Leptonic CP violation phases, quark–lepton similarity and seesaw mechanism

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Abstract

We explore generic features of the leptonic CP violation in the framework of the seesaw type I mechanism with similarity of the Dirac lepton and quarks mass matrices m_D. For this, we elaborate on the standard parametrization conditions which allow to simultaneously obtain the Dirac and Majorana phases. If the only origin of CP violation is the left-handed (LH) transformation which diagonalizes m_D (similar to quarks), the leptonic CP violation is suppressed and the Dirac phase is close to π or to 0 with $\sin \delta_{CP} \approx (\sin \theta_{13}^q / \sin \theta_{13}) \cos \theta_{23} \sin \delta_q \sim \lambda^2 \sin \delta_q$. Here $\lambda \sim \theta_C$, is the Cabibbo mixing angle, and θ_{13}^q and θ_{13} are the 1–3 mixing angles of quarks and leptons respectively. The Majorana phases β_1 and β_2 are suppressed as $\lambda^3 \sin \delta_q$. For Majorana neutrinos implied by seesaw, the right-handed (RH) transformations are important. We explore the simplest extension inspired by Left–Right (L–R) symmetry with small CKM-type CP violation. In this case, seesaw enhancement of the CP violation occurs due to strong hierarchy of the eigenvalues of m_D leading to $\delta_{CP} \sim 1$. The enhancement is absent under the phase factorization conditions which require certain relations between parameters of the Majorana mass matrix of RH neutrinos.

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1. Introduction

Establishing the leptonic CP violation is one of the major experimental frontiers in neutrino physics. The Dirac and Majorana CP phases are among the few yet unknown parameters for which a prediction may still be made. So, we need to understand what particular values or intervals of the CP phases will imply for fundamental theory.

Indeed, there are numerous predictions of the phases which are based on broad spectrum of ideas, approaches, and models [1]. Some approaches that have been employed are (i) Neutrino and charged lepton mass matrices with certain properties such as – textures [2], symmetries, and symmetry violations, e.g., $\mu - \tau$ reflection or generalized symmetry [3]; (ii) Models with discrete flavor symmetries [4], which can realize geometric origins of the phases, the CP violation due to group structure or complex Clebsch–Gordan coefficients [5], or connect the phases and the mixing angles [6], etc.; (iii) Grand unification with seesaw type I and type II [7][1]; (vi) Radiative generation of CP violation [9]; (vii) Relating the leptonic CP phase to other physics, e.g., a solution to the strong CP problem wherein $\delta_{CP} = 0$ or π is predicted [10]. Many efforts have been devoted to obtain maximal CP violation, i.e., $\delta_{CP} = \pi/2$ [11], although other values essentially from 0 to π have also been found.

Can we really predict the leptonic CP phase, given that even in the quark sector, where all parameters are known, there is no unique and convincing explanation of the value of CP phase? Moreover, in the lepton sector the situation is expected to be more complicated due to presence of additional structures which are responsible for the smallness of neutrino masses. Can the lepton and quark CP phases be equal, or connected in some way? To address these questions it is instructive to represent the lepton mixing matrix in the form

$$U_{PMNS} = U_L U_X, \tag{1}$$

where U_L is somehow related to the quark CKM-mixing matrix and U_X reflects new physics responsible for smallness of neutrino mass and large mixing angles [12–18]. Here U_L and U_X can follow from diagonalization of mass matrices of the charged leptons, $U_L = U_l^\dagger$, and neutrinos, $U_X = U_\nu$, respectively. Origins of CP violation can be in U_l [19] and/or U_ν. The assumption $U_l \sim V_{CKM}$ corresponds to the Quark-Lepton Complementarity [13], so that $U_{PMNS} = V_{CKM}^\dagger U_X$. This possibility has been explored for $U_X = U_{BM}$ (bimaximal mixing matrix) [14] and $U_X = U_{TBM}$ (tribimaximal mixing matrix) referred as Cabibbo-TBM [20]. In these cases the origin of CP could be in V_{CKM} or in the diagonal phase matrix attached to U_X. In [14] the "correlation matrix" U_X has been taken in the form $U_X = P(\phi_l) U_{BM} Q(\phi_i)$, where $P(\phi_l)$ and $Q(\phi_i)$ are diagonal phase matrices. It was noticed that if $\phi_l = 0$, the Jarlskog invariant is very small [14,17, 15]: $J_{PMNS} = \sin\theta_{13}^q \sin\delta_{CKM}$, i.e., too small to be measured in future experiments.

The ansatz (1) can be naturally realized in the seesaw type I mechanism [21] which is the simplest and the most natural explanation of smallness of neutrino masses as well as large lepton mixing [22]. It is simplest because only RH neutrinos are added to the theory. It is natural in the sense that it allows to explain smallness of neutrino mass and the substantial difference between lepton mixing and quark mixing, while at the same time maximally implementing the quark–lepton similarity. The latter, in turn, is expected, e.g., in Grand unified theories. Seesaw type I mechanism with similar Dirac mass matrices for neutrinos and quarks defines the canonical seesaw mechanism.

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[1] The extreme possibility is that the mixing of quarks and leptons coincides at the GUT scale and the low energy difference is due to large renormalization group evolution for a quasi-degenerate mass spectrum [8].

In this paper we consider the leptonic CP phases that can arise from this canonical seesaw mechanism, which provides the closest possible connection of the quark and lepton sectors. We will further generalize the relation (1) assuming that U_L has similar to $V_{CKM}^\dagger$ structure but in general does not coincide with $V_{CKM}^\dagger$. For the matrix U_X we will not assume any special structure but restrict it only by the condition that the product (1) reproduces the experimentally observed values of the mixing angles. We will find the phases in the standard parametrization of the PMNS matrix. For this we formulate and use the standard parametrization conditions which allow us to obtain simultaneously both the Dirac and Majorana CP phases.

We find that if the only source of CP violation is the Kobayashi–Maskawa (KM)-type phase in U_L, it leads to a small δ_{CP}. In the seesaw mechanism due to the Majorana nature of neutrinos the CP violation in the RH sector become relevant for the PMNS CP phases. That includes the phases in the RH rotation U_R that diagonalizes the Dirac mass matrix m_D as well as in the Majorana mass matrix of RH neutrinos, M_R. We find that generically the seesaw mechanism enhances CP violation that appears in U_R, so that $\delta_{CP} = \mathcal{O}(1)$. Such an enhancement is absent and the CP phases are small (or close to π) if parameters of M_R satisfy certain (phase factorization) relations. We find relations between the Dirac and Majorana phases which can be used to test these scenarios. An observation of (large) leptonic CP violation in oscillation experiments and/or neutrinoless double beta decay would therefore be a signature that there is a new source of CP violation, beyond the leptonic analogue of KM-phase and coming from the RH sector, or that neutrino masses do not arise from a canonical seesaw.

We will argue that specific values of the leptonic CP phases are possible since the contribution of the Kobayashi–Maskawa type phase turns out to be suppressed or be close to $\delta_{CP} \sim \pi$ and the main contribution comes from the RH sector which can obey specific symmetries.

The paper is organized as follows: In Section 2 we present the formalism for calculating the CP phases. In Section 3 we derive the expressions for the CP phases when the only source of CP violation is a KM-like phase in the left-handed transformation that diagonalizes the Dirac mass matrix. The general case with CP violation in the RH sector is considered in Section 4. In Section 5 we explore a specific case of CP violation in the RH sector, which corresponds approximately to a L–R symmetry of the theory. In Section 5.2 we consider special conditions where the resulting CP phase is still small. We then show that in general the seesaw enhancement of CP violation occurs which leads to $\delta_{CP} \sim \mathcal{O}(1)$, even if CP violation in U_R is of KM type. We present some phenomenological consequences in Section 6 and conclude in Section 7.

2. Seesaw type I, CP violation, and standard parametrization

2.1. U_X matrix in seesaw type I

We introduce the Dirac matrix, m_D, in the flavor basis and Majorana mass matrix, M_R, according to the mass terms of the Lagrangian

$$\mathcal{L}_{mass} = -\bar{\nu}_L m_D \nu_R - \frac{1}{2} \nu_R^T C^\dagger M_R \nu_R + h.c.$$

Integrating out the RH neutrinos we obtain $\mathcal{L}_{mass} = -\bar{\nu}_L m_\nu \bar{\nu}_L^T + h.c.$, where the matrix of light neutrinos in the flavor basis equals

$$m_\nu = -m_D M_R^{-1} m_D^T.$$

The Dirac mass matrix can be represented in the flavor basis as

$$m_D = U_L m_D^{diag} U_R^\dagger, \tag{2}$$

where U_L and U_R are the transformations $\nu_L = U_L \nu_L^{diag}$, $\nu_R = U_R \nu_R^{diag}$, that diagonalize m_D, and $m_D^{diag} \equiv \mathrm{diag}(m_{1D}, m_{2D}, m_{3D})$. The light neutrino mass matrix in the flavor basis is

$$m_\nu = U_{PMNS} m_\nu^{diag} U_{PMNS}^T, \tag{3}$$

where

$$\nu_L = U_{PMNS} \nu_{mass},$$

are the light neutrino flavor states and $m_\nu^{diag} = \mathrm{diag}(m_1, m_2, m_3)$ is the diagonal matrix of real and positive neutrino mass eigenvalues.

