STRUCTURE OF HEREDITARY ORDERS
OVER LOCAL RINGS

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Let $R$ be a noetherian integral domain and $K$ its quotient field, and
$\Sigma$ a semi-simple $K$-algebra with finite degree over $K$. If $A$ is a subring
in $\Sigma$ which is finitely generated $R$-module and $AK=\Sigma$, then we call it an
order. If $A$ is a hereditary ring, we call it a hereditary order (briefly
$h$-order).

This order was defined in [1], and the author has substantially
studied properties of $h$-orders in [5], and shown that we may restrict
ourselves to the case where $R$ is a Dedekind domain, and $\Sigma$ is a central
simple $K$-algebra.

In this note, we shall obtain further results when $R$ is a discrete
rank one valuation ring. Let $R$ be such a ring, and $\Omega$ a maximal order
with radical $\mathfrak{N}$, and $\Omega/\mathfrak{N}=$ $\Delta_n$; $\Delta$ division ring. Then we shall show the
following results: 1) Every $h$-order contains minimal $h$-orders $\Lambda$ such
that $\Lambda/N(\Lambda)\approx \Sigma\oplus \Delta$, where $N(\Lambda)$ is the radical of $\Lambda$, (Section 3); 2) The
length of maximal chains for $h$-order is equal to $n$, and we can decide
all chains which pass a given $h$-order, (Section 5); 3) For two $h$-orders
$\Gamma_1$ and $\Gamma_2$, they are isomorphic if and only if they are of same form, (see
definition in Section 4); 4) The number of $h$-orders in a nonminimal
$h$-order is finite if and only if $R/p$ is a finite field, where $p$ is a maximal
ideal in $R$, (Section 6).

In order to obtain those results we shall use a fundamental property
of maximal two-sided ideals in $\Lambda$; \{$\mathfrak{N}$, $R^{-1}\mathfrak{N}$, $R^{-2}\mathfrak{N}$, ..., $R^{-r+1}\mathfrak{N}$\}
gives a complete set of maximal two-sided ideals in $\Lambda$, where $\mathfrak{N}=N(\Lambda)$,
(Section 2).

H. Higikata has also determined $h$-orders over local ring in [8] by
direct computation and the author owes his suggestions to rewrite this
paper, (Section 6). However, in this note we shall decide $h$-orders as a
ring, namely by making use of properties of idempotent ideals and
radical.

We only consider $h$-orders over local ring in this paper, except Section
1, and problems in the global case will be discussed in [7] and in a
special case, where Σ is the field of quaternions, we will be discussed in [6].

1. Notations and preliminary lemmas.

Throughout this note, we shall always assume that R is a discrete
rank one valuation ring and K is the quotient field of R, and that Λ, Γ',
Ω are h-orders over R in a central simple K-algebra Σ.

For two orders Λ, Γ', the left Γ'- and right Λ-module $C_Λ(Γ') = \{ x \in Σ, Γ'x \subseteq Λ \}$ is called "(right) conductor of Γ' over Λ". By [5], Theorem 1.7,
we obtain a one-to-one correspondence between order $Γ (\supseteq Λ)$ and two-sided
idempotent ideal $\mathfrak{A}$ in Λ as follows:

$$Γ = \text{Hom}_R(\mathfrak{A}, \mathfrak{A}) \quad \text{and} \quad C_Λ(Γ) = \mathfrak{A}.$$ 

Furthermore, we have a one-to-one correspondence between two-sided
idempotent ideals $\mathfrak{A}$ and two-sided ideals $\mathfrak{M}$ containing the radical $\mathfrak{R}$ of
an order $Λ$ by [5], Lemma 2.4:

$$\mathfrak{A} + \mathfrak{R} = \mathfrak{M}.$$ 

Let $Λ/\mathfrak{M} = Λ/\mathfrak{M}_1 \oplus \cdots \oplus Λ/\mathfrak{M}_n$, where the $\mathfrak{M}_i$'s are maximal two-sided
ideals in Λ. Then $\mathfrak{R}$ is uniquely as an intersection of some $\mathfrak{M}_i$'s, say $\mathfrak{M}_{i_1}, \mathfrak{M}_{i_2}, \ldots, \mathfrak{M}_{i_r}$. We shall denote those relations by

$$\mathfrak{A} = I(\mathfrak{M}) = I(\mathfrak{M}_{i_1}, \mathfrak{M}_{i_2}, \ldots, \mathfrak{M}_{i_r}).$$ 

Let $Λ/\mathfrak{M}_i = (Δ_i)_{ni}$; $Δ_i$ division ring. Then by [5], Theorem 4.6, we know that the $Δ_i$'s depend only on $Σ$, and we shall denote it by $Δ$. For
any order $Γ'$, we denote the radical of $Γ'$ by $N(Γ')$. Let $Γ \supseteq Λ$ be h-orders,
and $C(Γ') = I(\mathfrak{M}_{i_1}, \mathfrak{M}_{i_2}, \ldots, \mathfrak{M}_{i_r})$. Then $C(Γ')/C(Γ')\mathfrak{R} \cong Λ/\mathfrak{M}_{i_1} \oplus \cdots \oplus Λ/\mathfrak{M}_{i_r}$
$\oplus C(Γ')\cap \mathfrak{R} /C(Γ')\mathfrak{R}$ as a right Λ-module; $(i_1, i_2, \ldots, i_r, j_1, j_2, \ldots, j_{n-r}) \equiv (1, 2, \ldots, n)$. By [5], Theorem 4.6 and its proof, we have

**Lemma 1.1.** $Γ'/N(Γ') \cong \text{Hom}_{Σ/\mathfrak{R}}(C(Γ')/C(Γ')\mathfrak{R}, C(Γ')/C(Γ')\mathfrak{R})$, and every
simple component of $C(Γ')\cap \mathfrak{R} /C(Γ')\mathfrak{R}$ appears in some $Λ/\mathfrak{M}_{j_t}, t=1, \ldots, n-r$.

Let $\hat{R}$ be the completion of $R$ with respect to the maximal ideal $\mathfrak{p}$ in $R$, and $\hat{K}$ its quotient field. Then $Σ = Λ \otimes \hat{K}$ is also central simple $\hat{K}$
-algebra and $\hat{Λ} = Λ \otimes \hat{R}$ is an order over $\hat{R}$ in $Σ$. If $Ω$ is a maximal order
in $Σ$, then $Ω$ is also maximal in $Σ$ by [1], Proposition 2.5. Let $Γ'$ be
any order in $Ω$, then we can find some $n$ such that $Ω/\mathfrak{p}^{n}Ω \subseteq Γ'$. Since
$Ω/\mathfrak{p}^{n}Ω \cong Ω/\mathfrak{p}^{n}Ω$ as a ring, there exists an order $Γ$ in $Ω$ such that $Γ = Γ'$. Furthermore, since $\otimes \hat{R}$ is an exact functor, we have

**Proposition 1.1.** Let $Ω$ be a maximal order in $Σ$. Then there is a
one-to-one correspondence between orders $\Gamma$ in $\Omega$ and order $\hat{\Gamma}$ in $\hat{\Omega}$.

If $\Delta$ is an $h$-order then $\mathfrak{N}$ is $\Delta$-projective, and hence, $\hat{\mathfrak{N}}$ is $\hat{\Delta}$-projective. Therefore, by usual argument (cf. [2], p. 123, Exer. 11, and [5], Lemma 3.6), we have

**Corollary.** By the above correspondence $h$-orders in $\Omega$ correspond to those in $\hat{\Omega}$.

**Proposition 1.2.** Let $\Lambda, \Gamma$, and $\Omega$ be as above. If $\Delta = \alpha' \Gamma \alpha'^{-1}$ for a unit $\alpha'$ in $\hat{\Omega}$, then $\Lambda = \alpha \Gamma \alpha^{-1}$, and $\alpha$ is unit in $\Omega$.

**Proof.** Since $\hat{\Omega}/\mathfrak{p} \hat{\Omega} \approx \Omega/\mathfrak{p}^n \Omega$ for some $n$, and $\mathfrak{p}^n \Omega$ is contained in $\mathcal{N}(\Omega)$, it is clear.

From those propositions many results in $h$-orders over $R$ are obtained from those in $h$-orders in the ring of matrices of maximal order $\mathcal{O}$ in a division ring $\Lambda'$ over a complete field. Furthermore, all $h$-orders in $\mathcal{O}_n$ are decided by Higikata [8]. However, in this note, we shall discuss properties of $h$-orders as a hereditary ring, namely, by means of idempotent ideals and radical, except the following lemma and the last section.

Let $\mathcal{O}$ be as above. Then $\mathcal{O}$ contains a unique maximal ideal $(\pi)$, and every left or right ideal is two-sided and is equal to $(\pi^r)$. From those propositions many results in $h$-orders over $R$ are obtained from those in $h$-orders in the ring of matrices of maximal order $\mathcal{O}$ in a division ring $\Lambda'$ over a complete field. Furthermore, all $h$-orders in $\mathcal{O}_n$ are decided by Higikata [8]. However, in this note, we shall discuss properties of $h$-orders as a hereditary ring, namely, by means of idempotent ideals and radical, except the following lemma and the last section.

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Theorem 1.2. Let \( R \) be a Dedekind domain and \( P \) a finite set of primes in \( R \), and \( \Omega \) a maximal order over \( R \) in \( \Sigma \). For any given \( h \)-order \( \Lambda(p) \) in \( \Omega_p, p \in P \), there exists a unique \( h \)-order \( \Lambda \) in \( \Omega \) such that \( \Lambda_{p}=\Lambda(p) \) for \( p \in P \), and \( \Lambda_{q}=\Omega_{q} \) for \( q \notin P \).

Proof. First, we assume \( P = \{ p \} \). By [4], Theorem 3.3, \( \Lambda(p)=\Omega_{p} \cap \Omega_{r} \cap \cdots \cap \Omega_{i} : \Omega_{i} \) maximal order over \( R_{p} \). Let \( \Omega'_{i}=C_{\Omega}(\Omega'_{i}) \), then \( \Omega'_{i}=\text{Hom}_{R}(\Omega', \Omega) \) where \( \Omega'_{i}=\Omega_{p} \). Furthermore, \( \Omega'_{i} \supseteq \Omega_{p} \). Let \( \Omega_{i}=\mathcal{C}(\Omega'_{i}) \), then \( \mathcal{C}_{i}=\mathcal{C}_{i} \) and \( \mathcal{C}_{i}=\mathcal{C}_{i} \) since \( \mathcal{C}_{i} \supseteq \mathcal{C}_{i} \). Put \( \Omega_{i}=\text{Hom}_{R}(\mathcal{C}_{i}, \mathcal{C}_{i}) \) and \( \Lambda_{i}=\bigcap_{i} \). Then \( \Lambda_{p}=\bigcap_{i} \text{Hom}_{R}(\mathcal{C}_{i}, \mathcal{C}_{i}) = \bigcap_{i} \Omega_{i} \). Hence, \( \Lambda \) is a desired \( h \)-order. Let \( \Lambda_{p} \) be such an \( h \)-order as above for \( p \neq q \). Then \( \bigcap_{i} \Lambda_{p} \) has a property in the theorem.

By virtue of this theorem we shall study, in this paper, \( h \)-orders over a valuation ring.

2. Normal sequence.

Let \( \Lambda \) be an \( h \)-order and \( \mathcal{R} \) the radical of \( \Lambda \). Let \( \{ \mathcal{M}_{i} \}; i=1, \cdots, n \), be the set of maximal two-sided ideals in \( \Lambda \). Since \( \mathcal{R}^{-1}\mathcal{M}=\mathcal{M}^{-1} \) by [5], Theorem 6.1, \( \mathcal{M} \rightarrow \mathcal{M}^{-1} \) gives a one-to-one correspondence among two-sided ideals \( \mathcal{M} \) in \( \Lambda \), which preserves inclusion by [5], Proposition 4.1.

Theorem 2.1. Let \( \Lambda \) be an \( h \)-order with radical \( \mathcal{R} \) such that \( \Lambda/\mathcal{R} \supseteq \sum \mathcal{M}_{i} \). For any maximal two-sided ideal \( \mathcal{M} \) in \( \Lambda \), \( \{ \mathcal{M}, \mathcal{R}^{-1}\mathcal{M}, \mathcal{R}^{-2}\mathcal{M}, \cdots, \mathcal{R}^{-n+1}\mathcal{M} \} \) gives a complete set of maximal two-sided ideals in \( \Lambda \).

