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On the fundamental groups of knotted 2-manifolds in the 4-space

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1. Introduction

Let M be a 2-dimensional manifold imbedded in the 4-dimensional Euclidean space R^4. Let $\mathfrak{F}(M)$ be the fundamental group of R^4-M. In the case that M is a spinning sphere S, namely a sphere obtained by rotating an arc about a 2-dimensional plane, the group $\mathfrak{F}(S)$ was investigated by E. Artin [1], E. R. Van Kampen [2] and J. J. Andrews and M. L. Curtis [3].

The presentation of $\mathfrak{F}(S)$ was discussed by R.H. Fox [4] and S. Kinoshita [5], where S is a knotted 2-sphere in general. Their method, the so called moving picture method, concerned with the slice knots or the null-equivalent knots, which appear as an intersection of S and a 3-dimensional subspace of R^4.

This paper contains the method of the Wirtinger's presentation of $\mathfrak{F}(M)$ by the classical projection method as in the knot theory. In this direction the principle of the method has been given by S. Kinoshita [6].

As an application of this method, a parallelism between knots in R^3 and knotted 2-spheres in R^4 will be discussed.

2. Preliminaries

Let R^4 be the 4-dimensional Euclidean space with a coordinate system (x, y, z, u). Let R^3 be the 3-dimensional subspace of R^4 defined by $u=0$. With every point $P=(x, y, z, u)$ of a complex M in R^4, we associate the point $P^*=(x, y, z, 0)$ and $u=u(P)$. We call P^* the *trace* and u the *height* of a point P respectively and denote by $P=[P^*, u(P)]$. The set of traces of points of M will be denoted by M^*. The projection $\varphi: P \to P^*$ is defined as usual.

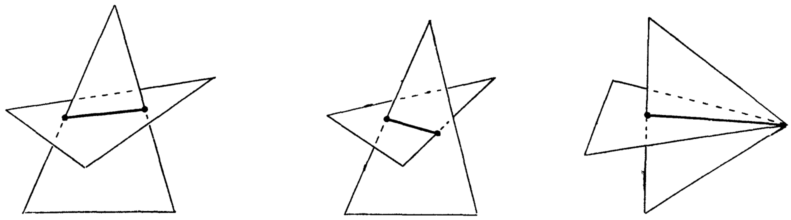
Throughout this paper terminologies are used in the semi-linear point of view. Hence complexes are polyhedral and mappings are simplicial.

Let M be a 2-dimensional closed orientable manifold. It is no loss of generality to assume the following condition:

(2.1) *If $P_1, \dots, P_m$ are vertices of M, then the system of points $(P_1^*, \dots, P_m^*)$ is in general position in R^3.*

Let $P^* \in M^*$. If there exist at least two points of M such that their traces coincide with P^*, then we say P^* a *cutting point* of M^*. The set of cutting points of M^* is denoted by $\Gamma(M^*)$, and called the *cutting* of M^*.

In virtue of (2.1), 2-dimensional simplexes of M^* have an intersection only in the following cases (Fig. 1).



(a_1) (a_2) (b)

Fig. 1

Hence $\Gamma(M^*)$ consists of segments, each of whose endpoints belongs to only one 1-dimensional simplex. Notice that the common vertex of two simplexes in Fig. 1,(b) is not a point of $\Gamma(M^*)$. We call such a point a *singular cutting point* of M^*.

We can also assume the following conditions by a slight modification of vertices of M.

(2.2) *A segment of $\Gamma(M^*)$ is the intersection of just two simplexes.*

(2.3) *There exist just three simplexes through a double point of $\Gamma(M^*)$.*

Since an endpoint of a segment of $\Gamma(M^*)$ belongs to only one 1-dimensional simplex as shown in Fig. 1, we have:

(2.4) *$\Gamma(M^*)$ consists of the following two kinds of polygons:*

(1) *closed polygons,*

(2) *polygonal arcs, whose endpoints are different or coincided singular cutting points.*

3. The linking

Let M be a 2-dimensional closed orientable manifold in R^4. Let f be a continuous mapping of the unit circle

$$c_1: \quad x^2+y^2=1$$

into R^4-M. Put $c=f(c_1)$. The vertices of c^* may be considered to be in general position in R^3.

If f can be extended to F which maps the unit disk

$$D_1: \quad x^2+y^2\leqq 1$$

into R^4-M, then we say that c does not link homotopically with M. Conversely, if such an extension does not exist, then we say that c links homotopically with M.