Inserting (3) and (2) into the seesaw expression (2) we obtain

$$U_{PMNS} m_\nu^{diag} U_{PMNS}^T = -U_L m_D^{diag} U_R^\dagger \frac{1}{M_R} U_R^* m_D^{diag} U_L^T. \tag{4}$$

The relationship in (4) can be re-expressed as

$$U_{PMNS} m_\nu^{diag} U_{PMNS}^T = U_L M_X U_L^T, \tag{5}$$

where

$$M_X \equiv -m_D^{diag} U_R^\dagger \frac{1}{M_R} U_R^* m_D^{diag}. \tag{6}$$

It is the structure of the matrix M_X that produces the difference in masses and mixing of quarks and leptons.

Since U_{PMNS} and U_L are unitary matrices, the eigenvalues on both sides of (5) should coincide. Therefore M_X can be rewritten as,

$$M_X = U_X m_\nu^{diag} U_X^T, \tag{7}$$

and the mixing matrix U_X is obtained by diagonalization of (6). From (5) and (7) we obtain

$$U_{PMNS} m_\nu^{diag} U_{PMNS}^T = U_L U_X m_\nu^{diag} U_X^T U_L^T,$$

which can be satisfied if and only if the matrix $U_L U_X$ coincides with U_{PMNS} up to a diagonal matrix $D = \mathrm{diag}[(-1)^n, (-1)^m, (-1)^k]$, where n, m, k are integers, which is the symmetry transformation of a generic diagonal Majorana mass matrix. Therefore, $U_{PMNS} = U_L U_X D$. In what follows, we will absorb D into the phase matrix of U_X.

Thus, within the seesaw paradigm we arrive at the relationship (1) with U_X being the matrix which diagonalizes M_X (6). Notice that U_L would be the lepton mixing matrix, if M_X is diagonal or there are no Majorana mass terms. Whereas, U_X encodes information about the eigenstates of the Dirac and Majorana mass matrices, as well as about mismatch of the ν_R transformations which diagonalize m_D and M_R. The matrix M_R can be written as $M_R = U_M M_R^{diag} U_M^T$, so that $M_R^{-1} = U_M^* (M_R^{diag})^{-1} U_M^\dagger$, and consequently, $M_X = -m_D^{diag} U_R^\dagger U_M^* (M_R^{diag})^{-1} U_M^\dagger U_R^* m_D^{diag}$. If $U_M = U_R^*$, then according to (6) M_X is diagonal.

In what follows we will explore the relationship expressed in (1) to derive predictions for the physical CP violating phases in U_{PMNS} in terms of the relevant parameters of the RH sector and U_L. Results of this section are general and can be applied to any mechanism which reproduces (1).

2.2. Standard parametrization conditions

Motivated by its widespread use, we will consider the CP phases that appear in the standard parametrization of the PMNS matrix U_{PMNS}^{std} [23]:

$$U_{PMNS}^{std} = R_{23}\Gamma_\delta R_{13}\Gamma_\delta^\dagger R_{12},$$

where $\Gamma_\delta \equiv \mathrm{diag}(1, 1, e^{i\delta_{CP}})$ and δ_{CP} is the Dirac CP violating phase. Usually to find the CP phase one computes the Jarlskog invariant of U_{li}, and uses the mixing parameters in the standard parametrization. We find that a more instructive and transparent way to find CP phases is to compute the mixing matrix directly and reduce it to the standard parametrization form by rephasing.

In general, the PMNS matrix can be written as

$$U_{PMNS} = D(\phi)U_{PMNS}^{std}(\delta_{CP})\Gamma_M(\beta). \tag{8}$$

Here $D(\phi) \equiv \mathrm{diag}(e^{i\phi_e}, e^{i\phi_\mu}, e^{i\phi_\tau})$ is the matrix of phases which can be eventually absorbed into the wave functions of charged leptons, and

$$\Gamma_M \equiv \mathrm{diag}\big(e^{i\beta_1}, e^{i\beta_2}, 1\big)$$

is the matrix of the Majorana phases.[2] We will use the standard parametrization also for the matrices on the RH side of (1):

$$U_L = D(\psi)U_L^{std}(\delta_L)D(\chi), \qquad U_X = D(y)U_X^{std}(\delta_X)D(z), \tag{9}$$

where $D(\psi) \equiv \mathrm{diag}(e^{i\psi_e}, e^{i\psi_\mu}, e^{i\psi_\tau})$, etc., U_L^{std} and U_X^{std} are the matrices in the standard parametrizations which contain a single CP phase each. Then the product of matrices in Eq. (1) equals

$$U_L U_X = D(\psi)U_L^{std}(\delta_q)D(\alpha)U_X^{std}(\delta_X)D(z), \quad \text{where } \alpha_l \equiv \chi + y. \tag{10}$$

Clearly, introduction of the two separate matrices $D(\chi)$ and $D(y)$ is irrelevant for the light neutrino mass matrix but it does matter for the structure of the RH sector.

Inserting expressions (8) and (10) into (1), and multiplying it by $D(\phi)^*$ and Γ_M^* from the left and the right sides respectively, we obtain

$$U_{PMNS}^{std} = D(\gamma)U_L^{std}(\delta_q)D(\alpha)U_X^{std}(\delta_X)D(\eta). \tag{11}$$

The phases

$$\gamma \equiv \psi - \phi, \qquad \eta \equiv z - \beta$$

should be selected in such a way that they bring the RHS of (11) to the standard parametrization form.

The conditions, that the matrix on the RH side of (11) is in the standard parametrization, are given by the following 5 equations

$$\mathrm{Arg}\{U_{e1}\} = \mathrm{Arg}\{U_{e2}\} = \mathrm{Arg}\{U_{\mu3}\} = \mathrm{Arg}\{U_{\tau3}\} = 0, \tag{12}$$

$$|U_{e1}|\,\mathrm{Im}\,U_{\mu2} = |U_{e2}|\,\mathrm{Im}\,U_{\mu1}. \tag{13}$$

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[2] An alternate parametrization of Γ_M is $\mathrm{diag}(1, e^{i\frac{\alpha_{21}}{2}}, e^{i\frac{\alpha_{31}}{2}})$, and we can recover these Majorana phases, $\alpha_{21} = 2(\beta_2 - \beta_1)$ and $\alpha_{31} = -2\beta_1$, by an overall rephasing of U_{PMNS} from the right side by $e^{-i\beta_1}$.

These conditions which we call the Standard Parametrization (SP) conditions fix 5 phases γ_e, γ_μ, γ_τ and η_1, η_2. Notice that conditions (12) determine the phases of the mixing matrix up to a rephasing: $U_{e3} \to e^{i\Phi} U_{e3}$, and $(U_{\mu 1}, U_{\mu 2}, U_{\tau 1}, U_{\tau 2}) \to e^{-i\Phi}(U_{\mu 1}, U_{\mu 2}, U_{\tau 1}, U_{\tau 2})$. This allows, in particular, to eliminate the phase of the 1–3 element. It is the condition (13) that fixes Φ and removes the ambiguity.

Once the SP-conditions are satisfied the phase of the 1–3 element of the obtained matrix will give

$$\delta_{CP} = -\operatorname{Arg}\{U_{e3}\}, \quad \text{or} \quad \sin\delta_{CP} = -\frac{\operatorname{Im} U_{e3}}{|U_{e3}|},$$

and the Majorana phases equal

$$\beta = z - \eta.$$

2.3. Quark-lepton similarity and general expression for the Dirac CP phase

The main assumption in this paper is that the Dirac mass matrix of neutrinos has similar structure to that of quarks: $m_D \sim m_u$ or $\sim m_d$, as can be motivated by the Grand unification or common flavor symmetry with the same charge assignments. Consequently, the mixing in leptonic sector which follows from the Dirac matrices is similar to that in the quark sector:

$$U_L^{std}(\delta_L) \sim V_{CKM}^\dagger(\delta_q). \tag{14}$$

Essentially, we will only assume that mixing matrix U_L has a hierarchical structure of elements, as the CKM matrix, i.e., $V_{ud} \gg V_{cd} \gg V_{td}$, etc., and express the smallness of these quantities by referring to the Wolfenstein parameter λ. We make no use on any other details of this similarity. In particular, the parameter λ does not have to be exactly the same as in the quark sector.

