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On the Uniform Topology of Bicompactifications.

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In this note we shall characterise the uniform space, which is a uniform subspace of its bicompactification $\beta(R)$ or $w(R)$[1], by introducing the notion of u-normality. Then we shall study some properties of u-normal spaces. We denote by R a uniform space having uniformity $\{\mathfrak{M}_x \mid \mathfrak{X}\}$.

Let A and B be subsets of R and $A \frown B = \phi$. When there exists $\mathfrak{M}_x \in \{\mathfrak{M}_x\}$ such that $S(A, \mathfrak{M}_x) \frown B = \phi$, we say that A and B are *u-separated.* It is easy to see that A and B are u-separated, when and only when there exists a uniformly continuous function φ such that

$$\varphi(a) = 0 \ (a \in A), \quad \varphi(a) = 1 \ (a \in B), \qquad 0 \leqq \varphi(a) \leqq 1.$$

A uniform space R is called a *Čech u-normal space,* when any disjoint completely closed sets[2] of R are u-separated, and is called a *u-normal space,* when any disjoint closed sets of R are u-separated.

Lemma 1. *In order that R is Čech u-normal, it is necessary and sufficient that every bounded continuous functions of R are uniformly continuous.*

Proof. Since for any disjoint completely closed sets F and G, there exists a bounded continuous function φ such that

$$\varphi(a) = 0 \ (a \in F), \quad \varphi(a) = 1 \ (a \in G), \qquad 0 \leqq \varphi(a) \leqq 1.$$

the sufficiency of the condition is obvious.

Conversely let R be Čech u-normal, then any finite open covering $\mathfrak{N} = \{N_i \mid i = 1, \dots, n\}$ is an element of $\{\mathfrak{M}_x\}$, if every N_i^c are completely closed.[3]

For put $N_i^c = F_i$, then $\bigcap_{i=1}^{n} F_i = \phi$ or $F_1 \frown (F_2 \frown \cdots \frown F_n) = \phi$, where F_1 and $F_2 \frown \cdots \frown F_n$ are completely closed. Hence there exists

$\mathfrak{M}_1 \in \{\mathfrak{M}_x\}$ such that

$$S(F_1, \mathfrak{M}_1) \frown S(F_2 \frown \cdots \frown F_n, \mathfrak{M}_1) = \phi .$$

Hence there exists a completely closed set H_1 such that

$$S(F_1, \mathfrak{M}_1) \subset S^c(F_2 \frown \cdots \frown F_n, \mathfrak{M}_1) \subset H_1 \subset (F_2 \frown \cdots \frown F_n)^c.$$

This implies that $H_1 \frown F_2 \frown \cdots \frown F_n = \phi$.

Assume that we get completely closed sets $H_1, \dots, H_i$ and elements $\mathfrak{M}_1, \dots, \mathfrak{M}_i$ of $\{\mathfrak{M}_x\}$ such that

$$H_1 \frown \cdots \frown H_i \frown F_{i+1} \frown \cdots \frown F_n = \phi, \; H_l \supset S(F_l, \mathfrak{M}_l) .$$

Since F_{i+1} and $H_1 \frown \cdots \frown H_i \frown F_{i+2} \frown \cdots \frown F_n$ are completely closed, there exists $\mathfrak{M}_{i+1} \in \{\mathfrak{M}_x\}$ such that

$$S(F_{i+1}, \mathfrak{M}_{i+1}) \frown S(H_1 \frown \cdots \frown H_i \frown F_{i+2} \frown \cdots \frown F_n, \mathfrak{M}_{i+1}) = \phi .$$

Hence there exists a completely closed set H_{i+1} such that

$$S(F_{i+1}, \mathfrak{M}_{i+1}) \subset S^c(H_1 \frown \cdots \frown H_i \frown F_{i+2} \frown \cdots \frown F_n, \mathfrak{M}_{i+1}) \subset H_{i+1}$$
$$\subset (H_1 \frown \cdots \frown H_i \frown F_{i+2} \frown \cdots \frown F_n)^c$$

This implies that $H_1 \frown \cdots \frown H_i \frown H_{i+1} \frown F_{i+2} \frown \cdots \frown F_n = \phi$.

Therefore by induction there exist $H_1, \dots, H_n$ such that $H_1 \frown \cdots \frown H_n = \phi$, $H_i \supset S(F_i, \mathfrak{M}_i)$.

Hence $\qquad \bigcap_{i=1}^{n} S(F_i, \mathfrak{M}_i) = \phi$ .... (S).

Put $\qquad \mathfrak{M}_1 \wedge \mathfrak{M}_2 \wedge \cdots \wedge \mathfrak{M}_n = \mathfrak{M} \in \{\mathfrak{M}_x\}$.