(3.1) *If $c^* \frown M^* = 0$, then c does not link homotopically with M.*

Proof. Let (Q, r) be the polar coordinate of D_1, where $Q \in c_1$ and $0 \leqq r \leqq 1$. Let $c_{1/2}$ be a circle of $r = 1/2$. Take a positive number h such that

$$h > \left| \max_{P \in M} u(P) - \min_{Q \in C_1} u(f(Q)) \right|.$$

Put

$$F(Q, r) = [f(Q)^*,\ u(f(Q)) + 2(1-r)h], \qquad 1/2 \leqq r \leqq 1.$$

Since $F(Q, r)^* = f(Q)^*$, F is a continuous mapping of $D_1 - D_{1/2}$ into $R^4 - M$. It is obvious that $F(c_{1/2})$ is null-homotopic in the half-space defined by $u \geqq h + \min_{Q \in C_1} u(f(Q))$. Hence c is null-homotopic in $R^4 - M$.

Consequently if c links with M homotopically, then we have $c^* \frown M^* \neq 0$. Suppose that $c^* \frown M^*$ consists of two points A^* and B^*. Let $A_1 \in c$ and $A_2 \in M$ be the points such that $A_1^* = A_2^* = A^*$. We define $\operatorname{sgn} A^*$ as follows:

$$\operatorname{sgn} A^* = \begin{cases} +1 & \text{if} \quad u(A_1) > u(A_2), \\ -1 & \text{if} \quad u(A_1) < u(A_2). \end{cases}$$

$\operatorname{sgn} B^*$ is defined in the same way.

(3.2) *If $\operatorname{sgn} A^* \cdot \operatorname{sgn} B^* = +1$, then c does not link with M homotopically.*

Proof. Suppose that $\operatorname{sgn} A^* = \operatorname{sgn} B^* = +1$, The same proof as (3.1) assures the statement.

In the case that $\operatorname{sgn} A^* = \operatorname{sgn} B^* = -1$, take a negaive number h' such that

$$h' < -\left| \min_{P \in M} u(P) - \max_{Q \in C_1} u(f(Q)) \right|$$

instead of h in the proof of (3.1).

(3.3) *$\operatorname{sgn} A^* \cdot \operatorname{sgn} B^* = -1$, then c links with M homotopically.*

Proof. Suppose that $\operatorname{sgn} A^* = +1$ and $\operatorname{sgn} B^* = -1$. Assume that c does not link homotopically with M. Then there exists an extension F of f over D_1 such that $F(D_1) \subset R^4 - M$. Since $c^* \frown M^*$ consists of two points A^* and B^*, $F(D_1)^* \frown M^*$ contains a cutting of polygonal arc, whose endpoints are A^* and B^*. Therefore there exist an arc a_1 connecting A_1 and B_1 on $F(D_1)$, and an arc a_2 connecting A_2 and B_2 on M such that $a_1^* = a_2^*$.

Let $P_1 \in a_1$ and $P_2 \in a_2$ be two variable points such that $P_1^* = P_2^*$. If $P_1 = A_1$ and $P_2 = A_2$, then we have $u(P_1) > u(P_2)$. If $P_1 = B_1$ and $P_2 = B_2$, then we have $u(P_1) < u(P_2)$. Therefore there exist points P_0^* on $a_1^* = a_2^*$ and $P_{01} \in a_1$, $P_{02} \in a_2$ such that $P_{01}^* = P_{02}^* = P_0^*$ and $u(P_{01}) = u(P_{02})$. Hence $P_{01} = P_{02}$. This contradicts the assumption.

We have the following corollary from (3.2).

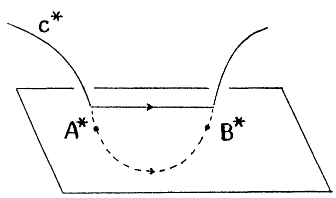


Fig. 2

(3.4) *Let c be a continuous image of an arc c_1 in $R^4 - M$. Let A^* and B^* be successive points of $c^* \frown M^*$ on c^*, where A^* and B^* can be connected by an arc on $M^* - \overline{\Gamma(M^*)}$. If $\operatorname{sgn} A^* = \operatorname{sgn} B^*$,* *then A^* and B^* can be cancelled, and vice versa.*

4: The fundamental groups

In virtue of the conditions in § 2, $\overline{\Gamma(M^*)}$ separates M^* into several domains $\Sigma_1^*, \dots, \Sigma_k^*$, each of which has an orientation induced by the orientation of M. We represent the orientation of Σ_i^* by a small vector $\mathbf{v}_i$ such that the direction of $\mathbf{v}_i$ coincides with the direction of a right-handed screw twisting along the orientation of Σ_i^*.

