Cross section to multiplicity ratios at very high energy

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\section{Introduction}
Some time ago, one of us suggested [1]\textsuperscript{1} that, at very high energy in elementary particle collisions, the cross section and the multiplicity should vary in the same way with energy. This idea arises in a picture, inspired by calculations in (massive photon) QED [2], where the incoming particle has an expanding radius induced by a series of \( N \) independent emissions of secondary particles. This leads to a random walk in the transverse dimension \( R \), so that one has \( R \sim \sqrt{N} \), or using \( \sigma \sim R^2 \).

\[
\sigma \propto N. \tag{1}
\]

The square root of the constant of the proportionality would have the interpretation of the step length in the random walk. While in Ref. [2] \( N \) is the number of (massive) photons emitted, in hadron reactions we take it as proportional to the multiplicity, which in the following we also call \( N \) (see Section 5).

Although in contemporary language one would not speak of QED for hadronic interactions, the situation with QCD is not so much different, and the idea is simple and general, so one may wonder if the relation does not indeed hold.

The picture is motivated [1] by the fact that coupling a particle to a light boson field will inescapably lead to a narrowing of its purely elastic scattering peak. This however implies scattering at high impact parameter, showing that the boson field delocalizes the particle. Assuming this delocalization takes place in a series of independent emissions, there is a diffusion in the transverse dimension, leading to \( R \sim \sqrt{N} \). This occurs in the model of Ref. [2] as the mass of the boson field is reduced, and we speculated there is an analogous behavior in very high energy hadron physics.\textsuperscript{2}

\section{LHC information}
That Eq. (1) may be true is suggested by two aspects of recently available information from the LHC.

Firstly, good fits [4] to both the total and inelastic pp cross section are possible with asymptotic \( \ln^2 \sqrt{s} \) behavior, where \( \sqrt{s} \) is the total cms (center of mass system) energy.

Secondly, data available for \( dN/dy \), the charged particle multiplicity density in the rapidity \( y \) [5], show a rise with energy, and this rise resembles a \( \ln \sqrt{s} \) behavior (Fig. 1, below). Such multiplicity information at the LHC is only available for central rapidities, \( y \approx 0 \), at present. But, one may estimate the total multiplicity \( N \) by multiplying the central \( dN/dy \) by the total rapidity interval \( Y \) expected for the secondaries (mostly pions):

\[
N \approx \left. \frac{dN}{dy} \right|_{y=0} \times Y \tag{2}
\]

\( Y \) also grows as \( \ln \sqrt{s} \), therefore suggesting \( N \sim \ln^2 \sqrt{s} \). With both the cross section and \( N \) showing the same \( \ln^2 \sqrt{s} \), asymptotic growth, Eq. (1) would hold.

\textsuperscript{1} In reading this reference it should be kept in mind that at the time a \( \ln \) and not a \( \ln^2 \) behavior of the data was thought to obtain.

\textsuperscript{2} In his 1973 Winter Petersburg School lectures, Gribov [3] also discussed a diffusion in the transverse spatial dimensions. However, this was in the context of a parton equilibrium model, which led to a constant total cross section. Thus he did not suggest or discuss our Eq. (1).
2.1. N behavior

To substantiate these statements concerning the total multiplicity we show in Fig. 1 a logarithmic fit to central \(dN/dy\) data selected from Fig. 20 (top) of Ref. [5]. We use the data for charged pions (one charge). There is considerable scatter in the points at lower energies and in the interest of dealing with a smooth curve, we simply use the point at 200 GeV, that appears to connect smoothly to the higher energy CMS points. A fit of form \(A \ln(\sqrt{s}/B)\) yields \(A = 0.60\) and \(B = 48\) GeV. Thus we will use

\[
\frac{dN}{dy} \bigg|_{y=0} \approx 0.60 \times (\ln(\sqrt{s}/\text{GeV}) - 3.9) \tag{3}
\]

in our estimates. Clearly this selective choice of data is not useful for estimating errors or uncertainties. This result is for one charge of the pions, for "all charged" one must multiply approximately by 2, and for total multiplicity including \(\pi^0\)'s by 3. Kaons and heavier particles are at the 10% level or less.

The various logarithmic expressions we have to deal with are of the form \(\ln(\sqrt{s}/\mu)\) where \(\mu\) is some mass scale. Since \(\ln(\sqrt{s}/\mu) = \ln(\sqrt{s}/\mu) - \ln(\mu/\text{GeV})\), we can also write the expressions as \(\ln(\sqrt{s}/\text{GeV}) + \text{constant}\), as in Eq. (3). This remark also implies that in determining the asymptotic behavior of the expressions, only the coefficient of the logarithmic terms and not the scale \(\mu\) enters.