Let M be an arbitrary element of $\mathfrak{M}$, then for every i there exists $M_i \in \mathfrak{M}_i$ such that $M_i \supset M$. Therefore it must be $M \frown F_i = \phi$ for some i, because $M \frown F_i \neq \phi$ (for all i) contradicts the condition (S). This implies $\mathfrak{M} < \mathfrak{N}$, i. e. $\mathfrak{N} \in \{\mathfrak{M}_x\}$.

Now let φ be an arbitrary bounded continuous function of R such that $0 \leq \varphi \leq 1$. We write

$$\mathfrak{U}_n = \{U_k \mid k = 0, 1, \dots, 2^n - 2\},$$
$$U_k = \{a \mid k/2^n < \varphi(a) < k+2/2^n\}.$$

Then, since $\mathfrak{U}_n$ is a finite open covering of R and every U_k^c are completely closed, we get $\mathfrak{U}_n \in \{\mathfrak{M}_x\}$.

Since $a \in S(b, \mathfrak{U}_n)$ implies $|\varphi(a) - \varphi(b)| < 1/2^{n-1}$, φ is uniformly continuous.

Theorem 1. *In order that the uniform topology of $\beta(R)$ can be reduced to that of R, it is necessary and sufficient that R is Čech u-normal and totally bounded. (This shows a necessary and sufficient condition for that $\beta(R)$ and the completion R^* of R is identical.)*

Proof. Necessity: Since $\beta(R)$ is totally bounded, R is also totally bounded.

Let F and G be complete closed sets of R and $F \frown G = \phi$, then there exists a continuous function φ such that

$$\left.\begin{array}{l}\varphi(a) = 0 \ (a \in F), \\ \varphi(a) = 1 \ (a \in G),\end{array}\right. \quad 0 \leqq \varphi(a) \leqq 1.$$

When we inflate φ to the continuous function φ^* of $\beta(R)$, φ^* is uniformly continuous from the bicompactness of $\beta(R)$. This implies the uniform continuity of φ.

Hence F and G are u-separated, and R is Čech u-normal.

Sufficiency: Let R be homeomorphic to the subspace $R_0 = f(R)$ of its Tychonoff's parallelotope $P\{I^{\Gamma} \mid \Gamma\}$, (where Γ is the set of all continuous functions φ of R such that $0 \leqq \varphi \leqq 1$.)

We shall prove that R and R_0 with the weak uniform topology by means of bounded continuous functions are uniformly homeomorphic. When this fact is proved, the sufficiency of this condition is obvious.

We denote by $\mathfrak{N}(\varphi, \alpha)$ the uniform covering $\{N(\varphi(a), \alpha) \mid a \in R\}$ of R_0, where $N(\varphi(a), \alpha) = \{f(x) \| \varphi(a) - \varphi(x) \mid < \alpha\}$, ($f$ is the homeomorphism between R and R_0).

Now we consider an arbitrary uniform covering $\mathfrak{N} = \mathfrak{N}(\varphi_1, \alpha_1) \wedge \dots \wedge \mathfrak{N}(\varphi_k, \alpha_k)$ of R_0. Since by Lemma 1, φ_i is uniformly continuous, there exists $\mathfrak{M}_i \in \{\mathfrak{M}_\alpha\}$ such that $b \in S(a, \mathfrak{M}_i)$ implies $|\varphi_i(a) - \varphi_i(b)| < \alpha_i$. Putting $\mathfrak{M} = \mathfrak{M}_1 \wedge \dots \wedge \mathfrak{M}_k$, we get $\mathfrak{M} \in \{\mathfrak{M}_\alpha\}$ and $f(\mathfrak{M}) < \mathfrak{N}$.

Conversely consider an arbitrary element $\mathfrak{M}$ of $\{\mathfrak{M}_\alpha\}$, and let $\mathfrak{M}_1^{\Delta} < \mathfrak{M}$, $\mathfrak{M}_2^* < \mathfrak{M}_1$, and $\mathfrak{M}_1, \mathfrak{M}_2 \in \{\mathfrak{M}_\alpha\}$. Since R is totally bounded, there exists a finite subcovering $\{\mathfrak{M}_{2i} \mid i = 1, \dots, k\}$ of $\mathfrak{M}_2$. Let us assume that $S(M_{2i}, \mathfrak{M}_2) \subset M_{1i} \in \mathfrak{M}_1$ $(i = 1, \dots, k)$, then we can construct continuous functions φ_i of R such that

$$\left.\begin{array}{l}\varphi_i(a) = 0 \ (a \in M_{2i}), \\ \varphi_i(a) = 1 \ (a \in M_{1i}^c),\end{array}\right. \quad 0 \leqq \varphi_i \leqq 1.$$

Let $a, b \in R$, and $b \notin R(a, \mathfrak{M})$, then $a \in M_{2i}$ implies $b \notin S(M_{2i}, \mathfrak{M}_2)$, whence $\varphi_i(a) = 0$, $\varphi_i(b) = 1$. Hence putting $\mathfrak{N} = \mathfrak{M}(\varphi_1, \frac{1}{2}) \wedge \dots \wedge \mathfrak{M}(\varphi_k, \frac{1}{2})$, we get $f^{-1}(\mathfrak{N}) < \mathfrak{M}_1^{\Delta} < \mathfrak{M}$. Therefore R and R_0 are uniformly homeomorphic. Thus the proof is complete.