Let γ^* be a simple arc of $\Gamma(M^*)$. From (2.2) there exist domains $\Sigma_i^*, \Sigma_{i+1}^*$; $\Sigma_j^*, \Sigma_{j+1}^*$ such that γ^* is a common boundary of these domains. Suppose that $\bar{\Sigma}_i \cap \bar{\Sigma}_{i+1} = \gamma_i$, $\bar{\Sigma}_j \cap \bar{\Sigma}_{j+1} = \gamma_j$ are arcs in R_4 such that $\gamma_i^* = \gamma_j^* = \gamma^*$. If $u(\gamma_i) > u(\gamma_j)$, then we call $\overline{\Sigma_i \cup \Sigma_{i+1}}$ the *over surface*, and $\overline{\Sigma_j \cup \Sigma_{j+1}}$ the *under surface*. To represent the relation of these surface, we use the following notations, cancelling the vector of the under surface (Fig. 3).

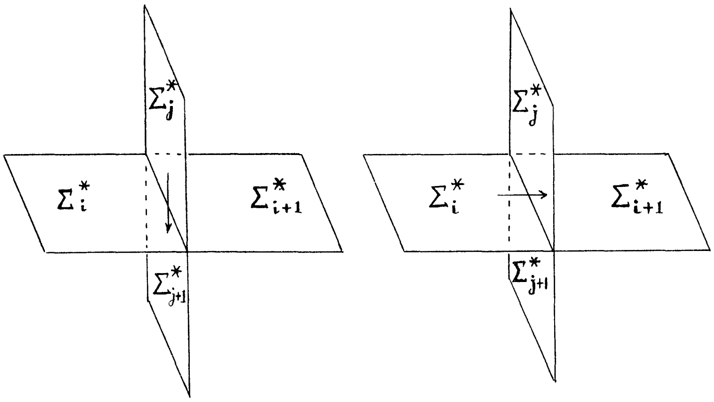


Fig. 3

The direction of the vector corresponds to the orientation of the over surface.

For each Σ_i^*, we take a small circle c_i^* such that $\Sigma_i^* \cap c_i^*$ consists of points A_i^*, B_i^* where sgn $A_i^* = +1$, sgn $B_i^* = -1$, and $\Sigma_j^* \cap c_i^* = 0$ for $j \neq i$. We define the orientation of c_i^* such that it coincides with the direction of $\mathbf{v}_i$ at the point A_i^*. It is obvious that each c_i^* defines a equivalent class of c_i in $R^4 - M$.

(4.1) *c_i and c_j are homotopic in $R^4 - M$.*

Proof. If $i = j$, then the statement is obvious. Let us prove that c_i and c_{i+1} in Fig. 3 are homotopic in $R^4 - M$.

Suppose that $\overline{\Sigma_i \cup \Sigma_{i+1}}$ is the *under* surface. Let T be a tube such that $\dot{T}^* = c_i^* - c_{i+1}^*$ and $T^* \cap \overline{\Sigma_i^* \cup \Sigma_{i+1}^*}$ consists of two longitudes $\alpha^* = A_i^* A_{i+1}^*$, $\beta^* = B_i^* B_{i+1}^*$. Put $\alpha_1 = \varphi^{-1}(\alpha^*) \cap T$, $\alpha_2 = \varphi^{-1}(\alpha^*) \cap \overline{\Sigma_i \cup \Sigma_{i+1}}$ and $\beta_1 = \varphi^{-1}(\beta^*) \cap T$, $\beta_2 = \varphi^{-1}(\beta^*) \cap \overline{\Sigma_i \cup \Sigma_{i+1}}$. Deform T such that $u(\alpha_1) > u(\alpha_2)$ and $u(\beta_1) < u(\beta_2)$. Then we have

$T \subset R^4 - \overline{\Sigma_i \cup \Sigma_{i+1}}$. Put $\delta^* = T^* \cap \overline{\Sigma_j^* \cup \Sigma_{j+1}^*}$ and $\delta_1 = \varphi^{-1}(\delta^*) \cap T$, $\delta_2 = \varphi^{-1}(\delta^*) \cap \overline{\Sigma_j \cup \Sigma_{j+1}}$. Deform T so far as $u(\delta_1) < u(\delta_2)$ but $u(\alpha_1) > u(\alpha_2)$. Then we have $T \subset R^4 - M$. Hence c_i and c_{i+1} are homotopic in $R^4 - M$. Other cases are proved successibly.

