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ALGEBRA OF STABLE HOMOTOPY OF MOORE SPACE

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0. Introduction

Let p denote an odd prime. A Moore space $M_p^n=M(n, Z_p)$ is a simply connected space with two non-vanishing (integral) homology groups $H_0(M_p^n)=Z$ and $H_n(M_p^n)=Z_p$. The mod p cohomology structure of M_p^n is as follows: $H^0(M_p^n; Z_p)=Z_p$, $H^n(M_p^n; Z_p)=Z_p=\{e^n\}$, $H^{n+1}(M_p^n; Z_p)=Z_p=\{e^{n+1}\}$, $H^i(M_p^n; Z_p)=0$, $i\neq 0$, n, $n+1$, and $\Delta e^n=e^{n+1}$ for the mod p Bockstein operator Δ, for $n\geqq 2$.

The m-th homotopy group $\pi_m(Z_p; n, Z_p)$ of the Moore space $M(n, Z_p)$ with the coefficient group Z_p (or, briefly, *the m-th mod p homotopy group of* $M(n, Z_p)$) is the set of homotopy classes of maps $M_p^m\to M_p^n$ with the track addition (See [3]).

The set $\pi_*=\sum_i \pi_{N+i}(Z_p; N, Z_p)$ (N denotes a sufficiently large integer) of the stable homotopy groups of the Moore space $M(N, Z_p)$ with the coefficient group Z_p (i.e., the stable mod p homotopy groups of $M(N, Z_p)$) admits a ring structrue with respect to the composition. Really, it forms an algebra over the field Z_p.

In this paper, we shall investigate its structure by means of the results and the methods of Toda [10], [11], [12].

For simplicity, we shall denote $\pi_{N+i}(Z_p; N, Z_p)$ by π_i and we shall say that an element of π_i is of dimension i.

Among the elements of π_*, δ denotes the element in π_{-1} such that $\delta^* e_2^N=(-1)^N e_1^N$ for the generators $e_1^N\in H^N(M_p^{N-1}; Z_p)$and $e_2^N\in H^N(M_p^N; Z_p)$; ι denotes the class of the identity map of M_p^N; α denotes the element in $\pi_{2(p-1)}$ such that $\mathcal{P}_\alpha^1 e^{N+1}=(-1)^{N+1}e^{N+2(p-1)}$ for the generators $e^{N+1}\in H^{N+1}(M_p^N; Z_p)$ and $e^{N+k}\in H^{N+k}(M_p^{N+k}; Z_p)$, $k=2(p-1)$, where $\mathcal{P}_\alpha^1$ is the functional cohomological operation with respect to $\mathcal{P}^1$ and α; β_1 denotes the element in $\pi_{2p(p-1)-1}$ such that $\alpha\beta_1=0$ and $\mathcal{P}_{\beta_1}^p e^{N+1}=(-1)^{N+1}e^{N+2p(p-1)}$ for the generators $e^{N+1}\in H^{N+1}(M_p^N; Z_p)$ and $e^{N+l}\in H^{N+l}(M_p^{N+l-1}; Z_p)$, $l=2p(p-1)$, where $\mathcal{P}_{\beta_1}^p$ is the functional cohomological operation with respect to $\mathcal{P}^p$ and β_1; and, β_s, $1<s<p$, denote the element in $\pi_{2(sp+s-1)(p-1)-1}$

defined inductively by the stable secondary composition $\langle\beta_{s-1}, \alpha, \beta_1\rangle$, respectively.

Then, our main theorems are

THEOREM I. *A set of additive bases for* π_* *is as follows in* $dim < 2p^2(p-1)-4$:

$$\begin{aligned}
&\delta,\ \iota,\\
&\alpha^t,\ \alpha^t\delta,\ \alpha^{t-1}\delta\alpha,\ \alpha^{t-1}\delta\alpha\delta, \quad \text{for } 1 \leqq t < p^2,\\
&(\beta_1\delta)^{r-1}\beta_1,\ \delta(\beta_1\delta)^{r-1}\beta_1,\ (\beta_1\delta)^r,\ \delta(\beta_1\delta)^r, \quad \text{for } 1 \leqq r \leqq p,\\
&\alpha(\delta\beta_1)^r,\ \delta\alpha(\delta\beta_1)^r,\ \alpha(\delta\beta_1)^r\delta,\ \delta\alpha(\delta\beta_1)^r\delta, \quad \text{for } 1 \leqq r < p,\\
&(\beta_1\delta)^r\beta_s,\ \delta(\beta_1\delta)^r\beta_s,\ (\beta_1\delta)^r\beta_s\delta,\ \delta(\beta_1\delta)^r\beta_s\delta,\\
&\alpha\delta(\beta_1\delta)^r\beta_s,\ \delta\alpha\delta(\beta_1\delta)^r\beta_s,\ \alpha\delta(\beta_1\delta)^r\beta_s\delta,\\
&\delta\alpha\delta(\beta_1\delta)^r\beta_s\delta, \quad \text{for } 0 \leqq r,\ 2 \leqq s < p, \text{ and } r+s < p.
\end{aligned}$$

THEOREM II. *The ring* π_*, *in* $dim < 2p^2(p-1)-4$, *is generated by* δ, α, *and* β_s, $1 \leqq s < p$, *with the following fundamental relations*:

(i) $\delta^2 = 0,\ \alpha\beta_s = \beta_s\alpha = 0,\ \beta_s\beta_t = 0$,

(ii) $2\alpha\delta\alpha = \alpha^2\delta + \delta\alpha^2$,

(iii) $\alpha\delta\beta_s = \beta_s\delta\alpha$,

(iv) $\beta_s\delta\beta_t = \dfrac{st}{s+t-1}\beta_1\delta\beta_{s+t-1}$.

THEOREM III. *The subring of* π_* *generated by* δ *and* α *has only two fundamental relations*: $\delta^2=0$ *and* $2\alpha\delta\alpha=\alpha^2\delta+\delta\alpha^2$.

Thus, the subalgebra generated by δ is an exterior algebra and the subalgebra generated by α is a polynomial algebra.

The relation $\beta_s\delta\beta_t = \dfrac{st}{s+t-1}\beta_1\delta\beta_{s+t-1}$, in π_*, implies the relation $\bar\beta_s\bar\beta_t = \dfrac{st}{s+t-1}\bar\beta_1\bar\beta_{s+t-1}$ in the stable homotopy ring G_* of sphere, for a suitable choice of the element $\bar\beta_s$ in G_*. This is an answer to a problem of Toda [10].

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1. Preliminaries

Throughout this paper, unless otherwise stated, all space are connected and have the homotopy type of a CW-complex. There is given a base point on each space, all maps take base point to base point and all homotopies keep base point fixed. All groups are finitely generated and abelian. We shall denote the additive group of integers by Z, and the additive group of integers modulo an odd prime p by Z_p. The closed interval $[0, 1]$ is denoted by I, and $f \simeq g$ denotes that two maps f and g are homotopic. Often a map and its homotopy class are denoted by the same letter.

Let X, Y be spaces with base points x_0, y_0, and $f: X \to Y$ a map. The mapping cylinder Y_f of f is the space obtained from the disjoint union $(X \times I) \cup Y$ by identifying $(x, 1) \in X \times I$ with $f(x) \in Y$ and shrinking $x_0 \times I$ to the base point y_0.

The mapping cone C_f of f is the space obtained from Y_f by shrinking $X \times 0$ to the base point y_0. The space Y_f has the same homotopy type as Y and we may regard X as a subspace of Y_f by the inclusion map $i_X: X \to Y_f$ defined by $i_X(x) = (x, 0)$. If $Y = x_0$ and $f: X \to x_0$ is the constant map, the mapping cylinder of f is the cone TX over X, and the mapping cone of f is the suspension SX of X. The iterated suspension $S^n X$ of X is defined inductively by $S^n X = S(S^{n-1} X)$. Note that $S(TX)$ and $T(TX)$ are homeomorphic to $T(SX)$. If $X = SX'$ for a space X', we can define a map $\varphi: X \to X \vee X$ $(= X \times x_0 \cup x_0 \times X)$ by

$$(1.1) \qquad \varphi(x', t) = \begin{cases} ((x', 2t), x_0) & 0 \leqq t \leqq 1/2, \\ (x_0, (x', 2t-1)) & 1/2 \leqq t \leqq 1, \end{cases} \quad x' \in X'.$$

We can also define a map $\varphi': C_f \to SX \vee C_f$ by

$$(1.1)' \quad \varphi'(y) = (x_0, y),\ y \in Y,\ \varphi'(x, t) = \begin{cases} (k(x, 2t), y_0) & 0 \leqq t \leqq 1/2, \\ (x_0, (x, 2t-1)) & 1/2 \leqq t \leqq 1, \end{cases} \quad x \in X,$$

where $k: C_f \to SX$ is the shrinking map of $Y \subset C_f$. We shall also denote the space C_f by $Y \bigcup_f TX$ or $Y \bigcup_\alpha TX$ where α is the homotopy class of f.

The loop space ΩX on X is the space of maps $\lambda: (I, \dot{I}) \to (X, x_0)$ with the compact open topology. The constant map $\lambda_0: I \to x_0$ is the base point of ΩX. The iterated loop space $\Omega^n X$ on X is defined inductively by $\Omega^n X = \Omega(\Omega^{n-1} X)$. Since X is assumed to have the homotopy type of a CW-complex, the loop space ΩX also has the homotopy type of a CW-complex [4].

We shall denote by $\pi(X, Y)$ the set of homotopy classes of maps $X \to Y$. For any pairs (X, A) and (Y, B) of spaces (i.e., $A \subset X$ and $B \subset Y$), the relative homotopy set $\pi(X, A; Y, B)$ is the set of homotopy classes of maps $f:(X, A)\rightarrow(Y, B)$ (i.e., $f: X\rightarrow Y$ and $f(A)\subset B$).

The following lemmas are well-known [3].

LEMMA 1.1. *If X is the suspension SX' of a space X', $\pi(X, Y)$ admits a natural group structure for any space Y. If X is the two-fold suspension S^2X'' of a space X'', the group $\pi(X, Y)$ is abelian.*

LEMMA 1.2. *There is a natural isomorphism $\pi(X, \Omega Y)\approx\pi(SX, Y)$, for any spaces X and Y.*

Let $\alpha\in\pi(X, Y)$ be the homotopy class of a map $f: X\rightarrow Y$, then the suspension $S\alpha\in\pi(SX, SY)$ of α is the homotopy class of the map Sf: $SX\rightarrow SY$ defined by $Sf(x, t)=(f(x), t)$, $x\in X$, $t\in I$, and the loop $\Omega\alpha\in\pi(\Omega X, \Omega Y)$ of α is the homotopy class of the map $\Omega f:\Omega X\rightarrow\Omega Y$ defined by $((\Omega f)(\lambda))(t)=f(\lambda(t))$, $\lambda\in\Omega X$, $t\in I$. They are well-defined and if $\pi(X, Y)$ admits a group structure, the correspondences $S_*:\pi(X, Y)\rightarrow\pi(SX, SY)$, $\Omega_*:\pi(X, Y)\rightarrow\pi(\Omega X, \Omega Y)$ defined by $S_*(\alpha)=S\alpha$, $\Omega_*(\alpha)=\Omega\alpha$ are homomorphisms.