Corollary 1. *A completely regular space R has one and only one Čech u-normal and totally bounded uniformity agreeing with its topology.*

Corollary 2. *In order that a uniform space R is uniformly homeomorphic to a subspace of its Tychonoff's parallelotope P, it is necessary and sufficient that R is Čech u-normal and totally bounded, where P has the weak uniform topology by means of bounded continuous functions of R.*

Corollary 3. *In order that the uniform topology of $w(R)$ can be reduced to that of R, it is necessary and sufficient that R is u-normal and totally bounded.*

Proof. The validity of this corollary is obvious from the fact that $\beta(R)$ and $w(R)$ are identioal, when and only when R is normal.

We can prove Corollary 3 as follows, too.

Lemma 2. *In order that two Cauchy filters $\mathfrak{F}=\{F_\alpha \mid A\}$ and $\mathfrak{G}=\{G_\beta \mid B\}$ of R are equivalent, it is necessary and sufficient that $S(F_\alpha, \mathfrak{M}_x) \frown G_\beta \neq \phi$ for all $F_\alpha \in \mathfrak{F}$, $G_\beta \in \mathfrak{G}$ and $\mathfrak{M}_x \in \{\mathfrak{M}_x\}$.*[4]

Proof. Necessity: Let $\mathfrak{F}$ and $\mathfrak{G}$ be equivalent, i. e. $\mathfrak{F} \smile \mathfrak{G}$ be a Cauchy filter, then for every $\mathfrak{M}_x \in \{\mathfrak{M}_x\}$, there exist two sets $F \in \mathfrak{F}$ and $G \in \mathfrak{G}$ such that $a, b \in F \smile G$ implies $b \in S(a, \mathfrak{M}_x)$.

Take arbitrary $F_\alpha \in \mathfrak{F}$ and $G_\beta \in \mathfrak{G}$, then $F_\alpha \frown F \neq \phi$, $G_\beta \frown G \neq \phi$. Therefore, when we take two points $a \in F_\alpha \frown F$ and $b \in G_\beta \frown G$, we see that

$$S(F_\alpha, \mathfrak{M}_x) \frown G_\beta \neq \phi \cdot$$

Sufficiency: For an arbitrary $\mathfrak{M}_x \in \{\mathfrak{M}_x\}$, we consider $\mathfrak{M}_y \in \{\mathfrak{M}_x\}$ such that $\mathfrak{M}_y^* < \mathfrak{M}_x$, and assume that $a, b \in F_\alpha$ or $a, b \in G_\beta$ implies $b \in S(a, \mathfrak{M}_y)$.

If $a \in F_\alpha$, $b \in G_\beta$, then by the assumption there exists $M \in \mathfrak{M}_y$ such that $M \frown F_\alpha \neq \phi$, $M \frown G_\beta \neq \phi$, and accordingly $L, N \in \mathfrak{M}_y$ such that $L \frown M \neq \phi$, $N \frown M \neq \phi$, $a \in L$, $b \in N$. Hence there exists $P \in \mathfrak{M}_x$ such that $a, b \in L \smile M \smile N \subset P \in \mathfrak{M}_x$, whence we get $b \in S(a, \mathfrak{M}_x)$. Therefore $\mathfrak{F} \smile \mathfrak{G}$ is a Cauchy filter, i. e. $\mathfrak{F}$ and $\mathfrak{G}$ are equivalent.

Let R be a uniform space with the uniform topology $\{\mathfrak{M}_x \mid \mathfrak{X}\}$. We classify all maximum Cauchy closed filters of R by equivalence, and denote by R^* the set of the classes $\{\mathfrak{F}\}$. We introduce a topology in R^* by means of the closed basis: $\{\{\mathfrak{F}\} \mid \exists \mathfrak{F}_0 \in \{\mathfrak{F}\} : F \in \mathfrak{F}_0\}$ ($\subset R^*$), where F is an arbitrary closed set of R, then it is obvious that R^* becomes a T_1-space.