Proof. We may assume that \( \mathcal{R}^{-i}\mathcal{M}=\mathcal{M} \). If \( i \leq n \), there exists an \( h \)-order \( \Omega \) such that \( C(\Omega) = I(\mathcal{M}, \mathcal{R}^{-1}\mathcal{M}, \cdots, \mathcal{R}^{-i-1}\mathcal{M}) \). Let \( \mathcal{C}=C(\Omega) \) and \( \mathcal{M}_{j}=\mathcal{R}^{-i}\mathcal{M}=\mathcal{M} \). \( \mathcal{R}^{-1}(\bigwedge_{j} \mathcal{M}_{j}) = \bigwedge_{j} \mathcal{M}_{j} \), and \( \mathcal{R}^{-1}\mathcal{C}=\mathcal{C} \) by the observation in Section 1. Since \( \mathcal{C} \cap \mathcal{R}/\mathcal{M}=\mathcal{C} \cap \mathcal{R}/\mathcal{M}, \mathcal{C}+\mathcal{M}/\mathcal{M} \) is contained in the annihilator of \( \mathcal{C} \cap \mathcal{R}/\mathcal{M} \) on \( \Lambda/\mathcal{M} \). However, by Lemma 1.1 \( \mathcal{C} \cap \mathcal{R}/\mathcal{M} \) contains only simple components which appear in \( \mathcal{C}+\mathcal{M}/\mathcal{M} \approx \Lambda/\mathcal{M}_{i} \cdots \mathcal{M}_{n-i} \) as a right \( \Lambda \)-module, which is a contradiction.

From this theorem we can find a sequence of maximal two-sided ideals \( \{ \mathcal{M}_{i} \}; i=1, \cdots, n \) in \( \Lambda \) such that \( \mathcal{R}^{-i}\mathcal{M}_{i}=\mathcal{M}_{i+1}, \mathcal{M}_{n+i}=\mathcal{M}_{i} \) for all \( i \). We shall call such a sequence \( \{ \mathcal{M}_{i} \} \) "a normal sequence".

Lemma 2.1. Let \( \Lambda \) be an \( h \)-order with radical \( \mathcal{R} \). If \( \Omega \) is an order containing properly \( \Lambda \), then \( \mathcal{R}^{-1}\mathcal{M} \) contains \( \Lambda \) and is not equal to \( \Omega \).
Proof. Let $C = C(\Omega)$. It is clear that $R^{-1}R$ is an order containing $\Lambda$, and that $C(R^{-1}R) = R^{-1}C$. Since $C = \Lambda$, $R^{-1}R = C$ by Theorem 2.1 and the observation in Section 1.

**Proposition 2.1.** Let $\Lambda, R$ be as above. For a two-sided ideal $\mathfrak{A}$ in $\Lambda$ $\mathfrak{A}$ is invertible in $\Lambda$ if and only if $\mathfrak{A}R = \mathfrak{A}R$.

Proof. If $\mathfrak{A}$ is invertible, then $\mathfrak{A} = \mathfrak{A}^*$ by [5], Theorem 6.1, and hence $\mathfrak{A}R = \mathfrak{A}R$. Conversely, let $\mathfrak{A}R = \mathfrak{A}R$, and $\Omega = \text{Hom}_\Lambda(\mathfrak{A}, \mathfrak{A}) = \text{Hom}_\Lambda(R^{-1}R, R^{-1}R) \supseteq R^{-1}R$. Since $\Omega, R^{-1}R$ contain same number of maximal two-sided ideals, $\Omega = R^{-1}R$. Therefore, $\Omega = \Lambda$ by Lemma 2.1, and hence $\mathfrak{A}$ is invertible by [5], Section 2.

**Lemma 2.2.** Let $\Lambda$ be an $h$-order, and $\{M_i\}_{i=1}^n$ the complete set of maximal two-sided ideals and $\mathfrak{A}$ a two-sided ideal in $\Lambda$. If $\mathfrak{A}M_i = \mathfrak{A}M_i$ for all $i$, then $\mathfrak{A}$ is principal, i.e., $\mathfrak{A} = \alpha\Lambda = \Lambda\alpha$.

Proof. Since $\mathfrak{A} = \bigcap M_i = \sum_{i_1 \ldots i_n} M_{i_1}M_{i_2} \ldots M_{i_n}$, $\mathfrak{A}R = \mathfrak{A}R$. Hence $\mathfrak{A}$ is invertible by Proposition 2.1, and $\Lambda = \text{Hom}_\Lambda(\mathfrak{A}, \mathfrak{A})$. Since $\mathfrak{A}$ is $\Lambda$-projective, we have a two-sided $\Lambda$-epimorphism $\psi: \Lambda \to \text{Hom}_\Lambda(\mathfrak{A}/\mathfrak{A}M_i, \mathfrak{A}/\mathfrak{A}M_i)$. Hence, $\mathfrak{A}/\mathfrak{A}M_i \subseteq \Sigma M_i$ as a right $\Lambda$-module. Therefore, $\mathfrak{A} = \alpha\Lambda$, and $\Lambda = \text{Hom}_\Lambda(\alpha\Lambda, \alpha\Lambda) = \Lambda\alpha^{-1}$.

In any $h$-order $\Lambda$, we have $N(\Lambda)^m = p\Lambda$ for some $m$, we call $m$ “the ramification index of $\Lambda$”, and $\Lambda$ “unramified” if $m = 1$.

**Theorem 2.2.** Let $\Lambda$ be an $h$-order with radical $R$, and $\{M_i\}_{i=1}^n$ the set of maximal two-sided ideals. Then $\mathfrak{A}R$ is principal. For a two-sided ideal $\mathfrak{A}$, $\mathfrak{A}R = \mathfrak{A}R$ for all $i$ if and only if $\mathfrak{A} = \mathfrak{A}R^r$ for some $r$. Let $\Omega$ be an order containing $\Lambda$, and $s,t$ are ramification indices of $\Omega$ and $\Lambda$, respectively. Then $n \mid t$, and $t \mid sn$. Therefore, if $\Omega$ is unramified, then $n = t$, and $\mathfrak{A}M_i = \mathfrak{A}M_i$ for all $i$ if and only if $\mathfrak{A} = \mathfrak{A}R^l$ for some $l$, (cf. Proposition 6.2).

Proof. The first part is clear by Theorem 2.1 and Lemma 2.2. Let $\mathfrak{A}R = \alpha\Lambda = \Lambda\alpha$. Since $\alpha^{-1}M_i\alpha = M_i$ for all $i$ and $C = C(\Omega) = I(M_i, \ldots, M_i)$, $\alpha^{-1}\mathfrak{A}\alpha = \mathfrak{A}$. Therefore, $\Omega = \text{Hom}_\Lambda(\mathfrak{A}, \mathfrak{A}) = \text{Hom}_\Lambda(\alpha^{-1}\mathfrak{A}\alpha, \alpha^{-1}\mathfrak{A}\alpha) = \alpha^{-1}\Omega\alpha$. Thus $\alpha\Omega = \Omega\alpha$ is an invertible two-sided ideal in $\Omega$, and hence, $\alpha\Omega = N(\Omega)^t$ by [5], Theorem 6.1. It is clear by Theorem 2.1 that $n \mid t$.

1) We call $\mathfrak{A}$ invertible in $\Lambda$ if $\mathfrak{A}^{-1}\mathfrak{A}^{-1} = \mathfrak{A}^{-1}\mathfrak{A} = \Lambda$; $\mathfrak{A}^{-1} = \{x \in \Sigma, \mathfrak{A}x \subseteq \Lambda\}$.
Furthermore, \( \mathfrak{M}' = (\mathfrak{M}'_n)^{\varphi} = \alpha^{t/n} \mathfrak{A} = \varphi \mathfrak{A} \). Therefore, \( \alpha^{t/n} = N(\Omega)^t = \varphi \Omega \), and hence, \( t - (t/n) = s \).

As an analogy to Lemma 2.2,

**Proposition 2.2.** Let \( \alpha \) be a non-zero divisor in \( \Lambda \). If \( \alpha^{-1} \mathfrak{M} \alpha \) is a maximal ideal in \( \Lambda \) for a maximal ideal \( \mathfrak{M} \), then \( \Delta \alpha \Lambda \) is principal ideal in \( \Lambda \).

**Proof.** Let \( \alpha^{-1} \mathfrak{M} \alpha = \mathfrak{M}' \), then \( \mathfrak{M} \alpha = \mathfrak{M} \mathfrak{M}' \), and \( \alpha^{-1} \mathfrak{M} = \mathfrak{M} \alpha^{-1} \). If we set \( \mathfrak{A} = \Delta \alpha \Lambda \), \( \mathfrak{A}' = \Delta \alpha^{-1} \Lambda \), then \( \mathfrak{M} \mathfrak{A} = \mathfrak{M} \mathfrak{A}' \) and \( \mathfrak{M} \mathfrak{M}' = \mathfrak{M} \mathfrak{M}' \). Since \( \mathfrak{M} \mathfrak{M}' = \mathfrak{M} \mathfrak{M}' = \alpha \mathfrak{M} \mathfrak{M}' \mathfrak{A}^{-1} \mathfrak{M} = \mathfrak{M} \), \( \mathfrak{M} \subseteq \text{Hom}_\mathfrak{A}(\mathfrak{M}, \mathfrak{M}) \). Similarly, we obtain \( \mathfrak{M} \subseteq \text{Hom}_\mathfrak{A}(\mathfrak{M}, \mathfrak{M}) \). Therefore, \( \mathfrak{M} \subseteq \text{Hom}_\Lambda(\mathfrak{M}, \mathfrak{M}) \cap \text{Hom}_\Lambda(\mathfrak{M}, \mathfrak{M}) = \Lambda \) by [5], Corollary 1.9 and Theorem 3.3. It is clear that \( \mathfrak{M} \subseteq \Lambda \), and hence \( \mathfrak{M}' = \Lambda \). Since \( \alpha^{-1} \subseteq \mathfrak{M}' = \Lambda \), \( \mathfrak{M} \leq \alpha \Lambda \), which implies \( \mathfrak{A} = \alpha \Lambda = \Delta \alpha \).

Next, we shall consider normal sequences of \( h \)-orders \( \Gamma \) and \( \Lambda \leq \Gamma \).

Before discussing that, we shall quote the following notations. Let \( \{\mathfrak{M}_i\}_{i=1}^n \) be the normal sequence of \( \Lambda \). We divide \( S = \{\mathfrak{M}_i\} \) to the subsets \( S^1, \cdots, S_r \), such that \( \bigcup S_i = S \), \( S_i \cap S_j = \emptyset \), and for any \( \mathfrak{M}_i \in S_j \), \( M_\mathfrak{M}_j \leq S_i \), \( M_\mathfrak{M}_j \leq \mathfrak{M}_i \), \( l < t \) if \( i < j \). Let \( S'_i = \{\mathfrak{M}_i, \mathfrak{M}_{i+1}, \cdots, \mathfrak{M}_{i+m_i-1}\} \), \( S_i = S'_i - \{\mathfrak{M}_{i+m_i-1}\} \). Then we call \( m_i \) the length of \( S_i \) or \( S'_i \). Let \( \Gamma \) be \( h \)-order containing \( \Lambda \). Then \( C(\Gamma) = I(\mathfrak{M}_i, \cdots, \mathfrak{M}_{i}) \), and by the above definition, \( C(\Gamma) \) corresponds uniquely to \( S_1, \cdots, S_r \); for example if \( C(\Gamma) = I(\mathfrak{M}_1, \mathfrak{M}_2, \mathfrak{M}_3, \mathfrak{M}_4) \), then \( S_1 = \{\mathfrak{M}_1, \mathfrak{M}_2, \mathfrak{M}_3\} \), \( S_2 = \emptyset \), \( S_3 = \{\mathfrak{M}_4\} \), \( S_4 = \emptyset \), for \( i \geq 3 \). Let \( \mathfrak{M} = I(S_1, S_2, \cdots, S_r) \). Then \( \Omega = \text{Hom}_\Lambda(\mathfrak{M}, \mathfrak{M}) \) is an order such that there exist no orders between \( \Omega \) and \( \Gamma \) by [5], Theorem 3.3.

