrak{M}_\alpha \in L(R)$ *imply* $\bigvee_\alpha \mathfrak{M}_\alpha \in L(R)$,

(2) $\mathfrak{M} \in L(R)$ *implies* $\mathfrak{M}^* \in L(R)$,

(3) *if* $\mathfrak{M} \in L(R)$, $a \in R$, *and* $U(a)$ *is a neighbourhood of* a, *then for some uniform covering* $\mathfrak{U} = \{U_0, U\}$ *such that* $U_0 \subset U(a)$, $a \notin \bar{U}$, $\mathfrak{U} \wedge \mathfrak{M} \in L(R)$ *holds.*

Proof. Let $\mathfrak{M} \in L(R)$, $\mathfrak{M}^* \neq \{R\}$, then $S(S(M, \mathfrak{M}), \mathfrak{M}) \notin \mathfrak{M}$ for each $M \in \mathfrak{M}$. For if $S(S(M, \mathfrak{M}), \mathfrak{M}) \subset M_0 \in \mathfrak{M}$, then since $S(S(M, \mathfrak{M}), \mathfrak{M}) \subset S(M, \mathfrak{M})$, for each

4) $\mathfrak{U} \circ \mathfrak{M} = \{S(U_0, \mathfrak{M}), M \mid M \frown U_0 = \phi\} \wedge \mathfrak{U}$.

$M' \in \mathfrak{M}$, either $M' \frown S(M, \mathfrak{M}) = \phi$ or $M' \subset S(M, \mathfrak{M})$ holds, which contradicts the connectedness of R. Let $a \in N \in \mathfrak{M}^{**}$, and let $b, c \in N$; $b \neq c$; $b, c \neq a$. If we take $\mathfrak{M}(a)$ such that $c \notin S(b, \mathfrak{M}(a))$, then for $\mathfrak{M}'(a) = \mathfrak{M}(a) \wedge \mathfrak{M}$, and for $\mathfrak{M}'(U(a)) = \mathfrak{U} \wedge \mathfrak{M}^{***}$, $L(R)$ satisfies conditions 1), 2′), 3′).

Corollary 3. *If R is a complete uniform space without isolated point, then the uniform topology of R is characterized by any lattice $L(R)$ of uniform basis satisfying the following two conditions,*

(1) $\mathfrak{M}_x, \mathfrak{M}_y \in L(R)$ *implies* $\mathfrak{M}_x \wedge \mathfrak{M}_y \in L(R)$,
$\mathfrak{M}_\alpha \in L(R)$ *imply* $\bigvee_\alpha \mathfrak{M}_\alpha \in L(R)$,

(2) $M \in \mathfrak{M} \in L(R)$ *implies* $\mathfrak{S}(M, \mathfrak{M}) \in L(R)$.[5]

Proof. It is obvious that $L(R)$ satisfies conditions 1), 2′), 3′),

Corollary 4. *If R is a compleıe uniform space without isolated point, then the uniform topology of Ri s characterized by any lattice $L(R)$ of uniform basis satisfying the following three conditions,*

(1) $\mathfrak{M}_x, \mathfrak{M}_y \in L(R)$ *imply* $\mathfrak{M}_x \times \mathfrak{M}_y \in L(R)$,

(2) $M \in \mathfrak{M} \in L(R)$ *implies* $\mathfrak{S}(M, \mathfrak{M}) \in L(R)$,

(3) *if* $\mathfrak{M} \in L(R)$, *and if each element M of $\mathfrak{M}'$ is either an element of $\mathfrak{M}$ or a sum of two elements of, $\mathfrak{M}$, then $\mathfrak{M}' \in L(R)$.*

Proof. It is obvious that $L(R)$ satisfies conditions 1), 2), 3).

Applying the theorem to the case of topological spaces, we get the following.

Corollary 5. *In order that two fully normal spaces R_1 and R_2 are homeomorphic, it is necessary and sufficient that there exists a lattice-isomorphism between $L(R_1)$ and $L(R_2)$, lattices of bases of open coverings satisfying conditions* 1), 2), 3) *or conditions* 1), 2′), 3′) *of the theorem.*

Corollary 6. *The topology of a fully normal space R is characterized by any lattice $L(R)$ of basis of neighbourhood-finite open coverings satisfying the following tow conditions,*

(1) $\mathfrak{B}_x, \mathfrak{B}_y \in L(R)$ *implies* $\mathfrak{B}_x \times \mathfrak{B}_y \in L(R)$,

(2) *if* $\mathfrak{B} \in L(R), a \in R$, *and $U(a)$ is a neighbourhood of a, then for some open covering* $\mathfrak{U} = \{U_0, U\}$ *such that* $U_0 \subset U(a)$, $a \notin \bar{U}$, $\mathfrak{U} \wedge \mathfrak{B} \in L(R)$ *holds.*

Proof. Let $\mathfrak{B}$ be an arbitrary open covering of $L(R)$, and V be an element of $\mathfrak{B}$. We take $a \in V$ and a neighbourhood $N(a)$ of a such that $N(a)$ meets only a finite number of elemets of $\mathfrak{B}$, $V_1, \dots, V_k$. We assume that $\bar{V}_i \not\supset V$ holds only for $V_1, \dots, V_h$. By condition (2) there exists an open covering $\mathfrak{U} = \{U_0, U\}$ such that $\mathfrak{U} \wedge \mathfrak{B} \in L(R)$; $U_0 \subset N(a)$, $a \notin \bar{U}$, and there exist $\mathfrak{U}_i = \{U_{0i}, U_i\}$ $(i = 1, \dots, h)$ such that $\mathfrak{U}_i \wedge \mathfrak{B} \in L(R)$; $U_{0i} \subset V - \bar{V}_i$, $V - \bar{V}_i \ni a_i \notin \bar{U}_i$. Then $\mathfrak{M}(V, \mathfrak{B}) = \bigwedge_{i=1}^{h} \mathfrak{U}_i \wedge \mathfrak{U} \wedge \mathfrak{B} \in L(R)$ satisfies condition 2) of $L(R)$ in the theorem: $V \notin \mathfrak{M}(V, \mathfrak{B})$; $\bar{V}_0 \not\supset V$, $V_0 \in \mathfrak{B}$ imply $V_0 \in \mathfrak{M}(V, \mathfrak{B})$. We can see easily that conditions 3) is satisfied, too.

5) $\mathfrak{S}(M, \mathfrak{M}) = \{S(M, \mathfrak{M}), \cup\{M' \mid M' \frown M = \phi, M' \in \mathfrak{M}\}\}$.