Lemma 3. $\{\{\mathfrak{F}\} \mid \exists \mathfrak{F}_0 \in \{\mathfrak{F}\} : F \in \mathfrak{F}_0\} = \{\{\mathfrak{F}\} \mid (\forall \alpha) F_\alpha \in \mathfrak{F} \in \{\mathfrak{F}\} \to$
$S(F_\alpha, \mathfrak{M}_x) \frown F \neq \phi \, (\forall \mathfrak{M}_x \in \{\mathfrak{M}_x\})\}$

Proof. Assume that $F \in \mathfrak{F}_0 \in \{\mathfrak{F}\}$, then $\mathfrak{F} \in \{\mathfrak{F}\}$ implies $\mathfrak{F} \sim \mathfrak{F}_0$. Therefore by Lemma 2 we get $S(F_\alpha, \mathfrak{M}_x) \frown F \neq \phi$ for all $F_\alpha \in \mathfrak{F}$, $\mathfrak{M}_x \in \{\mathfrak{M}_x\}$.

Conversely assume that $F_\alpha \in \mathfrak{F} \in \{\mathfrak{F}\}$ implies $S(F_\alpha, \mathfrak{M}_x) \frown F \neq \phi$. Then $\mathfrak{G} = \{G \mid G \supset S(F_\alpha, \mathfrak{M}_x) \frown F, \ F_\alpha \in \mathfrak{F}, \ \mathfrak{M}_x \in \{\mathfrak{M}_x\}, \ G$ is closed.$\}$ is a closed filter containing F, where we consider a fixed $\mathfrak{F}$. Further we can see that $\mathfrak{G}$ is a Cauchy filter.

For, for an arbitrary $\mathfrak{M}_x \in \{\mathfrak{M}_x\}$, we take $\mathfrak{M}_y$ such that $\mathfrak{M}_y^{**} < \mathfrak{M}_x$, and assume that $a, b \in F_\alpha (\in \mathfrak{F})$ implies $a \in S(b, \mathfrak{M}_y)$. If $a, b \in \overline{S(F_\alpha, \mathfrak{M}_y) \frown F} \, (\in \mathfrak{G})$, then there exist elements M and N of $\mathfrak{M}_y$ such that $a \in M$, $b \in N$; $M \frown S(F_\alpha, \mathfrak{M}_y) \neq \phi$, $N \frown S(F_\alpha, \mathfrak{M}_y) \neq \phi$. Hence there exist $P, Q \in \mathfrak{M}_y$ such that

$$P \frown M \neq \phi, \ Q \frown N \neq \phi; \ P \frown F_\alpha \neq \phi, \ Q \frown F_\alpha \neq \phi,$$

and $L \in \mathfrak{M}_y$ such that $P \frown L \neq \phi$, $Q \frown L \neq \phi$. Therefore there exists K such that $M \smile P \smile L \smile Q \smile N \subset K \in \mathfrak{M}_x$. Thus from $a \in M$, $b \in N$, we conclude that $a \in S(b, \mathfrak{M}_x)$, i.e. $\mathfrak{G}$ is a Cauchy filter.

Now we denote by $\mathfrak{G}_0$ a maximum closed filter containing $\mathfrak{G}$, then $G \in \mathfrak{G}_0$ and $F_\alpha \in \mathfrak{F}$ imply $S(F_\alpha, \mathfrak{M}_x) \frown G \neq \phi$, for all $\mathfrak{M}_x \in \{\mathfrak{M}_x\}$. For assume the contrary, i.e. $S(F_\alpha, \mathfrak{M}_x) \frown G = \phi$, then for $\mathfrak{M}_y \in \{\mathfrak{M}_x\}$: $\mathfrak{M}_y^{\Delta} < \mathfrak{M}_x$, we get $\overline{S(F_\alpha, \mathfrak{M}_y)} \frown G = \phi$ from the fact that $\overline{S(F_\alpha, \mathfrak{M}_y)} \subset S(F_\alpha, \mathfrak{M}_x)$. But this contradicts $\overline{S(F_\alpha, \mathfrak{M}_y)} \in \mathfrak{G}_0$, $G \in \mathfrak{G}_0$. Thus by Lemma 2 we conclude that $\mathfrak{F} \sim \mathfrak{G}_0$. Therefore $\mathfrak{G}_0$ is an element of $\{\mathfrak{F}\}$ containing F.

Thus the proof of Lemma 3 is complete.

For each $\mathfrak{M}_x = \{M_\alpha \mid A\} \in \{\mathfrak{M}_x\}$ we put $\mathfrak{N}_x = \{(M_\alpha^c)^c \mid A\}$, where M_α^c means the closure of M_α^c in R^*. Then we can see easily from Lemma 3 that $\{\mathfrak{N}_x \mid \mathfrak{X}\}$ becomes a basis of a uniformity agreeing with the topology of R^*, and that $R^* = \bar{R}$ and R^* is complete.

If R is u-normal and totally bounded, from Lemma 2 and the method of introducing the topology into R^* we see easily that R^* is identical with $w(R)$. The converse is obvious.

Therefore we can reproduce Corollary 3 in such a way. Though we used the metod of Theorem 1 at first to study the relation between R and its Tychonoff's parallelotope, the last method has more generality, when we study uniform topologies of general bicompactifications.[5]

From now forth we study properties of u-normal spaces.