According to (14), we will suggestively denote the elements $(U_L^{std})_{li}$ by the elements of V_{ul}^*, where the charged lepton index $l = (e, \mu, \tau)$ here corresponds to the down quarks (d, s, b) in V_{CKM} and the neutrino index $i = (1, 2, 3)$ corresponds to up-quarks (u, c, t). Denoting the elements of the matrix U_X^{std} by X_{li} we obtain for the matrix elements on the RHS of (11)

$$U_{lj} = e^{i(\gamma_l + \eta_j)}\big[V_{ul}^* X_{ej} e^{i\alpha_e} + V_{cl}^* X_{\mu j} e^{i\alpha_\mu} + V_{tl}^* X_{\tau j} e^{i\alpha_\tau} \big], \tag{15}$$

where $l = e, \mu, \tau$ and $j = 1, 2, 3$. We remind that in V we replace $e \to d$, $\mu \to s$, $\tau \to b$.

Introducing ξ_{lj} – the phases of the expressions in the brackets of (15), we can rewrite the elements of the PMNS matrix (15) as

$$U_{lj} = e^{i(\gamma_l + \eta_j + \xi_{lj})} |U_{lj}|.$$

The phases γ_l and η_j should be determined from the conditions of the standard parametrization.

The elements V_{ud}, V_{us}, V_{cb}, and V_{tb} are real. The elements $V_{cd} = -|V_{cd}|e^{i\phi_{cd}}$ and $V_{ts} = -|V_{ts}|e^{i\phi_{ts}}$ have an overall negative sign, so that the phases ϕ_{cd} and ϕ_{ts} are of order λ^4 and λ^2, respectively. The other phases are defined as usual, $V_{ub} = |V_{ub}|e^{i\phi_{ub}}$, $V_{td} = |V_{td}|e^{i\phi_{td}}$, and $V_{cs} = |V_{cs}|e^{i\phi_{cs}}$. The phases ϕ_{ub} and ϕ_{td} are $\mathcal{O}(1)$, while ϕ_{cs} is of order λ^6 and can be neglected. All these phases are known in terms of the quark CP violating phase δ_q.

Consider the element U_{e3} which contains the Dirac CP phase:

$$U_{e3} = s_{13} e^{-i\delta_{CP}} = e^{i\gamma_e}\big[e^{i\alpha_e} V_{ud} X_{e3} - e^{i(\alpha_\mu - \phi_{cd})}|V_{cd}|X_{\mu 3} + e^{i(\alpha_\tau - \phi_{td})}|V_{td}|X_{\tau 3} \big]. \tag{16}$$

Modulus and argument of U_{e3} determine θ_{13} and δ_{CP}, respectively. Since $|U_{e3}| = s_{13}$, from (16) we obtain

$$\sin\delta_{CP} = -\frac{1}{s_{13}}\big[\sin(\alpha_e + \gamma_e - \delta_X)V_{ud}|X_{e3}| - \sin(\alpha_\mu + \gamma_e - \phi_{cd})|V_{cd}|X_{\mu 3}$$
$$+ \sin(\alpha_\tau + \gamma_e - \phi_{td})|V_{td}|X_{\tau 3}\big]. \tag{17}$$

Recall that the phases α_l and δ_X parametrize the CP violation which originates from the RH sector. The phase γ_e is fixed by the standard parametrization conditions: $\gamma_e = \gamma_e(\alpha_l, \delta_X, \delta_q)$. The phase $\eta_3 = 0$, as z_3 and β_3 can be chosen to be zero, and the above expressions do not explicitly depend on η. The important feature of the result (17) is that contribution of α_τ to δ_{CP} is always suppressed by $V_{td}/s_{13} \sim \lambda^2$, δ_X is suppressed by X_{e3}, whereas the contributions of α_e and α_μ are unsuppressed.

3. A CKM-type origin of the leptonic CP violation

Suppose that the only source of CP violation is $U_L^{std}(\delta_L) \approx V_{CKM}(\delta_q)$, i.e., the matrix of transformation of the LH neutrino components that diagonalizes m_D. This is a direct analogy to the Kobayashi–Maskawa mechanism in the quark sector, as previously considered e.g. in [14]. It corresponds to U_X being a real matrix, so that

$$\alpha_l = 0, \qquad z = \delta_X = 0.$$

The matrix in front of $U_L^{std}(\delta_L)$ can be absorbed into the phases of the charged leptons. This can be thought of as the minimal CP violation that we expect for leptons if their Dirac masses are similar to quarks. In the context of the seesaw mechanism such a situation can be realized if both U_R and M_R are real, and the diagonal phase matrices vanish or cancel with each other. The cancellation can be due certain symmetries for RH neutrino components. In this case according to (16) and (17):

$$U_{e3} = e^{i\gamma_e}\big[V_{ud}X_{e3} - |V_{cd}|X_{\mu 3} + e^{-i\phi_{td}}|V_{td}|X_{\tau 3}\big], \tag{18}$$

and

$$\sin\delta_{CP} = -\frac{1}{s_{13}}\big[\sin\gamma_e(V_{ud}X_{e3} - |V_{cd}|X_{\mu 3}) + \sin(\gamma_e - \phi_{td})|V_{td}|X_{\tau 3}\big] + \mathcal{O}(\lambda^4), \tag{19}$$

where ϕ_{cd} has been neglected. The absolute value of U_{e3} according to (18) equals

$$|U_{e3}| = s_{13} = |A| \equiv |V_{ud}X_{e3} - |V_{cd}|X_{\mu 3}| + \mathcal{O}(\lambda^3).$$

Therefore

$$\sin\delta_{CP} = -\text{sign}\{A\}\sin\gamma_e - \frac{1}{s_{13}}\sin(\gamma_e - \phi_{td})|V_{td}|X_{\tau 3}.$$

Thus, the CP phase is determined essentially by γ_e which we find (see Appendix A for details) by imposing the SP conditions (12, 13) to be

$$\gamma_e = \frac{X_{e1}^2 X_{\mu 2}X_{\tau 2} - X_{e2}^2 X_{\mu 1}X_{\tau 1}}{V_{ud}X_{e1}X_{e2}X_{\tau 3}}s_{13}^q\sin\delta_q + \mathcal{O}(\lambda^4), \tag{20}$$

where we used the result (40) and $|V_{td}|\sin\phi_{td} \equiv \text{Im}\,V_{td} = s_{13}^q\sin\delta_q$. Since $s_{13}^q = \lambda^3$ the expression (20) shows that $\sin\gamma_e = \mathcal{O}(\lambda^3)$.

Let us express the elements X_{li} in terms of the elements of U_{PMNS}^{std}. Using the relations (15), at $\mathcal{O}(1)$ we have $X_{lj} \approx |U_{lj}|/(V_{CKM})_{ll}$, while X_{e3} turns out to be of the order λ: $X_{e3} = \pm s_{13}/V_{ud} + s_{23}|V_{cd}|/(V_{ud}|V_{cs}|)$. With these expressions for X_{li} and γ_e, we obtain from (19)

$$\sin \delta_{CP} = -\sin \delta_q \frac{s_{13}^q}{s_{13}} c_{23}[1 + 2s_{13} \tan \theta_{23} \cot 2\theta_{12}] + \mathcal{O}(\lambda^4, \lambda^3 s_{13}). \tag{21}$$

Similarly according to (39) and (41), the Majorana phases are

$$\beta_1 = \frac{s_{23}c_{12}}{s_{12}} s_{13}^q \sin \delta_q + \mathcal{O}(\lambda^4),$$

$$\beta_2 = -\frac{s_{23}s_{12}}{c_{12}} s_{13}^q \sin \delta_q + \mathcal{O}(\lambda^4).$$

The following comments are in order.

1. The main term in (21) is of the order $\lambda^3/s_{13} \sim \lambda^2$, that is, suppressed by $\sim \lambda^2$. This agrees with results obtained previously (e.g., [14,17]). At leading order (21) can be rewritten as

$$s_{13} \sin \delta_{CP} = (-c_{23})s_{13}^q \sin \delta_q,$$

or $\mathrm{Im}\, U_{e3} = -c_{23}\, \mathrm{Im}\, V_{ub}$. So, the Dirac CP phase in the leptonic sector is suppressed because the mixing is relatively large, compared to quark mixing.

2. The subleading term in the Dirac CP phase is of the order λ^3, and it is proportional to deviation of the 2–3 mixing from maximal.

