OCEAN modelling

ISSUE 100

OCTOBER 1993

FEATURING:

AN ORTHOGONAL, CURVILINEAR COORDINATE SYSTEM FOR A WORLD OCEAN MODEL

E. Deleersnijder, J-P. Van Ypersele and J-M. Campin

ABSTRACT

A orthogonal, curvilinear grid system for World Ocean modelling is examined. It involves the coupling of two non-singular spherical sub-grids, avoiding the North Pole singularity of the standard, spherical coordinates. The two sub-grids are connected in the equatorial Atlantic. It is shown how to minimize the sudden variation in the grid size across the connection line. It is suggested that the two spherical coordinate systems be considered as a single orthogonal, curvilinear coordinate system, in which the metric coefficients and the OGCM governing equations are established.

INTRODUCTION

Although the surface and the volume of the polar oceans are small compared with the whole World Ocean, sea-ice dynamics has a very significant influence on the Earth's climate: the processes of melting or freezing of sea-ice involve large amounts of energy and play the role of a significant salt source or sink, which has a profound influence on the large-scale thermohaline circulation. Thus, ocean models to be used in global climate studies must be truly global, i.e., they are to encompass the polar regions, so that they can be coupled to sea-ice models.

The governing equations of most OGCMs are discretized according to the finite-difference method. The numerical grid is generally based on the standard spherical coordinate system, which has singularities at both the North and the South Pole. As those singularities are approached, the latitudinal grid size tends to zero, which may lead to numerical instabilities because the CFL condition is no longer met. Since the South Pole is located sufficiently far away from the nearest oceanic region, the reduction of the latitudinal grid size has no harmful effect. Thus, it is only in the Arctic Ocean that the grid or the numerical method has to be adapted to circumvent this problem of numerical instability.

Several methods to deal with the singularities of the spherical coordinates have been examined (see for example Williamson, 1979). Finite-difference AGCMs usually rely on the Fourier-filtering of the appropriate latitudinal components of the dependent variables, or of their time derivatives. This technique, which has proved very efficient in the atmosphere, might be less adapted to ocean grids, where not all the grid points along a latitudinal circle are active. Despite this potential drawback, Fourier-filtering has been performed for a long time in global ocean simulations.

Recently, however, the LODYC model (Marti *et al.*, 1990; Marti *et al.*, 1992) has been adapted to an orthogonal curvilinear grid obtained by shifting the northern singularity into a land region, located in northern Canada. This solution, yet extremely appealing, requires that the computer code be capable of dealing with a generalized, orthogonal, curvilinear coordinate system. This is obviously not a serious problem, because the equations when written in the generalized coordinate system are very similar to their spherical counterparts (Hughes and Gaylord, 1964). On the other hand, generating the curvilinear grid implies the use of a sophisticated grid generation software. The latter must be capable of producing a grid that is "sufficiently orthogonal", since even slight departures from orthogonality are likely to lead to serious numerical errors (Thompson *et al.*, 1982). Nevertheless, in the long run, the LODYC technique may be recognized as having compelling advantages.

Another modification of the standard spherical coordinate system has also been put forward. Although we don't know who should be credited for inventing it, it is K. Bryan who made one of us aware of this solution, which consists in combining two spherical sub-grids in such a way that their singularities are not close to their own wet grid points. The first sub-grid (hereafter referred to as "G"), covering the Southern Hemisphere and the Pacific Ocean up to the Bering Strait, is based on geographical spherical coordinates. The second sub-grid (hereafter "G'") is associated with spherical coordinates having their poles on the Equator and encompasses the Northern Hemisphere part of the Atlantic together with the Arctic Ocean. The two sub-grids, G and G', are connected in the equatorial Atlantic.

In the present note, we provide a theoretical discussion of the main characteristics of this grid system, which is being implemented in our OGCM, as well as in other models, such as that of the British OCCAM project.

THE SPHERICAL SUB-GRIDS

As stated above, the sub-grid G is based on the geographical spherical coordinates λ, ϕ, and z, which, respectively, represent the longitude, the latitude, and the altitude above sea level. The cartesian coordinates, X, Y, and Z (Figure 7), are related to the spherical coordinates by

$$(X,Y,Z) = (a+z)(\cos\lambda\cos\phi, \sin\lambda\cos\phi, \sin\phi),$$

where a denotes the Earth's radius.