Let $f: X\rightarrow Y$ be a map, then we have the Puppe sequence of f [5]:

$$X\xrightarrow{f}Y\xrightarrow{j}C_f\xrightarrow{k}SX\xrightarrow{Sf}SY\xrightarrow{Sj}SC_f\xrightarrow{Sk}S^2X\xrightarrow{S^2f}S^2Y\longrightarrow\cdots$$

such that the following sequence is exact for any space U:

$$(1.2)\qquad \pi(X, U)\xleftarrow{f^*}\pi(Y, U)\xleftarrow{j^*}\pi(C_f, U)\xleftarrow{k^*}\pi(SX, U)\xleftarrow{(Sf)^*}$$
$$\pi(SY, U)\xleftarrow{(Sk)^*}\pi(SC_f, U)\longleftarrow\cdots.$$

Given a group H and an integer $n\geqq 2$, the *Moore space* [3] $M=M(n, H)$ is a simply connected space having only two non-vanishing (integral) homology groups: $H_0(M)=Z$, $H_n(M)=H$. For a given pair of n and H, the space $M(n, H)$ is determined uniquely up to homotopy type, in particular, $M(n, Z)$ is the homotopy type of the n-sphere S^n, $M(n, Z_p)$ is the homotopy type of the cell complex $S^n\bigcup_{p\iota}e^{n+1}$ where $e^{n+1}=TS^n$ is an $(n+1)$-cell and ι denotes the class of the identity map of S^n. It is easily seen that the suspension of $M(n-1, H)$ is an $M(n, H)$ for $n>2$, so that the set $\pi(M(n, H), X)$ admits an abelian group structure for any space X and $n>3$. It is called the *n-th homotopy group of X with a coefficient group H* and denoted by $\pi_n(H; X)$. Similarly, for any pair (X, A), the *n-th relative homotopy group $\pi_n(H; X, A)$ of (X, A) with a coefficient group H* is defined as the relative homotopy set $\pi(TM(n-1, H), M(n-1, H); X, A)$, for $n>4$. We have the exact sequence

$$\cdots \longrightarrow \pi_{n+1}(H; X, A) \xrightarrow{d} \pi_n(H; A) \xrightarrow{i_*} \pi_n(H; X) \xrightarrow{j_*} \pi_n(H; X, A) \longrightarrow \cdots$$

for any pair (X, A), and the naturalities $df_* = (f|A)_* d$ and $S_* f_* = (Sf)_* S_*$

$$\begin{array}{ccc} \pi_{n+1}(H; X, A) & \xrightarrow{f_*} & \pi_{n+1}(H; Y, B) \\ d\downarrow & & \downarrow d \\ \pi_n(H; A) & \xrightarrow{(f|A)_*} & \pi_n(H; B) \end{array} \qquad \begin{array}{ccc} \pi_n(H; X) & \xrightarrow{f_*} & \pi_n(H; Y) \\ S_*\downarrow & & \downarrow S_* \\ \pi_{n+1}(H; SX) & \xrightarrow{(Sf)_*} & \pi_{n+1}(H; SY) \end{array}$$

for a map $f: (X, A) \to (Y, B)$, and the suspension homomorphism S_*.

Given a group π and an integer $n \geqq 1$, the Eilenberg-MacLane space $K = K(\pi, n)$ is a space having only one non-vanishing homotopy group: $\pi_n(K) = \pi$. For a given pair of n and π, the space $K(\pi, n)$ is determined uniquely up to homotopy type. It is easily seen that the loop space on $K(\pi, n+1)$ is a $K(\pi, n)$, so that the set $\pi(X, K(\pi, n))$ admits an abelian group structure. It is the n-th cohomology group $H^n(X; \pi)$ of X with a coefficient group π.

A space X is said to be n-connected if $\pi_i(X) = 0$ for $0 \leqq i \leqq n$.

The following lemmas are well-known [2], [7].

LEMMA 1. 3. *Let X be $(m-1)$-connected and Y be $(n-1)$-connected $(m, n > 1)$, and $f: X \to Y$ be a map. Then, $\psi^*: \pi_r(Y_f, X) \to \pi_r(C_f)$ are isomorphisms for $r < m+n-1$, where $\psi: (Y_f, X) \to (C_f, y_0)$ is the shrinking map of X.*

LEMMA 1. 4. *If X is an $(n-1)$-connected space $(n > 1)$, then*
(i) *the (homotopy) suspension homomorphisms $S_*: \pi_r(X) \to \pi_{r+1}(SX)$ are isomorphisms for $r < 2n-1$.*
(ii) *the cohomology suspension homomorphisms $\Omega_*: H^r(X; \pi) \to H^{r-1}(\Omega X; \pi)$ are isomorphisms for $r < 2n-1$.*

Let $f: S^2 \to S^2$ be a map of degree p, then the mapping cone C_f of f is an $M(2, Z_p)$. So that, by (1.2), the following sequence is exact for any space X:

$$\begin{aligned} &\cdots \longrightarrow \pi_{r+1}(X) \xrightarrow{(p\iota)^*} \pi_{r+1}(X) \xrightarrow{k^*} \pi_r(Z_p; X) \xrightarrow{j^*} \pi_r(X) \xrightarrow{(p\iota)^*} \\ (1.3) \qquad &\pi_r(X) \longrightarrow \cdots \longrightarrow \pi_3(X) \longrightarrow \pi_3(X) \longrightarrow \pi_2(Z_p; X) \longrightarrow \\ &\pi_2(X) \longrightarrow \pi_2(X). \end{aligned}$$

By the exactness of the above sequence and the five lemma, we have

COROLLARY 1. 1. *If X is an $(n-1)$-connected space $(n > 1)$, the suspension homomorphisms $S_*: \pi_r(Z_p; X) \to \pi_{r+1}(Z_p; SX)$ are isomorphisms for $r < 2n-2$.*

COROLLARY 1.2. *If X is $(m-1)$-connected, and Y is $(n-1)$-connected $(m, n>1)$, then the homomorphisms $\psi_* : \pi_r(Z_p ; Y_f, X) \to \pi_r(Z_p ; C_f)$ are isomorphisms for $r<m+n-2$.*

By the above corollary, we have an exact sequence

$$\begin{aligned} &\pi_{2s-3}(Z_p ; X) \xrightarrow{f_*} \pi_{2s-3}(Z_p ; Y) \xrightarrow{j_*} \pi_{2s-3}(Z_p ; C_f) \xrightarrow{\tau} \\ &\pi_{2s-4}(Z_p ; X) \longrightarrow \cdots \longrightarrow \pi_{r+1}(Z_p ; C_f) \xrightarrow{\tau} \pi_r(Z_p ; X) \xrightarrow{f_*} \\ &\pi_r(Z_p ; Y) \xrightarrow{j_*} \pi_r(Z_p ; C_f) \xrightarrow{\tau} \pi_{r-1}(Z_p ; X) \longrightarrow \cdots , \end{aligned} \tag{1.4}$$

where $\tau = S_*^{-1}k_*$ and $s = Min(m, n)$.

Since S^n is $(n-1)$-connected and $SS^n = S^{n+1}$, Lemma 1.4 implies $S_*^i : \pi_{n+k}(S^n) \approx \pi_{n+k+i}(S^{n+i})$ for $n>k+1$ and $i>0$. Therefore, we can define the *k-th stable homotopy group* G_k of sphere by

$$G_k = \text{dir. lim}\ \{\pi_{n+k}(S^n), S_*\} \qquad \text{for any integer } k.$$

Similarly, we can define the *k-th stable homotopy group* π_k of Moore space $M(n, Z_p)$ *with a coefficient group* Z_p by

$$\pi_k = \text{dir. lim}\ \{\pi_{n+k}(Z_p ; n, Z_p), S_*\} \qquad \text{for any integer } k.$$

While, since $K(\pi, n)$ is $(n-1)$-connected and $\Omega K(\pi, n) = K(\pi, n-1)$, by Lemma 1.4, $\Omega_*^i : H^{n+k+i}(\pi, n+i ; G) \approx H^{n+k}(\pi, n ; G)$ for $n>k+1$ and $i>0$. So that we can define the k-th stable cohomology group $A^k(\pi, G)$ of $K(\pi, n)$ with a coefficient group G by

$$A^k(\pi, G) = \text{inv. lim}\ \{H^{n+k}(\pi, n ; G), \Omega_*\} \qquad \text{for any integer } k.$$

Given $\alpha \in \pi_q(G ; n, H)$, the composition operation $\alpha_X : \pi_n(H ; X) \to \pi_q(G ; X)$ is a correspondence defined by $\alpha_X(\beta) = \beta \circ \alpha$ for any space X and $\beta \in \pi_n(H ; X)$. Obviously, it is natural with respect to any map $f : X \to Y$, i.e. $f_* \alpha_X = \alpha_Y f_*$.

Given $\theta \in H^q(\pi, n ; G)$, the composition operation $\theta_X : H^n(X ; \pi) \to H^q(X ; G)$ is a correspondence defined by $\theta_X(u) = \theta \circ u$ for any space X and $u \in H^n(X ; \pi)$. Obviously, it is natural with respet to any map $f : Y \to X$, i.e. $f^* \theta_X = \theta_Y f^*$.

A set $\alpha = \{\alpha_X\}$ of correspondences $\alpha_X : \pi_n(H ; X) \to \pi_q(G ; X)$ (resp. $\theta = \{\theta_X\}$ of $\theta_X : H^n(X ; \pi) \to H^q(X ; G)$) defined for any space X, is called homotopical operation of type $(n, q ; H, G)$ (resp. cohomological operation of type $(n, q ; \pi, G)$) if it satisfies the naturality. The following lemmas are well-known.

LEMMA 1.5. *There is a one-to-one correspondence between elements of*

$\pi_q(G:n, H)$ (resp. $H^q(\pi, n; G)$) and homotopical operations of type $(n, q; H, G)$ (resp. cohomological operations of type $(n, q; \pi, G)$).

LEMMA 1.6. *If $\alpha\in\pi_q(G; n, H)$ (resp. $\theta\in H^q(\pi, n; G)$) is a stable element, then the composition operation α_X (resp. θ_X) is a homomorphism.*

The direct sum $\sum_k G_k$ (resp. $\sum_k \pi_k$, $\sum_k A^k(\pi, \pi)$) of stable groups has a multiplicative structure defined by the composition. In particular, $\mathcal{S}=\sum_k A^k=\sum_k A^k(Z_p, Z_p)$ is the (mod p) Steenrod algebra.

The following lemma is well-known [1].

LEMMA 1.7. *The (mod p) Steenrod algebra $\mathcal{S}$ is generated by $\Delta\in A^1$ and $\mathcal{P}^{p^k}\in A^{2p^k(p-1)}$ ($k=0, 1, 2, \cdots$) satisfying the Adem's relations:*

$$\mathcal{P}^a\mathcal{P}^b = \sum_{i=0}^{[a/p]}(-1)^{a+i}\binom{(b-i)(p-1)-1}{a-pi}\mathcal{P}^{a+b-i}\mathcal{P}^i, \qquad \text{if } a<pb,$$

$$\mathcal{P}^a\Delta\mathcal{P}^b = \sum_{i=0}^{[a/p]}(-1)^{a+i}\binom{(b-i)(p-1)}{a-pi}\Delta\mathcal{P}^{a+b-i}\mathcal{P}^i + \sum_{i=0}^{[a/(p-1)]}(-1)^{a+i-1}\binom{(b-i)(p-1)-1}{a-pi-1}\mathcal{P}^{a+b-i}\Delta\mathcal{P}^i, \qquad \text{if } a\leqq pb.$$

Let $f:X\to Y$, $g:Y\to Z$ be map such that $gf\simeq O:X\to Z$ where O denotes the constant map $X\to z_0$. Then we have

LEMMA 1.8. *There is a map $\bar{g}:C_f\to Z$ such that the restriction $\bar{g}|Y$ of $\bar{g}$ on $Y\subset C_f$ is g. For any two such maps $\bar{g}_1$, $\bar{g}_2$, we can define a map $d(\bar{g}_1, \bar{g}_2):SX\to Z$ such that $\bar{g}_1\simeq(d\vee g_2)\varphi'$, where φ' is themap defined in (1.1)′. Conversely, for any map $h:SX\to Z$, the map $\bar{g}'=(h\vee\bar{g})\varphi'$ satisfies $\bar{g}'|Y=g$.*

Proof. Since $gf\simeq 0$, there is a homotopy $H:X\times I\to Z$ such that $H(x, 0)=z_0$, $H(x, 1)=g(f(x))$. Define the map $\bar{g}:C_f\to Z$ by

$$\bar{g}(y) = g(y),\ y\in Y, \quad \bar{g}(x, t) = H(x, t),\ x\in X,\ t\in I.$$

Then, since $H(x, 1)=g(f(x))$, $\bar{g}$ is well-defined and, by definition, $\bar{g}|Y=g$.

Let $\bar{g}_1$, $\bar{g}_2$ be two such maps, then we define $d=d(\bar{g}_1, \bar{g}_2)$ by

$$d(x, t) = \begin{cases} \bar{g}_1(x, 2t), & 0\leqq t\leqq 1/2, \\ \bar{g}_2(x, 2-2t), & 1/2\leqq t\leqq 1. \end{cases} \quad x\in X.$$

Since $\bar{g}_1(x, 1)=g(f(x))=\bar{g}_2(x, 1)$, d is well-defined. It is easily verified that $\bar{g}_1\simeq(d\vee\bar{g}_2)\varphi'$ (rel. Y). The last assertion is obvious.

We shall denote by $\bar{\gamma}$ the homotopy class of $\bar{g}$, when γ is the class of g.