Take a base point O in $R^3 - M^*$. Let w_i^* be an arbitrary path connecting O and an arbitrary point P_i^* of c_i^*. We define the *signs* of points $w_i^* \cap M^*$ be all $+1$.

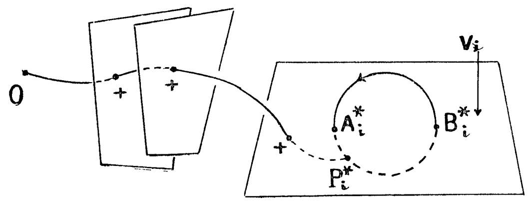


Fig. 4

Denote the closed path

$$O \underset{w_i^*}{\longrightarrow} P_i^* \underset{c_i^*}{\longrightarrow} P_i^* \underset{w_i^*}{\longrightarrow} O$$

by σ_i^*. It is obvious that the equivalent class of the closed path σ_i in $R^4 - M$ corresponding to σ_i^* does not depend on the choice of w_i^* and c_i^*.

(4.2) Theorem. $\sigma_1, \dots, \sigma_k$ *form a generator system of* $\mathfrak{F}(M)$ *with the base point O.*

Proof. Suppose that w is an arbitrary oriented closed path in $R^4 - M$ with the base point O. Let P^* be a point of $w^* \cap \Sigma_i^*$. We make σ_i correspond to P^* in the following manner:

1) If *sgn* $P^* = +1$, then $P^* \longrightarrow 1$,

2) If *sgn* $P^* = -1$ and the direction of v_i coincides with the direction of w^* at the point P^*, then $P^* \longrightarrow \sigma_i^{-1}$,

3) If *sgn* $P^* = -1$ and v_i and w^* have the opposite directions at the point P^*, then $P^* \longrightarrow \sigma_i$.

Thus a word $w(\sigma)$ corresponds to w. It is obvious from (3.4) that a representative of $w(\sigma)$ is equivalent to w.

(4.3) *If* $\overline{\Sigma_j \cup \Sigma_{j+1}}$ *is the over surface, then we have the following relations:*

(1) $\sigma_j^{-1}\sigma_{j+1} = 1$,

(2) $\sigma_{i+1}^{-1}\sigma_j^{\varepsilon}\sigma_i\sigma_j^{-\varepsilon} = 1$,

where $\varepsilon = +1$ *or* -1 *according as the direction of the vector of the over surface coincides or not with the direction* $\Sigma_i^* \longrightarrow \Sigma_{i+1}^*$.

Proof. (1) is obvious from (4.1). Let us prove (2) in the case of $\varepsilon = +1$. Let T be the tube in the proof of (4.1). Take a curve $w_{i,i+1}$ connecting P_i and P_{i+1} on T. The closed path

$$O \xrightarrow{w_{i+1}} P_{i+1} \xrightarrow{w_{i,\,i+1}} P_i \xrightarrow{c_i} P_i \xrightarrow{w_{i,i+1}} P_{i+1} \xrightarrow{w_{i+1}} O$$

is represented by $\sigma_j\sigma_i\sigma_j^{-1}$. It is obvious that this closed path is homotopic to σ_{i+1} (Fig. 5). Hence $\sigma_{i+1}=\sigma_j\sigma_i\sigma_j^{-1}$.