The North Pole of G' is located on the geographical equator at longitude θ. The spherical coordinates λ', ϕ', and $z' = z$, and the corresponding cartesian coordinates, X', Y', and Z' (Figure 7), satisfy

$$(X', Y', Z') =$$

$$(a + z')(\cos\lambda'\cos\phi', \sin\lambda'\cos\phi', \sin\phi').$$

It is assumed that the Atlantic part of the Equator corresponds to the $\lambda' = \pi/2$ meridian of the modified spherical coordinates. It is assumed that $\theta > 0$. It may be seen that $(X, Y, Z) = Y'\sin\theta + Z'\cos\theta, -Y'\cos\theta + Z'\sin\theta, X')$, which implies

$$(\lambda - \theta, \phi) = [-\text{atan}(\sin\lambda'\cos\phi),\ \text{asin}(\cos\lambda'\cos\phi)]\,. \tag{1}$$

The meridians of G are connected to the parallels of G' in the equatorial Atlantic—with equal tangents, but unequal curvatures. For G and G' to be compatible, it is necessary that the grid sizes be such that $\Delta\lambda = \Delta\phi'$ (Figure 8).

For numerical accuracy, it is desirable that the grid size variation be as "slow" as possible (Thompson *et al.*, 1985; Castro and Jones, 1987). It is across the connection line of G and G' that the rate of variation of the grid size is the highest, as is illustrated below.

Consider the grid point A located on the Equator at longitude λ_A and latitude $\phi_A = 0$. In the Southern Hemisphere, the grid point that is nearest to A is B, the position of which is given by $(\lambda_B, \phi_B) = (\lambda_A, -\Delta\phi)$. To the North, the neighbour of A is called C and is located at $(\lambda'_C, \phi'_C) = (\pi/2 - \Delta\lambda', \lambda_A + \pi/2 - \theta)$. If G were extended to the North of the Equator, C would be at location D, *i.e.*, at $(\lambda_D, \phi_D) = (\lambda_A, \Delta\phi)$. In a regular grid, C would be equivalent to D. This is however not the case. It is suggested to measure the departure from a regular grid by $(d_\lambda, d_\phi) = (\lambda_C - \lambda_D, \phi_D - \phi_C)$ (Figure 9). Using (1), one obtains

$$\begin{aligned} d_\lambda &= \pi/2 - \phi'_C\text{atan}(\cos\Delta\lambda'\cot\phi'_C) \\ &= (1/4)\sin(2\phi'_C)\,\Delta\lambda'^2 + O\left(\Delta\lambda'^4\right), \end{aligned} \tag{2}$$

$$\begin{aligned} d_\phi &= \Delta\phi - \text{asin}(\sin\Delta\lambda'\cos O'_C) \\ &= \Delta\phi - \Delta\lambda'\cos\phi'_C + O\left(\Delta\lambda'^3\right)\,. \end{aligned} \tag{3}$$

Let us assume, for the moment, that $\Delta\phi = \Delta\lambda'$. Hence, minimizing $|\, d_\lambda \,|$ and $|\, d_\phi \,|$ requires that $|\, \phi'_C \,|$ be kept as small as possible, implying that the connection of G and G' must be carried out in the vicinity of the equator of G'. In the Atlantic, the Equator approximately extends from the mouth of the Amazon ($\lambda \approx -50^\circ$ to Libreville (in Gabon) ($\lambda \approx 10^\circ$). The width of the connection is thus 60° of latitude. Therefore, by setting the North Pole of G' at $\lambda = \theta = 70^\circ$, *i.e.*, in the Macintosh II Indian Ocean (near the Maldives), one has $-30^\circ \leq \phi'_{<} 30^\circ$. The South Pole of G' is located at $\lambda = -110^\circ$, in the Pacific Ocean.

Clearly, $|\, d_\lambda \,| << |\, d_\phi \,|$. We thus concentrate on d_ϕ and we suggest looking for the value of $\Delta\lambda'$ minimizing the integral of $(d_\phi)^2$ along the connection line. This elementary least squares analysis is performed with the asymptotic expansion of d_ϕ for small $\Delta\lambda'$, *i.e.*, $d_\phi \sim \Delta\lambda'\cos\phi'_C$. It yields $\Delta\lambda' \approx 1.05\Delta\phi$. Hence, it may safely be assumed that $\Delta\lambda' = \Delta\phi$, which is obviously the simplest grid configuration. Nevertheless, at the ends of the connection line, where $\phi'_C \approx \pm 30^\circ$, $d_\phi/\Delta\phi \approx 1 - \cos\phi'_C$ is as small as 0.8. This represents a somewhat sudden, but probably acceptable (Castro and Jones, 1987), transition from G to G'.