Similarly, we have

LEMMA 1.9. *There is a map* $\tilde{f}: SX \rightarrow C_g$ *such that*

$$\begin{array}{lll} \tilde{f}(x, t) = (f(x), 2t), & 0 \leqq t \leqq 1/2, & \\ \tilde{f}(x, t) \in Z, & 1/2 \leqq t \leqq 1, & x \in X, \end{array}$$

hence $k_g \tilde{f} \simeq Sf: SX \rightarrow SY$, *where* $k_g: C_g \rightarrow SY$ *is the shrinking map of* $Z \subset C_g$. *For any two such maps* $\tilde{f}_1, \tilde{f}_2$, *we can define a map* $d(\tilde{f}_1, \tilde{f}_2): SX \rightarrow Z$ *such that* $\tilde{f}_1 \simeq (d \vee \tilde{f}_2)\varphi: SX \rightarrow C_g$ *where* φ *is the map defined in* (1.1). *Conversely, for any map* $h: SX \rightarrow Z$, *the map* $\tilde{f}' = (h \vee \tilde{f})\varphi$ *satisfies the same condition as* $\tilde{f}$.

We shall denote by $\tilde{\alpha}$ the homotopy class $\tilde{f}$, when α is the class of f.

2. Functional operations

In the remainder of this paper, unless otherwise stated, we shall be concerned with only stable elements (of homotopy and cohomology groups).

Let $\alpha \in \pi_q(G; n, H)$, $\beta \in \pi_n(H; X)$, and $\gamma \in \pi(X, Y)$ be elements such that $\alpha_X(\beta) = 0$ and $\gamma_*(\beta) = 0$. Then, by Lemma 1.8 and 1.9, there are elements $\tilde{\alpha} \in \pi_{q+1}(G; C_\beta)/j_{\beta*}\pi_{q+1}(G; X)$ and $\bar{\gamma} \in \pi(C_\beta, Y)/k_\beta^*\pi_{n+1}(H; Y)$ such that $j_\beta^* \bar{\gamma} = \gamma$, $k_{\beta*}\tilde{\alpha} = S\alpha$ where $j_\beta: X \rightarrow C_\beta$ is the injection and $k_\beta: C_\beta \rightarrow M(n+1, H)$ is the shrinking map of X. So that we can define an element $\{\gamma, \beta, \alpha\}$ in $\pi_{q+1}(G; Y)$ modulo $\gamma_*\pi_{q+1}(G; X) + (S\alpha)^*\pi_{n+1}(H; Y)$ to be the coset of $\bar{\gamma} \cdot \tilde{\alpha}$. It is called the secondary composition of γ, β, α [10; IV], [12; Chap. 1].

The following properties of the secondary compositions are well-known [10], [12].

LEMMA 2.1. (i) $\{\gamma_1 + \gamma_2, \beta, \alpha\} \equiv \{\gamma_1, \beta, \alpha\} + \{\gamma_2, \beta, \alpha\}$ mod. $\mathrm{Im}\, \gamma_{1*} + \mathrm{Im}\, \gamma_{2*} + \mathrm{Im}\, (S\alpha)^*$,

(ii) $\{\gamma, \beta_1 + \beta_2, \alpha\} \equiv \{\gamma, \beta_1, \alpha\} + \{\gamma, \beta_2, \alpha\}$ mod. $\mathrm{Im}\, \gamma_* + \mathrm{Im}\, (S\alpha)^*$,

(iii) $\{\gamma, \beta, \alpha_1 + \alpha_2\} \equiv \{\gamma, \beta, \alpha_1\} + \{\gamma, \beta, \alpha_2\}$ mod. $\mathrm{Im}\, \gamma_* + \mathrm{Im}\, (S\alpha_1)^* + \mathrm{Im}\, (S\alpha_2)^*$.

LEMMA 2.2. (i) $S\{\gamma, \beta, \alpha\} \equiv -\{S\gamma, S\beta, S\alpha\}$ mod. $\mathrm{Im}\, (S\gamma)_* + \mathrm{Im}\, (S^2\alpha)^*$,

(ii) *if* $\delta \cdot \gamma = 0$, $\delta\{\gamma, \beta, \alpha\} \equiv -\{\delta, \gamma, \beta\} \cdot (S\alpha)$ mod. $\mathrm{Im}\, \delta_*(S\alpha)^*$.

When $q = N+k+l+m$, $n = N+k+l$, $X = M(N+k, H)$ $Y = M(N, H)$ for a sufficiently large integer N and $G = H$, we can define a stable secondary composition $\langle\gamma, \beta, \alpha\rangle$ such that $\langle\gamma, \beta, \alpha\rangle = (-1)^N\{\gamma, \beta, \alpha\}$ ([10], [12]).

Let $\theta \in H^q(\pi, n; G)$, $f: X \rightarrow Y$. A functional cohomological operation

[6], $\theta_f : H^n(Y;\pi) \cap \operatorname{Ker} f^* \cap \operatorname{Ker}\theta_Y \to H^{q-1}(X;G)/f^*H^{q-1}(Y;G) + (\Omega\theta)_X H^{n-1}(X;\pi)$ is defined by

$$\theta_f(u) = \text{the coset of } \Delta^{*-1}\theta_{(Y,X)} j^{*-1}(u)$$

for $u \in H^n(Y;\pi) \cap \operatorname{Ker} f^* \cap \operatorname{Ker}\theta_Y$ (i.e. $f^*(u)=0$, $\theta_Y(u)=0$).

$$\begin{array}{ccccccccc}
 & & H^{n-1}(X;\pi) & \xrightarrow{\Delta^*} & H^n(Y_f, X;\pi) & \xrightarrow{j^*} & H^n(Y;\pi) & \xrightarrow{f^*} & H^n(X;\pi) \\
 & & \downarrow{\scriptstyle (\Omega\theta)_X} & & \downarrow{\scriptstyle \theta_{(Y,X)}} & & \downarrow{\scriptstyle \theta_Y} & & \\
H^{q-1}(Y;G) & \xrightarrow{f^*} & H^{q-1}(X;G) & \xrightarrow{\Delta^*} & H^q(Y_f, X;G) & \xrightarrow{j^*} & H^q(Y;G) & &
\end{array}$$

The functional cohomological operation θ_f is also denoted by θ_α, where α is the homotopy class of f.

The following properties of the functional cohomological operations are well-known [6], [9].

LEMMA 2.3. *Let $f: X \to Y$, $\theta \in H^{n'}(\pi, n; \pi')$, $\theta' \in H^q(\pi', n'; G)$, then*

(i) *for $u \in H^n(Y;\pi) \cap \operatorname{Ker} f^* \cap \operatorname{Ker}\theta_Y$,*

$$(\theta'\theta)_f(u) \equiv \theta'(\theta_f(u)) \qquad \text{mod. Im}\,(\Omega(\theta'\theta))_* + \text{Im}\, f^*,$$

(ii) *for $u \in H^n(Y;\pi) \cap \operatorname{Ker} f^* \cap \operatorname{Ker}(\theta'\theta)_Y$,*

$$(\theta'\theta)_f(u) \equiv \theta'_f(\theta(u)) \qquad \text{mod. Im}\,(\Omega\theta')_* + \text{Im}\, f^*.$$

LEMMA 2.4. *Let $f: X \to Y$, $g: Y \to Z$ and $\theta \in H^q(\pi, n; G)$, then*

(i) *for $u \in H^n(Z;\pi) \cap \operatorname{Ker}\theta_Z \cap \operatorname{Ker} g^*$,*

$$\theta_{gf}(u) \equiv f^*(\theta_g(u)) \qquad \text{mod. Im}\,(\Omega\theta)_* + \text{Im}\, f^*g^*,$$

(ii) *for $u \in H^n(Z;\pi) \cap \operatorname{Ker}\theta_Z \cap \operatorname{Ker} f^*g^*$,*

$$\theta_{gf}(u) \equiv \theta_f(g^*(u)) \qquad \text{mod. Im}\,(\Omega\theta)_* + \text{Im}\, f^*.$$

LEMMA 2.5. *Let X be the suspension of a space X', and $\alpha, \beta \in \pi(X, Y)$, $\theta \in H^q(\pi, n; G)$. Then,*

$$\theta_{\alpha+\beta}(u) = \theta_\alpha(u) + \theta_\beta(u) \qquad \text{mod. Im}\,(\Omega\theta)_* + \text{Im}\,\alpha^* + \text{Im}\,\beta^*,$$

for $u \in H^n(Y;\pi) \cap \operatorname{Ker}\theta_Y \cap \operatorname{Ker}\alpha^* \cap \operatorname{Ker}\beta^*$.

Let $\alpha \in \pi_q(G; n, H)$, $\beta \in \pi_n(H; X)$, $\gamma \in \pi(X, Y)$ be elements such that $\beta\alpha = 0$, $\gamma\cdot\beta = 0$. Then $\{\gamma, \beta, \alpha\}$ is defined to be the coset of $\bar{\gamma}\cdot\tilde{\alpha}$. If $\{\gamma, \beta, \alpha\} \equiv 0$, there are $\tilde{\alpha}_0$ and $\bar{\gamma}_0$ such that $\bar{\gamma}_0\tilde{\alpha}_0 = 0$. Hence, we have

LEMMA 2.6. *If $\{\gamma, \beta, \alpha\} \equiv 0$, there are elements $\tilde{\tilde{\alpha}}_0 \in \pi_{q+2}(G; C_{\bar{\gamma}_0}) / j_{\gamma_0 *}\pi_{q+2}(G;Y)$ and $\bar{\bar{\gamma}}_0 \in \pi(c_{\tilde{\alpha}_0}, Y)/k^*_{\tilde{\alpha}_0}\pi_{q+2}(G;Y)$ such that $j^*_{\tilde{\alpha}_0}\bar{\bar{\gamma}}_0 = \bar{\gamma}_0$, $k_{\bar{\gamma}_0 *}\tilde{\tilde{\alpha}}_0 = S\tilde{\alpha}_0$, (i.e. $j^*_\beta j^*_{\tilde{\alpha}_0}\bar{\bar{\gamma}}_0 = \gamma$, $k_{\beta *}k_{\gamma_0 *}\tilde{\tilde{\alpha}}_0 = S^2\alpha$), where $j_{\tilde{\alpha}_0}: C_\beta \to C_{\tilde{\alpha}_0}$, $j_{\bar{\gamma}_0}: Y \to C_{\bar{\gamma}_0}$ are injections and $k_{\tilde{\alpha}_0}: C_{\tilde{\alpha}_0} \to M(q+2, G)$, $k_{\bar{\gamma}_0}: C_{\bar{\gamma}_0} \to SC_\beta$ are shrinking maps of C_β and Y respectively.*

The following is a direct consequence of the definition of θ_f.

LEEMA 2.7. *Let $f: X \to Y$ be a map, and $\theta \in H^q(\pi, n; G)$. If $\theta_f(u) \equiv v$ for $u \in H^n(Y; \pi)$ and $v \in H^{q-1}(X; G)$, then $\theta(j^{*-1}(u)) \equiv \Delta^*(v)$* mod. $\Delta^*(\Omega\theta_X)H^{n-1}(X; \pi)$.

3. p-component of homotopy groups of sphere [10]

In the following, p denotes an odd prime.

In [10], Toda determined the generators and the relations of the p-primary component $G_k(p)$ of the k-th stable homotopy groups of sphere for $k < 2p^2(p-1)-3$.

In this section we recall briefly his results.