**Lemma 2.3.** Let \( \Gamma \), \( \Lambda \), \( \mathfrak{M}_i \), and \( S_i \) be as above, then \( \{\mathfrak{M}_i \} \) is the set of maximal two-sided ideals in \( \Gamma \) if \( \Gamma \) is not maximal.

**Proof.** Since \( \mathfrak{M}_i \Gamma = C(\mathfrak{M}_i) \) by [5], Proposition 3.1, we may prove by [5], Theorem 1.7 that every maximal two-sided ideal \( \mathfrak{M}_i \) in \( \Gamma \) is idempotent. Since \( \mathfrak{M}_i \neq N(\Gamma) \), \( \mathfrak{M}_i \) is not inversible, and hence, \( \tau^{(\mathfrak{M}_i \Gamma)} \mathfrak{M}_i = \mathfrak{M}_i \) by [5], Section 2. Therefore, \( \mathfrak{M}_i \) is idempotent by [5], Lemma 1.5.

By Lemma 1.1, we obtain that \( \mathfrak{M} / \mathfrak{M}_i \mathfrak{M} \cong \mathfrak{M}_1 \oplus \mathfrak{M}_2 \cdots \oplus \mathfrak{M}_i \) as a right \( \Lambda \)-module, where \( \mathfrak{M}_i \) is a direct sum of simple components in \( \Lambda / \mathfrak{M}_i \mathfrak{M}_i \).

**Lemma 2.4.** Let \( \Lambda \), \( \Gamma \), \( \mathfrak{M}_i \) and \( \mathfrak{M} / \mathfrak{M}_i \mathfrak{M} \) be as above. Then by the isomorphism \( \varphi \) in Lemma 1.1: \( \Gamma / N(\Gamma) \cong \text{Hom}_\Lambda(\mathfrak{M} / \mathfrak{M}_i \mathfrak{M}, \mathfrak{M} / \mathfrak{M}_i \mathfrak{M}) \). Therefore, \( \mathfrak{M}_i \mathfrak{M} / N(\Gamma) \) corresponds to \( \text{Hom}_\Lambda(\mathfrak{M}_i \mathfrak{M} / \mathfrak{M}_i \mathfrak{M}, \mathfrak{M}_i \mathfrak{M} / \mathfrak{M}_i \mathfrak{M}) \).

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2) \( \tau^{(\mathfrak{M}_i \Gamma)}(\mathfrak{M}) \) means the two-sided ideal in \( \Gamma \) generated images of \( f \); \( f \in \text{Hom}_\Gamma(\mathfrak{M}, \Gamma) \).
Proof. Since $\mathcal{C}_i\Gamma/N(\Gamma)$ is a maximal two-sided ideal in $\Gamma/N(\Gamma)$, $\mathcal{C}_i\Gamma/N(\Gamma)$ is characterized by the image of $\mathcal{C}/\mathcal{C}\mathcal{R}$ by $\varphi(\mathcal{C}_i\Gamma/N(\Gamma))$. $\mathcal{C}/\mathcal{C}\mathcal{R}=\Lambda/\mathcal{M}_{t_1+m_1-1}\oplus \cdots \oplus \Lambda/\mathcal{M}_{t_r+m_r-1}\oplus \mathcal{C}/\mathcal{R}/\mathcal{C}\mathcal{R}$, and $\mathcal{C}_i\Gamma(\mathcal{C}/\mathcal{C}\mathcal{R})=\mathcal{C}_i\mathcal{C}$ + $\mathcal{C}/\mathcal{C}\mathcal{R}=\mathcal{C}_i+\mathcal{C}/\mathcal{C}\mathcal{R} \supseteq \Lambda/\mathcal{M}_{t_1+m_1-1}\oplus \cdots \oplus \cdots \oplus \Lambda/\mathcal{M}_{t_r+m_r-1}$, which implies the lemma.

**Lemma 2.5.** Let $\Lambda$ be an h-order with radical $\mathcal{R}$ and normal sequence $\mathcal{M}_i$, $i=1, \ldots, r$. Then $\mathcal{M}_i/\mathcal{M}_i\mathcal{R} \cong \Lambda/\mathcal{M}_i \oplus \cdots \oplus \Lambda/\mathcal{M}_r \oplus \mathcal{O}_{i+1}$ as a right $\Lambda$-module. Hence, $\Omega_i/N(\Omega_i) = \Delta_{m_1} \oplus \cdots \oplus \Delta_{m_{i-1}} \oplus \Delta_{m_i+r_i+1} \oplus \Delta_{m_i+2} \cdots \oplus \Delta_{m_r}$, where $\mathcal{O}_{i+1}$ is a direct sum of $m_i$ simple components of $\Lambda/\mathcal{M}_{i+1}$, and $\Lambda/\mathcal{M}_i = \Delta_{m_i}$ and $\Omega_i = \text{Hom}_\Lambda(\mathcal{M}_i, \mathcal{M}_i)$.

Proof. We obtain similarly to the proof of Lemma 2.2 that $\Lambda/\mathcal{M}_i \cong \text{Hom}_\Lambda(\mathcal{M}_{i+1}, \mathcal{M}_{i+1})$, since $\Lambda = \text{Hom}_\Lambda(\mathcal{R}, \mathcal{R})$ and $\mathcal{M}_i \mathcal{R} = \mathcal{M}_i \mathcal{R}_{i+1}$. Furthermore, since $\mathcal{M}_i = \text{C}(\text{Hom}_\Lambda(\mathcal{M}_i, \mathcal{M}_i))$, and $\mathcal{M}_i \mathcal{R} = \mathcal{M}_i \mathcal{R}_{i+1}$, we have the lemma by Lemma 1.1.

**Corollary.** Let $\Lambda$ be an h-order with radical $\mathcal{R}$ such that $\Lambda/\mathcal{R} \cong \sum_{i=1}^r \Delta_{m_i}$, then $\sum_{i=1}^r m_i$ does not depend on $\Lambda$, and the length of maximal chain for h-orders in $\Sigma$ does not exceed $n = \sum_{i=1}^r m_i$.

Proof. Since, every maximal order is isomorphic, $\Sigma m_i$ does not depend on $\Lambda$. Since $n = \Sigma m_i \geq r$, the second part is clear by [5], Theorem 3.3.

**Remark.** We shall show that every length of maximal chain is equal to $n$ in the following section.

Before proving one of the main theorems in this section we shall consider a special situation of Lemma 2.3. Let $\Gamma = \text{Hom}_\Lambda(\mathcal{M}_i, \mathcal{M}_i)$. Then $\mathcal{C}_i = I(\mathcal{M}_i, \mathcal{M}_i)$.

**Lemma 2.6.** Let $\Gamma$, $\Lambda$ and $\mathcal{C}_i$ be as above. Then $\{\mathcal{C}_i\Gamma\}$ $i=2, \ldots, r$ is the normal sequence in $\Gamma$.

Proof. Let $\mathcal{C}_1 = \mathcal{C}_i\Gamma$. Then $\Omega = \text{Hom}_\Lambda(\mathcal{C}_2, \mathcal{C}_2) = \text{Hom}_\Lambda(\mathcal{R}_2, \mathcal{R}_2)$. If $\Omega$ is maximal, then $\Gamma$ contains only two maximal ideals, and hence, we have nothing to prove. Thus, we may assume $r \geq 4$. We denote $N(\Gamma), N(\Omega)$, $N(\Lambda)$ by $\mathcal{R}$, $\mathcal{R}', \mathcal{R}''$, respectively. Let $\Gamma_i = \text{Hom}_\Lambda(\mathcal{M}_i, \mathcal{M}_i) \subseteq \Omega$. Then $\mathcal{M}_i/\mathcal{M}_i\mathcal{R}'' = \Lambda/\mathcal{M}_i \oplus \Lambda/\mathcal{M}_i \oplus \Lambda/\mathcal{M}_i \oplus \cdots \oplus \Lambda/\mathcal{M}_i$ and $\mathcal{C}_2 + \mathcal{R}' / \mathcal{R}'' = \Lambda/\mathcal{M}_i \oplus \cdots \oplus \Lambda/\mathcal{M}_i$, and $\mathcal{C}_2\Gamma_i/N(\Gamma_i)$ is a maximal two-sided ideal, we obtain $\mathcal{C}_2 +$

3) $\downarrow$ means that we omit $i$th component.
We consider a natural right \( \Lambda \)-homomorphism \( \varphi : C_2 / C_2 R^r \to M_i / M_i R^r \). Then \( \varphi (C_2 / C_2 R^r) = C_2 + M_2 R^r / M_2 R^r = \Lambda / M_2 \oplus \ldots \oplus \Lambda / M_r \). On the other hand \( C_2 / C_2 R^r = \Lambda / M_2 \oplus \ldots \oplus \Lambda / M_r \). Hence, \( C_2 \cap R' / C_2 R^r \) contains a direct sum of simple components which appear in \( \Lambda / M_i \). Let \( \{ D_i = I(C_2, C_i) \} \ i = 3, \ldots, r \) be the set of maximal ideals in \( \Omega \). Since \( \Omega = \text{Hom}_{r}(C_2, C_2) \), we obtain by Lemmas 2.4, 2.5, \( D_i / R_i \approx \Lambda / R_i \) as a ring for \( i \geq 3 \) except one \( k \) of indices \( i \). However, we have shown that \( C_2 / C_2 R^r \geq \Lambda / M_2 \oplus \ldots \oplus \Lambda / M_r \), and hence, we know \( k = 3 \). Therefore, by Lemma 2.5 we obtain \( R^{-1} R = C_3 \). Similarly, we can prove \( R^{-1} R = C_{i+1} \) for \( i \leq n-1 \). Therefore, we have proved the lemma by Theorem 2.1.

Now, we can prove the following theorem.

**Theorem 2.3.** Let \( \Lambda \) be an \( h \)-order with normal sequence \( \{ M_i \} i = 1, \ldots, n \). Then for an order \( \Gamma \) corresponding to a sequence \( \{ S_i \} i = 1, \ldots, r \), \( \{ C_i \} i = 1, \ldots, r \) is the normal sequence in \( \Gamma \). Furthermore, \( C(\Gamma) / C(\Gamma) R \approx R \oplus R R' \oplus \ldots \oplus R R^{(r)} \). Hence, \( \Gamma / N(\Gamma) \approx \Delta_1 \oplus \ldots \oplus \Delta_r \), where \( R_i \) is a simple component in \( \Lambda / M_{i+1} \), and \( l_i = \sum_{i=1}^{i+m_i-1} s_i \), and \( \Lambda / M_i = \Delta_i \), \( \mathcal{E}_i = I(S_i, \ldots, S_r) \).

**Proof.** We shall prove the theorem by induction on the number \( r \) of maximal two-sided ideals in \( \Gamma \). If \( r = n \), then \( \Lambda = \Gamma \). If \( r = n-1 \), then the theorem is true by Lemma 2.6. We assume \( r < n-1 \). Let \( \Gamma' \) be an order between \( \Lambda \) and \( \Gamma \) such that \( C(\Gamma) = \Gamma \{ S_0, S_1, \ldots, S_r \} \), and \( \{ S_0, S_1 \} = S_i, S_i = S_i \) for \( i \geq 2 \). Then \( \{ I(S_0, \ldots, S_i, \ldots, S_r) \Gamma \} i = 0, \ldots, r \) is the normal sequence in \( \Gamma' \) by induction hypothesis. Let \( \mathcal{E}_i = I(S_0, \ldots, S_i, \ldots, S_r) \Gamma \). Since \( S_0 = C(\Gamma) \Gamma \), \( \Gamma = \text{Hom}_{r}(\mathcal{E}_0, \mathcal{E}_0) \). Therefore, by Lemma 2.6, \( \{ I_\Gamma (\mathcal{E}_0, \mathcal{E}_0) \Gamma \} i = 1, \ldots, r \) is the normal sequence in \( \Gamma \). Since \( S_i = \{ S_0, S_i \} \), \( I_\Gamma (\mathcal{E}_0, \mathcal{E}_0) \Gamma = I(S_0, \ldots, S_i, \ldots, S_r) \Gamma \). Furthermore, \( \Gamma / N(\Gamma) \approx \Delta_1 \oplus \Delta_{i+1} \oplus \ldots \oplus \Delta_{r+1} \), where \( \Gamma' / N(\Gamma') \approx \Delta_{i'} \oplus \Delta_{i'} \oplus \ldots \oplus \Delta_{r'} \); \( l_i = l_i' \) for \( i \geq 2 \). Since \( \sum_{i=1}^{r} l_i = l_0 + l_1 + \ldots + l_r \). Thus we have proved the second part by Lemma 2.4.