A SINGLE COORDINATE SYSTEM

Most OGCM computer codes assume that the numerical grid is based on a single coordinate system. As we suggest to resort to 2 different coordinate systems, modifications will have to be brought about in the computer code. Various approaches may be thought of, and it is difficult to determine a *priori* which one is preferable. Here we suggest considering the two spherical coordinate systems as a single, orthogonal, curvilinear coordinate system. This way, the computational domain doesn't have to be viewed as a set of two sub-domains in which the computations have to be carried out separately—with an appropriate matching on the connection line. Furthermore, as will be seen, a natural treatment is possible of the sudden grid size decrease across the connection line. On the other hand, however, the governing equations must be written in a curvilinear coordinate system—as is done in the LODYC model (Marti *et al*, 1990; Marti *et al.*, 1992)—that is slightly more complicated than the original spherical one.

The independent variables x, y, and z, of our general coordinate system satisfy $(\partial/\partial x,\ \partial/\partial y) = a^{-1}(\partial/\partial\lambda, \partial/\partial\phi)$ in G and $(\partial/\partial x,\ \partial/\partial y) = a^{-1}(\partial/\partial\phi', -\partial/\partial\lambda)$ in G'. The horizontal grid sizes are $\Delta x = a\Delta\lambda = a\Delta\phi'$ and $\Delta y = a\Delta\phi = a\Delta\lambda'$. As for the spherical coordinate used in most OGCMs, it is considered that the metric coefficients are such that $\partial h_x/\partial z = 0 = \partial h_y/\partial z$, and that $h_z = 1$. It follows that Macintosh II $(h_x, h_y) = (\cos\phi,\ 1)$ in G and $= (1, \cos\phi')$ in G'.

Unlike h_x, the metric coefficient h_y is discontinuous across the connection line of G and G'. Indeed, on the northern side of this line h_y is equal to $\cos\phi'$, whereas it is equal to 1 in the Southern Hemisphere. If analytic calculations were carried out, this difficulty would be dealt with by simply matching the values of the dependent variables and the appropriate fluxes across the connection line. For the purposes of numerical calculations, a single value of h_y is obviously required at $\phi = 0$.

Let us consider a grid box straddling the Equator. Its area is about $a^2\Delta\lambda(\Delta\phi/2 + \cos\phi'\Delta\lambda'/2)$, which must be equivalent to $h_x h_y \Delta x \Delta y$. Hence, $h_y = (1 + \cos\phi')/2$ on the connection line.

Here, it should be pointed out that the metric coefficients should not be estimated from analytic expressions. In fact, to preseve the order of accuracy of the numerical scheme, the metric coefficients should be computed from finite-difference expressions that are consistent with those used to solve the governing equations of the model (Thompson *et al.*, 1985). However, with spherical coordinate it is so tempting to analytically derive the values of the metric coefficients that most modellers, if not all of them, proceed this way. This is the reason that we also adopted this method.

To complete the present study, we outline the OGCM equations in the coordinate system described above. We define the following operators:

$$A(\alpha) = \frac{\partial}{\partial t}(h_x h_y \alpha) + \frac{\partial}{\partial x}(h_y u \alpha) + \frac{\partial}{\partial y}(h_x v \alpha) + (h_x h_y w \alpha), \tag{4}$$

$$D(\alpha) = \frac{\partial}{\partial x}\left(A_H^\alpha \frac{h_y}{h_x}\frac{\partial \alpha}{\partial x}\right) + \frac{\partial}{\partial y}\left(A_H^\alpha \frac{h_x}{h_y}\frac{\partial \alpha}{\partial y}\right) + \frac{\partial}{\partial z}\left(A_V^\alpha h_x h_y \frac{\partial \alpha}{\partial z}\right), \tag{5}$$

where t denotes time; u, v, and w represent the velocity components along the $x-$, $y-$, and $z-$axis; A_H^α and A_V^α are appropriate horizontal and vertical diffusivities of the quantity α.