THEOREM 3.1. [10; Theorem 4.15]

$$
\begin{aligned}
G_{2rp(p-1)-1}(p) &= Z_{p^2} = \{\alpha'_{rp}\} && \textit{for } 1 \leqq r < p-1, \\
&= Z_{p^2} + Z_p = \{\alpha'_{(p-1)p}\} + \{\alpha_1\beta_1^{p-1}\} && \textit{for } r = p-1, \\
G_{2t(p-1)-1}(p) &= Z_p = \{\alpha_t\} && \textit{for } 1 \leqq t < p^2 \\
& && \textit{and } t \not\equiv 0 \ (\textit{mod } p), \\
G_{2(rp+s)(p-1)-2(r-s)}(p) &= Z_p = \{\beta_1^{r-s-1}\beta_{s+1}\} && \textit{for } 0 \leqq s < r \leqq p-1, \\
G_{2(rp+s+1)(p-1)-2(r-s)-1}(p) &= Z_p = \{\alpha_1\beta_1^{r-s-1}\beta_{s+1}\} && \textit{for } 0 \leqq s < r \leqq p-1 \\
& && \textit{and } r-s \neq p-1, \\
G_{2p(p-1)-2p}(p) &= Z_p = \{\beta_1^p\} && \\
G_k(p) &= 0 && \textit{otherwise for } k < 2p^2(p-1)-3.
\end{aligned}
$$

There is a sequence of spaces $X_0, X_1, \cdots, X_k, \cdots$ such that X_{k+1} is a fibre space over X_k with the fibre $K(\pi_{N+k}(S^N), N+k-1)$ for a sufficiently large integer N, and with the projection $p_{k+1}: X_{k+1} \to X_k$, where $X_0 = S^N$. It is easily seen that X_k is also a fibre space over S^N with the projection $p'_k = p_1 \cdots p_{k-1}p_k$, and that X_k is $(N+k-1)$-connected and $p'_{k*}: \pi_i(X_k) \approx \pi_i(S^N)$ for $i \geqq N+k$.

From (3.10) and Theorem 3.11 of [10], we have

THEOREM 3.2. *$H^{N+k}(X_k; Z_p)$ is generated by an element*

$$
\begin{aligned}
&a_t && \textit{for } k = 2t(p-1)-1,\ 1 \leqq t < p^2 \textit{ and } t \neq (p-1)p, \\
&b_s^{(s-1)} && \textit{for } k = 2(sp+s-1)(p-1)-2 \textit{ and } 1 \leqq s < p,
\end{aligned}
$$

and $a_{(p-1)p}$ is a generator of a subgroup of $H^{N+k}(X_k; Z_p) = Z_p + Z_p$ for $k = 2(p-1)p(p-1)-1$. They satisfy the relations:

$$
\begin{aligned}
&R_t a_t = 0 \quad \textit{for } 1 \leqq t < p^2 \qquad \textit{and} \qquad \Delta\mathcal{P}^1\Delta a_{rp-1} = 0 \quad \textit{for } 1 \leqq r < p, \\
&\mathcal{P}^1 b_s^{(s-1)} = 0 \textit{ and } W_s b_s^{(s-1)} = 0 \quad \textit{for } 1 \leqq s < p,
\end{aligned}
$$

where $R_t=(t+1)\mathcal{P}^1\Delta-t\Delta\mathcal{P}^1$, $W_s=(s+1)\mathcal{P}^p\mathcal{P}^1\Delta-s\mathcal{P}^{p+1}\Delta+(s-1)\Delta\mathcal{P}^{p+1}$.

Let $K'=S^N\bigcup_{p_k'}TX_k$, $k=2p(p-1)-2$, then $\Delta^*:H^i(X_k;Z_p)\approx H^{i+1}(K';Z_p)$ $(i\neq N)$, and $j^*:H^N(K';Z_p)\approx H^N(S^N;Z_p)$, where Δ^* is the coboundary homomorphism and $j:S^N\to K'$ is the injection. Put $a_0=j^*(e^N)\in H^N(K';Z_p)$ for the fundamental class $e^N\in H^N(S^N;Z_p)$, then by (3.12) of [10]

$$\Delta b_1=\mathcal{P}^p a_0 \tag{3.1}$$

in $H^{N+2p(p-1)}(K';Z_p)$ where $b_1=\Delta^*(b_1^{(0)})$.

Finally, (4.10) of [10] implies

LEMMA 3.1. *Let K be an $(N+k-1)$-connected finite cell-complex, $f:K\to S^N$ a map. Then there is a map $g:K\to X_k$ such that $f=p_k'g$.*

4. Additive structure of π_k

Let N denote a sufficiently large integer.

Since the mapping cone of $p\iota\in\pi_N(S^N)$ is an $M(N,Z_p)$ by (1.4) and (1.3), we have the following exact sequences:

$$\cdots\longrightarrow G_k\xrightarrow{(p\iota)_*}G_k\xrightarrow{j_*}\pi_{N+k}(M_p^N)\xrightarrow{\tau}G_{k-1}\xrightarrow{(p\iota)_*}G_{k-1}\longrightarrow\cdots \tag{4.1}$$

$$\cdots\longrightarrow\bar{G}_{k+1}\xrightarrow{(p\iota)^*}\bar{G}_{k+1}\xrightarrow{k^*}\pi_k\xrightarrow{j^*}\bar{G}_k\xrightarrow{(p\iota)^*}\bar{G}_k\longrightarrow\cdots \tag{4.2}$$

where $M_p^N=M(N,Z_p)$ and $\bar{G}_k=\pi_{N+k}(M_p^N)$.

Let G_k' be the subgroup of G_k consisting of the elements whose orders are finite and prime to p, then $(p\iota)_*:G_k'\to G_k'$ is an isomorphism. So that they have no influences upon $\pi_{N+k}(M_p^N)$ in (4.1). Thus, we may replace G_k by $G_k(p)$ in (4.1) for $k>0$.

Since p is an odd prime, the following is a direct consequence of (4.1) and (4.2).

PROPOSITION 4.1. $\pi_{-1}=Z_p=\{k^*j_*(\iota)\}$ *and* $\pi_0=Z_p=\{j^{*-1}j_*(\iota)\}$, *where* $\iota\in G_0$ *is the class of the identity map of* S^N.

It is readily seen that the set $\pi_*=\sum_k\pi_k$ admits a ring structure with respect to the composition such that $\pi_i\circ\pi_j\subset\pi_{i+j}$. In particular, we have

COROLLARY 4.1. *For any k, π_k is a π_0-module.*

Since $\pi_0=Z_p$ is a field, π_k is a vector space over Z_p. Hence we have

COROLLARY 4.2. *π_* is an algebra over Z_p.*

PROPOSITION 4.2. $\pi_k=G_{k+1}\otimes Z_p+G_k\otimes Z_p+G_k*Z_p+G_{k-1}*Z_p$, *where* $\otimes$ *and* $*$ *denote the tensor and torsion product, respectively.*

Proof. We may identify $(G_k)_p = G_k/pG_k$ with $G_k \otimes Z_p$ and $_p(G_k) = \{g \in G_k | pg=0\}$ with $G_k * Z_p$. So, by the exactness of (4.1) and since p is an odd prime,

$$0 \longrightarrow G_k \otimes Z_p \xrightarrow{j_*} \bar{G}_k \xrightarrow{\tau} G_{k-1} * Z_p \longrightarrow 0$$

is a split exact sequence. Hence, we have $\bar{G}_k = j_*(G_k \otimes Z_p) + \tau^{-1}(G_{k-1} * Z_p) = G_k \otimes Z_p + G_{k-1} * Z_p$. By (4.2), we have also a split exact sequence

$$0 \longrightarrow \bar{G}_{k+1} \otimes Z_p \xrightarrow{k^*} \pi_k \xrightarrow{j^*} \bar{G}_k * Z_p \longrightarrow 0 .$$

We shall denote a right inverse of τ (or j^*) of an element α by $\tau^{-1}(\alpha)$ (or $j^{*-1}(\alpha)$), though it is not uniquely determined. Since $(G \otimes Z_p) \otimes Z_p = (G \otimes Z_p) * Z_p = G \otimes Z_p$ and $(G * Z_p) \otimes Z_p = (G * Z_p) * Z_p = G * Z_p$, we have

$$\begin{aligned} \pi_k &= k^* j_*(G_{k+1} \otimes Z_p) + j^{*-1} j_*(G_k \otimes Z_p) + k^* \tau^{-1}(G_k * Z_p) + j^{*-1} \tau^{-1}(G_{k-1} * Z_p) \\ &= G_{k+1} \otimes Z_p + G_k \otimes Z_p + G_k * Z_p + G_{k-1} * Z_p . \end{aligned}$$

Put $\delta = k^* j_*(\iota) \in \pi_{-1}$, then $\delta^2 \in \pi_{-1} \cdot \pi_{-1} \subset \pi_{-2} = 0$, and hence $\delta^2 = 0$.

Let R_δ, $L_\delta : \pi_k \to \pi_{k-1}$ be homomorphisms defined by $R_\delta(\gamma) = \gamma \cdot \delta$, $L_\delta(\gamma) = \delta \cdot \gamma$, for $\gamma \in \pi_k$, respectively. Then, we have

PROPOSITION 4.3. *The homomorphism R_δ maps $j^{*-1}\tau^{-1}(G_k * Z_p) \subset \pi_{k+1}$ isomorphically onto $k^*\tau^{-1}(G_k * Z_p) \subset \pi_k$ in such a way that $R_\delta(j^{*-1}\tau^{-1}(\psi)) = k^*\tau^{-1}(\psi)$, for $\psi \in G_k * Z_p$. It also maps $j^{*-1}j_*(G_k \otimes Z_p) \subset \pi_k$ isomorphically onto $k^* j_*(G_k \otimes Z_p) \subset \pi_{k-1}$ in such a way that $R_\delta(j^{*-1}j_*(\varphi)) = k^* j_*(\varphi)$, for $\varphi \in G_k \otimes Z_p$.*

Proof. It suffice to prove the equalities. By the definition of δ, we have

$$\begin{aligned} R_\delta(j^{*-1}\tau^{-1}(\psi)) &= (j^{*-1}\tau^{-1}(\psi))(k^* j_*(\iota)) = (k^* j_*)^*(j^{*-1}\tau^{-1}(\psi)) \\ &= k^*(j^* j^{*-1}(\tau^{-1}(\psi))) = k^*\tau^{-1}(\psi) , \end{aligned}$$

and

$$\begin{aligned} R_\delta(j^{*-1}j_*(\varphi)) &= (j^{*-1}j_*(\varphi))(k^* j_*(\iota)) = (k^* j_*)^*(j^{*-1}j_*(\varphi)) \\ &= k^*(j^* j^{*-1}(j_*(\varphi))) = k^* j_*(\varphi) . \end{aligned}$$

PROPOSITION 4.4. *Assume that an element $\psi \in G_k$ satisfies $p\psi = 0$ and is not divisible by p, then $L_\delta(j^{*-1}\tau^{-1}(\psi)) = j^{*-1}j_*(\psi)$.*

Proof. Since $p\psi = 0$ and ψ is not divisible by p, $\psi \in G_k * Z_p$ and $\psi \in G_k \otimes Z_p$. So that, $j^{*-1}\tau^{-1}(\psi)$ and $j^{*-1}j_*(\psi)$ are well-defined. While, since we are only concerned with the stable elements, we may regard τ as k_*. Hence

$$L_\delta(j^{*-1}\tau^{-1}(\psi)) = (k^* j_*(\iota))(j^{*-1}\tau^{-1}(\psi)) = (k^* j_*)_*(j^{*-1}k_*^{-1}(\psi))$$
$$= j_*(k_* k_*^{-1}(j^{*-1}(\psi))) = j^{*-1}j_*(\psi).$$

COROLLARY 4.3. *If* $pG_k=0$, *then* L_δ *maps* $j^{*-1}\tau^{-1}(G_k{*}Z_p)\subset\pi_{k+1}$ *isomorphically onto* $j^{*-1}j_*(G_k\otimes Z_p)\subset\pi_k$.

COROLLARY 4.4. *Let* $\psi\in G_k$ *be an element of order p and not divisible by p, then* $\psi'\delta$ *and* $\delta\psi'$ *are linearly independent, and* $\delta\psi'\delta\neq 0$ *where* $\psi'=j^{*-1}\tau^{-1}(\psi)\in\pi_{k+1}$.

LEMMA 4.1. *Let* $\psi'=j^{*-1}\tau^{-1}(\psi)$ *and* $\varphi'=j^{*-1}\tau^{-1}(\varphi)$ *for* $\psi,\ \varphi\in G_*{*}Z_p$. *Then we have* $j^{*-1}\tau^{-1}(\psi\varphi)=\psi'\delta\varphi'$, *if* $\psi\cdot\varphi\neq 0$.

Proof. By the definition of δ, ψ', and φ',

$$\psi'\delta\varphi' = (j^{*-1}\tau^{-1}(\psi))(k^* j_*(\iota))(j^{*-1}\tau^{-1}(\varphi)) = j^*(j^{*-1}\tau^{-1}(\psi))\cdot k_*(j^{*-1}\tau^{-1}(\varphi))$$
$$= \tau^{-1}(j^* j^{*-1}(\psi))\cdot j^{*-1}(k_* k_*^{-1}(\varphi)) = j^{*-1}\tau^{-1}(\psi\varphi).$$

LEMMA 4.2. *Let* $\psi'=j^{*-1}\tau^{-1}(\psi)$, $\varphi'=j^{*-1}\tau^{-1}(\varphi)$, *for* $\psi,\ \varphi\in G_*{*}Z_p$. *Then, we have* $\psi'\ \varphi'\equiv j^{*-1}\tau^{-1}(\langle\psi, p\iota, \varphi\rangle)$ *mod.* $j^{*-1}\tau^{-1}(\psi\cdot G_*+G_*\cdot\varphi)$.