3. Numerically we have $\sin \delta_{CP} \approx -0.05 \sin \delta_q = -0.046$, as $\delta_q = 1.2 \pm 0.08$ radian. To determine the phase itself we should also estimate $\cos \delta_{CP}$. Since $\sin \delta_{CP} \ll 1$, we have $\cos \delta \approx \pm 1$. Therefore according to (18) $\cos \delta_{CP} = \mathrm{sign}\{A\}$, which corresponds to either

$$\delta_{CP} \approx -\delta \quad \text{or} \quad \delta_{CP} \approx \pi + \delta,$$

where the deviation $\delta \approx (s_{13}^q/s_{13})c_{23} \sin \delta_q$, is of the order λ^2.

4. The Majorana phases are smaller and suppressed as λ^3. Numerically one finds that $\beta_1 \approx 0.01$ and $\beta_2 \approx -0.005$. Notice that these are the "induced" phases by the Dirac quark phase δ_q and SP conditions. Indeed, the phase δ_q appears in a mixing matrix that is not in the standard form, and β_i are the phases obtained in rephasing procedure to bring this matrix to the standard form.

5. As we remarked before, the Dirac phase can be obtained from the Jarlskog invariant in the standard parametrization:

$$J_{CP} \equiv \mathrm{Im}[U_{e1}^* U_{\mu 3}^* U_{e3} U_{\mu 1}] = \frac{1}{8} \sin 2\theta_{13} \sin 2\theta_{13} \sin 2\theta_{13} \cos \theta_{13} \sin \delta_{CP}. \tag{22}$$

Using expressions (15) for the elements in the LHS of this equality taken for all zero phases but δ_q we obtain in the lowest order

$$J_{CP} = -V_{cs}^2 V_{ud} X_{\mu 1} X_{\mu 2} X_{\mu 3}\, \mathrm{Im}\, V_{td} \approx -X_{\mu 1} X_{\mu 2} X_{\mu 3}\, \mathrm{Im}\, V_{td}.$$

Expressions (47) and (48) in Appendix B, for $X_{\mu i}$ in terms of PMNS mixing angles allow to rewrite this as

$$J_{CP} = s_{12}c_{12}s_{23}c_{23}^2\, \mathrm{Im}\, V_{td} = s_{12}c_{12}s_{23}c_{23}^2 s_{13}^q \sin \delta_q.$$

Finally, inserting this into LHS of Eq. (22) we find $\sin \delta_{CP} = -c_{23}(s_{13}^q/s_{13}) \sin \delta_q$ which coincides with the lowest order term in Eq. (21).

6. The results obtained in this section do not actually depend on mechanism of neutrino mass generation. They are based on a general parametrization of the PMNS matrix (1), with the assumption that $U_L \sim V_{CKM}^\dagger$ is the only source of the CP violation and requirement that the product (1) reproduces the observed lepton mixing angles. Although we have motivated this ansatz in the context of seesaw type I, any model that satisfies $U_L \sim V_{CKM}^\dagger$ and has no other source of CP violation leads to the same result.

4. General case of CP violation

In general the assumption made in the previous section, that the left transformation is the only source of CP violation, is not valid for Majorana neutrinos implied by seesaw. In the case of Majorana neutrinos, phases of the RH sector become important for PMNS mixing. In particular, the CP phase in the right matrix U_R will contribute to δ_{CP}. The CP violation in RH sector doesn't affect the CP violation in the CKM matrix because quarks do not have a Majorana mass term. In this sense, the analogy between the lepton and quark sector cannot be exact even if Dirac matrices are the same – the matrix U_R has physical consequences for neutrinos.

Consider the most general possibility, when CP violating parameters exist in both the Dirac and Majorana mass matrices involved in the seesaw. Neglecting terms of the order λ^3 we obtain from (17)

$$\sin\delta_{CP} = -\frac{1}{s_{13}}\Big[\sin(\alpha_e + \gamma_e - \delta_X)V_{ud}|X_{e3}| - \sin(\alpha_\mu + \gamma_e)|V_{cd}|X_{\mu3}\Big]. \tag{23}$$

In the leading order in λ the conditions of standard parametrization (12) give

$$\eta_1 + \alpha_e + \gamma_e = 0, \qquad \eta_2 + \alpha_e + \gamma_e = 0,$$
$$\gamma_\mu = -\alpha_\mu, \qquad \gamma_\tau = -\alpha_\tau, \tag{24}$$

and the 5th condition reads:

$$\sin(\gamma_\mu + \eta_2) = r\sin(\gamma_\mu + \eta_1). \tag{25}$$

(Notice that $r \approx -2$, because U_X is close to being U_{TBM}). The only solution of this system is the following: $\eta_1 = \eta_2 \equiv \eta$ from (24), then $\gamma_\mu = -\eta$ from (25), and then $\alpha_\mu = \eta$, $\gamma_e + \alpha_e = -\eta$. Inserting these expressions into (23) we obtain

$$\sin\delta_{CP} = \frac{1}{s_{13}}\Big[V_{ud}|X_{e3}|\sin(\alpha_\mu + \delta_X) - |V_{cd}|X_{\mu3}\sin\alpha_e\Big].$$

All three phases δ_X, α_e, and α_μ are free parameters and one can obtain any value of the CP phase. In specific cases, some of these phases can be removed or fixed resulting in a more precise prediction, e.g., if $X_{e3} = 0$, we get $\sin\delta_{CP} \approx -\sin\alpha_e$. For $\alpha_e = \alpha_\mu = \delta_X = 0$ we obtain $\delta_{CP} = 0$, in agreement with our consideration in Section 3 at this order.

If $\alpha_\mu \neq 0$ and $\alpha_e = \alpha_\tau = 0$, we obtain by using the standard parametrization conditions

$$\sin\delta_{CP} = \sin\alpha_\mu \frac{V_{cd}X_{\mu3}}{|-V_{cd}X_{\mu3}e^{i\alpha_\mu} + V_{ud}X_{e3}|}.$$

According to this expression δ_{CP} can be of the order 1 if α_μ is unsuppressed.

The Majorana phases equal

$$\beta_1 = z_1 - \alpha_\mu, \qquad \beta_2 = z_2 - \alpha_\mu,$$

which gives $\beta_1 - \beta_2 = z_1 - z_2$, where z_i are also unknown parameters, which can be fixed once M_R is determined. So, in general, all leptonic CP phases are unconstrained and can be large.

5. CP violation from U_R and seesaw enhancement of the CP phase

5.1. CP phases in the left–right symmetric case

Here we explore the minimal extension of the CKM case that includes effect of the RH sector. In the spirit of L–R symmetric models we assume that

$$U_R \approx U_L \sim V_{CKM}^\dagger,$$

and there is no CP violation in M_R in the L–R symmetry basis. So,

$$M_X \equiv -m_D^{diag} V_{CKM} M_R^{-1} V_{CKM}^T m_D^{diag}, \tag{26}$$

where now M_X is a complex symmetric matrix. The CP violation in $U_R \sim U_L$ is very small, being suppressed by λ^3.

To elucidate the role of CP violation from U_R and effect of seesaw we assume that M_R has the following form:

$$M_R^{-1} = V_{CKM}^{0T} (m_D^{diag})^{-1} \tilde{M}_{TBM} (m_D^{diag})^{-1} V_{CKM}^0, \tag{27}$$

where $V_{CKM}^0 = V_{CKM}(\delta = 0)$ is the CKM-like matrix with zero value of the CP phase and $\tilde{M}_{TBM} \approx M_{TBM}$. The latter ensures that matrix U_X is close to U_{TBM}, which leads to the observed PMNS mixing angles.

Inserting expression (27) into (26) we can represent M_X as

$$M_X = -K \tilde{M}_{TBM} K^T,$$

where

$$K \equiv m_D^{diag} V_{CKM} V_{CKM}^{0T} (m_D^{diag})^{-1}$$

is the correction matrix that captures the effect of a non-zero CP phase. Indeed, for $\delta_q = 0$, $K = I$ the above would provide $M_X \approx -\tilde{M}_{TBM}$.

Computing explicitly, we find

$$V_{CKM} V_{CKM}^{0T} = I + \begin{pmatrix} 0 & 0 & -V_{td}^* \\ 0 & 0 & 0 \\ V_{td} & 0 & 0 \end{pmatrix},$$

and $V_{td} \approx \lambda^3(1 - e^{i\delta}) \equiv \lambda^3 \xi$. Let us take $m_D = m_{3D} \, \text{diag}(\lambda^m, \lambda^n, 1)$. We can also include coefficients of order one here, but they will not change final conclusion. Then

$$K = \begin{pmatrix} 1 & 0 & -V_{td}^* \lambda^m \\ 0 & 1 & 0 \\ V_{td} \lambda^{-m} & 0 & 1 \end{pmatrix} \approx \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \xi \lambda^{3-m} & 0 & 1 \end{pmatrix}. \tag{28}$$

For $m \geq 4$, the 3–1 element of the correction matrix is large, i.e., enhanced as $\geq \xi \lambda^{-1}$. It is this factor, related to the strong hierarchy of the eigenvalues of the Dirac matrix, that can lead to enhancement of the CP violation. Note that the correction in (28) does not depend on the second eigenvalue λ^n.