Let \( \Lambda \) be an \( h \)-order with \( \{ M_i \} i = 1, \ldots, r \). If \( \Lambda / M_i = \Delta_m \), then \( (m_1, \ldots, m_r) \) is uniquely determined by \( \Lambda \) up to cyclic permutation. We call it a *form of \( \Lambda \). Furthermore, we know that \( (m_1, \ldots, m_r) \) is a nonzero integral solution of

\[
\sum_{i=1}^{r} X_i = n.
\]

4) For any right \( \Lambda \)-module \( M \), \( M' \)-means a direct sum of \( t \) copies of \( M \).
Corollary. If $\Lambda$ is a minimal $h$-order in $\Sigma$ with normal sequence $\{\mathcal{M}_i\}_{i=1,\ldots,n}$ then for any nonzero integral solution $(m_1,\ldots,m_r)$ of (1) there exists an $h$-order $\Gamma$, whose form is $(m_1,\ldots,m_r)$.

Proof. We associate a solution $(m_1,\ldots,m_r)$ to a set $\{S_1,\ldots,S_r\}$, $S_i = \{\mathcal{M}_{t_i},\ldots,\mathcal{M}_{t_i+m_i-1}\}$, where $t_i = m_i + \cdots + m_{i-1}$, $m_0 = 1$. Then $\Gamma = \text{Hom}^\ast_{\Omega}(\mathcal{I}(S_1,\ldots,S_r), \mathcal{I}(S_1,\ldots,S_r))$ is a desired order by the theorem.


By Theorem 1.1, we know that there exist minimal $h$-orders $\Lambda$ in the central simple $K$-algebra, namely $\Lambda/N(\Lambda) = \Delta \oplus \cdots \oplus \Delta$. In this section, we shall show that every $h$-order contains minimal $h$-orders.

Lemma 3.1. Let $\Gamma$ be an $h$-order and $\Lambda$, $\Lambda'$ be $h$-orders in $\Gamma$ such that there exist no orders between $\Gamma$ and $\Lambda$, $\Lambda'$, respectively. If $\mathcal{C} = \mathcal{C} \Delta(\Gamma)/\mathcal{N}$, then $\Lambda$ is isomorphic to $\Lambda'$ by an inner-automorphism of unit element in $\Gamma$, where $\mathcal{N} = N(\Gamma)$.

Proof. Let $\mathcal{C} = \mathcal{C} \Delta(\Gamma,l)$, $\mathcal{C}' = \mathcal{C} \Delta(\Gamma')$. Since $\mathcal{C}/\mathcal{N} \cong \mathcal{C}'/\mathcal{N}$, there exists a unit element $e$ in $\Gamma$ such that $\mathcal{C} = e^{-1} \mathcal{C}' e$. $\Gamma' = \text{Hom}^\ast_{\Delta}(\mathcal{C}, \mathcal{C}) = \text{Hom}^\ast_{\Delta}(e^{-1} \mathcal{C}' e, e^{-1} \mathcal{C}' e) = e^{-1} \text{Hom}^\ast_{\Delta}(\mathcal{C}', \mathcal{C}) e = e^{-1} \Gamma'' e$, where $\Gamma'' = \text{Hom}^\ast_{\Delta}(\mathcal{C}', \mathcal{C})$. On the other hand, by Theorem 2.3, we obtain that $\Gamma'$ and $\Gamma''$ contains the same number of maximal two-sided ideals as those in $\Gamma$. Hence, $\Gamma' = e^{-1} \Gamma'' e$ by [5], Theorem 3.3. Furthermore, $\Lambda = \Delta \cap \Gamma' \cap \Gamma'' = e^{-1} (\Delta \cap \Gamma'' e) e = e^{-1} \Lambda' e$.

Lemma 3.2. Let $\Gamma \supseteq \Delta$ be $h$-orders, then $N(\Lambda) \supseteq N(\Gamma)$.

Proof. Let $\mathcal{N} = N(\Lambda)$, and $\mathcal{N}' = N(\Gamma)$. We may assume that there are no orders between $\Delta$ and $\Gamma$. Then $\mathcal{C} \Delta(\Gamma) = \mathcal{N}$ is a maximal two-sided ideal in $\Delta$ by Lemma 3.4. Hence, we obtain by Lemma 1.1 that $\mathcal{N}' \mathcal{N} \subseteq \mathcal{N} \mathcal{N}' \subseteq \mathcal{N}' \mathcal{N} \subseteq \mathcal{N}$. Therefore, $\mathcal{N}' = \mathcal{N} \mathcal{N} \subseteq \mathcal{N} \mathcal{N} = \mathcal{N}$. For any maximal two-sided ideal $\mathcal{N}' = \mathcal{N}$ in $\Lambda$, we have $\mathcal{N}' = \mathcal{N}'(\mathcal{M} + \mathcal{M}') \subseteq \mathcal{N} + \mathcal{M} \mathcal{M}' \subseteq \mathcal{M}'$ since $\Lambda = \mathcal{M} + \mathcal{M}'$. Therefore, $\mathcal{N}' \subseteq \mathcal{N} = \mathcal{N}$.

Theorem 3.1. Every $h$-order contains minimal $h$-orders.

Proof. We obtain a minimal $h$-order $\Delta$ by Theorem 1.1. Let $\Gamma$ be $h$-order. Since every maximal order is isomorphic, we may assume $\Delta$ and $\Gamma$ are contained in a maximal order. Let $\{\mathcal{M}_i\}_{i=1,\ldots,r}$ be the normal sequence of $\Gamma$ with form $(m_1,\ldots,m_r)$, and $\Omega = \text{Hom}^\ast_{\Delta}(\mathcal{M}_i, \mathcal{M}_i)$. We assume that $\Omega \supseteq \Delta$. Let $\mathcal{N} = N(\Omega)$, and $\mathcal{N}' = N(\Gamma)$. Since $\mathcal{N}' \supseteq \mathcal{N}$, $\mathcal{N} \supseteq \mathcal{N}$. Now, we consider a left ideal $\mathcal{M}_i/\mathcal{M}'$ in $\Omega/\mathcal{N} = \text{Hom}^\ast_{\mathcal{N}}(\mathcal{M}_i/\mathcal{M}, \mathcal{M})$. \(\square\)
\[ \mathfrak{M}_i / \mathfrak{M}_i \mathfrak{N}' \]. Since \((\mathfrak{M}_i, \mathfrak{M}_j) = 1 \) if \( i \neq j \), there exist \( m \) in \( \mathfrak{M}_1 \) and \( y \) in \( \mathfrak{M}_2 \cdots \mathfrak{M}_r \), such that \( 1 = m + y, m^2 - m = m(m - 1) \in \mathfrak{M}_1 \mathfrak{M}_2 \cdots \mathfrak{M}_r = \mathfrak{M}_1 (\mathfrak{M}_2 \cdots \mathfrak{M}_r) \subset \mathfrak{M}_1 \mathfrak{N}' \). Therefore, \( \mathfrak{M}_i / \mathfrak{M}_i \mathfrak{N}' = m\Lambda + \mathfrak{M}_i \mathfrak{N}' / \mathfrak{M}_i \mathfrak{N} \mathfrak{N}' / \mathfrak{M}_i \mathfrak{N} \). It is clear that \( \mathfrak{M}_i (\mathfrak{N}' / \mathfrak{M}_i \mathfrak{N}) = (0) \). Hence, \( \mathfrak{M}_i / \mathfrak{R} = (\mathfrak{N} / \mathfrak{R}) \mathfrak{M}_1 \mathfrak{N} / \mathfrak{N} \mathfrak{M}_i = \mathfrak{R} \mathfrak{M}_1 \). Therefore, \( \mathfrak{M}_i / \mathfrak{M}_i = \mathfrak{M}_i / \mathfrak{M}_i \mathfrak{N} \) by the above observation. Hence, \( \Gamma \) is isomorphic to \( \Gamma' \) which contains \( \Delta \). We can prove the theorem by induction.

**Corollary.** Every minimal \( h \)-order is isomorphic. If two minimal \( h \)-orders are contained in an order \( \Gamma \), then this isomorphism is given by a unit element in \( \Gamma \).

**Proof.** In the above, we use the fact that any \( h \)-order is isomorphic to an order containing a fixed minimal \( h \)-order, which implies the first part of the corollary. The second part is clear from the proof of the theorem.

**Theorem 3.2.** Let \( \Omega \) be a maximal order such that \( \Omega / N(\Omega) = \Delta_n \). Then every length of maximal chain for \( h \)-orders is equal to \( n \).

**Proof.** It is clear from Theorems 1.1 and 3.1.

**4. Isomorphisms of \( h \)-orders.**

In this section, we shall discuss isomorphisms over \( R \) among \( h \)-orders. For this purpose, we shall use the following definition. Let \( \Gamma_1, \Gamma_2 \) be \( h \)-orders containing an \( h \)-order \( \Lambda \). If there exists an isomorphism \( \theta \) of \( \Gamma_1 \) to \( \Gamma_2 \) such that \( \theta(\Lambda) = \Lambda \), we call \( \theta \) "isomorphism over \( \Lambda \)" and "\( \Gamma_1, \Gamma_2 \) are isomorphic over \( \Lambda \)". Let \( \Lambda \) be an \( h \)-order with normal sequence \( \{ \mathfrak{M}_i \} i = 1, \ldots, r \). Then we shall call that \( \Lambda \) is \( r \)th order, and the rank of \( \Lambda \) is \( r \). 1st order is nothing but maximal order \( \Omega \), and \( n \)th order is minimal if \( \Omega / N(\Omega) = \Delta_n \).

We have introduced an equation

\[ \sum_{i=1}^{r} X_i = n \]

in Section 2. We shall only consider nonzero integral solutions of (1). Hence, by solution we mean always such solutions. We shall define a relation among solutions \((a_1, \ldots, a_r)\) as follows: \((a_1, \ldots, a_r) \equiv (a'_1, \ldots, a'_r)\) if they are only different by a cyclic permutation. We shall denote the
number of classes of solutions by $\varphi(n, r)$. It is clear that $\varphi(n, r) = \varphi(n, n-r)$, and that $\varphi(n, 2) = [n/2]$, and $\varphi(p, r) = \left(\frac{p}{r}\right) / p$, where $p$ is prime and $\left[ \frac{\cdot}{\cdot} \right]$ Gauss' number.

We note that every isomorphism is given by an inner-automorphism in $\Sigma$.

Let $\Lambda$ be an $h$-order with radical $\mathfrak{R}$. If $\mathfrak{R}$ is principal, we call $\Lambda$ "a principal $h$-order". Every maximal order and minimal order are principal.

**Theorem 4.1.** Let $\Lambda$ be an $h$-order with form $(m_1, \ldots, m_r)$. Then $\Lambda$ is principal if and only if $m_1 = \cdots = m_r$, (cf. [9], Theorem 1).

**Proof.** If $m_1 = \cdots = m_r$, $\Lambda$ is principal by the fact $\Lambda = \text{Hom}_\mathfrak{R}(\mathfrak{R}, \mathfrak{R}) = \text{Hom}_\mathfrak{R}(\mathfrak{R}, \mathfrak{R})$ and by [5], Corollary 4.5. Conversely, if $N = \alpha \Lambda = \Delta \alpha$, then $\alpha^{-1}(\Lambda/\mathfrak{R}) \alpha = \Lambda/\alpha^{-1}\mathfrak{R}_t \alpha$ by Theorem 2.1, and hence, $m_i = m_{i+1}$ for all $i$.

**Proposition 4.1.** Let $\Lambda$ be an $h$-order with radical $\mathfrak{R}$, and $\Gamma_1, \Gamma_2$ orders containing $\Lambda$. If $\Gamma_1, \Gamma_2$ are isomorphic over $\Lambda$, then this isomorphism is given by an element in $\mathfrak{R}$. In this case $C(\Gamma_i) = \mathfrak{R}^{-t} C(\Gamma_i) \mathfrak{R}^t$ for some $t$.