Proof. Let $\psi\in G_k$, $\varphi\in G_l$ and $\Phi: S^{N+k+l+1}\to S^N$ represent an element in $\langle\psi, p\iota, \varphi\rangle$. Then, by the definition of secondary composition, it decomposes into a composition of two maps $S^{N+k+l+1}\xrightarrow{\tilde\varphi} M_p^{N+l}\xrightarrow{\bar\psi} S^N$ where $\tilde\varphi$ is an element such that $k_*(\tilde\varphi)=\varphi$, $\bar\psi$ is an element such that $j^*(\bar\psi)=\psi$, for the shrinking map $k: M_p^{N+l}\to S^{N+l+1}$ of $S^{N+l}\subset M_p^{N+l}$, and the injection $j: S^{N+l}\to M_p^{N+l}$. So that, $j^{*-1}\tau^{-1}(\Phi): M_p^{N+k+l+1}\to M_p^{N-1}$ decomposes into a composition $M_p^{N+k+l+1}\xrightarrow{\varphi''} M_p^{N+l}\xrightarrow{\psi''} M_p^{N-1}$. But, by the definition, $\varphi''=j^{*-1}k_*^{-1}(\varphi)=j^{*-1}\tau^{-1}(\varphi)=\varphi'$ and $\psi''=j^{*-1}\tau^{-1}(\psi)=\psi'$. Hence, we have $j^{*-1}\tau^{-1}(\langle\psi, p\iota, \varphi\rangle)\equiv\psi'\varphi'$.

Toda [11] proved that there are elements $\alpha_t\in\langle\alpha_{t-1}, p\iota, \alpha_1\rangle$ of order p in $G_{2t(p-1)-1}$ for all integers $t\geqq 1$. Recently, he has also proved that the element α_t is not divisible by p if $t\not\equiv 0 \pmod p$.

Therefore, by Lemma 4.2 and Corollary 4.4, we have

PROPOSITION 4.5. *For all* $t\geqq 1$, $\alpha^t=j^{*-1}\tau^{-1}(\alpha_t)\neq 0$, *and for* $t\not\equiv 0$ $(\bmod p)$, $\delta\alpha^t\delta\neq 0$.

Now we shall show some examples of the additive structures of π_k.

EXAMPLE 4.1. By Theorem 3.1 and the above Proposition, $\pi_{2t(p-1)} =\{\alpha^t\}$, for $1\leqq t<p^2$, $t\neq(p-1)p$, and $\alpha^{(p-1)p}$ generates a subgroup contained in $\pi_{2(p-1)p(p-1)}$.

EXAMPLE 4.2. By Corollary 4.3 and the above Proposition, $\alpha^t\delta$ and

$\delta\alpha^t$ are linearly independent in $\pi_{2t(p-1)-1}$ for $t\not\equiv 0 \pmod p$. In particular, $\pi_{4(p-1)-1}=\{\alpha^2\delta\}+\{\delta\alpha^2\}$.

EXAMPLE 4. 3. By Theorem 3.1 and Propositions 4.3, 4.4, $\pi_{2(sp+s-1)(p-1)-1} = \{\alpha^{sp+s-1}\delta\}+\{\delta\alpha^{sp+s-1}\}+\{\beta'_s\}$, for $1\leqq s<p$, where $\beta'_s=j^{*-1}\tau^{-1}(\beta_s)$, $\beta_s\in G_{2(sp+s-1)(p-1)-2}$. Since β_s is of order p and not divisible by p, by Corollary 4. 4, $\beta'_s\delta$ and $\delta\beta'_s$ are linearly independent and $\delta\beta'_s\delta\neq 0$.

EXAMPLE 4. 4. Since $\alpha_1\beta_1^r\beta_s$ is of order p and not divisible by p in G_* if $2\leqq s<p$, $0\leqq r$, and $r+s<p$, we have $\delta\psi'\delta\neq 0$ for $\psi'=j^{*-1}\tau^{-1}(\alpha_1\beta_1^r\beta_s)$. While, by Lemma 4. 1, we have $j^{*-1}\tau^{-1}(\alpha_1\beta_1^r\beta_s)=\alpha\delta(\beta'_1\delta)^r\beta'_s$ for β'_s in the above example. Hence, $\delta\alpha\delta(\beta'_1\delta)^r\beta_s\delta\neq 0$ if $2\leqq s<p$, $0\leqq r$, and $r+s<p$.

EXAMPLE 4. 5. Similarly, $\delta(\beta'_1\delta)^p\neq 0$ because β_1^p is of order p and not divisible by p in G_*.

5. Some base of π_*

The following is easily verified.

LEMMA 5. 1. *$H^i(M_p^N;Z_p)=Z_p$ for $i=0$, N, $N+1$. Let e^N and e^{N+1} be generators, then $\Delta e^N=e^{N+1}$ for the mod p Backstein operator Δ ; $H^i(M_p^N;Z_p)=0$ for $i\neq 0$, N, $N+1$.*

For $\delta=k^*j_*(\iota)$ $k^*j_*(\iota)\in\pi_{-1}$, it is clear that

$$\delta^* e_2^N=(-1)^N e_1^N \quad\text{and}\quad \delta^* e^{N+1}=0 \tag{5.1}$$

where $e_1^N\in H^N(M_p^{N-1};Z_p)$, $e_2^N\in H^N(M_p^N;Z_p)$ and $e^{N+1}\in H^{N+1}(M_p^N;Z_p)$ are the generators.

Conversely, it is easily seen that the element $\delta'\in\pi_{-1}$ satisfying (5. 1) is uniquely determined and coincides with δ.

LEMMA 5. 2. *Let $\psi'=j^{*-1}\tau^{-1}(\psi)$, $\psi\in G_k*Z_p$, and $\theta\in A^{k-1}$ be a cohomological operation. Then, $\theta_\psi e^N=(-1)^N x e^{N+k}$, if and only if $\theta_{\psi'}\bar{e}^{N+1}=(-1)^{N+1}x\bar{e}^{N+k+1}$ where $x\in Z_p$, and $e^N\in H^N(S^N;Z_p)$, $e^{N+k}\in H^{N+k}(S^{N+k};Z_p)$, $\bar{e}^{N+1}\in H^{N+1}(M_p^{N+1};Z_p)$ and $\bar{e}^{N+k+1}\in H^{N+k+1}(M^{N+k+1};Z_p)$ are the generators.*

Proof. Note that if $j:S^N\to M_p^N$ is the injection and $k:M_p^N\to S^{N+1}$ is the map shrinking $S^N\subset M_p^N$, then $j^*(\bar{e}^N)=e^N$ and $k^*(e^{N+1})=\bar{e}^{N+1}$.

Let $\theta_\psi e^N=(-1)^N x e^{N+k}$ then since τ may be regarded as k_* Lemma 2. 4 implies,

$$j^*\theta_{\psi'}\bar{e}^{N+1}=j^*(\theta_{\psi'}k^*(e^{N+1}))\equiv\theta_{j^*k_*(\psi')}(e^{N+1})=\theta_\psi(e^{N+1})=(-1)^{N+1}xe^{N+k+1}.$$

While, $j^*:H^{N+k+1}(M_p^{N+k+1};Z_p)\to H^{N+k+1}(S^{N+k+1};Z_p)$ is an isomorphism and the modulus groups are contained in $\theta_* H^{N+1}(M_p^{N+k+1}; Z_p) + \psi'^* H^{N+k+1}(M_p^N; Z_p) = 0$, so we have $\theta_{\psi'}\bar{e}^{N+1} = (-1)^{N+1} x \bar{e}^{N+k+1}$. Conversely, let $\theta_{\psi'}\bar{e}^{N+1} = (-1)^{N+1} x \bar{e}^{N+k+1}$ and $\theta_\psi e^N = (-1)^N y e^{N+k}$, then the above argument shows $x = y$.

Next, let $\alpha_1 \in G_{2p-3}$ be the element defined [10; (4.5)] which satisfies $\mathcal{P}^1_{\alpha_1} e^N = (-1)^N e^{N+2p-3}$ for the functional operation $\mathcal{P}^1_{\alpha_1}$. Therefore for $\alpha = j^{*-1}\tau^{-1}(\alpha_1) \in \pi_{2p-2}$ we have by the above lemma

(5.2) $$\mathcal{P}^1_\alpha e^{N+1} = (-1)^{N+1} e^{N+2p-2}$$

where $e^{N+1} \in H^{N+1}(M_p^N; Z_p)$ and $e^{N+2p-2} \in H^{N+2p-2}(M_p^{N+2p-2}; Z_p)$ are the generators.

Conversely, again by the above lemma, the element $\alpha' \in \pi_{2p-2}$ satisfying (5.2) is uniquely determined and coincides with α.

PROPOSITION 5.1. $2\alpha\delta\alpha = \alpha^2\delta + \delta\alpha^2$.

Proof. By Example 4.2, $\pi_{4(p-1)-1} = \{\alpha^2\delta\} + \{\delta\alpha^2\}$, so we may put $\alpha\delta\alpha = x\alpha^2\delta + y\delta\alpha^2$ with some coefficients x and y. Since $\delta^2 = 0$, $\delta\alpha\delta\alpha - x\delta\alpha^2\delta = \delta\alpha(\delta\alpha - x\alpha\delta) = 0$. Hence, by Lemma 1.8, there is a map $f: K \to M_p^N$ such that the class of the restriction $f | M_p^{N+2(p-1)-1}$ is $\delta\alpha$, where $K = M_p^{N+2p-3} \bigcup_\psi TM_p^{N+4p-6}$, $\psi = \delta\alpha - x\alpha\delta$. Hence, by Lemmas 2.4, 2.5, 2.7, 5.1, and (5.1), (5.2), we have

$$\mathcal{P}^1_f e^N = e^{N+2p-3} \quad \text{for } e^N \in H^N(M_p^N; Z_p) \text{ and } e^{N+2p-3} \in H^{N+2p-3}(K; Z_p),$$
$$\mathcal{P}^1 e^{N+k} = e^{N+k+2p-2}, \quad \mathcal{P}^1 e^{N+k+1} = x e^{N+k+2p-1} \quad \text{for } e^{N+k+i} \in H^{N+k+i}(K; Z_p),$$

where $k = 2p-3$ and $i = 0, 1, 2p-2, 2p-1$. By Lemma 1.7 (i.e. the Adem's relation), $2\mathcal{P}^1\Delta\mathcal{P}^1 = \mathcal{P}^1\mathcal{P}^1\Delta + \Delta\mathcal{P}^1\mathcal{P}^1$, and by Lemmas 2.3 and 2.5,

$$2xe^{N+k+2p-1} = 2\mathcal{P}^1\Delta\mathcal{P}^1_f e^N = \mathcal{P}^1\mathcal{P}^1_f\Delta e^N + \Delta\mathcal{P}^1\mathcal{P}^1_f e^N = e^{N+k+2p-1}.$$

So that $x = 1/2$. Similarly, $y = 1/2$ is deduced from $(\alpha\delta - y\delta\alpha)\alpha\delta = 0$.

COROLLARY 5.1. (i) $\alpha^s\delta\alpha^t = t\,\alpha^{s+t-1}\delta\alpha + (1-t)\alpha^{s+t}\delta$, *for* $s, t \geqq 0$.
(ii) $(s+t)\alpha^s\delta\alpha^t = s\alpha^{s+t}\delta + t\delta\alpha^{s+t}$, *for* $s, t \geqq 0$.
(iii) $\delta\alpha^{rp} = \alpha^{rp}\delta$, *and* $\delta\alpha^{rp}\delta = 0$, *for* $r \geqq 1$.
(iv) $\alpha^s\delta\alpha^t\delta = \delta\alpha^t\delta\alpha^s = t\alpha^{s+t-1}\delta\alpha\delta$, *for* $s, t \geqq 0$.

