

OCEAN modelling

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EDITORIAL



To determine the stability conditions of a numerical scheme, it is customary to examine a linearized version in an unbounded domain. This is usually referred to as the von Neuman analysis. Although this technique resorts to many simplifying assumptions, it is generally capable of providing a satisfactory approximation to the stability condition of the original, non-idealized, numerical algorithm. Here, we present a simple geophysical fluid problem where the von Neuman analysis provides a stability limit that is completely irrelevant and, hence, useless.

We consider linear, damped inertia-gravity waves, which obey the following equations:

$$\frac{\partial \eta}{\partial t} + h \nabla \cdot \mathbf{u} = 0, \quad (1)$$

$$\frac{\partial \mathbf{u}}{\partial t} + f \mathbf{e} \times \mathbf{u} = -g \nabla \eta + A_H \nabla^2 \mathbf{u}, \quad (2)$$

where t is time; ∇ denotes the horizontal gradient operator; η and \mathbf{u} represent the sea surface elevation and the horizontal velocity; \mathbf{e} , h , f , g , and A_H are the vertical unit vector, the (constant) sea depth, the (constant) Coriolis parameter, the gravitational acceleration, and the horizontal viscosity, respectively.

The domain of interest is assumed to be closed so that, on its boundary, \mathbf{u} must vanish. Hence, if $A_H > 0$, the total—potential + kinetic—energy of the flow must decrease until a state of rest is attained. If A_H is zero, only the component of \mathbf{u} normal to the boundary is prescribed to be zero and the total energy remains constant as the time increases. It is desirable that any numerical solution of (1)–(2) exhibit the same properties.

We define all variables on the B-grid and we apply the well known forward-backward scheme to (1)–(2), *i.e.*, all time derivatives are discretized by forward differences, the other terms are taken explicitly, with the exception of the pressure term that is evaluated at instant $t + \Delta t$, instead of t , Δt being the time step. In addition, the Coriolis term is numerically evaluated by

$$f \mathbf{e} \times \mathbf{u} \rightarrow f[(1 - \alpha) \mathbf{e} \times \mathbf{u}(t) + \alpha \mathbf{e} \times \mathbf{u}(t + \Delta t)], \quad 0 \leq \alpha \leq 1, \quad (3)$$

where α is the “rate of implicitness” of the time discretization of $f \mathbf{e} \times \mathbf{u}$. One may believe that, when

$\alpha > 1/2$, artificial damping is added to the scheme, which should then become more stable. This seems to be confirmed by the von Neuman analysis, which for $A_H = 0$, leads to the following stability conditions:

$$\alpha = 1/2: \quad \Delta t \leq \frac{\Delta s}{c}, \quad (4)$$

$$\alpha = 1: \quad \Delta t \leq \frac{\Delta s}{c} \frac{1}{\sqrt{\max(0, 1 - \frac{\Delta s^2 f^2}{4c^2})}}, \quad (5)$$

with $c^2 = gh$ and $\Delta s = \min(\Delta x, \Delta y) - \Delta x$ and Δy denoting the space increments in the direction of the x - and y -axis, respectively.

As can be seen from (4)–(5), taking $\alpha = 1$ should improve the scheme’s stability and, if $\Delta s^2 f^2 / (4c^2) \geq 1$, the numerical algorithm should be unconditionally stable. One may believe that introducing a non-zero viscosity should not render the scheme less stable.

We have carried out a series of numerical experiments in a square domain of 50×50 grid points bounded by impermeable walls. Very surprisingly, the numerical results are in marked contrast with what could be expected from the theoretical stability analysis!

For $A_H = 0$ and $\alpha = 1$, the domain of stability of the scheme should be delimited by

$$f^2 \Delta t^2 \geq 4 \left(\frac{c^2 \Delta t^2}{\Delta s^2} - 1 \right). \quad (6)$$

The numerical simulations indicate a completely different stability domain (Figure 1a). If some diffusion is present, the stability domain somewhat expands but remains much smaller than that corresponding to (6) (Figure 1b).