**Proof.** If $\beta^{-1} \Gamma_1 \beta = \Gamma_2$, and $\beta \Lambda \beta^{-1} = \Delta$ for $\beta \in \Sigma$, then we may assume that $\beta \in \Delta$. Since $\beta \Lambda = \Delta \beta$ is invertible two-sided ideal in $\Lambda$, $\beta \Lambda = \mathfrak{R}^t$ for some $t \geq 0$. It is clear that $C(\Gamma_i) = \beta^{-1} C(\Gamma_i) \beta = \mathfrak{R}^{-t} C(\Gamma_i) \mathfrak{R}^t$.

**Corollary.** If $\Lambda$ is principal, then $\Gamma_1$ and $\Gamma_2$ are isomorphic over $\Lambda$ if and only if $\mathfrak{R}(\Gamma_i) = \mathfrak{R}^{-t} C(\Gamma_i) \mathfrak{R}^t$ for some $t$, where $\mathfrak{R} = \mathfrak{R}(\Lambda)$.

**Theorem 4.2.** Let $\Lambda$ be a principal $h$-order of a form $(s, \ldots, s)$. Then the following statements are true:

1) $\Gamma_1, \Gamma_2$ are isomorphic if and only if $\Gamma_1, \Gamma_2$ are isomorphic over $\Lambda$.
2) The number of classes of isomorphic $m-r$th orders containing $\Lambda$ is equal to $\varphi(m, r)$.
3) Those isomorphisms are given by inner-automorphisms of $\alpha^i$ for some $i$, where $N(\Lambda) = \alpha \Lambda = \Delta \alpha$.
4) Let $\Lambda_1, \Lambda_2$ be $h$-orders. Then $\Lambda_1$ and $\Lambda_2$ are isomorphic if and only if they are of same form.

**Proof.** Let $\Gamma_1$ and $\Gamma_2$ be $m-r$th orders and $C_{i} = C(\Omega_i)$ $i=1, 2$. Let

$C_{i} = I(\mathfrak{M}_{i_1}, \mathfrak{M}_{i_2}, \ldots, \mathfrak{M}_{i_{r}})$, $C_{2} = I(\mathfrak{M}_{j_1}, \mathfrak{M}_{j_2}, \ldots, \mathfrak{M}_{j_{r}})$, $i_1 < i_2 < \cdots < i_r$, $j_1 < j_2 < \cdots < j_r$, and $\{\mathfrak{M}_{i}\} i=1, \ldots, m$ the normal sequence of $\Lambda$. If $\Gamma_1$ and $\Gamma_2$
are isomorphic over $\Lambda$, then $\mathbb{C}=\alpha^{-t}\mathbb{C}\alpha^t$ for some $t$ by the above corollary. Furthermore, $\alpha^{-t}\mathbb{M}_{i+t}\alpha^t=\mathbb{M}_{(i+t)}$, where $(i_{i_1}+t)\equiv i_1+t \mod m$, and $0<(i+t)\leq m$. Therefore, $(i_{i_1}+t), (i_{i_2}+t), \ldots, (i_{i_t}+t), (i_{i_{t+1}}+t), \ldots, (i_{i_{t+(r-s)}}+t))
\equiv (j_1, j_2, \ldots, j_r)$. We shall associate the set $(j_1, j_2, \ldots, j_r)$ to a class of solution of (1) as follows: $x_1=j_2-j_1, \ldots, x_{r-1}=j_r-j_{r-1}, x_r=j_1+m-j_r$. Then $(j_1, \ldots, j_r)$ and $(i_1, \ldots, i_r)$ correspond to the same class. Conversely, for any $m-r$ th $h$-orders $\Gamma_1$ and $\Gamma_2$ if $(j_1, (j_i))$ correspond to the same class, then there exists some $t$ such that $((i_i+t))= (j_i)$. Hence, let $(x_1, \ldots, x_r)$ be any solution of (1). Let $C=I(\mathbb{M}_1, \mathbb{M}_{1+x_1}, \ldots, \mathbb{M}_{1+x_1+\cdots+x_{r-1}})$, then $\Gamma=\text{Hom}_\Lambda(C, \mathbb{C})$ is an $h$-order containing $\Lambda$ and $\Gamma$ corresponds to $(x_1, \ldots, x_r)$ by the above mapping, which implies 2).
Next, we shall consider $r$ th order $\Gamma_i$ $(i=1, 2)$ containing $\Lambda$. If $\Gamma_1$ and $\Gamma_2$ are isomorphic, then they are of same form $(st_1, st_2, \ldots, st_r)$. If we associate $(t_1, t_2, \ldots, t_r)$ to $\Gamma_i$, then $\Gamma_1$ and $\Gamma_2$ correspond to the same class of solution of (1) replacing $n$ by $m$. Conversely, for any solution $(t_j)$ of (1), we can find an order $\Gamma_i(\geq \Lambda)$ of a form $(st_1, \ldots, st_r)$ by Theorem 2.3. Hence, the number of classes of isomorphic $r$ th orders is equal to or larger than $\varphi(m, r)$. On the other hand, that number does not exceed the number of classes of isomorphic $r$ th orders over $\Lambda$, which is equal to $\varphi(m, m-r)\geq \varphi(m, r)$ by 2). Therefore, we have proved 1). 3) is clear by 1) and Proposition 4.1. 4) is clear from the above and Theorem 3.1.

Corollary 4.1. Let $\Gamma_1$ and $\Gamma_2$ be isomorphic over $\Lambda$, then they are isomorphic over any principal $h$-orders $\Lambda'$ contained in $\Lambda$. In this case the form of $\Lambda$ has a periodicity.5)

Proof. The first part is clear by the theorem, and the isomorphism is given by $\alpha^t$, where $\mathbb{R}=N(\Lambda')=\alpha \Lambda'$. Hence, $\alpha^{-t}\Lambda\alpha^t=\Lambda$, which means $C_\Lambda(\Lambda)=\mathbb{R}^{-t}C_\Lambda(\Lambda)\mathbb{R}$.

Corollary 4.2. Let $\Gamma_1$ and $\Gamma_2$ be $h$-orders contained in an order $\Omega$, and which are isomorphic, then this isomorphism is given by a unit element in $\Omega$ and an element $\alpha^t$, where $\alpha$ is a generator of radical of minimal $h$-order contained in $\Gamma_1$.

It is clear by Theorem 4.2 and Corollary to Theorem 3.1.

Corollary 4.3. For principal $h$-orders $\Gamma_1, \Gamma_2$, the following statements are equivalent:

5) If a form is the following type: $(m_1, m_2, \ldots, m_1, m_2, \ldots)$, then we call the form has a periodicity.
1) \( \Gamma_1 \) and \( \Gamma_2 \) are isomorphic,
2) \( \Gamma_1/N(\Gamma_1) \) and \( \Gamma_2/N(\Gamma_2) \) are isomorphic,
3) \( \Gamma_1 \) and \( \Gamma_2 \) are of the same rank.

**Remark.** The above corollary is not true for any \( h \)-order. For instance, let \( \{M_1, M_2, \ldots, M_s\} \) be the normal sequence of a minimal \( h \)-order \( \Lambda \) in \( K_s \), and \( \mathcal{E}_1=I(M_1, M_4, M_6), \mathcal{E}_2=I(M_1, M_4, M_6) \). Then \( \Gamma_1=\text{Hom}^A_\alpha(\mathcal{E}_1, \mathcal{E}_2) \) and \( \Gamma_2=\text{Hom}^A_\alpha(\mathcal{E}_2, \mathcal{E}_3) \) have different form \((1, 2, 3)\) and \((2, 1, 3)\), but \( \Gamma_2/N(\Gamma_1) \cong \Gamma_2/N(\Gamma_2) \).

**Corollary 4.4.** Let \( \Gamma_1 \) and \( \Gamma_2 \) be \( h \)-orders containing principal \( h \)-orders \( \Lambda_1 \), and \( \Lambda_2 \) such that there exist no orders between \( \Gamma_1 \) and \( \Lambda_1 \). Then the statements in Corollary 4.3 are true.

**Proof.** Every \( \Gamma \) containing \( \Lambda \) which satisfies the condition of the corollary is isomorphic by Theorems 2.3 and 4.2. Hence, the corollary is true by Corollary 4.2.

**Corollary 4.5.** Let \( n \) be the length of maximal chain for \( h \)-orders. If \( n \leq 5 \), (1) and 2) in Corollary 4.3 are equivalent for any orders. If \( n \leq 3 \), (1), (2), and 3) in Corollary 4.3. are equivalent for any orders.

We shall recall the definition of same type in [5], Section 4. If there exists a left \( \Gamma_1 \) and right \( \Gamma_2 \) ideal \( \mathfrak{A} \) in \( \Sigma \) for two orders \( \Gamma_1 \) and \( \Gamma_2 \) such that \( \Gamma_1=\text{Hom}^\alpha_2(\mathfrak{A}, \mathfrak{A}), \Gamma_2=\text{Hom}^\alpha_1(\mathfrak{A}, \mathfrak{A}) \), we call \( \Gamma_1 \) and \( \Gamma_2 \) belong to the same type”.

**Lemma 4.1.** Let \( \Lambda_1 \) and \( \Lambda_2 \) be \( h \)-orders which belong to the same type, and \( \Omega_1, \Omega_2 \) containing \( \Lambda_1, \Lambda_2 \), respectively. Then \( \Omega_1, \Omega_2 \) belong to the same type if and only if \( \Omega_1 \), \( \Omega_2 \) are of same rank.

**Proof.** By the assumption, we have a left \( \Lambda_1 \) and right \( \Lambda_2 \) ideal \( \mathfrak{A} \) such that \( \mathfrak{A}=\text{Hom}^\alpha_2(\mathfrak{A}, \mathfrak{A}), \Lambda_2=\text{Hom}^\alpha_1(\mathfrak{A}, \mathfrak{A}) \). Then \( \mathfrak{A}^{-1}=\Lambda_1, \mathfrak{A}^{-1}=\Lambda_2, \) and hence, \( \mathfrak{A}^{-1}\Lambda_1\mathfrak{A}=\Lambda_2, \mathfrak{A}\Lambda_2\mathfrak{A}^{-1}=\Lambda_1 \) by [5], Section 4. Let \( \mathcal{E}=C_\Lambda(\Omega_1). \) Then \( \Omega_1=\text{Hom}^\alpha_1(\mathcal{E}, \mathcal{E}) \). It is clear that \( \Omega_1=\text{Hom}^\alpha_1(\mathcal{E}, \mathcal{E})=\text{Hom}^{\alpha^{-1}}_\Lambda(\mathcal{E}\mathfrak{A}, \mathcal{E}\mathfrak{A})=\text{Hom}^\alpha_2(\mathcal{E}\mathfrak{A}, \mathcal{E}\mathfrak{A}). \) Let \( \Omega_2=\text{Hom}^\alpha_1(\mathcal{E}\mathfrak{A}, \mathcal{E}\mathfrak{A}) \), then \( \Omega_2 \geq \Lambda_2. \) Since \( \Omega_1, \Omega_2 \) belong to the same type, they are of same rank. Therefore, \( \Omega_2, \Omega_2 \) belong to the same type by [5], Theorem 4.2. Hence, \( \Omega_1, \Omega_2 \) belong to the same type.

The following theorem is a generalization of [5], Theorem 4.3.

**Theorem 4.3.** Let \( \Gamma_1, \Gamma_2 \) be orders in \( \Sigma \). Then \( \Gamma_1 \) and \( \Gamma_2 \) belong to the same type if and only if \( \Gamma_1 \) and \( \Gamma_2 \) are of same rank.
Proof. Let $\Lambda_1, \Lambda_2$ be minimal $h$-orders in $\Gamma_1, \Gamma_2$, respectively. Then $\Lambda_i = e \Lambda_i e^{-1}$ by Corollary to Theorem 3.1. Hence, $\Lambda_1 = \text{Hom}_A^e (e \Lambda_2, e \Lambda_2)$, and $\Lambda_2 = \text{Hom}_A^e (\Lambda_1 e, \Lambda_2 e)$. Thus, we obtain the theorem by Lemma 4.1.

5. Chain of $h$-orders.

In this section, we shall study by making use of arguments in the proof of Theorem 3.1 how we can find maximal chains of $h$-orders which pass a given $h$-order $\Gamma$. We have already known by [5], Theorem 3.3 how we can construct chains of $h$-orders containing $\Gamma$, which is determined by the structure of $\Gamma/\mathbb{N}(\Gamma)$.

First, we shall study a relation between left conductor $D(\ )$ and right conductor $C(\ )$.