Proof. (i) Since $2\alpha\delta\alpha = \alpha^2\delta + \delta\alpha^2$, $2\alpha^2\delta\alpha = \alpha^3\delta + \alpha\delta\alpha^2$, so $\alpha\delta\alpha^2 = 2\alpha^2\delta\alpha - \alpha^3\delta$. Thus, inductively we have $\alpha\delta\alpha^t = t\alpha^t\delta\alpha + (1-t)\alpha^{t+1}\delta$. Hence, by multiplying α^{s-1} to the left, $\alpha^s\delta\alpha^t = t\alpha^{s+t-1}\delta\alpha + (1-t)\alpha^{s+t}\delta$.
(ii) By (i), we have $\delta\alpha^{s+t} = (s+t)\alpha^{s+t-1}\delta\alpha + (1-(s+t))\alpha^{s+t}\delta$, so that $\alpha^{s+t-1}\delta\alpha = \frac{1}{s+t}(\delta\alpha^{s+t} - (1-s-t)\alpha^{s+t}\delta)$ for $s+t \not\equiv 0 \pmod p$, and $\delta\alpha^{s+t} = \alpha^{s+t}\delta$ for

$s+t\equiv 0 \pmod p$. Hence, by (i), we have $(s+t)\ \alpha^s\delta\alpha^t = s\alpha^{s+t}\delta + t\delta\alpha^{s+t}$. (iii) follows immediately from (ii), and (iv) follows immediately from (i) and (ii).

COROLLARY 5.2. *For $t\geqq 1$, $\alpha^t\delta$ and $\alpha^{t-1}\delta\alpha$ are linearly independent, and $\alpha^{t-1}\delta\alpha\delta \neq 0$.*

Proof. Let $x\alpha^t\delta + y\alpha^{t-1}\delta\alpha = 0$, then, by multiplying δ to the right, we have $y\alpha^{t-1}\delta\alpha\delta = 0$. If $\alpha^{t-1}\delta\alpha\delta = 0$, $\alpha^s\delta\alpha\delta = 0$ for $s\geqq t-1$, so we may suppose $t\not\equiv 0 \pmod p$. Then, by (ii) of the above Corollary, $t\alpha^{t-1}\delta\alpha\delta = \delta\alpha^t\delta$. But, by Proposition 4.6, $\delta\alpha^t\delta \neq 0$, which is a contradiction. Hence that, $y=0$ and $x=0$.

Put $\beta_1' = j^{*-1}\tau^{-1}(\beta_1) \in \pi_{2p(p-1)-1}$, for $\beta_1 \in G_{2p(p-1)-2}$. Let $f: M_p^{N+2p(p-1)-2} \to M_p^{N-1}$ represent β_1', then $f' = \bar{k}f \not\simeq 0 : M_p^{N+2p(p-1)-2} \to M_p^N \to S^N$ where $\bar{k}$ is the map shrinking $S^{N-1}\subset M_p^{N-1}$. Since $M_p^{N+2p(p-1)-2}$ is $(N+2p(p-1)-3)$-connected, by Lemma 3.1, there is a map $g: M_p^{N+k}\to X_k$ such that $f' = p_k' g$, $k = 2p(p-1)-2$, and $g_* : \pi_i(M_p^{N+k}) \to \pi_i(X_k)$ is an isomorphism for $i < N+k$ and an epimorphism for $i = N+k$. By the theorem of Whitehead [8; p. 276], $g^* : H^i(X_k; Z_p) \to H^i(M_p^{N+k}; Z_p)$ is a monomorphism for $i = N+k$. So $g^*(b_1^{(0)}) = (-1)^N x e^{N+k}$ for the generators $e^{N+k}\in H^{N+k}(M_p^{N+k}; Z_p)$, $b_1^{(0)} \in H^{N+k}(X_k; Z_p)$, and a coefficient $x\not\equiv 0 \pmod p$.

Let $K = S^N \cup_{f'} TM_p^{N+k}$, $K' = S^N \cup_{p_k'} TX_k$ and $j: S^N\to K$, $j': S^N \to K'$ be the injections. Then, there is a map $\bar{g}: K\to K'$ such that $\bar{g}j = j'$, and hence the following diagram is commutative:

$$\begin{array}{ccccccc} & & H^i(X_k; Z_p) & \xrightarrow{\Delta^*} & H^{i+1}(K'; Z_p) & & \\ & \overset{p_k'^*}{\nearrow} & \downarrow g^* & & \downarrow \bar{g}^* & \overset{j'^*}{\searrow} & \\ H^i(S^N; Z_p) & & & & & & H^{i+1}(S^N; Z_p). \\ & \underset{f'^*}{\searrow} & & & & \underset{j'^*}{\nearrow} & \\ & & H^i(M_p^{N+k}; Z_p) & \xrightarrow{\Delta^*} & H^{i+1}(K; Z_p) & & \end{array}$$

So, easily we have $g^*(a_0) = e^N$ for the generators $a_0 \in H^N(K'; Z_p)$ and $e^N\in H^N(K; Z_p)$. While, by (3.1), $\Delta b_1 = \mathcal{P}^p a_0$ for $b_1 = \Delta^*(b_1^{(0)}) \in H^{N+k+1}(K'; Z_p)$. Hence, for $e^{N+k+1} = \Delta^*(e^{N+k}) \in H^{N+k+1}(K; Z_p)$, we have

$$\begin{aligned} \Delta x e^{N+k+1} &= \Delta(\Delta^*(xe^{N+k})) = (-1)^N\Delta(\Delta^* g^*(b_1^{(0)})) \\ &= (-1)^N\Delta(\bar{g}^*\Delta^*(b_1^{(0)})) = (-1)^N \Delta\bar{g}^*(b_1) \\ &= (-1)^N\bar{g}^*(\Delta b_1) = (-1)^N\bar{g}^*(\mathcal{P}^p a_0) = (-1)^N \mathcal{P}^p e^N. \end{aligned}$$

It is easily seen that the coefficient x does not depend on N. Since $x\not\equiv 0 \pmod p$ and Z_p is a field, there is a number $x'\in Z_p$ such that $xx'\equiv 1 \pmod p$.

Let $\beta_1'' = \beta_1'\cdot(x'\iota) \in \pi_{2p(p-1)-1}$, then we have $\Delta e^{N+k+1} = (-1)^{N+1}\mathcal{P}^p_{\beta_1''} e^{N+1}$

for the generators $e^{N+1} \in H^{N+1}(M_p^N ; Z_p)$ and $e^{N+k+1}(M_p^{N+k+1} ; Z_p)$, $k=2p(p-1)-2$.

By Corollary 5.2, $\pi_{2(p+1)(p-1)-1} = \{\alpha^{p+1}\delta\} + \{\alpha^p\delta\alpha\}$, so we may put $\alpha\beta_1'' = x\alpha^{p+1}\delta + y\alpha^p\delta\alpha$ for some coefficients x and y. Hence, putting

$$\beta_1 = \beta_1'' - x\alpha^p\delta - y\alpha^{p-1}\delta\alpha\,,$$

we have

$$\alpha\beta_1 = \alpha\beta_1'' - x\alpha^{p+1}\delta - y\alpha^p\delta\alpha = 0\,.$$

It is easily verified that

$$\begin{aligned}\mathcal{P}^p_{\beta_1} e^{N+1} &= \mathcal{P}^p_{\beta_1''} e^{N+1} - x\mathcal{P}^p_{\alpha^p\delta} e^{N+1} - y\mathcal{P}^p_{\alpha^{p-1}\delta_\alpha} e^{N+1} \\ &= (-1)^{N+1}\Delta e^{N+2p(p-1)-1}\,.\end{aligned}$$

By Theorem 3.1 and Corollary 5.2,

$$\alpha^* : \pi_{2(p+1)(p-1)-1} \to \pi_{2(p+2)(p-1)-1}$$

is an isomorphism. While, $\alpha_*(\beta_1\alpha) = \alpha\beta_1\alpha = 0$, so $\beta_1\alpha = 0$. Thus, we have

PROPOSITION 5.2. *There is an element $\beta_1 \in \pi_{2p(p-1)-1}$ such that*

$$\mathcal{P}^p_{\beta_1} e^{N+1} = (-1)^{N+1}\Delta e^{N+2p(p-1)-1} \tag{5.3}$$

for the generators $e^{N+1} \in H^{N+1}(M_p^N ; Z_p)$ and $e^{N+k} \in H^{N+k}(M_p^{N+k} ; Z_p)$, $k=2p(p-1)-1$, and $\alpha\beta_1 = \beta_1\alpha = 0$.

Conversely, it is easily seen that the element $\tilde{\beta}_1 \in \pi_{2p(p-1)-1}$ satisfying (5.3) and $\alpha\tilde{\beta}_1 = \tilde{\beta}_1\alpha = 0$ is uniquely determined and coincides with β_1.

Since $\alpha\beta_1 = \beta_1\alpha = 0$, we can define the secondary composition $\{\beta_1, \alpha, \beta_1\}$. If $\{\beta_1, \alpha, \beta_1\} \equiv 0$, by Lemma 2.6, there is a map $f : K \to M_p^{N-1}$ such that the class of the restriction $f | M_p^{N+k}$ is β_1 where $K = M_p^{N+k} \bigcup_\alpha TM_p^{N+k+2(p-1)} \bigcup_{\beta_1} TM_p^{N+k'}$, $k=2p(p-1)-2$, and $k'=2(2p+1)(p-1)-2$. So $\bar{k}f \neq 0 : K \to M_p^{N-1} \to S^N$ where $\bar{k}$ is the map shrinking $S^{N-1} \subset M_p^{N-1}$. Since K is $(N+k-1)$-connected, similarly as in the proof of Proposition 5.1, there is a map $g : K \to X_k$ such that $g^* : H^{N+k}(X_k ; Z_p) \to H^{N+k}(K ; Z_p)$ is a monomorphism. Hence $g^*(b_1^{(0)}) = x e^{N+k}$ for the generators $b_1^{(0)} \in H^{N+k}(X_k ; Z_p)$, $e^{N+k} \in H^{N+k}(K ; Z_p)$ and a coefficient $x \not\equiv 0 \pmod p$. While, by Lemma 2.7 and (5.1), (5.3), we have

$$\begin{aligned}&\Delta e^{N+k} = e^{N+k+1},\ \mathcal{P}^1 e^{N+k+1} = \pm e^{N+k+2p-1},\ \Delta e^{N+k+2p-1} = e^{N+k+2p}, \\ &\mathcal{P}^p e^{N+k+2p} = \pm\Delta e^{N+k'+1} = \pm e^{N+k'+2}\,,\end{aligned} \tag{5.4}$$

for the generators $e^{N+i} \in H^{N+i}(K ; Z_p)$, where $i=k,\ k+1,\ k+2p-1,\ k+2p$,

$k'+2$, $k'+3$. By Theorem 3.2, $W_1 b_1^{(0)}=0$, so, by the Adem's relation $\mathcal{P}^p\Delta-\Delta\mathcal{P}^p=\mathcal{P}^1\Delta\mathcal{P}^{p-1}$, we have

$$\begin{aligned} 0 &= g^*(\Delta W_1 b_1^{(0)}) = \Delta W_1 g^*(b_1^{(0)}) = (2\Delta\mathcal{P}^p\mathcal{P}^1\Delta-\Delta\mathcal{P}^{p+1}\Delta)(x e^{N+k}) \\ &= 2x\Delta\mathcal{P}^p\mathcal{P}^1\Delta e^{N+k} = 2x(\mathcal{P}^p\Delta\mathcal{P}^1\Delta-\mathcal{P}^1\Delta\mathcal{P}^{p-1}\mathcal{P}^1\Delta)e^{N+k} \\ &= 2x\mathcal{P}^p\Delta\mathcal{P}^1\Delta e^{N+k} = \pm 2x e^{N+k'+2} \neq 0, \end{aligned}$$

which is a contradiction. Thus, $\{\beta_1, \alpha, \beta_1\}\not\equiv 0$.

PROPOSITION 5.3. *For $1\leqq s<p-1$, there is an element $\beta_s\in \pi_{2(sp+s-1)(p-1)-1}$ such that $\alpha\beta_s=\beta_s\alpha=0$ and $\{\beta_s, \alpha, \beta_1\}\not\equiv 0$.*

Proof. By the above argument, this assertion is true for $s=1$. Suppose, inductively, that β_{s-1} $(s>1)$ satisfying the condition is defined then the stable secondary composition $\langle\beta_{s-1}, \alpha, \beta_1\rangle$ defines a unique non-trivial element β_s in $\pi_{2(sp+s-1)(p-1)-1}$, because $\beta_{s-1}\cdot\pi_{2(p+1)(p-1)}=0$ and $\pi_{2(s-1)(p+1)(p-1)}\cdot\beta_1=0$.