Theorem 2. *Any topological space R admits at most one u-normal metric (or enumerable uniformity), which is the finest uniform topology (so called a-structure).*

Proof. Since the u-separation of a u-normal space is difined by means of its topology, and by virtue of the Lemma obtained by the author,[6] the uniform topology of a metric space is defined by the notion of u-separation, we get the uniqueness of such a uniform topology.

We denote by $\{\mathfrak{M}_n \mid n=1, 2, \dots\}$ a basis of such a uniformity of R. If we assume that there exists an open coverng $\mathfrak{N}$ such that $\mathfrak{M}_n \not< \mathfrak{N}$ for all n, then taking $\mathfrak{N}=\mathfrak{N}_1, \mathfrak{N}_2, \mathfrak{N}_3, \dots\dots$ such that $\mathfrak{N}_1 > \mathfrak{N}_2^{\Delta}, \mathfrak{N}_2 > \mathfrak{N}_3^{\Delta}, \dots$, we see that the metric $\{\mathfrak{N}_n \wedge \mathfrak{M}_n \mid n=1, 2, \dots\}$ is u-normal and is not equivalent with $\{\mathfrak{M}_n\}$. But the last result contradicts the above mentioned uniqueness of such a uniformity. Therefore $\{\mathfrak{M}_n\}$ must be the finest uniform topology of R.

Theorem 3. *In order that a metric space R is u-normal, it is necessary and sufficient that the set H of all points of accumulation of R is bicompact, and for all $\varepsilon > 0$ there exists $\delta > 0$ such that $S_\varepsilon^c(H) \subset A_\delta$, where*

$$S_\varepsilon(H)=\{a \mid \rho(a\,b) < \varepsilon,\ b \in H\},$$
$$A_\delta = \{a \mid \rho(a\,b) > \delta\ (\forall\, b \neq a)\}.$$

Proof. Necessity: Assume that H is not bicompact, and an open covering $\mathfrak{M}$ of H has no finite subcovering. Since H is fully normal, there exists an open covering $\mathfrak{N}$ such that $\mathfrak{N}^{**} < \mathfrak{M}$.

If there exist $N_1, N_2, \dots, N_k \in \mathfrak{N}$ such that for every $N \in \mathfrak{N}$, $S(N, \mathfrak{N}) \frown S(N_i, \mathfrak{N}) \neq \phi$ holds for some i, then there exists M_i such that $S(N, \mathfrak{N}) \smile S(N_i, \mathfrak{N}) \subset M_i \in \mathfrak{M}$, and this shows that $\mathfrak{M}$ has a finite subcovering $\{M_i \mid i=1, \dots, k\}$, which contradicts the assumption. Therefore we can select $N_1, N_2, \dots$, such that

$$S(N_i, \mathfrak{N}) \frown S(N_j, \mathfrak{N}) = \phi \qquad (i \neq j).$$

Let $a_i, b_i \in N_i$; $\rho(a_i\,b_i) < \frac{1}{i}, \neq 0$ then from the method of selecting

N_i it must be $\overline{\{a_i\}}=\{a_i\}$, $\overline{\{b_i\}}=\{b_i\}$, whence they are disjoint, but not u-separated, which contradicts the u-normality of R. Hence H must be bicompact.

Next, assume that $S^c_\varepsilon(H) \not\subset A_\delta$ for all δ, then there exist $a_1 \in S^c_\varepsilon(H)$ and $b_1 \neq a_1$ such that $\rho(a_1 b_1) < \frac{\varepsilon}{2}$.

We denote by $\varepsilon(a)$ $\inf\{\rho(a x) \mid x \neq a\}$, then, as $a_1, b_1 \notin H$, it must be $\varepsilon(a_1) > 0$, $\varepsilon(b_1) > 0$. Hence we can select $a_2 \in S^c_\varepsilon(H)$ and $b_2 \neq a_2$ such that $\rho(a_2 b_2) < \inf(\varepsilon(a_1), \varepsilon(b_1), \varepsilon/3)$, then a_1, b_1, a_2 and b_2 are different from the others. We can select in the same way a_3, b_3; a_4, b_4; ... such that $\{a_i\} \frown \{b_i\} = \phi$, $\rho(a_i b_i) < 1/i+1$ and $\{a_i\} \subset S^c_\varepsilon(H)$. Since every points of $S^c_\varepsilon(H)$ are isolated, $\overline{\{a_i\}}=\{a_i\}$. Since $\rho(b_i, x) > \varepsilon/2\ (x \in H)$, $\overline{\{b_i\}}=\{b_i\}$. But $\{a_i\}$ and $\{b_i\}$ are not u-separated, which contradicts the u-normality of R.