Theorem 5.1. Let $\Gamma \triangleright \Lambda$ be $h$-orders. Then $C(\Gamma) = R D(\Gamma) R^{-1}$, where $R = N(\Lambda)$.

Proof. Let $\{M_i\} i = 1, \ldots, r$ be the normal sequence in $\Lambda$, and let $\Gamma = \text{Hom}_A^e (M_2, M_1)$, then $D(\Gamma) = M_2$. There exists some $M_i$ such that $\Gamma = \text{Hom}_A^e (M_i, M_i)$, and hence, $\{I(M_i, M_i)\} i = j$ is the normal sequence in $\Gamma$. Since $M_2/\mathbb{R} M_1 \Lambda M_1 \oplus \Lambda M_2 \oplus \ldots \oplus \Lambda M_r \oplus \mathbb{R} M_2$, where $\mathbb{R} = R/R M_2$ is a direct sum of $m_2$ simple components which appear in $\Lambda/\mathbb{R} M_2$, $M_i I(M_i, M_i) \Lambda M_i + R M_2/R M_2 = \Lambda M_1 \oplus \Lambda M_2 \oplus \ldots \oplus \Lambda M_r + R I(M_i, M_i) M_i \Lambda M_2$. Hence, if $i \neq j$, $\Gamma /I(M_i, M_i) \Lambda M_i \cong \Delta_{m_i}$ or $\Delta_{m_{i+1}}$ by Lemma 2.1. However, $\Gamma /I(M_i, M_i) \Lambda M_i = \Delta_{m_{i+1}}$ by Lemma 2.5, which is a contradiction. If $i = n$, then $\mathbb{R} M_n(I(M_n, M_n) M_n) = (0)$, and hence, $M_n I(M_n, M_n) M_n + R M_n/R M_2 = \Lambda M_1 \oplus \Lambda M_2 \oplus \ldots \oplus \Lambda M_{n-1}$, which also contradicts the fact that $I(M_n, M_n)$ is a maximal two-sided ideal. Let $\mathbb{R} = I(M_n, \ldots, M_1)$ and $\mathbb{D} = I(M_n, \ldots, M_1)$, then $\mathbb{R} = \mathbb{R}^{-1} \mathbb{D} \mathbb{R}$. We assume that $\Gamma = \text{Hom}_A^e (\mathbb{C}, \mathbb{C}) = \text{Hom}_A^e (\mathbb{D}, \mathbb{D})$. Then $\Omega = \text{Hom}_A^e (I(\mathbb{C}, M_{n+1}), I(\mathbb{C}, M_{n+1})) = \text{Hom}_A^e (I(\mathbb{C}, M_{n+1}), I(\mathbb{C}, M_{n+1})) = \text{Hom}_A^e (I(\mathbb{C}, M_{n+1}), I(\mathbb{C}, M_{n+1})) = \text{Hom}_A^e (I(\mathbb{C}, M_{n+1}), I(\mathbb{C}, M_{n+1})) = \text{Hom}_A^e (I(\mathbb{C}, M_{n+1}), I(\mathbb{C}, M_{n+1}))$, by the first part. Hence, $\Omega = \text{Hom}_A^e (I(\mathbb{C}, M_{n+1}), I(\mathbb{C}, M_{n+1})) = \text{Hom}_A^e (I(\mathbb{C}, M_{n+1}), I(\mathbb{C}, M_{n+1})) \mathbb{R}^{-1}$, $\mathbb{R}^{-1} C(\Gamma) \mathbb{R}^{-1} \subset \mathbb{C}(\Gamma) \mathbb{R}^{-1}$, since $\mathbb{C}(\Gamma) = \mathbb{R} \mathbb{D} \mathbb{R}^{-1} \mathbb{D} \mathbb{R}^{-1}$.

Theorem 5.2. Let $\Lambda$ be a principal $h$-order and $\Gamma$ an order containing $\Lambda$. Then every $h$-order containing $\Lambda$ which is isomorphic to $\Gamma$ is written as $T(\Gamma)$, where $T$ is the following functor: for $\Omega \triangleright \Lambda \Gamma(\Omega) = \text{Hom}_A^e (\mathbb{C}(\Omega), \mathbb{C}(\Omega))$, and $T'(\Omega) = T(T'(\Omega))$.

Proof. It is clear by Theorems 4.2 and 5.1, and Proposition 4.1.
We note that for two \( h \)-orders \( \Lambda \supseteq \Gamma \), \( C_r(\Lambda) \supseteq N(\Gamma) \) by Lemma 3.2.

**Lemma 5.3.** Let \( \Gamma \) be an \( r \)-th order with radical \( R \) and \( \mathcal{S} \) a left ideal containing \( R \) in \( \Gamma \) such that \( \mathcal{S}/R \cong \Delta_{m_1} \otimes \cdots \otimes \Delta_{m_l} \otimes I \otimes \Delta_{m_{l+1}} \otimes \cdots \otimes \Delta_{m_r} \); I a proper left ideal in \( \Delta_{m_1} \). Then \( \Gamma = \text{Hom}_h(\mathcal{S}, \mathcal{S}) \cap \text{Hom}_h(R, \mathcal{S}) = \Gamma \cap \text{Hom}_h(\mathcal{S}, \mathcal{S}) \) is an \( r+1 \)-th \( h \)-order and \( C(\Gamma) = \mathcal{S} \). Hence, \( \Gamma \) is uniquely determined by the rank and conductor. Furthermore, every \( r+1 \)-th \( h \)-order in \( \Gamma \) is expressed as above.

**Proof.** Since \( \mathcal{S}/R = \Gamma \), \( r(\mathcal{S}) = \Gamma \). If we put \( \Gamma' = \text{Hom}_h(\mathcal{S}, \mathcal{S}) \), then \( \Gamma' = \text{Hom}_h(\mathcal{S}, \mathcal{S}) \) by [1], Theorem A 2. By the same argument in the proof of Theorem 3.1, we can find an \( r+1 \)-th \( h \)-order \( \Lambda' \) such that \( C_{r+1}(\Gamma')/R \cong \mathcal{S}/R \). Hence, there exists a unit element \( e \in \Gamma \) such that \( C_{r+1}(\Gamma') = \mathcal{S}/R \). Furthermore, \( \Lambda' = \Gamma ' \cap \text{Hom}_h(\mathcal{S}, \mathcal{S}) \cap \text{Hom}_h(\mathcal{S}, \mathcal{S}) = \Gamma ' \cap \text{Hom}_h(\mathcal{S}, \mathcal{S}) = \mathcal{S}^{-1}(\Gamma ' \cap \mathcal{S}) \). Therefore, \( \Lambda = \Gamma ' \cap \mathcal{S} = \mathcal{S} \). Hence \( \Lambda = \Lambda' \). The last part is clear.

Let \( \Lambda \) be an \( h \)-order of form \( (m_1, m_2, \ldots, m_r); \Lambda/N(\Lambda) = \Delta_{m_1} \oplus \cdots \oplus \Delta_{m_r} \), and \( \mathcal{S}_{i,j} \) a left ideal in \( \Lambda \) such that \( \mathcal{S}_{i,j} \supseteq \mathcal{S}_i \), and \( \mathcal{S}_{i,j}/R = \Delta_{m_1} \oplus \cdots \oplus \Delta_{m_r} \), \( I_{i,j} \) a non-zero left ideal in \( \Delta_{m_i} \). We denote \( \text{Hom}_h(\mathcal{S}_{i,j}, \mathcal{S}_{i,j}) \) by \( \Delta(\mathcal{S}_{i,j}) \) and \( I_{i,j} \) by \( I(I_{i,j}) \). Let \( k(I_{i,j}) \) be the length of composition series of \( I_{i,j} \) as a left \( \Lambda \)-module.

**Theorem 5.3.** Let \( \Lambda \), \( \mathcal{S}_{i,j} \) be as above. Then \( \Gamma = \bigcap_{j=1}^{t, i=1} \Delta(\mathcal{S}_{i,j}) \) is an \( h \)-order if and only if \( I(I_{i,j}) \) is linearly ordered by inclusion for all \( i \). Every \( r+s(i) \)-th \( h \)-order in \( \Lambda \) is uniquely written as above.

**Proof.** We assume that \( \Gamma \) is an \( h \)-order and \( \Lambda \) is a minimal \( h \)-order in \( \Gamma \). Let \( S_i = \{ \mathcal{M}_{i_1}, \mathcal{M}_{i_2}, \ldots, \mathcal{M}_{i_{i+1}} \} \) be a set of maximal two-sided ideals in \( \Lambda \), such that \( C_{\Lambda_0}(\Lambda) = I(S_i, S_{i-1}, \ldots, S_1) \), (cf. Section 2). We denote \( \Lambda \cap \Delta(\mathcal{S}_{i,j}) \) by \( \Gamma_j \). Since \( \Gamma_j \) is an \( r+1 \)-th order from Lemma 2.5 we obtain \( C_{\Lambda_0}(\Gamma_j) = I(S_i, S_{i-1}, \ldots, S_1) \). We assume \( \rho(j_i) < \rho(j_1) \). Let \( \mathcal{G}_i = S_i - \{ \mathcal{M}_{p,j_1}, \mathcal{M}_{p,j_2} \} \), \( \mathcal{C} = I(S_i, S_{i-1}, \ldots, S_1) \). Then \( \Gamma' = \text{Hom}(\mathcal{C}, \mathcal{C}) \) is an \( r+2 \)-th \( h \)-order and \( \Gamma' = \Gamma_1 \cap \Gamma_2 \). Let \( \mathcal{R}_1 = I(S_i, S_{i-1}, S_i - \{ \mathcal{M}_{p,j_2} \}, \ldots, S_1) \) and \( \mathcal{R}_2 = I(S_i, \ldots, S_1 - \{ \mathcal{M}_{p,j_1} \}) \), then we obtain a normal sequence \( \{ \mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3, \ldots \} \) in \( \Gamma' \) by Theorem 2.3, and \( C_{\Gamma'}(\Gamma_j) = \mathcal{R}_2 \). Since \( C_{\Gamma'}(\Lambda) = I(\mathcal{R}_1, \mathcal{R}_2) \), \( C_{\Gamma'}(\Lambda) = I(\mathcal{R}_1, \mathcal{R}_2) \) by the usual argument in Sections 2 and 3, where \( \Gamma' = \mathcal{R}_2 \), and \( I \) is a simple left ideal in \( \Delta_{m_1} \). On the other hand, since \( \mathcal{S}_{i,j} = C_{\Gamma_j}(\Lambda) = I(\mathcal{R}_1, \mathcal{R}_2) \Gamma_{j_2} \), and \( I(\mathcal{R}_1, \mathcal{R}_2) \Gamma_{j_2} \),
$I(\mathfrak{A}_1, \mathfrak{A}_2) \Gamma_{f_2}, \ldots$ is a normal sequence in $\Gamma_{f_2}$, we obtain $\mathfrak{A}_{i,f_2}/A(\Lambda) = \Delta_{m_1} \oplus \ldots \oplus \Delta_{m_{i-1}} \oplus I \oplus \Delta_{m_{i+1}} \oplus \ldots \approx C_{\phi}(\Lambda)/N(\Lambda)$. However, $\mathfrak{A}_{i,f_2} \supseteq C_{\phi}(\Lambda)$, and hence $\mathfrak{A}_{i,f_2} = C_{\phi}(\Lambda) \subseteq \mathfrak{A}_{i,j}$. Thus we have proved that $\{(\mathfrak{A}_{i,j})\}_j$ is linearly ordered for any $i$. Conversely, we assume that $\{(\mathfrak{A}_{i,j})\}_j$ is linearly ordered for all $i$, and $k(I_{i,j}) > k(I_{i,j}) > k(I_{i,j})$. Let $\Delta_i$ be a minimal order in $\Delta$ and $\{S_i\}$ be as above. If we denote $I(S_i, \ldots, S_i - M_{i+1} - k_{i,j}, \ldots, S_{i+1}, \ldots)$ by $S_{i,j}$, then $\Gamma_{i,j} = \text{Hom}_{\phi}(\mathfrak{A}_{i,j}, \mathfrak{A}_{i,j})$ is an $r + 1$-th order in $\Lambda$ and $\mathfrak{A}_{i,j} = C_{\phi}(\Lambda) \approx \mathfrak{A}_{i,j}$. Furthermore, we know by the above argument that $\{(\mathfrak{A}_{i,j})\}_j$ is linearly ordered. Therefore, there exists a unit element $\mathfrak{A}$ in $\mathfrak{A}$ such that $\mathfrak{A}_{i,j} = \mathfrak{A}$ for all $i, j$. Hence $\Gamma = \Lambda \cap \bigcap_{i,j} \Delta(\mathfrak{A}_{i,j}) = \Lambda \cap \bigcap_{i,j} \mathfrak{A} = \mathfrak{A}$. The second part is clear from the proof.