We take

$$
\tilde{M}_{TBM} \sim m_0 \begin{pmatrix} a\lambda^p & b\lambda & f\lambda \\ \dots & 1 & g \\ \dots & \dots & h \end{pmatrix}, \tag{29}
$$

where a, b, g, h are real coefficients of the order 1. Then using the correction matrix (28) we obtain

$$
M_X \propto \begin{pmatrix} a\lambda^p & b\lambda & a\xi\lambda^{-m+p+3} + f\lambda \\ \dots & 1 & b\xi\lambda^{-m+4} + g \\ \dots & \dots & a\xi^2\lambda^{-2m+p+6} + 2f\xi\lambda^{-m+4} + h \end{pmatrix}.
$$

The only possibility to have M_X be an approximate TBM mass matrix is $m \le 4$ and $p \ge 2$. That is, the hierarchy of the Dirac mass matrix is strongly restricted by the condition that correct PMNS mixing is reproduced. If the hierarchy of the eigenstates of the Dirac mass matrix is too strong, i.e., $m > 4$, no solution which gives correct mixing angles exists in the presence of a CP violating phase. At the same time a solution always exists for arbitrarily strong hierarchies if there is no CP phases in U_R.

Taking $m = 4$ and $p = 2$, we obtain

$$
M_X \propto \begin{pmatrix} a\lambda^2 & b\lambda & \lambda(a\xi + f) \\ \dots & 1 & b\xi + g \\ \dots & \dots & a\xi^2 + 2f\xi + h \end{pmatrix}, \tag{30}
$$

and corrections to all the elements are suppressed by at least λ^2. Now the problem is to find phases of the matrix U_X (y_l, δ_X, z_i) that diagonalizes (30).

5.2. Factorization of phases

The phases of U_X can be found immediately if the phases are factorized from M_X. Under the conditions

$$
\mathrm{Arg}(a\xi + f) = \mathrm{Arg}(b\xi + g) = \frac{1}{2}\mathrm{Arg}(a\xi^2 + 2f\xi + h) \equiv \phi_F,
$$

which we will call the phases factorizations conditions, the matrix (30) can be written as

$$
M_X = D(\phi_F)M_X^0 D(\phi_F),
$$

where $D(\phi_F) = (1, 1, e^{i\phi_F})$ and

$$
M_X^0 \propto \begin{pmatrix} a\lambda^2 & b\lambda & \lambda a|F| \\ \dots & 1 & b|F| \\ \dots & \dots & a|F|^2 \end{pmatrix}. \tag{31}
$$

Here $F \equiv |F|e^{i\phi_F} \equiv \xi + f/a$.

The factorization conditions can be satisfied if

$$
\frac{f}{a} = \frac{g}{b}, \qquad f^2 = ah.
$$

Since M_X^0 is real and, in general, can be diagonalized by real matrix O, we have $U_X = D(\phi_F)O$. So that in the notation of (9), $y_e = y_\mu = \delta_X = z = 0$ and $y_\tau = \phi_F$. Furthermore,

since in the L–R symmetric case $D(\xi)$ is irrelevant, $\alpha_l = y_l$ and $\alpha_\tau = y_\tau = \phi_F$. Thus, $D(\alpha) =$ diag$(1, 1, e^{i\alpha_\tau})$ and the factorization phase is determined by

$$\tan \alpha_\tau = \frac{\sin \delta}{1 + f/a - \cos \delta}.$$

Furthermore,

$$|F| = |\xi + f/a| = \left[(1 + f/a)^2 - 2(1 + f/a) \cos \delta + 1\right]^2.$$

If $f = -1$ and $a = 1$ we have $\alpha_\tau = -\delta$ and $|F| = 1$. More generally, for the interval $a = 1 - \lambda$ to $1 + \lambda$ we obtain that α_τ changes from $-74°$ to $-86°$. For $f > 0$ the interval for the phase α_τ is $27°$–$30°$. In both cases α_τ differs from $\phi_{td} \approx 50°$. Solving the SP conditions (see Appendix A), we find

$$\sin \delta_{CP} \approx - \text{sign}\{A\} \, 2 \sin \phi_{td} \, V_{ts} \cot 2\theta_{23} - \frac{1}{s_{13}} \sin(\alpha_\tau - \phi_{td}) |V_{td}| X_{\tau 3}. \tag{32}$$

Thus, in the case of factorization with only $\alpha_\tau \neq 0$, the final value of the CP phase is still small, being suppressed by λ^2. The reason is that α_τ enters the expression for $\sin \delta_{CP}$ with small factor V_{td}. The Majorana phases (which appear as by-product of the standard parametrization conditions) equal (see Appendix B)

$$\sin \beta_1 \approx \sin \beta_2 \approx - \text{sign}\{A\} 2 \sin \phi_{td} V_{ts} \cot 2\theta_{23}.$$

For $\alpha_\tau = \phi_{td}$ all three CP phases are equal.

The matrix (31) does not satisfy the exact TBM conditions:

$$(M_X)_{12} = -(M_X)_{13}, \qquad (M_X)_{13} = (M_X)_{33},$$
$$(M_X)_{22} - (M_X)_{23} = (M_X)_{11} + (M_X)_{12},$$

which for (31) take the form

$$b = -a|F|, \qquad a|F|^2 = 1, \qquad 1 - b|F| \approx b\lambda.$$

Indeed, from the first and second equalities we have $b|F| = -1$ and from the last one: $b|F| \approx 1 - b\lambda \approx 1$. The deviation of M_X^0 from the TBM form leads, in particular, to a non-zero 1–3 mixing:

$$X_{e3} \approx \tan \theta_{13}^X \sim \frac{\lambda}{\sqrt{2}} \frac{1}{|F|}$$

which can be in agreement with data.

5.3. Seesaw enhancement of CP violation

In general in the absence of factorization the mass matrix M_X will generate a non-zero α_e, α_μ, and δ_X, and consequently a large δ_{CP}. Expressions for phases of U_X in the three generation case are very complicated and difficult to analyze. Therefore to show effect of enhancement of the CP phases we will consider the two leptonic generations. In the case of a hierarchical neutrino mass spectrum the 2–3 block of elements in the mass matrix is dominant, with elements of the first row and column being suppressed by $m_2/m_3 \sim \lambda$ as in (30). Therefore we consider the second and third neutrinos. Results obtained in this approximation are expected to receive corrections of the order λ when mixing with the first neutrino is turned on.

The matrix M_X can be written as

$$M_X = D(\Phi_H) M_X^0 D(\Phi_H),$$

where $D(\Phi_H) = \mathrm{diag}(1, e^{i\Phi_H/2})$ and

$$M_X^0 = m_0 \begin{pmatrix} 1 & Ge^{i\psi} \\ \dots & H \end{pmatrix}. \tag{33}$$

Here $Ge^{i\Phi_G} \equiv b\xi + g$, $He^{i\Phi_H} \equiv a(\xi^2 + 2f\xi + h)$, and $\psi \equiv \Phi_G - \Phi_H/2$. It is easy to show that selecting parameters a, b, g, f, h one can get any value of ψ from zero to $\mathcal{O}(1)$.