Sufficiency: Assume that closed sets F and G of R are disjoint but not u-separated, then there exist two sequences of points $\{a_i\}$ and $\{b_i\}$such that $a_i \in G$, $b_i \in G$ and $\rho(a_i b_i) < 1/i$. If $\{a_i\}$ is cofinal in H, there exists a cluster point a of $\{a_i\}$ from the bicompactness of H, and it becomes at the same time a cluster point of $\{b_i\}$, which means $a \in F \frown G$, and this is a contradiction.

If $\{a_i\}$ is cofinal in H^c, since $\rho(a_i b_i) < 1/i$, $\{a_i\}$ is residual in every A^c_δ and accordingly in every $S_\varepsilon(H)$ by assumption. Hence we can assume that $a_i \in S_{\varepsilon_i}(H)$ and $\varepsilon_i \to 0\ (i \to \infty)$, i.e. we can select $h_i \in H$ such that $\rho(a_i h_i) < \varepsilon_i$. From the bicompactness of H, $\{h_i\}$ has a cluster point h, which is at the same time a cluster point of $\{a_i\}$ and $\{b_i\}$. Therefore $h \in F \frown G$, which contradicts the assumption. This contradiction completes the proof of Theorem 3.

Corollary 4. *A u-normal metric space R is bicompact, if every $A_\delta\ (\delta > 0)$ are finite sets,* (for example when R has no isolated point)

Corollary 5. *Any u-normal metric space is complete.*

Theorem 4. *In order that a metrizable space R is homeomorphic to a u-normal metric space, it is necessary and sufficient that the set H of all points of accumulation of R is bicompact.*

Proof. Necessity of the condition is obvious from Theorem 3.

Sufficiency: We shall show that the enumerable set $\mathfrak{S}_n = \{S_{1/n}(a), x \mid a \in H, x \in H^c\}$ of open coverings is a basis for the a-structure of

R, where $S_{1/n}(a)=\{x \mid \rho(x\,a)<1/n\}$.

Let M be an arbitrary open covering of R and

$$H \subset \bigcup_{i=1}^{k} M_i \quad (M_i \in \mathfrak{M}).$$

Let us select k closed sets $\bar{N}_i$ such that $\bar{N}_i \subset M_i$, $\bigcup_{i=1}^{k} \bar{N}_i = H$, then $\min_{i=1\cdots k} d(\bar{N}_i, M_i^c) > \frac{1}{n}$.

For, if $d(\bar{N}_i, M_i^c)=0$, then there exist $a_j \in \bar{N}_i$ and $b_j \in M_i^c$ such that $\rho(a_j b_j)<1/j$. Since $\bar{N}_i$ is bicompct, $\{a_j\}$ has a cluster point a, which is a cluster point of $\{b_j\}$ at the same time. Therefore $a \in \bar{N}_i \frown M_i^c$, which contradicts the assumption, i. e. it must be $d(\bar{N}_i, M_i^c)>0$ and accordingly $\min_{i=1\ldots k} d(\bar{N}_i, M_i^c) > \frac{1}{n}$.

Then it is easy to see that $\mathfrak{S}_n < \mathfrak{M}$ for this n; hence $\{\mathfrak{S}_n\}$ is a basis for the a-structure of R, and the metric by means of $\{\mathfrak{S}_n\}$ is u-normal.

Now let us consider the case of non-metric spaces. If R is u-normal, then it is obvious that any Cauchy sequence of points converges, but, when we concern ourslves with general directed sets of points, it is difficult to study relations between u-normality and completeness.

In the general case we get the following

Theorem 5. *If a fully normal space R is a u-normal, locally complete and uniformly connected space*[7], *then R is complete.*

Proof. Let R be a uniform space satisfying the assumption, and let $\{\mathfrak{M}_x\}$ be a basis for its uniformity such that every $\mathfrak{M}_x$ consist of connected sets only.

1. Since R is fully normal and locally complete, by the result of A. H. Stone[8] we get a neighbourhood-finite open covering $\mathfrak{U}=\{U_\alpha \mid A\}$ such that U_α are complete and an open covering $\mathfrak{V}=\{V_\alpha\}$ such that $\bar{V}_\alpha \subset U_\alpha$. For these coverings we construct continuous functions f_α such that

$$\begin{aligned} f_\alpha(a) &= 1 \quad (a \in V_\alpha), \\ f_\alpha(a) &= 0 \quad (a \in U_\alpha^c), \end{aligned} \qquad 0 \leq f_\alpha(a) \leq 1.$$

We denote by $\Delta=\{\delta\}$ the collection of finite sets consisting of α, and by $n(\delta)$ the number of δ. Put

$$N_k=\{a \mid \exists\gamma : f_\alpha(a)>\gamma\ (\alpha\in\delta),\ f_\alpha(a)<\gamma\ (\alpha\notin\delta)$$

for some δ such that $n(\delta)=k\}$, then $\{N_k \mid k=1, 2 \dots\}$ is an open covering of R. Put

$$M_\delta=\{a \mid \exists\gamma : f_\alpha(a)>\gamma\ (\alpha\in\delta),\ f_\alpha(a)<\gamma\ (\alpha\notin\delta)\},$$

then M_δ is an open set, and $\smile\{M_\delta^{(k)} \mid \delta\in\Delta,\ n(\delta)=k\}=N_k$, where $M_\delta^{(k)}=M_\delta \frown N_k$.