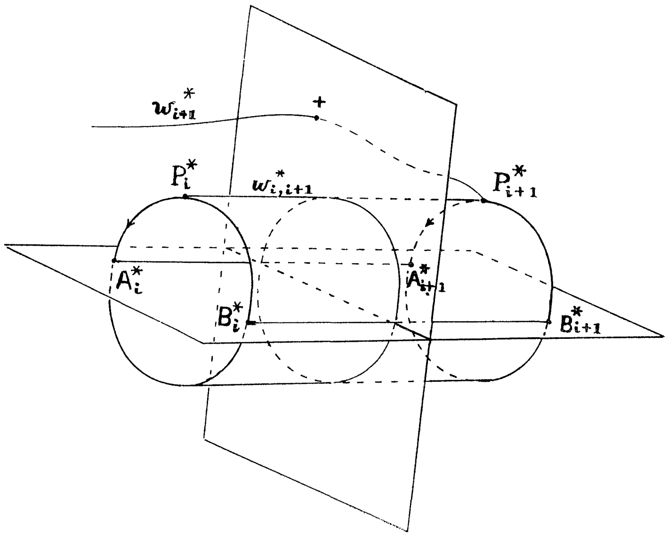


Fig. 5

(4.4) Theorem. *The relations* (4.3) *corresponding to all arcs of* $\Gamma(M^*)$ *form a system of defining relations of* $F(M)$.

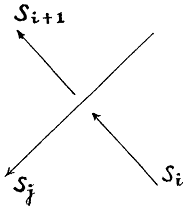
Proof. If there exist no singular cutting points, then the statement is obvious. Suppose that there exist some singular cutting points. Let σ be a closed path which is null-homotopic in R^4-M. There exist continuous mappings f, F such that $f(c_1)=\sigma$, $F(D_1)\subset R^4-M$ and $F|c_1=f$, where c_1 and D_1 are as in §3. By a slight modification of F we have $F(D_1)^*\subset R^3-(\overline{\Gamma(M^*)}-\Gamma(M^*))$. Hence σ can be represented as a consequence of relations (4.3).

5. Spheres in $\mathbf{R}^4$

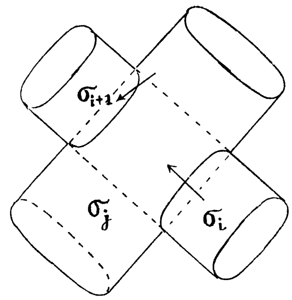
Let k be a knot in R^3. A construction of a 2-sphere S in R^4, whose fundamental group $\mathfrak{F}(S)$ is isomorpic to $\mathfrak{F}(k)$, was given in [3] by rotating an arc along a plane in R^4. Let us discuss the same problem by the projection method.

(5.1) *Let k be a knot in R^3. There exists a torus T_k in R^4 such that $\mathfrak{F}(T_k)$ is isomorphic to $\mathfrak{F}(k)$.*

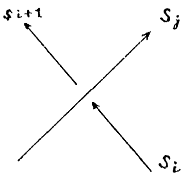
Proof. In the Wirtinger's presentation, the defining relations of $\mathfrak{F}(M)$ are given in the same form as the defining relations of $\mathfrak{F}(k)$. So we construct T_k in the following correspondence (Fig. 6), where tubes represented by dotted lines, which show that they go through the other tubes, correspond to the cross points of the under-going arcs of k. The inessential generators are omitted.



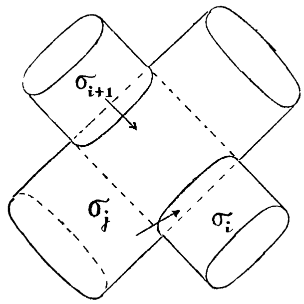
$s_{i+1}=s_j s_i s_j^{-1}$



$\sigma_{i+1}=\sigma_j\sigma_i\sigma_j^{-1}$



$s_{i+1}=s_j^{-1}s_i s_j$



$\sigma_i=\sigma_j\sigma_{i+1}\sigma_j^{-1}$

Fig. 6

It is obvious that $\mathfrak{F}(T_k)$ is isomorphic to $\mathfrak{F}(k)$.

Now let us construct a knotted 2-sphere S_k in R^4 from T_k as follows. Let P be an arbitrary point of k. Take a meridian circle c on T_k corresponding to the point P. Cut the torus T_k into a tube T_k' by a plane through c, and add two disks to the terminals of T_k'. Then we get a knotted sphere S_k in R^4. We say that T_k and S_k are *similar* to k.