We will diagonalize M_X^0 (33) with $U_X^0 = D(y^0) R_X(\theta) D(z)$, where $R_X(\theta)$ is a 2×2 rotation matrix, $D(y) = \mathrm{diag}(e^{iy_\mu}, e^{iy_\tau^0})$, and $D(z) = \mathrm{diag}(e^{i\beta_2}, 1)$ are the phase matrices. Then $U_X = D(\Phi_H) U_X^0$. The diagonalization condition $U_X^{0\dagger} M_X^0 U_X^{0*} = m_\nu^{diag}$, can be written as

$$R_X^T(\theta) D(\Delta) M_X^0 D(\Delta) R_X(\theta) = e^{2iy_\mu} D(z) m_\nu^{diag} D(z), \tag{34}$$

where $D(\Delta) \equiv \mathrm{diag}(1, e^{i\Delta})$ and $\Delta \equiv y_\mu - y_\tau^0$. From (34) we obtain the relations which determine the phases y_μ, y_τ^0, β_2:

$$\frac{1}{2} \sin 2\theta (1 - He^{i2\Delta}) + \cos 2\theta\, Ge^{i(\psi+\Delta)} = 0,$$

$$c^2 + s^2 He^{i2\Delta} - \sin 2\theta\, Ge^{i(\psi+\Delta)} = \frac{m_2}{m_0} e^{i2(y_\mu+\beta_2)},$$

$$s^2 + c^2 He^{i2\Delta} + \sin 2\theta\, Ge^{i(\psi+\Delta)} = \frac{m_3}{m_0} e^{i2y_\mu}. \tag{35}$$

The solution is very simple in the case of maximal mixing: $\cos 2\theta = 0$, when the first equation in (35) is satisfied for $H = 1$ and $\Delta = 0$, so that $y_\mu = y_\tau^0$. The two other equations give

$$1 - Ge^{i\psi} = \frac{m_2}{m_0} e^{i2(y_\mu+\beta_2)}, \qquad 1 + Ge^{i\psi} = \frac{m_3}{m_0} e^{i2y_\mu}.$$

From these equations we obtain

$$\sin 2y_\mu = G \frac{m_3}{m_0} \sin \psi = \frac{G \sin \psi}{\sqrt{1 + 2G \cos \psi + G^2}}, \tag{36}$$

and G determines the mass hierarchy:

$$\frac{m_2}{m_3} = \sqrt{\frac{1 - 2G \cos \psi + G^2}{1 + 2G \cos \psi + G^2}}.$$

The equality (36) implies that $\sin 2y_\mu$ is of the order $\sin \psi$. And since ψ can be $\mathcal{O}(1)$, can have a large $\alpha_\mu = y_\mu$, and consequently, a large δ_{CP}. Furthermore, by selecting G the correct mass hierarchy can be obtained.

In the case of deviation of 2–3 mixing from maximal, $H \neq 1$, one obtains in general corrections to the obtained results of the order $(H - 1)$. In special case $\cos \psi \approx 0$ the corrections can be enhanced.

5.4. CP phases with other assumptions on M_R

Similar results can be obtained with other ansatzes for M_R^{-1}.

(1) Consider

$$M_R^{-1} = \left(m_D^{diag}\right)^{-1} \tilde{M}_{TBM} \left(m_D^{diag}\right)^{-1},$$

with $\tilde{M}_{TBM}$ given in (29). It differs from the ansatz in Section 5.3 by the absence of the rotation V_{CKM}^0. Taking $m_D = \mathrm{diag}(\lambda^4, \lambda^2, 1)$, which the only possibility which can lead to nearly TBM mass matrix for M_X, we obtain

$$M_X \propto \begin{pmatrix} a\lambda^p & -a\lambda^p + b\lambda & a\lambda^{p-1}\xi + \lambda(f-b) \\ \dots & a\lambda^{p-2} - 2b + 1 & -a\lambda^{p-2} + b\xi - (f-b) + g - 1 \\ \dots & \dots & a\lambda^{p-2}\xi^2 + 2\xi(f-b) - 2g + h + 1 \end{pmatrix}.$$

In contrast to the previous case, now it is possible to have $p = 1$, leading to dominance of terms with a. That is, the whole matrix at the lowest order is generated by the 1–1 element of $\tilde{M}_{TBM}$:

$$M_X \propto \frac{a}{\lambda} \begin{pmatrix} \lambda^2 & -\lambda & \lambda\xi \\ \dots & 1 & -\xi \\ \dots & \dots & \xi^2 \end{pmatrix} + \begin{pmatrix} 0 & -ab\lambda & \lambda(f-b) \\ \dots & -2b+1 & b\xi - (f-b) + g - 1 \\ \dots & \dots & 2\xi(f-b) - 2g + h + 1 \end{pmatrix}.$$

At the lowest order (the first term) phase factorization occurs automatically and the matrix M_X is close to TBM, having only one nonzero mass eigenvalue. The factorization phase equals $\alpha_\tau = \mathrm{Arg}\,\xi = \phi_{td}$, and according to (32)

$$\sin\delta_{CP} = -\,\mathrm{sign}\{A\}2\sin\phi_{td}\,V_{ts}\cot 2\theta_{23}.$$

Corrections of the order λ then generate lighter masses giving naturally $m_2/m_3 = O(\lambda)$ as well as modify CP phases. Selecting g and h one can achieve phase factorization of the whole matrix. In this case the elements of the third column become

$$(M_X)_{e\tau} = a\xi', \qquad (M_X)_{\mu\tau} = -\left(\frac{a}{\lambda} - b\right)\xi', \qquad (M_X)_{\tau\tau} = \frac{a}{\lambda}\xi',$$

with

$$\xi' = \xi + \lambda\frac{f-b}{a}.$$

The latter gives $\alpha_\tau = \phi_{td} + O(\lambda)$.

(2) Instead of $U_R = V_{CKM}^\dagger$ we could use a more general expression $U_R = D^*(\kappa)V_{CKM}^\dagger D(\kappa)$, where $D(\kappa) = \mathrm{diag}(e^{i\kappa_1}, e^{i\kappa_2}, e^{i\kappa_3})$. We can fix κ_i in such a way that the 3–1 element in the matrix $V_{CKM}D(\kappa)V_{CKM}^{0T}$, which led to the seesaw enhancement, is zero. For $\kappa_1 = \kappa_2 = 0$ and $\kappa_3 = \delta$ we obtain

$$V_{CKM}D(\kappa)V_{CKM}^{0T} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -\lambda^2\xi \\ 0 & -\lambda^2\xi & e^{i\delta} \end{pmatrix}.$$

Through this rephasing we moved the CP phase from the 1–3 to the 2–3 element. For the correction matrix we find

$$K = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -\lambda^{n+2}\xi \\ 0 & -\lambda^{-n+2}\xi & e^{i\delta} \end{pmatrix}.$$

Notice that now the second eigenvalue of m_D matters. Finally, with $\tilde{M}_{TBM}$ from (29) we obtain

$$M_X = K\tilde{M}_{TBM}K^{-1} \propto \begin{pmatrix} a\lambda^p & b\lambda & -b\xi\lambda^{-n+3} + f\lambda e^{i\delta} \\ ... & 1 & -\xi\lambda^{-n+2} + ge^{i\delta} \\ ... & ... & \xi^2\lambda^{-2n+4} - 2ge^{i\delta}\xi\lambda^{-n+2} + he^{i2\delta} \end{pmatrix}.$$

$M_X \sim M_{TBM}$ can be obtained for $n = 2$. In this case

$$M_X \propto \begin{pmatrix} a\lambda^p & b\lambda & -b\lambda\xi + f\lambda e^{i\delta} \\ ... & 1 & -\xi + ge^{i\delta} \\ ... & ... & \xi^2 - 2ge^{i\delta}\xi + he^{2i\delta} \end{pmatrix}.$$

The factorization is absent, in general, but it can be achieved by imposing relations $g^2 = h$, $f/b = g$. As a result,

$$M_X \propto \begin{pmatrix} a\lambda^p & b\lambda & -b\lambda\xi'' \\ ... & 1 & -\xi'' \\ ... & ... & (\xi'')^2 \end{pmatrix},$$

where $\xi'' \equiv \xi - ge^{i\delta}$. If $g = -1$, we have $\xi'' = 1$. In this case the contribution to the CP phase from the RH sector disappears and we revert to the situation described in Section 3 with CKM origin of CP violation.

Three main results emerge from this analysis of CP violation under the assumptions that $U_L \approx U_R \sim V_{CKM}^\dagger$ and there is no CP violation in M_R in the L–R symmetric basis:

1. The hierarchy of Dirac masses of neutrinos cannot be too strong, i.e., $m_{1D}/m_{3D} \leq \lambda^4$ and $m_{2D}/m_{3D} \leq \lambda^2$. The observed mixing angles of U_{PMNS} impose this requirement. This is significantly weaker than the mass hierarchy of up quarks.
2. The CP phases can in general be large, even if the only sources of CP violation are the Dirac phases in U_L^{std} and U_R^{std}, where the CP phase effect is suppressed by λ^3. This enhancement originates from seesaw and the hierarchy of Dirac masses of neutrinos.
3. If parameters of M_R satisfy certain relations – the phase factorization conditions (which could be a consequence of some symmetry), the phases can factor out from M_X. Furthermore, the only non-vanishing phase which enters the phase factors is α_τ. This is related to certain pattern of CP violation in CKM matrix. In this case no enhancement occurs and δ_{CP} turns out of the order λ^2.

M_X deviates from M_{TBM} since the correction in K is relatively large: being of the order λ^3, which is still larger than the hierarchy of masses in m_D.

5.5. Enhancement of a small phase in U_R

In the previous examples large δ_{CP} has been obtained at the cost of deviation of U_X from U_{TBM}. With decrease of δ_q, correction to the matrix M_X due to CP violation (given by K) decreases and $M_X \to \tilde{M}_{TBM}$. So, M_X can coincide with M_{TBM} up to small corrections. (This

however implies that we depart from L–R symmetry or quark–lepton similarity, assuming smaller values of δ_q.)