It is striking to note that, for $\alpha = 1/2$, the numerical experiments (Figure 1c and 1d) lead to a stability domain that seems to be approximately in agreement with (4). As a matter of fact, it is only for $\alpha = 1/2$ that the scheme presents a large area of stability. Thus, for all practical purposes, it is clear that one should only consider $\alpha = 1/2$.

Can anyone provide a theoretical explanation for that odd behaviour?

FIGURE 1
(Deleersnijder and Campin)

Stability domain (indicated by the crosses) found in the numerical experiments with $\alpha = 1$ and $A_H \Delta t / \Delta s^2 = 0$ (a), $\alpha = 1$ and $A_H \Delta t / \Delta s^2 = 0.1$ (b), $\alpha = 1/2$ and $A_H \Delta t / \Delta s^2 = 0$ (c), $\alpha = 1/2$ and $A_H \Delta t / \Delta s^2 = 0.1$ (d). The thick line is the limit between the stable and the unstable domain for $A_H = 0$.

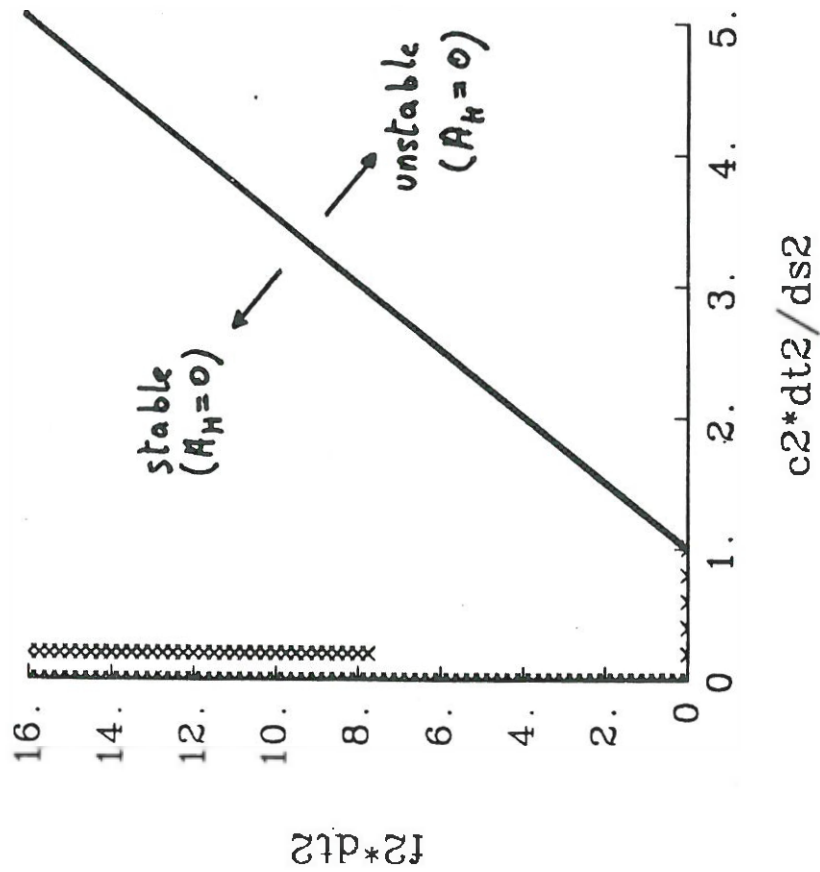


FIGURE 1a
(Deleersnijder and Campin)
 $\alpha = 1$ and $A_H \Delta t / \Delta s^2 = 0$

