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Delaunay Mesh Generation for Oceanic Computations

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1 Introduction

The global ocean is one of the principal components of the climatic system. For instance, the oceanic poleward heat transport is of the same order of magnitude as the atmospheric one, i.e. 10^{15} W [15]. Due to the non-linear behaviour of the climatic system, numerical models are the only serious tools to understand thoroughly the interaction of its components and to predict its evolution.

Nowadays, most global ocean circulation models (OGCM) are based on finite difference codes based on structured grids. However, there does not exist curvilinear coordinate systems on the sphere without singularities and meridian convergence zone — what limits the stability of the explicit models. Despite all efforts to bypass these drawbacks [3, 7, 11, 12], two main disadvantages due to the rigidness of the structured grids are inescapable. Firstly, the rigidness of these grids combined with the expensive CPU-cost of the OGCM prevent the specific refinement of the grids without nesting or adaptive mesh refinement [4]. Secondly, the piece-wise constant shape coastlines drawn by the structured grids of the global ocean exert some spurious form stress on model boundary currents [1] and the alternative grid generation method based on the the boundary-fitted coordinates [9] only works for regional applications. Weaver, Marshall or Haidvogel [1, 10, 13] suggested that the finite element methods could be a promising alternative approach for OGCM codes.

In this paper, we present an automatic mesh generator for the global ocean — a spherical domain. To take advantage of robust and well known algorithms, we have chosen to subdivide the world ocean into a conform triangulation. Renka [16] already implemented an automatic triangulation generator of the sphere which, unlike the generator described below, did not take into account boundaries like coastlines.

2 Finite Elements for OGCM

Let $\mathbf{u}$ be the exact solution of a general circulation model. The block-field vector $\mathbf{u}$ includes all variables of an oceanic circulation model : the velocity components, the pressure, the temperature and the salinity. Let $\mathbf{u}^h$ be the approximation of $\mathbf{u}$ so that

$$(1) \qquad \mathbf{u}^h(\mathbf{x},t) = \sum_{i=1}^{N} \mathbf{U}_i(t)\phi_i(\mathbf{x})$$

where ϕ_i are typically piecewise polynomial shape functions and where $\mathbf{U}_i$ are unknown nodal values.

For a coercive discrete operator, the following interpolation property holds [6] :

$$(2) \qquad ||\mathbf{u} - \mathbf{u}^h||_{H^1(\Omega)} \leq C\frac{h^{p+1}}{\rho}||\mathbf{u}||_{H^2(\Omega)}$$

where $||.||_{H^1(\Omega)}$ and $||.||_{H^2(\Omega)}$ are usual Sobolev norms, p is the order of the shape functions, h is a typical mesh size, ρ is a typical inscribed circle radius of the triangulation elements, C is a triangulation-independent constant. The quality factor $\frac{\rho}{h}$ — the inverse of the shape factor $\frac{h}{\rho}$ — is commonly used to quantify the quality of a triangulation. It belongs to the interval $[0,1]$. The quality factor of a degenerated triangle vanishes while the one of an equilateral triangle is equal to the unity.

Due to the equation (2), the fact that the triangulation exhibits great quality factors is a crucial issue for the accuracy of the finite element approximation of the solution. Typically, it is accepted that a triangulation is a good one if all its triangles have just acute angles. That means the quality factor of the worst triangle must be greater than 0.5. The so-called Delaunay triangulation generally exhibits good shape factors.

3 Delaunay triangulation

Let $X = \{x_1, x_2, \ldots, x_3\}$ be a set of points, called sites. Now, partition the plane in Voronoi regions where each point are nearer a site than the others. The Voronoi diagram is defined as the set of points belonging to more than one Voronoi diagram. The Delaunay triangulation of the sites X is the dual graph of the Voronoi diagram and is obtained by drawing connecting lines between sites perpendicular to the edges of the diagram. An important property of the Delaunay diagram is the following: the open circle circumscribed to any triangle of the Delaunay triangle does not contain any triangle vertex.

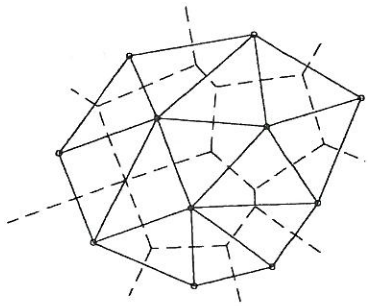

Figure 1: The Delaunay triangulation (continuous lines) is obtained by drawing connecting lines between sites perpendicular to the edges of the Voronoi diagram (dash lines). Sometimes quadrangular elements are produced and can be correctly divided into two triangles.

To adapt the Delaunay triangulation to the sphere, we have replaced the usual distance with the geodesic one, i.e. the length of the unique great circle arc passing trough two points of the sphere. Therefore, we can define the Voronoi diagram on the sphere. It can be demonstrated that the dual triangulation of this diagram exhibits in most cases good shape factors. Such a triangulation is characterized by the Delaunay criterion: if S is the surface to triangulate, the open sphere circumscribed to any triangle of the triangulation and whose center lies on the surface of S does not contain any triangle vertex.

Delaunay triangulations are used for several applications and stimulated researchers to develop a variety of algorithms to obtain it. As state-of-the-art approaches, let us cite the intersection of halfplanes, the divide and conquer , the Fortune's algorithm or the incremental construction [8, 17].

As our problem consists in creating a good triangulation from a given spherical domain but without knowing a priori all the sites x_i, we have chosen an incremental construction. Indeed, the principle of the incremental method is the construction of a Delaunay triangulation $\mathcal{T}_{k+1}$ of $k+1$ sites from a Delaunay triangulation $\mathcal{T}_k$. The method we have used can be described as follows:

Creation of an initial triangulation

The initial triangulation must be a Delaunay triangulation, which can be easily implemented. For symmetry reasons we have chosen an initial triangulation with 5 sites (two on the poles and the remaining three on the equator) and six triangles. However, this is not the only possible choice.