Suppose $\delta_q = \epsilon \lambda^2$, where $|\epsilon| \leq 1$. In this case $\xi \approx -i\delta_q = -i\epsilon\lambda^2$ and

$$K = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -i\epsilon\lambda & 0 & 1 \end{pmatrix}.$$

Here the correction is suppressed by λ^2 in comparison with that in (28). Let us take for definiteness the parameters of $\tilde{M}_{TBM}$ to be $a = b = f = -g = h = 1$ which ensures the exact TBM mixing in the lowest order with vanishing lowest neutrino mass. Then

$$M_X = m_0 \begin{pmatrix} \lambda & \lambda & \lambda - i\epsilon\lambda^2 \\ ... & 1 + \lambda & -1 + \lambda - i\epsilon\lambda^2 \\ ... & ... & 1 + \lambda - 2i\epsilon\lambda^2 \end{pmatrix},$$

where $\lambda \approx \sqrt{\Delta m_{21}^2 / \Delta m_{31}^2}$ leads to the correct neutrino masses. The additional imaginary terms give corrections to the TBM values of the 1–2 and 2–3 mixing angles proportional to $\epsilon\lambda^2$. They also generate small 1–3 mixing: $X_{e3} \approx \epsilon\lambda^2$ and

$$\delta_X \approx \frac{\pi}{2} + \mathcal{O}(\epsilon\lambda^2).$$

All the other induced phases are close to 0 or to π, i.e., $D(y) = \text{diag}(1, -1, 1)$ and $D(z) = \text{diag}(-1, -1, 0)$, with corrections as $\epsilon\lambda^2$. According to (17) this contributes to the PMNS phase as

$$\sin\delta_{CP} \approx -\frac{X_{e3}}{s_{13}} \approx -\frac{\epsilon\lambda^2}{s_{13}} \sim \epsilon\lambda.$$

So, seesaw can convert a tiny CP phase $\delta_q \equiv \epsilon\lambda^2$ in U_R to a maximal CP phase $\delta_X \approx \pi/2$ in U_X. This happens because of the large hierarchy of Dirac masses and seesaw.

6. Remarks on phenomenology

Our results have the following phenomenological consequences:

1. For the scenarios with CKM type CP violation and in the L–R symmetric case with phase factorization the value of $\sin\delta_{CP}$ is expected to be small, and the phase is close to π or zero. This agrees with the result of a global fit in [24]:

$$\delta_{CP} = \left(1.39^{+0.33}_{-0.27}\right)\pi \quad \text{(NH)}, \qquad \left(1.35^{+0.24}_{-0.39}\right)\pi \quad \text{(IH)},$$

   although statistical significance of this indication is low. At a 2σ-level, δ_{CP} is also consistent with zero because of a second local minimum at that value (in both hierarchies). The value $\pi/2$, however, is disfavored in both cases.
   Observation of $\delta_{CP} \sim \pi$ would be some indication of the CKM scenario or L–R scenario with phase factorization.

2. Observation of a large value, $\delta_{CP} \gg \lambda^2$, in experiments will rule out these scenarios and imply that either there are other sources of CP violation besides the CKM-like phase in U_L or that the considered framework (canonical seesaw) is invalid, e.g., Dirac mass matrices are non-hierarchical, or seesaw type I is not the mechanism for generating neutrino masses.

3. In our notation, the effective Majorana mass of the electron neutrino is

$$m_{ee} = \left| \sum_i m_i e^{2i\beta_i} U_{ei}^2 \right|,$$

which, for inverted mass ordering in the limit of hierarchical masses, is mainly sensitive to $\beta_1 - \beta_2$. Since $\beta_1 - \beta_2 = \mathcal{O}(\lambda^3)$, no cancellation of contributions to m_{ee} from the first two mass eigenstates is expected and m_{ee} is expected to be relatively large. For normal ordering m_{ee} depends mainly on the combination $\delta_{CP} + \beta_2$. Measuring the Majorana phases (or their differences) will be challenging for scenarios described above.

4. Future precise measurements of the phases may allow to disentangle the possibilities: CP in the left rotations only and L–R symmetric case. In the former, one expects $\sin\delta_{CP} \gg \beta_{1,2}$, whereas the latter predicts all three phases to be equal in the specific case of factorization.

5. If the baryon asymmetry of the Universe is generated via leptogenesis (decays of the RH neutrinos in our case), this imposes certain restrictions on structure of the RH sector of seesaw; see, e.g., [25] and [26] for recent reviews. In particular, successful leptogenesis gives the bounds on mass of the lightest RH neutrino (in most of the cases we require a strongly hierarchical spectrum) and on combinations

$$\frac{1}{[U_M^T U_R (m_D^{diag})^2 U_R^\dagger U_M^*]_{ii}}$$
$$\times \operatorname{Im}\left\{ \left[U_M^T U_R \left(m_D^{diag} \right)^2 U_R^\dagger U_M^* \right]_{ij} \left[U_L m_D^{diag} U_R^\dagger U_M^* \right]_{\alpha i} \left[U_L m_D^{diag} U_R^\dagger U_M^* \right]_{\alpha j}^* \right\},$$

where $\alpha = e, \mu, \tau$ is the flavor index and i, j are indices of the RH neutrino mass eigenstates. The combinations determine the lepton asymmetries in the lepton channel α. In the case of unflavored leptogenesis a summation over α proceeds, and the dependence on U_L disappears. So, leptogenesis would require complex phases in U_R and/or U_M. This is not necessary in the flavored case [25].

7. Conclusions

We have studied the Dirac and Majorana CP violating phases in context of the seesaw type I mechanism with similar Dirac mass matrices for quarks and leptons. In this case a relationship $U_{PMNS} = U_L U_X$ is realized with $U_L \sim V_{CKM}^\dagger$. We formulated the standard parametrization conditions for the mixing matrix to obtain simultaneously both the Dirac and Majorana CP phases. Possible connections of the Dirac CP violating phases in the quark and lepton sectors have been explored.

The main results that we obtained are:

1. If the Dirac CP phase in U_L is the only source of CP violation (which is similar to what happens in quark sector with Kobayashi–Maskawa mechanism), and there is no CP violation in the RH sector, the leptonic CP violation is very small $\sin\delta_{CP} = \mathcal{O}(\lambda^2)$. The phase itself is either close to zero or to π with the deviation of the order of λ^2. The Majorana phases are expected to be even smaller: $\beta_1 \approx \beta_2 = \mathcal{O}(\lambda^3)$.

2. If the Dirac mass matrices are symmetric so that $U_L = U_R \sim V_{CKM}(\delta_q)$ and the Majorana mass matrix of the RH fields is real in the L–R symmetric basis, δ_{CP} is in general enhanced by the seesaw mechanism. Furthermore, the Dirac masses of the neutrinos are constrained to be not strongly hierarchical. to reproduce the correct mixing.

3. The seesaw enhancement of phase is absent if M_R has a specific form that leads to the phase factorization in M_X. In this case, $\beta_1 \approx \beta_2 = \mathcal{O}(\lambda^2)$ $\alpha_\tau = \mathcal{O}(\lambda^2)$ and $\sin \delta_{CP} = \mathcal{O}(\lambda^2)$. In particular case $\alpha_\tau = \phi_{td}$ three phases are equal and small $\beta_1 \approx \beta_2 \approx \sin \delta_{CP} = \mathcal{O}(\lambda^2)$. Thus, the presence of the CP violation in the RH sector in the factorization case enhances the Majorana phases, but keeps the Dirac phase at the same order for this scenario.

4. Generic CP violation in the RH sector can lead to arbitrary and independent values of all three phases for arbitrary hierarchy of the eigenvalues of m_D. We identify that the observable CP phase depends mainly on α_e, α_μ, and δ_X, if it is measured to be large.

The formalism developed here allows to explore implications of measurements of the CP phases for the RH sector. For example, if a large CP phase is observed, the observable CP phases will mainly depend on three unknown phases in the RH sector: α_e, α_μ, and δ_X. Thus, determination of δ_{CP} and the Majorana phase may provide information on these parameters.

We may also get some direct hints about the flavor symmetry and quark–lepton unification, if special values of the CP violating phases are observed or if certain correlations between them are seen. Coming back to the initial question about the quark and leptonic CP phases, even in the context in which quarks and leptons are maximally related (quark–lepton symmetry, seesaw type I) one cannot expect equality of the quark and lepton Dirac phases. The phases are related but, generically, strongly different. The difference can be related to different mixing angles (especially 1–3 mixing angle) and to seesaw mechanism itself.