The mass conservation equation reads (Hughes and Gaylord, 1964):

$$A(1) = 0\cdot \tag{6}$$

If Q^c stands for the rate of production/destruction of the scalar quantity c, it is easily shown that the latter obeys (Hughes and Gaylord, 1964)

$$A(c) = h_x h_y Q^c + D(c)\cdot \tag{7}$$

With the above notations, the horizontal momentum equations may be written as (Hughes and Gaylord, 1964):

$$A(u) - v^2 \frac{\partial h_y}{\partial x} + uv \frac{\partial h_x}{\partial y} = h_x h_y f v - \frac{h_y}{\rho_0}\frac{\partial p}{\partial x} + D(u), \tag{8}$$

$$A(v) - u^2 \frac{\partial h_x}{\partial y} = -h_x h_y f u - \frac{h_x}{\rho_0}\frac{\partial p}{\partial y} + D(v), \tag{9}$$

where f, ρ_0, and p, denote the Coriolis parameter, the reference density, and the pressure, respectively. In the horizontal diffusion terms of the momentum equations, the horizontal velocity is considered a scalar quantity, meaning that we have neglected the numerous, additional terms stemming from the derivatives of the base vectors. This simplifying assumption, which may not be universally accepted, rests on two main arguments.

First, the horizontal diffusion terms are introduced to represent the effect of sub-grid scale advection processes and to damp small scale computational modes which are ill represented by the numerical scheme. It is far from guaranteed that those terms should be parameterized by resorting to the "true Laplacian" of the horizontal velocity vector. Thus, any term that mostly damp the smallest scale motions may be a good candidate.

Second, by not using the Laplacian of the horizontal velocity vectors, we drop terms that involve, at most, first derivatives of the velocity components. They should thus be smaller than those, which are retained in (8) and (9), involving second derivatives of the velocity components, since the Laplacian operator is dominated by the smallest scale motions.

It is worth pointing out that the last two terms in the left-hand side of (8) and (9) stem from the derivatives of the base vectors. Whether or not they could be neglected is not clear. The reasoning applied to similar terms in the diffusive parameterizations may not hold valid for them, chiefly because the advective processes are likely to be larger than diffusive ones. The terms in question appear as source/sink terms although they come from the budget of the momentum fluxes, which means that they may be cast into a conservative form. To do so, several methods are suggested in the literature (Anderson *et al.*, 1968; Vinokur, 1974; Eiseman and Stone, 1980), but they involve complications that are probably unnecessary in the scope of our grid system, which is fairly regular, rendering it less necessary to seek truly conservative forms.

CONCLUSIONS

An orthogonal, curvilinear grid is examined for application in OGCMs. The properties of the grid are studied on a theoretical basis and it is shown that our grid is most probably acceptable. This will hopefully be confirmed by the runs of our OGCM that will soon be available on the global grid system discussed above.

ACKNOWLEDGEMENTS

The development of ASTR's OGCM is carried out within the scope of the impulse programme "Global Change" (Belgian State, Contract GC/10/013, the "Conventions d'Actions de Recherches Concertées" No 92/97–154 with the Communauté Française de Belgique, and the impulse programme "Information Technology" (Belgian State, Contract IT/SC/20). Financial support for Jean-Michel Campin is provided by an EC grant.

REFERENCES

ANDERSON, J.L., PREISER, S., and RUBIN, E.L., 1968: Conservation form of the equations of hydrodynamics in curvilinear coordinate systems. *J. Comput. Phys.*, **2**, 279–287.

CASTRO, I.P., and JONES, J.M., 1987: Studies in numerical computations of recirculating flows, *Int. J. Numer. Methods Fluids*, **7**, 793–823.

EISEMAN, P.R., and STONE, A.P., 1980: Conservation laws of fluid dynamics—A survey. *SIAM Rev.*, **22**, 12–27.

HUGHES, W.F., and GAYLORD, E.B., 1964: Basic Equations of Engineering Science, *McGraw-Hill*, 163pp.

MARTI, O., CHARTIER, M., and DELECLUSE, P., 1990: A World Ocean model in curvilinear coordinates. In: Journées Numériques 1990—Courants Océaniques (Eds.: J.M. Crolet and P. Lesaint), Publications Mathématiques de Besançon, Besançon, 17–25.