We notice that $M_\delta^{(k)}\frown M_{\delta'}^{(k)}=\phi(n(\delta)=n(\delta')=k,\ \delta\neq\delta')$, and $M_\delta^{(k)}\frown V_\alpha=\phi$ $(\alpha\notin\delta)$, i, e, $M_\delta^{(k)}$ meets a finite number of V_α only. Since $\mathfrak{N}=\{N_k\}$ is enumerable, by K. Morita's theorem it has a star-finite refinement $\mathfrak{P}'=\{P_i' \mid i=1, 2, \dots\}$.[9]

Let $\mathfrak{P}=\{P_i \mid i=1, 2, \dots\}$, $\mathfrak{Q}=\{Q_i \mid i=1, 2 \dots\}$ be open coverings such that $\overline{P}_i\subset P_i'$, $\overline{Q}_i\subset P_i$.

If $\overline{P}_i\subset N_k$, then $\mathfrak{W}_i=\{M_\delta^{(k)}\frown V_\alpha\frown\overline{P}_i \mid \delta\in\Delta,\ n(\delta)=k,\ \alpha\in A\}$ is a star-finite refinement of $\mathfrak{V}$ in $\overline{P}_i$.

2. When we construct $S_i^\infty=\bigcup_{n=1}^{\infty} S^n(Q_i,\ \mathfrak{Q})$, then S_i^∞ and S_j^∞ are disjoint or identical: hence every elements of any $\mathfrak{M}_x\in\{\mathfrak{M}_x\}$ are contained in some S_i^∞, because they are connected.

Let $\varphi(t \mid \mathfrak{T})$ be an arbitrary Cauchy directed set of points, then φ is residual in some S_i^∞ by the above fact.

We shall prove that there exists $\mathfrak{M}_x\in\{\mathfrak{M}_x\}$ such that for some m, every elements M of $\mathfrak{M}_x$, in which φ is residual, is contained in $\bigcup_{n=1}^{m} S^n(Q_i,\ \mathfrak{Q})=S^m$.

To see this, assume that the contrary holds. Let $\mathfrak{M}_x$ be an arbitrary element of $\{\mathfrak{M}_x\}$, and let $\mathfrak{M}_y^*<\mathfrak{M}_x$, $\mathfrak{M}_y\in\{\mathfrak{M}_x\}$. If φ is residual in $M_y\in\mathfrak{M}_y$, then every elements of $\mathfrak{M}_y$, in which φ is residual, meet M_y. Hence when $S(M_y,\ \mathfrak{M}_y)\subset M_x\in\mathfrak{M}_x$, from the assumption we get $M_x\not\subset S^m(=\bigcup_{n=1}^{m} S^n(Q_i,\ \mathfrak{Q}))$ for all m.

Since M_x is connected, $M_x\frown(\overline{S^n}-S^n)\neq\phi$ $(n\geq n_0)$, when $M_x\frown S^{n_0}\neq\phi$. Therefore two closed sets $F=\bigcup_{n=1}^{\infty}(\overline{S}^{2n-1}-S^{2n-1})$ and $G=\bigcup_{n=1}^{\infty}(\overline{S}^{2n}-S^{2n})$ are disjoint but not u-separated, which contradicts the u-normality of R.

3. We denote by $\mathfrak{M}_x$ an element of $\{\mathfrak{M}_x\}$ such that every elements of $\mathfrak{M}_x$, in which φ is residual, are contained in S^m for a definite m.

If $S^m = \bigcup_{i=1}^{h} Q_i$, then $\mathfrak{P}_0 = \{\overline{S^m} \frown P_i \mid i = 1 \dots h\}$ is a finite open covering of $\overline{S^m}$. Hence from the u-normality of $\overline{S^m}$, we get a refinement $\mathfrak{M}_x \in \{\mathfrak{M}_x\}$ of $\mathfrak{P}_0$ in $\overline{S^m}$.

Let $\mathfrak{M}_z^* < \mathfrak{M}_x \wedge \mathfrak{M}_y$, $\mathfrak{M}_z \in \{\mathfrak{M}_x\}$, and let $\varphi(t \mid \mathfrak{T})$ be residual in a definite element M_0 of $\mathfrak{M}_z$. Let M be an arbitrary element of $\mathfrak{M}_z$, in which $\varphi(t \mid \mathfrak{T})$ is residual, then $M \subset S(M_0, \mathfrak{M}_z) \subset M_x \in \mathfrak{M}_x$; $M \subset S(M_0, \mathfrak{M}_z) \subset M_y \in \mathfrak{M}_y$, for some M_x and M_y. Since $\varphi(t \mid \mathfrak{T})$ is residual in M_x, we get $M_x \subset \overline{S^m}$.