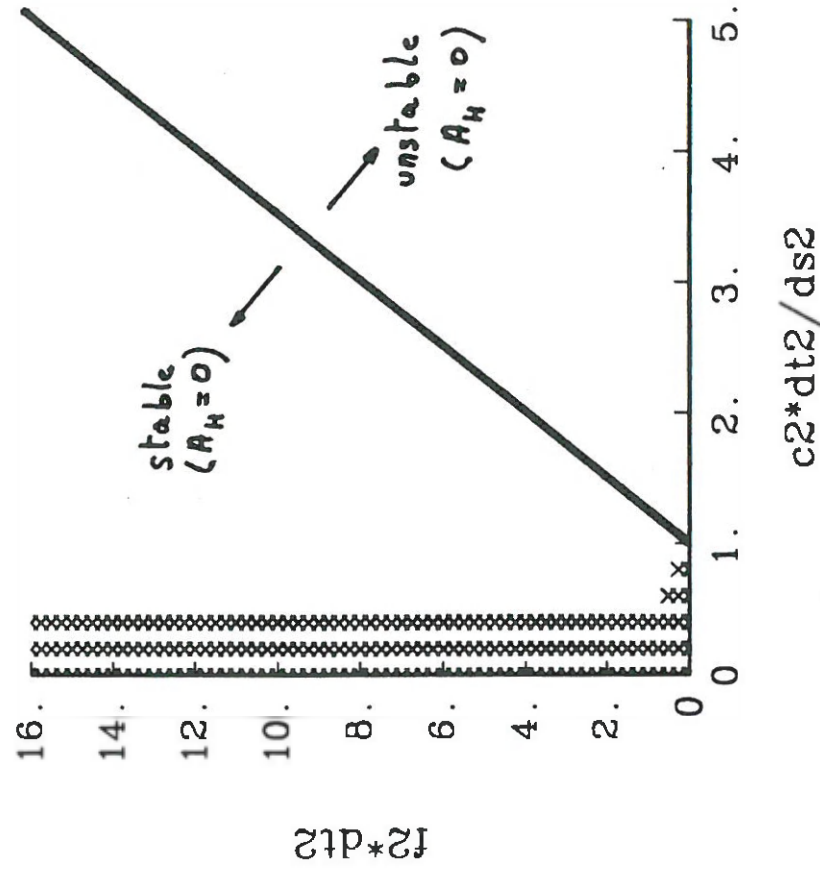


FIGURE 1b
(Deleersnijder and Campin)
 $\alpha = 1$ and $A_H \Delta t / \Delta s^2 = 0.1$