MARTI, O., MADEC, G., and DELECLUSE, P., 1992: Comment on "Net diffusivity in ocean general circulation models with nonuniform grids" by F.L. Yin and I.Y. Fung, *J. Geophys. Res.*, **97**, 12,763–12,766.

THOMPSON, J.F., WARSI, Z.U.A., and MASTIN, C.W., 1982: Boundary-fitted coordinate systems for numerical solution of partial differential equations—A review. *J. Comput. Phys.*, **47**, 1–108.

THOMPSON, J.F., WARSI, Z.U.A., and MASTIN, C.W., 1985: *Numerical Grid Generation—Foundations and Applications*, North-Holland, 483pp.

VINOKUR, M., 1974: Conservation equations of gasdynamics in curvilinear coordinate systems. *J. Comput. Phys.*, **14**, 105–125.

WILLIAMSON, D.L., 1979: Difference approximation for fluid flow on a sphere. In: *Numerical Methods used in Atmospheric Models* (Vol. II), GARP Publications Series No. 17, WMO-ICSU, 51–120.

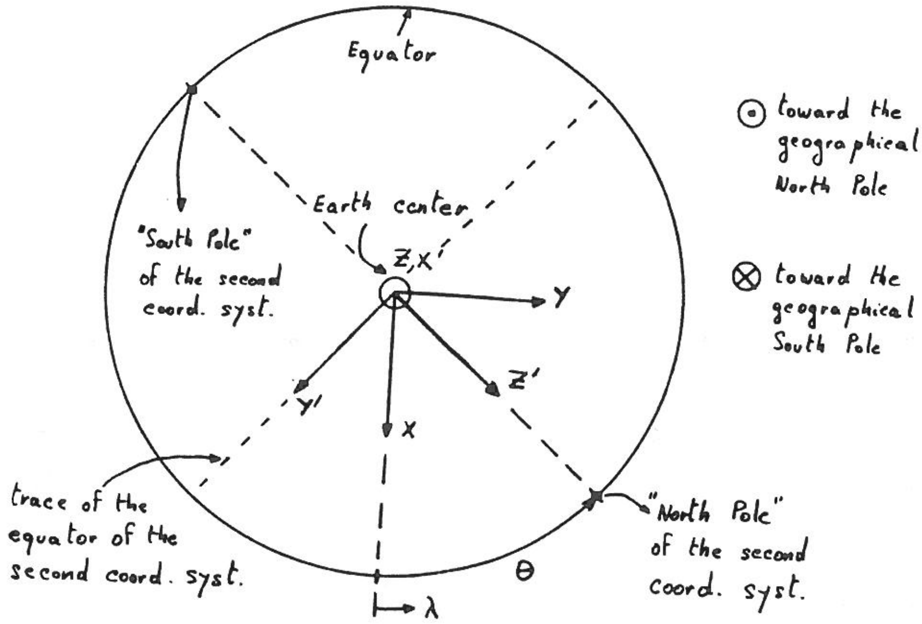
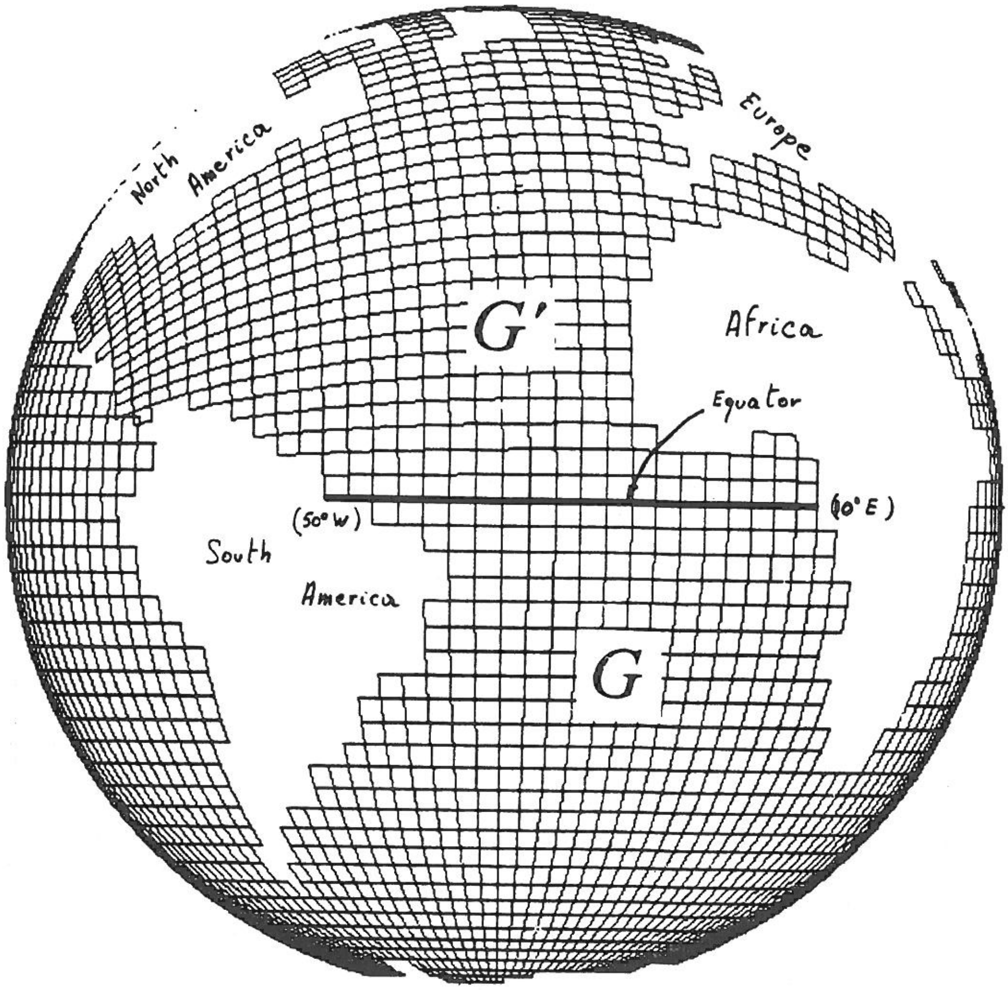