On the other hand from $M_y \in \mathfrak{M}_y$ we get $M_y \frown \overline{S^m} \subset P_i \frown \overline{S^m}$, whence

$$M \subset M_x \frown M_y = M_x \frown \overline{S^m} \frown M_y \subset P_i \frown \overline{S^m} \subset P_i$$

for a definite P_i and for every $M \in \mathfrak{M}_z$, in which $\varphi(t \mid \mathfrak{T})$ is residual.

4. Since from 3 for every $\mathfrak{M}_u < \mathfrak{M}_z$, $\mathfrak{M}_u \in \{\mathfrak{M}_x\}$ and for every $M \in \mathfrak{M}_u$, in which $\varphi(t \mid \mathfrak{T})$ is residual, we get $M \subset \overline{P_i}$, it must be for a definite $M_{\delta,\alpha}$, $M \subset \bigcup_{n=1}^{\infty} S^n(M_{\delta,\alpha}, \mathfrak{B}_i)$ for such M, where

$$M_{\delta,\alpha} = M_\delta^{(k)} \frown V_\alpha \frown \overline{P}_i \in \mathfrak{B}_i, \quad \overline{P}_i \subset N_k.$$

(See the last part of 1.)

Since $\overline{P}_i$ is u-normal, by the same method as in 2 we can prove that there exists $M \in \mathfrak{M}_u < \mathfrak{M}_z$ such that $\varphi(t \mid \mathfrak{T})$ is residual in M, and $M \subset \bigcup_{n=1}^{l} S^n(M_{\delta,\alpha}, \mathfrak{B}_i)$ for some l. Since $\mathfrak{B}_i$ is star-finite, $\bigcup_{n=1}^{l} S^n(M_{\delta,\alpha}, \mathfrak{B}_i)$ is a union of some finite number of $M_{\delta,\alpha}$.

Since $\varphi(t \mid \mathfrak{T})$ is residual in $\bigcup_{n=1}^{l} S^n(M_{\delta,\alpha}, \mathfrak{B}_i)$, it must be residual in a definite element $M_{\delta,\alpha}$ of $\mathfrak{B}_i$. From the fact that $\overline{M}_{\delta,\alpha} \subset \overline{V}_\alpha$, we get the completeness of $\overline{M}_{\delta,\alpha}$, whence $\varphi(t \mid \mathfrak{T})$ must converge. Thus the proof of Theorem 5 is complete.

Notes.

1) We denote by $\beta(R)$ Čech's bicompactification of R and by $w(R)$ Wallman's one.

2) A closed set F of R is called a completely closed set, when there exists a continuous function φ of R such that

$$F = \{a \mid \varphi(a) = 0\}.$$

3) We denote by N^c the complement of N.

4) In this note we consider filters consisting of closed sets only.

5) See N. A. Shanin, On Special Extensions of Topological Spaces, and his other papers, C. R. URSS, 38 (1943) No. 1, No. 4, No. 5-6.

6) On the Lattices of Functions on Topological Spaces and of Functions on Uniform Spaces, Osaka Math. Journal, 1 (1949) No. 2, Lemma 2.

7) We mean by a uniformly connected space a uniform space, which has a basis $\{\mathfrak{M}_\alpha\}$ for its uniformity such that every $\mathfrak{M}_\alpha$ consist of connected sets only.

8) A. H. Stone, On Paracompactness and Product Space, Bull. of Amer. Math. 54 (1948) No. 10.

An open covering $\mathfrak{U}$ of R is called neighbourhood-finite, when each point x of R has a neighbourhood $V(x)$, which meets only finite number of elements of $\mathfrak{U}$.

$\mathfrak{U}$ is called star-finite, when each element of $\mathfrak{U}$ meets only finite number of elemets of $\mathfrak{U}$.

9) K. Morita, Star-Finite Converings and the Star-Finite Property, Math. Japonicae, Vol. 1, No. 2.

To see this in this case, for example consider an open covering $\mathfrak{M} = \{\overline{M}_k\}$, $M_k \subset N_k$. and construct continuous functions f_k such that

$$\left.\begin{array}{l} f_k(a) = k \ (a \in \overline{M}_k), \\ f_k(a) = 0 \ (a \notin N_k), \end{array}\right. \quad 0 \leqq f_k(a) \leqq k.$$

Putting $f = \sup f_k$, $L_n = \{a \mid n-1 < f(a) < n+1\}$, $\mathfrak{L} = \{L_n \mid n = 0, 1, 2, \ldots\ldots\}$, we get a star-finite refinement $\mathfrak{L} \wedge \mathfrak{M}$ of $\mathfrak{M}$.

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