By Lemma 2.2,

$$\langle\beta_{s-1}, \alpha, \beta_1\rangle\cdot\alpha \equiv \beta_{s-1}\langle\alpha, \beta_1, \alpha\rangle \text{ modulo } \beta_{s-1}\cdot\pi_{2(p+1)(p-1)}\cdot\alpha,$$

and since $\beta_{s-1}\cdot\langle\alpha, \beta_1, \alpha\rangle\subset\beta_{s-1}\cdot\pi_{2(p+2)(p-1)}=\beta_{s-1}\cdot\pi_{2(p+1)(p-1)}\cdot\alpha=0$, we have $\beta_s\alpha=0$. Also, we have

$$\alpha\beta_s = \alpha\cdot\langle\beta_{s-1}, \alpha, \beta_1\rangle = -\langle\alpha, \beta_{s-1}, \alpha\rangle\cdot\beta_1 = 0.$$

Hence, $\{\beta_s, \alpha, \beta_1\}$ can be defined. If $\{\beta_s, \alpha, \beta_1\}\equiv 0$, by Lemma 2.6, there is a map $f: K\to M_p^{N-1}$ such that the class of the restriction $f|M_p^{N+k}$ is β_s, where $K=M_p^{N+k}\cup_\alpha TM_p^{N+k+2p-2}\cup_{\beta_1} TM_p^{N+k'}$, $k=2(sp+s-1)(p-1)-2$ and $k'=2((s+1)p+s)(p-1)-2$. So that, there is a map $g: K\to X_k$ such that $g^*: H^{N+k}(X_k; Z_p)\to H^{N+k}(K; Z_p)$ is a monomorphism. Hence $g^*(b_s^{(s-1)})=xe^{N+k}$ for the generators $b_s^{(s-1)}\in H^{N+k}(X_k; Z_p)$, $e^{N+k}\in H^{N+k}(K; Z_p)$ and a coefficient $x\not\equiv 0 \pmod p$. Since $W_s b_s^{(s-1)}=0$ and the relation (5.4) holds in $H^*(K; Z_p)$, we have

$$\begin{aligned} 0 &= g^*(\Delta W_s b_s^{(s-1)}) = \Delta W_s g^*(b_s^{(s-1)}) = \Delta W_s(x e^{N+k}) \\ &= \pm(s+1)x e^{N+k'+2} \neq 0, \end{aligned}$$

which is a contradiction.

In the case when $s=p-1$, similarly to the above, we have an element $\beta_{p-1}\in\pi_{2(p-2)(p-1)-1}$ such that

$$\alpha\beta_{p-1} = \beta_{p-1}\alpha = 0. \tag{5.5}$$

COROLLARY 5.3. *For* $1 \leqq s < p$, β_s, $\alpha^{sp+s-1}\delta$ *and* $\alpha^{sp+s-2}\delta\alpha$ are linearly independent.

Proof. Let $x\alpha^{sp+s-1}\delta + y\alpha^{sp+s-2}\delta\alpha + z\beta_s = 0$, then, by multiplying α to the left, we have $x\alpha^{sp+s}\delta + y\alpha^{sp+s-1}\delta\alpha = 0$. But, by Corollary 5.2, $x=y=0$, so $z=0$.

COROLLARY 5.4. (i) *For* $1 \leqq s < p$, $\delta\beta_s\delta \neq 0$.
(ii) *For* $2 \leqq s < p$, $0 \leqq r$ *and* $r+s < p$, $\delta\alpha\delta(\beta_1\delta)^r\beta_s\delta \neq 0$.
(iii) $\delta(\beta_1\delta)^p \neq 0$.

Proof. (i) By Example 4.3, $\beta_s \in \pi_{2(sp+s-1)(p-1)-1} = \{\alpha^{sp+s-1}\delta\} + \{\alpha^{sp+s-2}\delta\alpha\} + \{\beta'_s\}$, where $\beta'_s = j^{*-1}\tau^{-1}(\beta_s)$, $\beta_s \in G_{2(sp+s-1)(p-1)-2}$, so we may put β_s $x_s\alpha^{sp+s-1}\delta + y_s\alpha^{sp+s-2}\delta\alpha + z_s\beta'_s$. Since β_s, $\alpha^{sp+s-2}\delta$ and $\alpha^{sp+s-1}\delta\alpha$ are linearly independent, we have $z_s \not\equiv 0 \pmod p$. Hence, by Corollary 5.1, (iv), and Example 4.3, we have $\delta\beta_s\delta = z_s\delta\beta'_s\delta \neq 0$. Similarly $\delta\alpha\delta(\beta_1\delta)^r\beta_s\delta = z_1^r z_s\delta\alpha\delta(\beta'_1\delta)^r\beta'_s\delta \neq 0$ by Example 4.4, and $\delta(\beta_1\delta)^p = z_1^p\delta(\beta'_1\delta)^p \neq 0$ by Example 4.5.

6. Multiplicative structure of π_*

Now, we shall study some relations among δ, α and β_s.

LEMMA 6.1. *Let* $\psi' \equiv xj^{*-1}\tau^{-1}(\psi) \bmod \pi_{k+1}/\{j^{*-1}\tau^{-1}(\psi)\}$ *for* $\psi \in G_k * Z_p$, *and* $\varphi' \equiv yj^{*-1}\tau^{-1}(\varphi) \bmod \pi_{l+1}/\{j^{*-1}\tau^{-1}(\varphi)\}$ *for* $\varphi \in G_l * Z_p$, *where* x, y *are some coefficients* $\not\equiv 0 \pmod p$, $\psi \cdot \varphi \neq 0$, *and* $kl \equiv 0 \pmod 2$. *Then,*

$$\psi'\delta\varphi' \equiv \varphi'\delta\psi' \bmod \pi_{k+l+1}/\{j^{*-1}\tau^{-1}(\psi\varphi)\}.$$

Proof. By Lemma 4.1, we have $(j^{*-1}\tau^{-1}(\psi))\delta(j^{*-1}\tau^{-1}(\varphi)) = j^{*-1}\tau^{-1}(\psi\varphi)$ and $(j^{*-1}\tau^{-1}(\varphi))\delta(j^{*-1}\tau^{-1}(\psi)) = j^{*-1}\tau^{-1}(\varphi\psi)$. While, since $kl \equiv 0 \pmod 2$, we have $\psi\varphi = \varphi\psi$ in G_*. So that,

$$\begin{aligned}\psi'\delta\varphi' &\equiv xy(j^{*-1}\tau^{-1}(\psi))\delta(j^{*-1}\tau^{-1}(\varphi)) = xyj^{*-1}\tau^{-1}(\psi\varphi) \\ &= xyj^{*-1}\tau^{-1}(\varphi\psi) \equiv \varphi'\delta\psi',\end{aligned}$$

modulo $(\pi_{k+1}/\{j^{*-1}\tau^{-1}(\psi)\})\delta(\pi_{l+1}/\{j^{*-1}\tau^{-1}(\varphi)\}) \subset \pi_{k+l+1}/\{j^{*-1}\tau^{-1}(\psi\varphi)\}$.

PROPOSITION 6.1. (i) *For* $1 \leqq s < p$, $\alpha\delta\beta_s = \beta_s\delta\alpha$.
(ii) *For* $1 \leqq s < p-1$, $\beta_1\delta\beta_s = \beta_s\delta\beta_1$.
(iii) *For* $1 \leqq s$, $t < p$ *and* $s+t < p$, $\beta_s\beta_t = 0$.

Proof. (i) By the definition,

$$\alpha = j^{*-1}\tau^{-1}(\alpha_1) \text{ and } \beta_s \equiv xj^{*-1}\tau^{-1}(\beta_s) \bmod \pi_k/\{j^{*-1}\tau^{-1}(\beta_s)\},$$

for $\alpha_1 \in G_{2p-3}$, $\beta_s \in G_k$, $k = 2(sp+s-1)(p-1)-2$, and a coefficient $x \not\equiv 0$

(mod p). So, by Lemma 6. 1, $\alpha\delta\beta_s \equiv \beta_s\delta\alpha$ mod $\pi_{2s(p+1)(p-1)-2}/\{j^{*-1}\tau^{-1}(\alpha_1\beta_s)\}$. While, by Theorem 3. 1 andP roposition 4. 2, $\pi_{2s(p+1)(p-1)-2}/\{j^{*-1}\tau^{-1}(\alpha_1\beta_s)\} = \{\alpha^{sp+s-1}\delta\alpha\delta\}$, so we may put $\alpha\delta\beta_s = \beta_s\delta\alpha + x\alpha^{sp+s-1}\delta\alpha\delta$ for a coefficient x. But

$$0 = \alpha^2\delta\beta_s\alpha = \alpha\beta_s\delta\alpha^2 + x\alpha^{sp+s}\delta\alpha\delta\alpha = x\alpha^{sp+s}\delta\alpha\delta\alpha\,,$$

and $\alpha^{sp+s}\delta\alpha\delta\alpha \neq 0$, so that $x=0$,

(ii) Similarly to (i), $\beta_1\delta\beta_s = \beta_s\delta\beta_1$ mod $\pi_{2((s+1)p+s-1)(p-1)-3}/\{\beta_1\delta\beta_s\}$. While, by Theorem 3. 1 and Proposition 4. 2, $\pi_{2((s+1)p+s-1)(p-1)-3} = Z_p = \{\beta_1\delta\beta_s\}$. So that $\beta_1\delta\beta_s = \beta_s\delta\beta_1$.

(iii) Since $\beta_s\beta_t \in \pi_{2k(p-1)-2} = \{\alpha^{k-1}\delta\alpha\delta\}$, $k=(s+t)p+s+t-2$, we may put $\beta_s\beta_t = x\alpha^{k-1}\delta\alpha\delta$. But, $0 = \alpha\beta_s\beta_t = x\alpha^k\delta\alpha\delta$ and $\alpha^k\delta\alpha\delta \neq 0$, so $x=0$.

Added in proof. When $p=3$, $\beta_1\beta_1 \in \pi_{22} = \{\alpha^5\delta\alpha\delta\} + \{\delta\alpha\delta(\beta_1\ \delta)^2\}$, so the above argument is not valid. We have no idea to know whether $\beta_1\beta_1=0$ or not. So, Theorem II (ii) should be understood that $\beta_s\beta_t=0$ *if* $p\neq 3$, and also propositions in section 7 are valid under the assumption $p\neq 3$.

COROLLARY 6. 1. *For* $1 \leqq s < p$, $\alpha^2\delta\beta_s = \alpha\delta\beta_s\delta\alpha = 0$.

Now, we have the following theorems:

THEOREM I. *A set of additive bases for* π_*, *is as follows in dim* $< 2p^2(p-1)-4$:

δ, ι,
α^t, $\alpha^t\delta$, $\alpha^{t-1}\delta\alpha$, $\alpha^{t-1}\delta\alpha\delta$, *for* $1 \leqq t < p^2$,
$(\beta_1\delta)^{r-1}\beta_1$, $\delta(\beta_1\delta)^{r-1}\beta_1$, $(\beta_1\delta)^r$, $\delta(\beta_1\delta)^r$, *for* $1 \leqq r \leqq p$,
$\alpha(\delta\beta_1)^r$, $\delta\alpha(\delta\beta_1)^r$, $\alpha(\delta\beta_1)^r\delta$, $\delta\alpha(\delta\beta_1)^r\delta$, *for* $1 \leqq r < p$,
$(\beta_1\delta)^r\beta_s$, $\delta(\beta_1\delta)^r\beta_s$, $(\beta_1\delta)^r\beta_s\delta$, $\delta(\beta_1\delta)^r\beta_s\delta$,
$\alpha\delta(\beta_1\delta)^r\beta_s$, $\delta\alpha\delta(\beta_1\delta)^r\beta_s$, $\alpha\delta(\beta_1\delta)^r\beta_s\delta$.
$\delta\alpha\delta(\beta_1\delta)^r\beta_s\delta$, *for* $0 \leqq r$, $2 \leqq s < p$, *and* $r+s < p$.