FIGURE 7
(Deleersnijder, Van Ypersele and Campin)

Section through the equatorial plane showing the axes of the cartesian coordinates associated with the two spherical coordinate systems considered in the present study.



(Jacques Haus, ASTR)

FIGURE 8
(Deleersnijder, Van Ypersele and Campin)
Perspective view of our grid system. The thick line represents the intersection of G and G'.

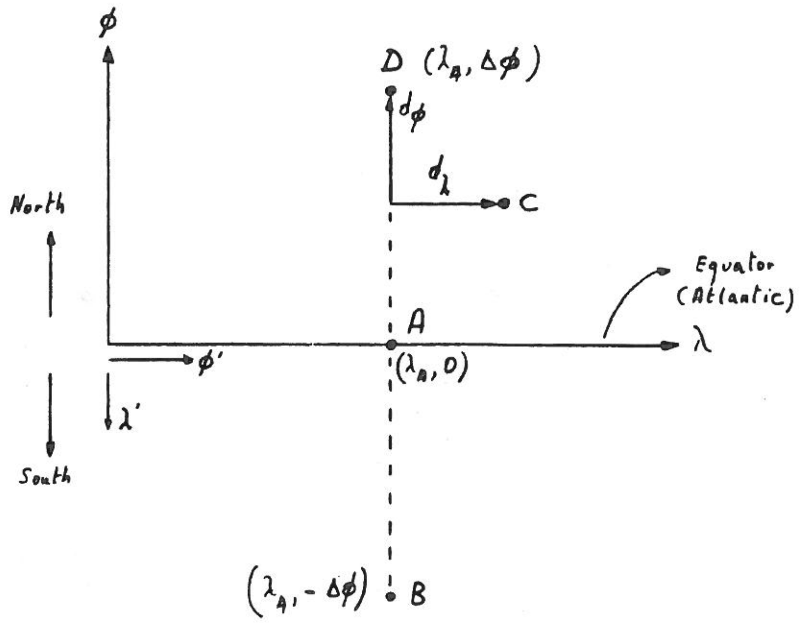


FIGURE 9

(Deleersnijder, Van Ypersele and Campin)

Locations of the points A, B, and C, and D, in the equatorial Atlantic used to define the distances d_λ and d_ϕ.