Some results of this paper can be modified by the RGE effects. Since the light neutrino spectra we have considered are hierarchical, the renormalization correction are small and they will not affect our conclusions. The threshold effects due to possible large hierarchy of masses of the RH neutrinos are important when implications for M_R are considered but this is beyond the scope of this paper.

Acknowledgement

Appendix A. Solution of the standard parametrization conditions

In this appendix we provide details of computations of the CP phases using the standard parametrization conditions.

A.1. CP violation from CKM only

Using explicit expressions for U_{e1} and U_{e2} in (15), we obtain from the conditions $\mathrm{Arg}\{U_{e1}\} = \mathrm{Arg}\{U_{e2}\} = 0$ that

$$\beta_1 = \gamma_e + \xi_{e1}, \qquad \beta_2 = \gamma_e + \xi_{e2}, \tag{37}$$

where ξ_{ei} are given by

$$\xi_{e1} = -\frac{|V_{td}| X_{\tau 1}}{V_{ud} X_{e1}} \sin \phi_{td} + \mathcal{O}(\lambda^4),$$

$$\xi_{e2} = -\frac{|V_{td}| X_{\tau 2}}{V_{ud} X_{e2}} \sin \phi_{td} + \mathcal{O}(\lambda^4). \tag{38}$$

We see that $\xi_{ei} = \mathcal{O}(\lambda^3)$, which means $\sin(\eta_i + \gamma_e)$ is of the order λ^3. The reason behind this is that the CP violation originates from the Kobayashi–Maskawa phase associated with the element suppressed by λ^3, while one of real terms in (15) is always of the order 1. Similarly, using (15), with $\alpha_l = \delta_X = 0$, and the conditions $\text{Arg}\{U_{\mu 3}\} = \text{Arg}\{U_{\tau 3}\} = 0$, we find

$$\gamma_\mu = \mathcal{O}(\lambda^4)$$
$$\gamma_\tau = \frac{|V_{ub}|X_{e3}}{V_{tb}X_{\tau 3}} \sin\phi_{ub} + \mathcal{O}(\lambda^4).$$

As we will show, $X_{e3} \leq \mathcal{O}(\lambda)$, so that $\sin\gamma_\tau$ is also at most order λ^4.

Neglecting phases γ_μ and γ_τ, in the lowest order the 5th condition (13) becomes

$$X_{e1}X_{\mu 2}\sin\beta_1 = X_{e2}X_{\mu 1}\sin\beta_2.$$

Then it follows using (37), (38) that

$$\gamma_e = \frac{r\xi_{e1} - \xi_{e2}}{1-r}, \qquad r \equiv \frac{X_{e2}X_{\mu 1}}{X_{e1}X_{\mu 2}}, \tag{39}$$

and explicitly

$$\gamma_e = \frac{|V_{td}|(X_{e1}^2 X_{\mu 2}X_{\tau 2} - X_{e2}^2 X_{\mu 1}X_{\tau 1})\sin\phi_{td}}{V_{ud}X_{e1}X_{e2}X_{\tau 3}} + \mathcal{O}(\lambda^4) \tag{40}$$

which shows that $\sin\gamma_e = \mathcal{O}(\lambda^3)$. For Majorana phases we have

$$\beta_1 = \frac{\xi_{e1} - \xi_{e2}}{1-r}, \qquad \beta_2 = r\beta_1. \tag{41}$$

A.2. Left–right symmetry with factorization

Let us consider δ_{CP} in the presence of $\alpha_\tau \neq 0$. From (16) we have $s_{13} = |U_{e3}| \approx |A'|$, where

$$A' \equiv V_{ud}X_{e3} - |V_{cd}|X_{\mu 3} + |V_{td}|X_{\tau 3} \approx A. \tag{42}$$

We can then rewrite Eq. (17) neglecting ϕ_{cd} as

$$\sin\delta_{CP} = -\text{sign}\{A\}\sin\gamma_e - \frac{1}{s_{13}}\sin(\alpha_\tau - \phi_{td})V_{td}X_{\tau 3}. \tag{43}$$

Nonzero α_τ modifies the phases in (38),

$$\xi_{e1} = \frac{|V_{td}|X_{\tau 1}}{V_{ud}X_{e1}}\sin(\alpha_\tau - \phi_{td}), \qquad \xi_{e2} = \frac{|V_{td}|X_{\tau 2}}{V_{ud}X_{e2}}\sin(\alpha_\tau - \phi_{td}).$$

So, with high accuracy $\beta_1 = \beta_2 \equiv \beta$, and consequently, $\gamma_e - \beta = O(\lambda^3)$.

From the conditions $\text{Im}\,U_{\mu 3} = 0$ we obtain

$$\gamma_\mu + \xi_{\mu 3} = 0, \qquad \xi_{\mu 3} = -\frac{V_{ts}X_{\tau 3}}{|V_{cs}|X_{\mu 3}}\sin\alpha_\tau, \tag{44}$$

so that $\xi_{\mu 3} = \mathcal{O}(\lambda^2)$. The equality $\text{Im}\,U_{\tau 3} = 0$ gives $\gamma_\tau + \alpha_\tau = |V_{cb}|X_{\mu 3}/(|V_{tb}|X_{\tau 3})\sin\alpha_\tau$. The 5th SP condition (13), gives at the leading order

$$\left[|V_{cs}|X_{\mu 1}\sin(\gamma_\mu - \beta) + |V_{ts}|X_{\tau 1}\sin\alpha_\tau\right]X_{e2}$$
$$= \left[|V_{cs}|X_{\mu 2}\sin(\gamma_\mu - \beta) + |V_{ts}|X_{\tau 2}\sin\alpha_\tau\right]X_{e1},$$

which leads to

$$\sin(\gamma_\mu - \beta) = \sin\alpha_\tau \frac{|V_{ts}|X_{\mu 3}}{|V_{cs}|X_{\tau 3}}.$$

Using expression for γ_μ from (44) we obtain

$$\sin\beta = \sin\gamma_e = \sin\alpha_\tau \frac{|V_{ts}|}{|V_{cs}|} \frac{X_{\tau 3}^2 - X_{\mu 3}^2}{X_{\tau 3}X_{\mu 3}} = \sin\alpha_\tau \frac{|V_{ts}|}{|V_{cs}|} \frac{2\cos 2\theta_{23}}{\sin 2\theta_{23}}. \tag{45}$$

Thus, $\beta = \mathcal{O}(\lambda^2)$, and consequently, $\gamma_e = \mathcal{O}(\lambda^2)$ or smaller. Inserting $\sin\gamma_e$ from (45) into (43) we obtain

$$\sin\delta_{CP} = -\operatorname{sign}\{A\}\sin\phi_{td} \frac{|V_{ts}|}{|V_{cs}|} \frac{2\cos 2\theta_{23}}{\sin 2\theta_{23}} - \frac{1}{s_{13}}\sin(\alpha_\tau - \phi_{td})V_{td}X_{\tau 3}. \tag{46}$$

According to (46) effect of non-zero α_τ, i.e. from the RH sector, is of the same order as the result for the CKM phase only. If $X_{\mu 3} = X_{\tau 3}$, that is the 2–3 mixing in U_X is maximal $\beta = 0$, but

$$\delta_{CP} = -\frac{|V_{td}|}{\sqrt{2}s_{13}}\sin(\alpha_\tau - \phi_{td})V_{td}.$$

Appendix B. Expressions for elements of the U_X matrix

For a real U_X, using the relations (15) we obtain at the lowest order

$$X_{e1} = c_{12}/V_{ud} + \mathcal{O}(\lambda),$$
$$X_{e2} = s_{12}/V_{ud} + \mathcal{O}(\lambda),$$
$$X_{\mu 3} = s_{23}/|V_{cs}| + \mathcal{O}(\lambda^2),$$
$$X_{\tau 3} = c_{23}/V_{tb} + \mathcal{O}(\lambda^2). \tag{47}$$

Using smallness of X_{e3} the elements $X_{\mu 1}$, $X_{\mu 2}$, $X_{\tau 1}$, and $X_{\tau 2}$ are expressed in terms of the above 4 elements and X_{e3} as

$$X_{\mu 1} = -X_{e2}X_{\tau 3} - X_{e1}X_{\mu 3}X_{e3} + \mathcal{O}(\lambda^2),$$
$$X_{\mu 2} = X_{e1}X_{\tau 3} - X_{e2}X_{\mu 3}X_{e3} + \mathcal{O}(\lambda^2),$$
$$X_{\tau 1} = X_{e2}X_{\mu 3} - X_{e1}X_{\tau 3}X_{e3} + \mathcal{O}(\lambda^2),$$
$$X_{\tau 2} = -X_{e1}X_{\mu 3} - X_{e2}X_{\tau 3}X_{e3} + \mathcal{O}(\lambda^2). \tag{48}$$

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