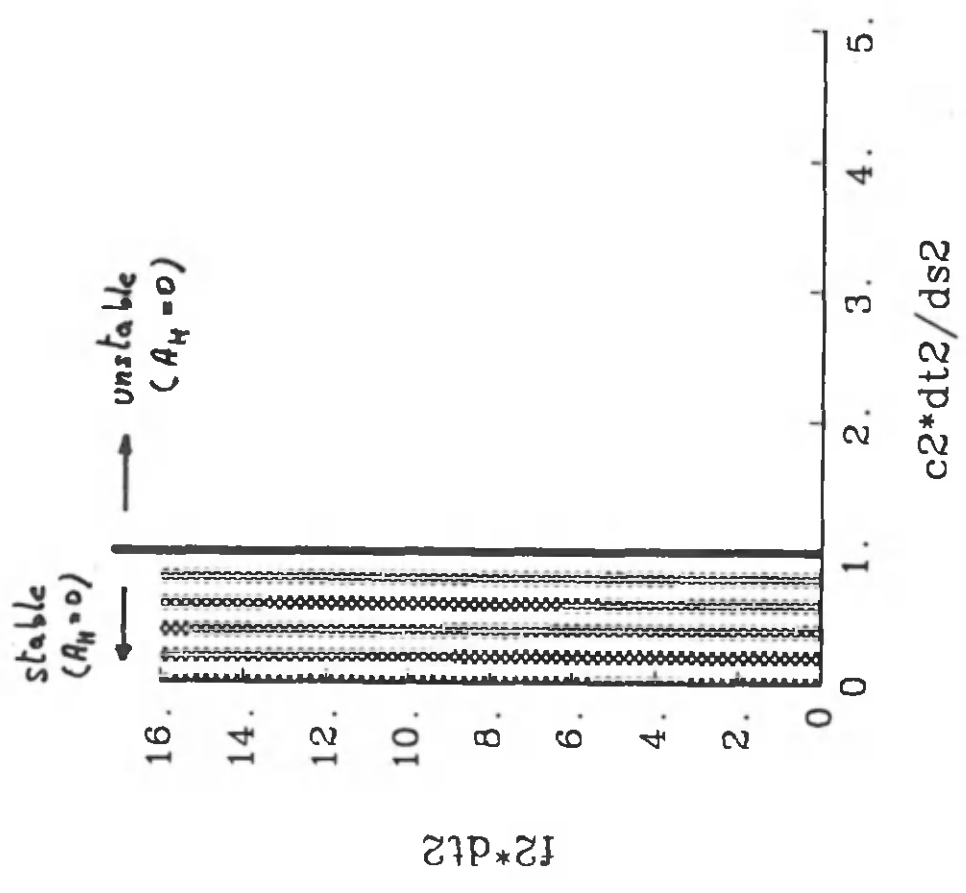


FIGURE 1c
(Deleersnijder and Campin)
 $\alpha = 1/2$ and $A_H \Delta t / \Delta s^2 = 0$

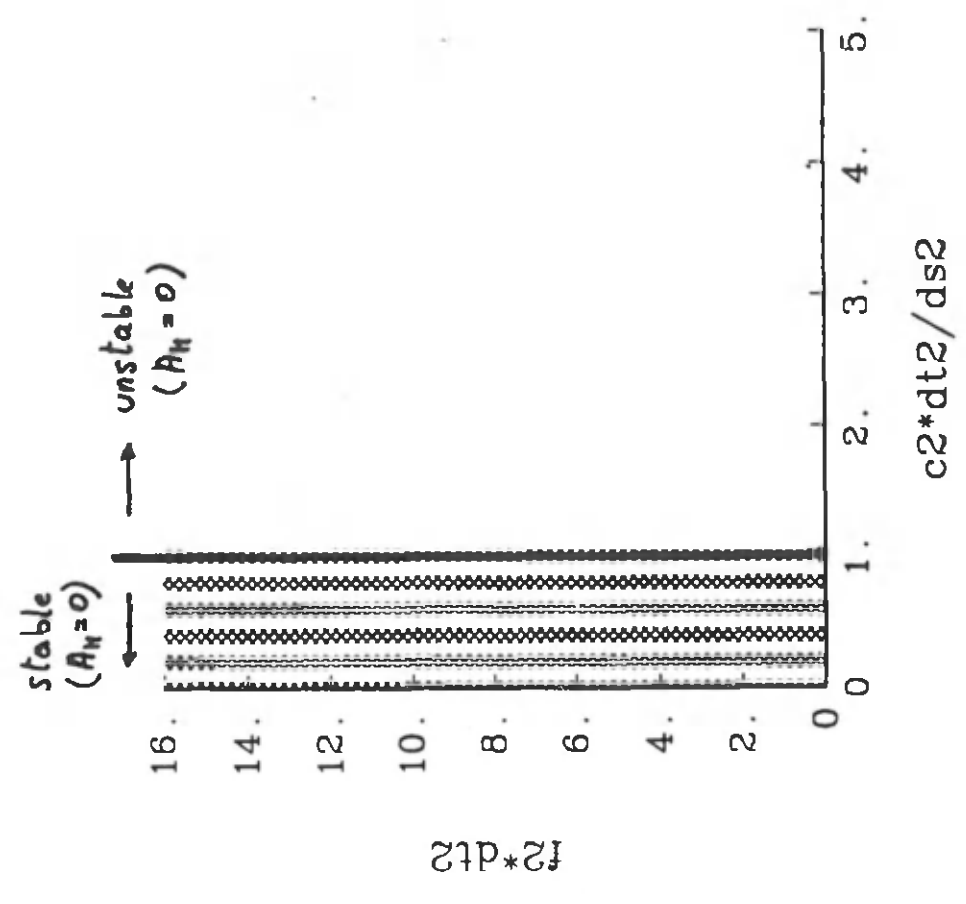


FIGURE 1d
(Deleersnijder and Campin)
 $\alpha = 1/2$ and $A_H \Delta t / \Delta s^2 = 0.1$