(5. 2) Theorem. *If S_k is similar to k, then $\mathfrak{F}(S_k)$ is isomorphic to $\mathfrak{F}(k)$.*

Proof. Suppose that the presentation of $\mathfrak{F}(k)$ is given as follows:

Generators: $(s_1, \ldots, s_n)$

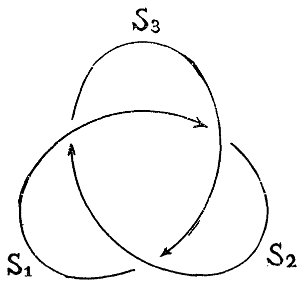
Relations: $(R_k)\begin{cases} s_1=s_{i_1}^{\varepsilon_1}s_2 s_{i_1}^{-\varepsilon_1} \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ s_n=s_{i_n}^{\varepsilon_n}s_1 s_{i_n}^{-\varepsilon_n} \end{cases}$ $(\varepsilon_i=\pm 1)$

Let P be a point of a segment s_m of the projection of k, and Q, R be the endpoints of s_m. Let s_m' and s_m'' be the subsegment of s_m such that $s_m'=QP$ and

$s''_m = PR$. If we take a system of generators $(s_1, \ldots, s'_m, s''_m, \ldots, s_n)$ instead of $(s_1, \ldots, s_m, \ldots, s_n)$, then we have relations (R'_k) replacing s_m in (R_k) by s'_m or s''_m and a new relation $s'_m = s''_m$ as a system of defining relations of $\mathfrak{F}(k)$. By a geometrical consideration, we can prove that the relation $s'_m = s''_m$ is an induced relation of the relations of (R'_k).

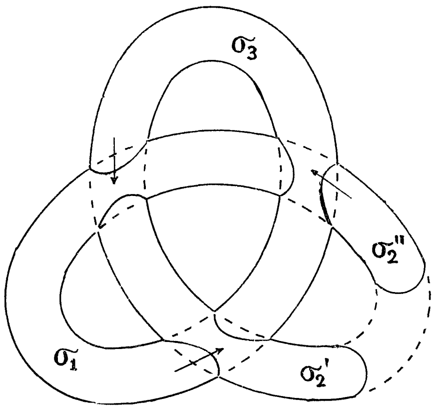
On the other hand the presentation of $\mathfrak{F}(S_k)$ is given by the generators $(\sigma_1, \ldots, \sigma'_m, \sigma''_m, \ldots, \sigma_n)$ and relations corresponding to (R'_k). Hence $\mathfrak{F}(S_k)$ is isomorphic to $\mathfrak{F}(k)$.

Example 1.



Generators: s_1, s_2, s_3.

Relations: $\begin{cases} s_1 = s_3 s_2 s_3^{-1} \\ s_2 = s_1 s_3 s_1^{-1} \\ s_3 = s_2 s_1 s_2^{-1} \end{cases}$



Generators: $\sigma_1, \sigma'_2, \sigma''_2, \sigma_3$

Relations: $\begin{cases} \sigma_1 = \sigma_3 \sigma''_2 \sigma_3^{-1} \\ \sigma'_2 = \sigma_1 \sigma_3 \sigma_1^{-1} \\ \sigma_3 = \sigma'_2 \sigma_1 \sigma'^{-1}_2 \end{cases}$

Fig. 7

The first relation of $\mathfrak{F}(S)$ in Fig. 7 is cancelled. We can prove that the projection S^* in Fig. 7 is deformed into the projection S'^* in Fig. 8 by a deformation of S into S' in R^4.

It is worthy of notice that if T is not a similar torus of knots, then $\mathfrak{F}(S)$ is not always isomorphic to $\mathfrak{F}(T)$ as shown in Example 2.

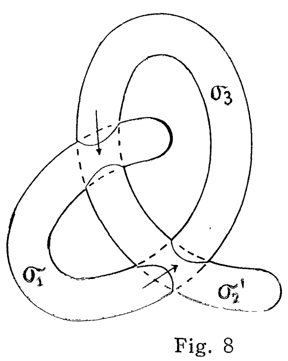


Fig. 8

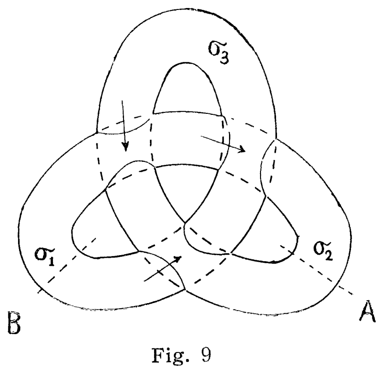


Fig. 9

Example 2. We get the torus T in Fig. 9 by changing the relation of heights of the torus in Fig. 7. If we cut the torus T by the plane A, then we get the same sphere as in Fig. 8. But if we cut T by the plane B, then we get a sphere which is the same as Example 10, p. 135 in [4]. Obviously the fundamental groups of these spheres are not coincide.

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