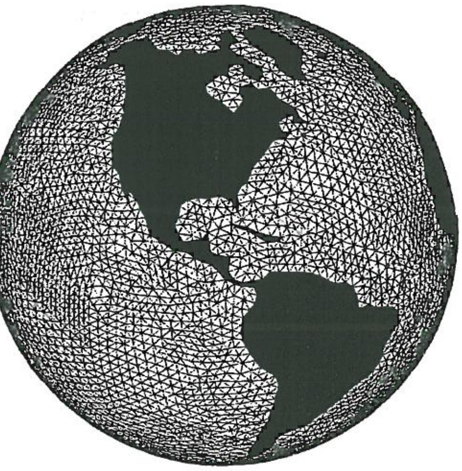

Figure 2: A uniform Delaunay triangulation of the ocean (12000 triangles).

Insertion of boundary sites

To have an easy representation of continents and islands, known boundary sites are firstly inserted one by one in the Delaunay triangulation.

The creation of a Delaunay triangulation $\mathcal{T}_{k+1}$ of $k+1$ sites from a Delaunay triangulation $\mathcal{T}_k$ of k sites needs two steps. The first one consists in inserting the new site p in the old triangulation $\mathcal{T}_k$: the triangle which contains p is searched and replaced by three new triangles whose vertices are the new site p and the vertices of the old triangle. The second step transforms the new triangulation into a Delaunay one. Only a limited number of triangles are involved by such a transformation. We proceed by swapping the common segment of two triangles which do not satisfies the Delaunay criterion [5].

Destruction of triangles

The insertion of boundary sites in the triangulation generates triangles outside the oceans. To avoid the creation of non-desired inner sites, those are destroyed immediately after the insertion of the boundary sites.

Creation of inner sites

Finally, new sites are created in a dynamic fashion at the middle of well chosen segments in order to reach a preset level of local mesh size. Uniform distribution of triangles shapes is obtained by an iterative procedure that consists in inserting a new site in the middle of the longest segment of the triangulation. Non-uniform triangulations are obtained by using a weighted distance to identify the longest segment. Illustrations of uniform and graded meshes along Western boundary and Equator are respectively in Figure 2 and Figure 3.

Figure 3: A graded Delaunay triangulation mesh of the global ocean refining the Gulf Stream and the equatorial regions (7000 triangles).

4 Results

The boundary design and the segments weight govern essentially the quality of the grid — both controlled by the user.

The boundary design must be adapted to the desired mesh size: a too fine boundary design leads to flat triangles. The stopping condition of the incremental algorithm must be based on an a priori specified mesh size, what allows an automatic fit of the boundary design. As the small-scale features of the coastlines (fjords, bays, etc.) do not significantly influence the large-scale pattern of the oceanic circulation and as they are limiting factors for an OGCM, we treat the boundary design in two steps.

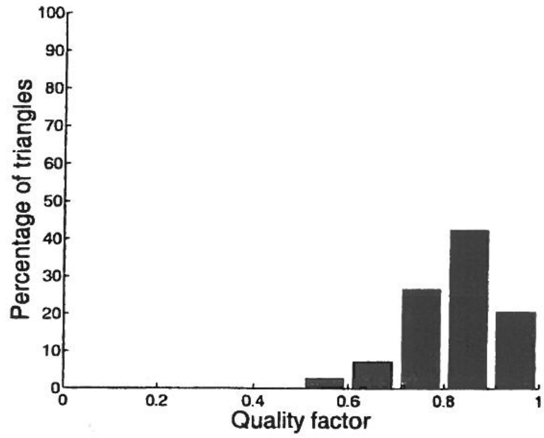

Figure 4: Distribution of the quality factors of the triangulation displayed in Figure 2.

In the first step, a physical representation of the coastlines is obtained by inserting boundary sites with exact geographical coordinates. The choice of those nodes depends mainly on modeling issues. In the second step, such a representation is enriched by adding new boundary sites with interpolated coördinates and not exact geographical coordinates. In fact, we filter small scale details of the coastlines. Those details would be totally irrelevant for our calculations and would moreover introduce critical numerical difficulties. Figure 4 shows the distribution of the quality factor of Figure 2. More than 90 % of the triangles of the grid have quality factor greater than 0.7 and no triangles have quality factor smaller than 0.5. The worst quality factor is equal to 0.53.

Finally, strong difference in weight distribution would lead to highly graded meshes that perform poorly from a numerical and geometrical point of view. Indeed, sharp gradients in weight distribution mean that two regions with different mesh size are juxtaposed so that at the interface flat triangles could be generated.

5 Conclusion

We have implemented an incremental algorithm to generate unstructured mesh for a global ocean circulation model. Needing only the specification of a coastline and segments weight, a triangulation with good shape factors can automatically be created. In particular, no singularities or uncontrolled convergence zones are created. The generator appears to be able to refine at a correct scale the topological and dynamical features which are key points for a globally well-resolved ocean circulation model. Among these are equatorial dynamic, western boundary currents, mesoscale eddies, ridges, continental slopes, channels or straits (figure 3).

The next step of our work will be the development of an ocean general circulation model based on the Finite Element method to compare the efficiency of unstructured grids with classical approaches. Finally, the mesh generator is a general purpose tool that could be useful in other fields of geophysics. Let us just cite the interpretation of scattered measurements on the Earth [14] or the analysis of the geography influence in paleoclimates [2].

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