From the above proof we have

**Corollary 5.1.** Let $\Gamma = \Lambda \cap \bigcap_{i,j} \Delta(\mathfrak{A}_{i,j})$, and $k(i, j) = k((\mathfrak{A}_{i,j}))$. If $k(i, j) > k(i, j')$, for $j < j'$, $\Gamma$ is of a form $(m_i - k_{i,j}, k_{i,1} - k_{i,2}, \ldots, k_{i,s(i)}, \ldots, m_i - k_{i,j}, k_{i,j} - k_{i,j'}, \ldots, k_{i,s(i)}, \ldots)$.

**Corollary 5.2.** Let $\{\Omega_i\}_{i=1}^n$ be $\h$-orders. Then $\bigcap_i \Omega_i$ is an $\h$-order if and only if intersection of any two of the $\Omega_i$'s is an $\h$-order.

**Proof.** Since every $\h$-order is written as an intersection of maximal orders, we may assume that the $\Omega_i$'s are maximal. If $\Omega_i \cap \Omega_i$ is an $\h$-order, then $\Omega_i = \text{Hom}_{\phi}(\mathfrak{A}_i, \mathfrak{A}_i)$, for a left ideal $\mathfrak{A}_i \supseteq \bigcap N(\Omega_i)$ in $\Omega_i$. Let $\mathfrak{A}_i + \mathfrak{A}_j = \mathfrak{A}$. Then $\Omega_i \cap \Omega_j = \text{Hom}_{\phi}(\mathfrak{A}, \mathfrak{A})$. Hence $\Omega_i$ or $\Omega_j$ is equal to $\text{Hom}_{\phi}(\mathfrak{A}, \mathfrak{A})$ by [5], Theorem 3.3. Therefore, $\mathfrak{A} = \mathfrak{A}_i$ or $\mathfrak{A}_j$ which shows that $\{(\mathfrak{A}_i)\}$ is linearly ordered. Hence $\bigcap \Omega_i$ is an $\h$-order by the theorem. Converse is clear by [5], Corollary 1.4.

**Proposition 5.1.** Let $\Lambda$ be an $\h$-order and $\mathfrak{A}$ a left ideal containing $N(\Lambda)$ such that $\mathfrak{A} \Lambda = \Lambda$. Then $\Gamma = \Lambda \cap \bigcap \mathfrak{A}$ is a unique maximal order among orders $\Gamma$ in $\Lambda$ such that $C_{\phi}(\Lambda) = \mathfrak{A}$. Hence $\mathfrak{A}$ is idempotent.

**Proof.** Let $\mathfrak{A} = \bigcap \mathfrak{A}_i$; $\mathfrak{A}_i/\mathfrak{A} = \Delta_{m_1} \oplus \ldots \oplus \Delta_{m_r} \oplus \ldots \Delta_{m_r}$. Then $\Gamma = \Lambda \cap \bigcap_i \Delta(\mathfrak{A}_i)$. Hence, $C_{\phi}(\Lambda) \subseteq \bigcap C_{\phi}(\mathfrak{A}_i)(\Lambda) = \bigcap \mathfrak{A}_i = \mathfrak{A}$. It is clear that $C_{\phi}(\Lambda) \supseteq \mathfrak{A}$. If $C_{\phi}(\Lambda) = \mathfrak{A}$ for an $\h$-order $\Gamma \subseteq \mathfrak{A}$. Then $\Gamma \subseteq \Lambda \cap \text{Hom}_{\phi}(\mathfrak{A}, \mathfrak{A}) = \mathfrak{A}$, since $C_{\phi}(\Lambda)$ is a two-sided ideal in $\Gamma$.

**Corollary 5.3.** Let $\Gamma = \Lambda \cap \bigcap \mathfrak{A}$, then $C_{\phi}(\Lambda) = \bigcap \mathfrak{A}_i$.

**Proof.** Let $C_{\phi}(\Lambda) = \bigcap \mathfrak{A}_i$, where the $\mathfrak{A}_i$'s are as in the proof of...
Corollary 5.2. \( \Gamma' = \Lambda \cap \text{Hom}_s(C_s(\Lambda), C_s(\Lambda)) \subseteq \Gamma \) and \( \Gamma' = \Lambda \cap \bigcap_i \Delta(Q_i) \).

Since \( \Delta(Q_i) \supseteq \Gamma' \), \( Q_i = Q_{k,j} \) for some \( k, j \). Hence \( \Delta(Q) = \bigcap_i Q_{i,j} \).

PROPOSITION 5.2. Let \( \Lambda \) be a principal \( h \)-order and \( \mathcal{Q} \) a left ideal in \( \Lambda \). Then \( \mathcal{Q} \) is principal if and only if \( \tau'_h(Q) = \Lambda \) and \( \Delta(Q) \) is principal.

Proof. If \( \mathcal{Q} = \Delta_\alpha \), then \( \Delta(Q) = \alpha^{-1} \Delta_\alpha \), and hence \( \Delta(Q) \) is principal, and \( \tau'_h(Q) = \Delta_\alpha^{-1} \Delta_\alpha = \Lambda \). If \( \tau'_h(Q) = \Lambda \), \( \Lambda = \text{Hom}_{\Delta(Q)}(Q, Q) \). Furthermore if \( \Delta(Q) \) is principal, \( \Lambda \) and \( \Delta(Q) \) have the same form, and hence \( \mathcal{Q} \) is principal by [5], Corollary 4.5.

We shall discuss further properties of one-sided ideals in the forthcoming paper [7].

PROPOSITION 5.3. For any \( r \)th order \( \Gamma \), there exist \( n-r+1 \) minimal \( h \)-orders \( \Lambda_i \) such that \( \Gamma = \bigcup \Lambda_i \), where \( n \) is the length of maximal chain for \( h \)-orders in \( \Sigma \).

Proof. We prove the proposition by induction on rank \( r \) of orders. If \( r = n \), then \( \Gamma \) is minimal. If \( \Gamma \) is an \( r \)th order \( (r < n) \), then \( \Gamma / N(\Gamma) = \Delta_{m_1} \oplus \cdots \oplus \Delta_{m_r} \), and \( m_i > 1 \) for some \( i \). Therefore, there exist two distinct left ideals \( \mathcal{Q}_1 \) and \( \mathcal{Q}_2 \) in \( \Gamma \) by Theorem 5.3 such that \( L_i = C_{\alpha_i}(\Gamma) \), and \( C_{\alpha_2}(\Gamma) = \mathcal{Q}_2 \) for some \( r+1 \)th orders \( \Omega_1 \) and \( \Omega_2 \). Since \( \Omega_1 \neq \Omega_2 \), \( \Gamma = \Omega_1 \cup \Omega_2 \). By induction hypothesis we obtain that \( \Omega_i = \bigcup_{j=1}^{r} \Lambda_{i,j} \), where the \( \Lambda_{i,j} \)'s are minimal \( h \)-orders. Since \( \Omega_1 \neq \Omega_2 \), there exists \( \Lambda_{2,j} \subset \Omega_1 \). Hence \( \Gamma = \Omega_1 \cup \Lambda_{2,j} = \bigcup_{i=1}^{r+1} \Lambda_i \).

6. Numbers of \( h \)-orders.

We shall count numbers of \( h \)-orders in an \( h \)-order.

LEMMA 6.1. Let \( \Gamma \supseteq \Delta \) be \( h \)-orders and \( \varepsilon \) a unit in \( \Gamma \). If \( \varepsilon^{-1} \Delta \varepsilon = \Delta \) then \( \varepsilon \in \Delta \).

Proof. Since \( \varepsilon \Delta = \Delta \varepsilon \) is a two-sided inversible ideal with respect to \( \Delta \) in \( \Sigma \), \( \Delta \varepsilon = \mathcal{R}^\rho \) by [5], Theorem 6.1, where \( \mathcal{R} = N(\Lambda) \). Let \( \mathcal{R}' = p \Delta \), then \( \Delta \varepsilon = \mathcal{R}^\rho = p^\rho \Delta \). Hence, \( \varepsilon^{-1} p^\rho \) is a unit in \( \Delta \), and hence in \( \Gamma \). Therefore, \( \rho = 0 \), which implies \( \Delta \varepsilon = \Delta \).

PROPOSITION 6.1. Let \( \Omega \) be an \( h \)-order. If \( \Gamma_1 \) and \( \Gamma_2 \) are isomorphic by an inner-automorphism in \( \Omega \) for \( \Gamma_1 \subseteq \Omega \) \((i = 1, 2)\), and \( \Gamma_1 \neq \Gamma_2 \), then \( \Gamma_1 \cap \Gamma_2 \) is not an \( h \)-order.

Proof. If \( \Gamma_1 \cap \Gamma_2 \) is \( h \)-order, there exists a minimal \( h \)-order \( \Lambda \) in
Tl and Fj. Since Ti and are isomorphic by an inner-automorphism in 
\Omega, they are isomorphic over \Lambda by Theorem 4.2. Hence, £\Lambda \varepsilon^{-1} = \Lambda. 
Therefore, \varepsilon is a unit in \Lambda, and in \Gamma_i, which is a contradiction to the 
fact \Gamma_1 = \Gamma_2.

**Corollary 6.1.** Let \Omega be a maximal order and \Gamma_1, \Gamma_2 nonmaximal 
distinct principal h-orders of same rank in \Omega, then \Gamma_1 \cap \Gamma_2 is not an h-
order.

**Proof.** Let \Lambda_1 and \Lambda_2 be minimal h-orders contained in \Gamma_1 and \Gamma_2, 
respectively. Then \Lambda_i = \varepsilon^{-1} \Lambda_\varepsilon \varepsilon; \varepsilon unit in \Omega by Corollary to Theorem 3.1. 
However, by Theorems 2.3 and 4.1, \Gamma_1 = \varepsilon^{-1} \Gamma_\varepsilon \varepsilon.

**Corollary 6.2.** Let \Omega be an h-order, and \{\Gamma_i\} the set of r th h-orders 
between \Omega and a fixed minimal h-order \Lambda in \Omega. Then every r th order in 
\Omega is isomorphic by inner-automorphism in \Omega to some \Gamma_i, and those iso-
morphic classes by units in \Omega do not meet each other.

It is clear by the proof of Theorem 3.1 and the proposition.

**Theorem 6.1.** The following conditions are equivalent:
1) The number of h-orders in a maximal order is finite,
2) The number of h-orders in a nonminimal h-order is finite.
3) \(R/\wp\) is a finite field.

To prove this we use the following elementary property.

**Lemma 6.2.** Let \(B = \Delta_n\) be a simple ring and \(L = B \oplus \cdots \oplus B\).
then for any unit element \(\varepsilon\) in \(B\) \(L \varepsilon = L\) if and only if
\[\varepsilon = \begin{pmatrix} \varepsilon_1 & 0 \\ C & \varepsilon_2 \end{pmatrix}\]
\(\varepsilon_1, \varepsilon_2\) are units in \(\Delta_r\) and \(\Delta_{n-r}\), and \(C\) is an arbitrary element in \((n-r) \times r\) 
matrices over \(\Delta\).

**Proof of Theorem 6.1.** Let \(\Gamma\) be a nonminimal r th h-order. By Theo-
rem 5.3 \(r+1\) th h-orders contained in \(\Gamma\) correspond uniquely to left ideals
\(\mathcal{S}_i; \mathcal{S}_i/N(\Gamma) = \Delta_{m_1} \oplus \cdots \oplus \Delta_{m_r}\). Hence, the number of \(r+1\) h-
orders in \(\Gamma\) is equal to the number of those left ideals. The number of left ideals in \(\Gamma/N(\Gamma)\) which are isomorphic to \(\mathcal{S}_i/N(\Gamma)\) is equal to 
\([\Gamma/N(\Gamma)]^* : 1]/[E(\mathcal{S}_i) : 1]\), where \(*\) means the group of units and 
\(E(\mathcal{S}_i) = \{\varepsilon \in (\Gamma/N(\Gamma))^*; (\mathcal{S}_i/N(\Gamma)) \subseteq \mathcal{S}_i/N(\Gamma)\}\). Since \([\Delta : R/\wp] < \infty\), 
\([\Gamma/N(\Gamma)]^* : 1]/[E(\mathcal{S}_i) : 1] < \infty\) if and only if \([R/\wp : 1] < \infty\) by Lemma 6.1. Thus, we
obtain 2) \Rightarrow 3). Since the length of maximal chain is finite, we have 1) \( \Rightarrow 2 \).