THEOREM II. *The ring* π_*, *in dim* $< 2p^2(p-1)-4$, *is generated by* γ, α, *and* β_s, $1 \leqq s < p$, *with the following fundamental relations*:

(i) $\delta^2 = 0$, $\alpha\beta_s = \beta_s\alpha = 0$, $\beta_s\beta_t = 0$,
(ii) $2\alpha\delta\alpha = \alpha^2\delta + \delta\alpha^2$,
(iii) $\alpha\delta\beta_s = \beta_s\delta\alpha$,
(iv) $\beta_s\delta\beta_t = \dfrac{st}{s+t-1}\beta_1\delta\beta_{s+t+1}$.

REMARK. The class ι of the identity map $M_p^N \to M_p^N$ is the identity element of π_*.

The proof of (iv) will be given in the next section.

THEOREM III. *The subring of* π_* *generated by* δ *and* α *has only two fundamental relations*: $\delta^2=0$ *and* $2\alpha\delta\alpha=\alpha^2\delta+\delta\alpha^2$.

7. Secondary compositions

Following the method of Toda [10; IV], we shall study some secondary compositions.

Similarly to Lemma 4. 8, iii) of [10], we have

LEMMA 7. 1. *Let* $\alpha\in\pi_q(G;n,H)$, $\alpha'\in\pi_q(G;n',H')$, $\beta\in\pi_n(H;X)$, $\beta'\in\pi_{n'}(H';X)$, *and* $\gamma\in\pi(X,Y)$ *be elements such that* $\beta\alpha=\beta'\alpha'=0$, $\gamma\beta=\gamma\beta'=0$ *and* $\{\gamma,\beta,\alpha\}+\{\gamma,\beta',\alpha'\}\equiv 0$. *Then there is an element* $\bar{\gamma}\in\pi(K,Y)$ *such that* $j^*(\bar{\gamma})=\gamma$ *where* $K=X\bigcup_{(\beta,\beta')}T(M(n,H)\vee M(n',H'))\bigcup_{\alpha\vee\alpha'}TM(q+1,G)$, *and* $j:X\rightarrow K$ *is the injection.*

Also, similarly to Theorem 4. 3, ii) of [10], we have

LEMMA 7. 2. *Let* $\alpha\in\pi_h$, $\beta\in\pi_k$, $\gamma\in\pi_l$, $\delta\in\pi_m$, $\varepsilon\in\pi_n$ *be stable elements such that* $\beta\alpha=\gamma\beta=\delta\gamma=\varepsilon\delta=0$, $\langle\delta,\gamma,\beta\rangle\circ\alpha\equiv 0$ *and* $\varepsilon\circ\langle\delta,\gamma,\beta\rangle\equiv 0$. *Then,*

$$\langle\langle\varepsilon,\delta,\gamma\rangle,\beta,\alpha\rangle+(-1)^h\langle\varepsilon,\langle\delta,\gamma,\beta\rangle,\alpha\rangle+(-1)^{k+h}\langle\varepsilon,\delta,\langle\gamma,\beta,\alpha\rangle\rangle\equiv 0$$
$$\text{modulo } \operatorname{Im}\varepsilon_*+\operatorname{Im}\alpha^*+\operatorname{Im}\langle\varepsilon,\delta,\gamma\rangle_*+\operatorname{Im}\langle\gamma,\beta,\alpha\rangle^*.$$

PROPOSITION 7. 1. *For* $1\leqq s<p-1$, $\langle\beta_s,\beta_1,\alpha\rangle\equiv-\dfrac{s}{s+1}\beta_{s+1}$ *modulo* $\{\alpha^{(s+1)p+s}\delta\}+\{\alpha^{(s+1)p+s-1}\delta\alpha\}$.

Proof. Since $\langle\beta_s,\beta_1,\alpha\rangle\subset\pi_{2((s+1)p+s)(p-1)-1}$, we may put $\langle\beta_s,\beta_1,\alpha\rangle\equiv x\beta_{s+1}$. So that $\langle\beta_s,\beta_1,\alpha\rangle-x\langle\beta_s,\alpha,\beta_1\rangle\equiv 0$. By Lemma 7. 1, we have a map $f:K\rightarrow M_p^{N-1}$ such that the class of $f|M_p^{N+k}$ is β_s where $K=M_p^{N+k}\cup T(M_p^{N+k+l}\vee M_p^{N+k+m})\cup TM_p^{N+k+l+m+1}$, $k=2(sp+s-1)(p-1)-2$, $l=2(p-1)$, $m=2p(p-1)-1$. Therefore, similarly to the proof of Proposition 5. 3, there is a map $g;K\rightarrow X_k$ such that $g^*(b_s^{(s-1)})=ye^{N+k}$, $y\not\equiv 0 \pmod p$. While, as is easily seen, the following relations hold in $H^*(K;Z_p)$:

$$\Delta e^{N+k}=e^{N+k+1},\ \mathcal{P}^p e^{N+k}=(-1)^{N+k}e^{N+k+m+1},\ \mathcal{P}^p e^{N+k+1}=(-1)^{N+k+1}e^{N+k+m+2},$$
$$\mathcal{P}^1 e^{N+k+1}=(-1)^{N+k+1}e^{N+k+l+1},\ \mathcal{P}^1 e^{N+k+m+1}=0,$$
$$\mathcal{P}^p e^{N+k+l+1}=(-1)^{N+k+l+2}x\,e^{N+k+l+m+2},\ \mathcal{P}^1 e^{N+k+m+2}=(-1)^{N+k+m+2}e^{N+k+l+m+2}.$$

By Theorem 3. 2,

$$\begin{aligned}0&=g^*(W_sb_s^{(s-1)})=W_sg^*(b_s^{(s-1)})=yW_se^{N+k}\\&=y((s+1)\mathcal{P}^p\mathcal{P}^1\Delta-s\mathcal{P}^1\mathcal{P}^p\Delta+(s-1)\Delta\mathcal{P}^1\mathcal{P}^p)e^{N+k}\\&=y((s+1)(-1)^{l+3}x\,e^{N+k+l+m+2}-s(-1)^{m+3}e^{N+k+l+m+2}),\end{aligned}$$

since $y \neq 0$, we have $(s+1)x+s=0$, so that $x=-\frac{s}{s+1}$.

PROPOSITION 7.2. *For* $1 \leqq s < p-1$, $\langle \beta_1\, \alpha, \beta_s \rangle = \beta_{s+1}$.

Proof. By Lemma 7.2, we have

$$\langle\langle \beta_1, \alpha, \beta_{s-1}\rangle, \alpha, \beta_1\rangle + (-1)^h \langle \beta_1, \langle \alpha, \beta_{s-1}, \alpha\rangle, \beta_1\rangle + (-1)^{h+k}\langle \beta_1, \alpha, \langle \beta_{s-1}, \alpha, \beta_1\rangle\rangle \equiv 0\,,$$

modulo $\operatorname{Im}\beta_{1*}+\operatorname{Im}\beta_1^*+\operatorname{Im}\beta_{s*}+\operatorname{Im}\beta_s^*=0$, where $h=2p(p-1)-1$ and $k=2(p-1)$. Since for $s=1$, $\langle \beta_1, \alpha, \beta_1\rangle = \beta_2$, so inductively we may assume that $\langle \beta_1, \alpha, \beta_{s-1}\rangle = \beta_s$. While, $\langle \beta_s, \alpha, \beta_1\rangle = \beta_{s+1}$, $\langle \alpha, \beta_{s-1}, \alpha\rangle = x\alpha^{(s+1)p+s}$, and $\langle \beta_1, x\alpha^{(+1)p+s}, \beta_1\rangle = \langle \beta_1\alpha, x\alpha^{(s+1)p+s-1}, \beta_1\rangle = 0$. Hence, $\beta_{s+1}-\langle \beta_1, \alpha, \beta_s\rangle = 0$.
Similarly we have

COROLLARY 7.1. *For* $s+t<p$, $\langle \beta_s, \alpha, \beta_t\rangle = \beta_{s+t}$.

COROLLARY 7.2. *For* $s+t<p$, $\langle \beta_s, \beta_t, \alpha\rangle \equiv (-1)^t \frac{s}{s+t}\beta_{s+t}$.

PROPOSITION 7.3. *For* $1 \leqq s < p-2$, $\langle \alpha, \beta_1, \beta_s\rangle = -\frac{s}{s+1}\beta_{s+1}$.

Proof. We may put $\langle \alpha, \beta_1, \beta_s\rangle = x\beta_{s+1}$, for a coefficient x. By Lemma 7.2, we have

$$\langle\langle \beta_1, \alpha, \beta_1\rangle, \beta_s, \alpha\rangle - \langle \beta_1, \langle \alpha, \beta_1, \beta_s\rangle, \alpha\rangle - \langle \beta_1, \alpha, \langle \beta_1, \beta_s, \alpha\rangle\rangle \equiv 0\,,$$

so that $\langle \beta_2, \beta_s, \alpha\rangle - x\langle \beta_1, \beta_{s+1}, \alpha\rangle - (-1)^s \frac{1}{s+1}\langle \beta_1, \alpha, \beta_{s+1}\rangle \equiv 0$. Thus, $(-1)^s\frac{2}{s+2}\beta_{s+2} - (-1)^{s+1}\frac{x}{s+2}\beta_{s+2} - (-1)^s\frac{1}{s+1}\beta_{s+2} \equiv 0$. Hence, we have $x=-\frac{s}{s+1}$.

COROLLARY 7.3. *For* $s+t<p-1$, $\langle \alpha, \beta_t, \beta_s\rangle \equiv (-1)^t \frac{s}{s+t}\beta_{s+t}$.

PROPOSITION 7.4. *For* $s+t<p$, $\beta_s\delta\beta_t = \frac{st}{s+t-1}\beta_1\delta\beta_{s+t-1}$.

Proof. Put $\beta_s\delta\beta_t = x_{s,t}\beta_1\delta\beta_{s+t-1}$ for a coefficient $x_{s,t}$. Then, by Proposition 6.1, (ii), $\beta_s\delta\beta_1 = \beta_1\delta\beta_s$, so $x_{s,1}=1$. For $t>1$, by Proposition 7.1,

$$\begin{aligned}\beta_s\delta\beta_t &\equiv \frac{t}{t-1}\beta_s\delta\langle \beta_{t-1}, \beta_1, \alpha\rangle \equiv \frac{t}{t-1}\langle \beta_s\delta\beta_{t-1}, \beta_1, \alpha\rangle \\ &= \frac{t}{t-1}\langle x_{s,t-1}\beta_1\delta\beta_{s+t-2}, \beta_1, \alpha\rangle \equiv \frac{t}{t-1}x_{s,t-1}\beta_1\delta\langle \beta_{s+t-2}, \beta_1, \alpha\rangle \\ &\equiv \frac{t}{t-1}x_{s,t-1}\frac{s+t-2}{s+t-1}\beta_1\delta\beta_{s+t-1}\,.\end{aligned}$$

So that,

$$x_{s,t}=\frac{t}{t-1}\frac{s+t-2}{s+t-1}x_{s,t-1}=\frac{t}{t-1}\frac{s+t-2}{s+t-1}\frac{t-1}{t-2}\frac{s+t-3}{s+t-2}\cdots\frac{2}{1}\frac{s}{s+1}x_{s,1}$$
$$=\frac{st}{s+t-1}.$$

Now, we must calculate the modulus groups. But, a simple calculation shows that all these groups are 0. Hence, we have $\beta_s\delta\beta_t=\dfrac{st}{s+t-1}\beta_1\delta\beta_{s+t-1}$.

COROLLARY 7.4. *For* $s+t=s'+t'<p$, $\dfrac{1}{ts}\beta_s\delta\beta_t=\dfrac{1}{s't'}\beta_{s'}\delta\beta_{t'}$.

REMARK. In [10; IV], Toda defined the element β_s as a non-trivial element of $G_{2(sp+s-1)(p-1)-2}=Z_p$. So, we may choose $\bar\beta_s\in G_*$ as $\bar\beta_s=\tau j^*(\beta_s)$ for $\beta_s\in\pi_{2(sp+s-1)(p-1)-1}$. By Lemma 4.1, we have that $\tau j^*(\beta_s\delta\beta_t)=\bar\beta_s\bar\beta_t$. So that, the relation $\beta_s\delta\beta_t=\dfrac{st}{s+t-1}\beta_1\delta\beta_{s+t-1}$ for $s+t<p$ implies that $\bar\beta_s\bar\beta_t=\dfrac{st}{s+t-1}\bar\beta_1\bar\beta_{s+t-1}$ for $s+t<p$ in G_*. This is an answer to the problem of Toda [10; IV, p. 326].

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