If we want to count the number of \( h \)-orders in \( \Gamma \), we may use the argument in the proof of Theorem 6.1. However, it is complicated a little. By virtue of Corollary 6.2, we may fix a minimal \( h \)-order in \( \Lambda \). From this point, we shall study the numbers of \( h \)-orders in the special case as follows.

In Section 1, we have noted that we may restrict \( R \) to the case of a complete, discrete valuation ring. By \( \wedge \) we mean completion with respect to the maximal ideal \( p \) in \( R \). Let \( \Omega \) be a maximal order with radical \( \mathcal{R} \); \( \Omega/\mathcal{R} = \Delta_n \). Let \( \mathcal{S} = T_n' \); \( T \) division ring, then \( \mathcal{O} = \mathcal{O}' \), where \( \mathcal{O} \) is a unique maximal order with radical \( (\pi) \) in \( T \). Since \( \Omega/\mathcal{R} = \mathcal{O}/\mathcal{R} \), \( n' = n \).

In order to decide all types of \( h \)-orders in \( \mathcal{S} \), we may consider \( h \)-orders containing a fixed minimal \( h \)-order by Theorem 3.1. By Lemma 1.2, we obtain a minimal \( A \)-order \( A \), which we shall fix in this section; namely

\[
\Lambda = \{(a_{i,j}) \in \Sigma, a_{i,j} \in \mathcal{O}, a_{i,j} \in (\pi) \text{ for } i > j\},
\]

\[
N(\Lambda) = \{(a_{i,j}) \in \Lambda, a_{i,j} \in (\pi)\} = \mathcal{R},
\]

\[
\mathcal{R}^{-1} = \{(a_{i,j}) \in \Sigma, a_{i,j} \in \mathcal{O} \text{ if } i = n, i = 1; a_{j,j} \in (\pi) \text{ if } i+1 < j \text{ and } a_{n,1} \in (1/\pi)\mathcal{O}\}.
\]

From now on we denote \( \mathcal{S}, \mathcal{O}, \mathcal{K} \) by \( \Sigma, \mathcal{O}, R \), respectively.

Let \( \mathcal{M}_i = \{(a_{i,j}) \in \Lambda, a_{ji} \in (\pi)\} \). Then the \( \mathcal{M}_i \)'s are the set of maximal two-sided ideals in \( \Lambda \). Since \( e_{i-1,1} \pi e_{i,1} e_{i-1,1} = \pi e_{i-1,1} e_{i,1} \in \mathcal{R}^{-1} \mathcal{M}_i \mathcal{R} \), we know that \( \mathcal{R}^{-1} \mathcal{M}_i \mathcal{R} = \mathcal{M}_{i-1} \). Hence, \( \{\mathcal{M}_n, \mathcal{M}_{n-1}, \ldots, \mathcal{M}_1\} \) is the normal sequence in \( \Lambda \). We can easily check that \( \Gamma_i = \text{Hom}_K(\mathcal{M}_i, \mathcal{M}_i) \) = the ring generated by \( \Lambda \) and \( e_{i-1,1} \) if \( i = 1 \), and that \( \Gamma_i = \text{Hom}_K(\mathcal{M}_i, \mathcal{M}_i) = \{(a_{i,j}) \in \Sigma, a_{i,j} \in (\pi) \text{ for } i < j, a_{i,j} \in \mathcal{O} \text{ for } i = n, j = 1, \text{ and } a_{n,1} \in (1/\pi)\mathcal{O}\} \). Hence, \( \{\Gamma_1, \ldots, \Gamma_n\} \) is a complete set of \( n-1 \)th order in \( \mathcal{O} \). For any order \( \Gamma \) between \( \Omega \) and \( \Lambda \), \( C(\Gamma) = I(\mathcal{M}_{i_1}, \ldots, \mathcal{M}_{i_p}) (i_j > 1) \). Then \( \Gamma \) is the ring generated by \( \Lambda \) and \( \{e_{j-1,1}\} \) \( j = i_1, \ldots, i_p \).

Summarizing the above, we have

**Theorem 6.2.** Every \( h \)-order in \( \Sigma \) is isomorphic to the following type

6) Those types are changed by the suggestion of Mr. Higikata.
where \( n = \sum m_i \), and \( \mathcal{O}(i \times j) \): all \((i \times j)\) matrices over \( \mathcal{O} \).

We shall return to problem of counting the number of \( h \)-orders. By virtue of Theorem 6.1, we may assume that \( \overline{\mathbb{R}}/\mathfrak{p} \) is a finite field and hence, \( \mathcal{O}/\pi = \text{GF}(p^m) \).

**Lemma 6.3.** Let \( \Gamma, \Omega \) be as above. Then the number of isomorphic classes of \( \Gamma \) by unit element in \( \Omega \) is equal to \( [(\Omega/\pi\Omega)^* : (\Gamma/\pi\Omega)^*] \).

**Proof.** By Lemma 6.1, this number is equal to \( [\Omega^* : 1^*] \), and by the above remark \( \pi\Gamma \leq N(\Gamma') \). Hence, we have \( (\Omega/\pi\Omega)^*/(\Gamma/\pi\Omega)^* \cong \Omega^*/1^* \).

**Lemma 6.4.** \( [(\Omega/\pi\Omega)^* : (\Gamma/\pi\Omega)^*] = (p^m - 1)(p^{m-1} - 1) \cdots (p^{m^{(n-1)}} - 1) / \prod_{i=1}^r (p^{m_i - m} - p^m - p^{m_i - m_i - 1}) \).

**Proof.** It is clear that \( / \mathcal{O}(\pi\Omega) = (\mathcal{O}/\pi)^\times \) and \( [(\mathcal{O}/\pi)^\times : 1] = [GL(n, p^m) : 1] = (p^m - 1)(p^{m-1} - 1) \cdots (p^{m^{(n-1)}} - 1) \) by [4], p. 77, Theorem 99. \( \Gamma/\pi\Omega = \left\{ \begin{array}{c} B_{1,1} \\ \vdots \end{array} \right\} \\ \left( \begin{array}{c} \ast \\ B_{r,r} \end{array} \right) \), and hence, \( r(\in \Gamma/\pi\Omega) \) is unit if and only if the \( B_{1,1}, \ldots, B_{r,r} \) are unit in \( (\mathcal{O}/\pi)m_i \). Therefore, \( [(\Gamma/\pi\Omega)^* : 1] = \prod_{i=1}^r (GL(m_i, p^m) : 1) p^{m^*} \) s = \( \sum_{i=1}^r m_i(n - m_i - m_2 - \cdots - m_i) \).

By Corollary 6.4, and Theorem 4.1, we have

**Theorem 6.3.** The number of \( r \)th \( h \)-orders in a maximal order is equal to

\[
\sum_{m_1 + m_2 + \cdots + m_r = m} \{ p^{m_1 - 1}(p^{m_2} - p^m) \cdots (p^{m_r} - p^{m^{(n-1)}}) / \prod_{i=1}^r (p^{m_1} - 1)(p^{m_i} - p^m) \cdots (p^{m_i - m_i - 1}) \}^{(p^{m_1} - p^{m_i}) \cdots (p^{m_i - m_i - 1})} \}
\]

The number of \( r \)th principal \( h \)-orders in \( r \)th principal \( h \)-order is equal to

\[
\{ p^{m_1} - 1(p^{m_2} - p^m) \cdots (p^{m_r} - p^{m^{(n-1)}}) \}^{r} / \\
\{ p^{m_1} - 1(p^{m_2} - p^m) \cdots (p^{m_r} - p^{m^{(n-1)}}) \}^{r} p^{(m_1^2 + r^2 - r^2 p^r)}. \]

Especially, the number of minimal \( h \)-orders in a maximal order is equal to
We shall describe \( \Lambda \) as follows:

\[
\Lambda = \begin{pmatrix}
\pi A_{1,1} & \pi A_{1,2} & \cdots & \pi A_{1,m} \\
A_{2,1} & A_{2,2} & \cdots & A_{2,m} \\
A_{m,1} & A_{m,2} & \cdots & A_{m,m}
\end{pmatrix}; \quad A_{i,j} \text{ is matrices of } m_i \times m_j \text{ over } \mathfrak{O}.
\]

Since

\[
N = \begin{pmatrix}
\pi A_{1,1} & \pi A_{1,2} & \cdots & \pi A_{1,m} \\
A_{2,1} & A_{2,2} & \cdots & A_{2,m} \\
A_{m,1} & A_{m,2} & \cdots & A_{m,m}
\end{pmatrix}; \quad N^n = \pi \Lambda.
\]

Let \( t \) be the ramification index of a maximal order, namely \( \pi^t = pe, \)

\( e \in \mathfrak{O}. \) Then we have a explicit result of Theorem 2.2.

**Proposition 6.2.** Let \( \Lambda \) be an \( r \)th order, then its ramification index is equal to \( tr. \)

**Proposition 6.3.** Let \( \Lambda \) be an \( r \)th principal order, and \( \alpha \) an element in \( \Lambda \) such that \( \Delta \alpha^{n/r} = N(\Lambda) \) for some \( n. \) Then \( \Gamma = \Lambda \cap \alpha^{-1} \Delta \alpha \cap \cdots \cap \alpha^{-(n/r)-1} \Delta \alpha^{i-(n/r)} \) is an \( n \)th principal order, and any \( n \)th principal order \( \Gamma \) in \( \Lambda \) is written as above and \( N(\Gamma) = \alpha \Gamma = \Gamma \alpha, \) where \( r | n. \)

**Proof.** If \( \Gamma \) is an \( n \)th principal order with \( N(\Gamma) = \alpha \Gamma \) in \( \Lambda, \) we can easily show, by Theorems 2.1 and 2.3, that \( \Delta \alpha^{n/r} = \Delta \alpha^{n/r} \) and \( \Gamma = \Lambda \cap \alpha^{-1} \Delta \alpha \cap \cdots \cap \alpha^{-(n/r)-1} \Delta \alpha^{i-(n/r)}. \) Since \( \alpha^{n/r} = \Delta \alpha^{n/r}, \) \( \alpha^{n/r} = N(\Lambda) \). However \( \alpha^{n/r} = \beta \Lambda, \) and hence \( i = 1 \) by Proposition 6.2. Therefore, \( \Delta \alpha^{n/r} = N(\Lambda). \) Conversely if \( \Delta \alpha^{n/r} = N(\Lambda), \) \( \Delta \alpha^{i} \) is a left ideal in \( \Lambda \) containing \( N(\Lambda) \) for \( i \leq n/r, \) and \( \Delta \alpha^{i}/\Delta \alpha^{i+1} \cong \Lambda/\alpha \) as a left \( \Lambda \)-module. If \( \Delta \alpha \Delta = \Lambda, \) \( \Lambda/\Lambda \alpha = \bigoplus_{i=1}^{n/r} I_{i} \), \( \Delta \alpha^{i}/N(\Lambda) = \bigoplus_{i=1}^{(n/r)-1} I_{i} \). Then \( \Gamma = \Lambda \cap \bigoplus_{i=1}^{(n/r)-1} \mathbf{Hom}_{\Lambda}(\Delta \alpha^{i}, \Delta \alpha^{j}) \cap \bigoplus_{i=1}^{(n/r)-1} \mathbf{Hom}_{\Lambda}(\Delta \alpha^{n/r-1}, \Delta \alpha^{n/r-1}) \) is a principal \( n \)th order by Corollary 5.1. It is clear that \( \alpha \Gamma = \Gamma \alpha. \) Hence \( \alpha \Gamma = N(\Gamma)/\beta \Gamma. \) However, \( \psi = (\alpha^{n/r})_{\alpha} = \beta \psi_{\alpha} \), where \( \epsilon, \epsilon' \) and \( \epsilon'' \) are units in \( \Lambda. \) Hence \( i = 1. \)
Bibliography

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