3. Asymptotic ratios

3.1. Asymptotic cross sections

According to the \(c_2\) coefficients of Ref. [4] the asymptotic behavior of the pp inelastic cross section is \(4c_2^{\text{inel}} \times \ln^2(\sqrt{s}/\text{GeV}) = 0.56 \times \ln^2(\sqrt{s}/\text{GeV})\), while as expected for a "black disc" one has \(4c_2 \times \ln^2(\sqrt{s}/\text{GeV}) = 1.1 \times \ln^2(\sqrt{s}/\text{GeV})\) for the total cross section.

3.2. Asymptotic multiplicity

In the fit of Section 2.1 we arrived at \(A = 0.6\) as the coefficient of the logarithm for \(dN/dy\) for a single charge of the pion. For \(Y\) we anticipate \(Y = 2 \times \ln(\sqrt{s}/\mu)\). The factor 2 is chosen so that

\[
N \approx 3.6 \times \ln^2(\sqrt{s}/\text{GeV}), \tag{4}
\]

for the asymptotic total multiplicity.

3.3. Asymptotic ratios

Combining Sections 3.1 and 3.2 we have

\[
\frac{\sigma^{\text{inel}}}{N} \approx 0.16 \text{ mb}, \quad \frac{\sigma^{\text{tot}}}{N} \approx 0.31 \text{ mb}, \tag{5}
\]

for the asymptotic ratios. The black disk limit obtains, where \(\sigma^{\text{elastic}}/N = \sigma^{\text{inel}}/N\) and \(\sigma^{\text{tot}}/N = 2\sigma^{\text{inel}}/N\).

The units are naturally mb, or one might like to say, mb per pion.

4. Approach to the limit

It is interesting to see how the limits Eq. (5) are approached and how close the limits are at present energies. Since even at LHC energies the black disk limit is remote and \(\sigma^{\text{inel}}\) and \(\sigma^{\text{tot}}\) do not yet have the same energy dependence (see Fig. 3 of Ref. [4]) the behavior of \(\sigma^{\text{inel}}/N\) and \(\sigma^{\text{tot}}/N\) and their difference \(\sigma^{\text{el}}/N\) will be somewhat different.

To examine the approach to the limit, we need not only the coefficient of the \(\ln^2\) term but the actual total multiplicity at present LHC energies. The PDG tables [6] give the total charged multiplicity up to \(\sqrt{s}\) of 900 GeV, using UA5 data. For the higher energies, we can make an estimate using Eq. (2). For \(dN/dy\) we have the fit Eq. (3). For \(Y\) it is now necessary to have the non-leading term, the constant \(C\) in \(Y = 2(\ln(\sqrt{s}/\text{GeV}) + C)\). By requiring that Eq. (2), with Eq. (3) multiplied by 2, fits to the UA5 points at 200, 546, and 900 GeV [8] with \(N = 21, 28\) and 36 for the charged multiplicity we obtain \(C = -1.3\). Fig. 2 shows this fit with the UA5 points.

The negative value for \(C\) indicates that the effective \(Y\) is somewhat less than that between the incoming protons; this may be a reflection of the fact that \(dN/dy\) falls off at large \(y\) so that using its central value throughout, as we do, tends to give an overestimate that must be compensated by a smaller \(Y\) [7].
We shall thus use for the charged pion multiplicity above 1000 GeV
\[ N \approx 3 \times (0.60 \times (\ln(\sqrt{s}/\text{GeV}) - 3.9)) \times (2(\ln(\sqrt{s}/\text{GeV}) - 1.3)) \]
\[ = 3.6 \times \ln^2(\sqrt{s}/\text{GeV}) - 19 \times \ln(\sqrt{s}/\text{GeV}) + 18 \]  
(6)

With this estimate for \( N \) and the fits for the \( \sigma \) from Ref. [4] we obtain the \( \sigma/N \) values shown in Fig. 3.

It is appears that, at the present upper LHC energy of about 7 TeV, the \( \sigma_{\text{tot}}/N \) and \( \sigma_{\text{inel}}/N \) are well above the asymptotic values of Eq. (5) and decreasing. The ratios are varying slowly, for example at 100 TeV, \( \sigma_{\text{tot}}/N \) will be about 0.33, still far from 0.16. It is interesting, however, that \( \sigma_{\text{tot}}/N \approx 0.2 \) near 7 TeV, is rather closer to the asymptotic value.

5. Discussion

It is intriguing that the \( \sigma_{\text{tot}}/N \) ratio in Fig. 3 is approximately constant and close to the asymptotic value, even though at these energies the \( \ln^2 \) term in Eq. (6) is not completely dominant. Apparently, \( \sigma_{\text{tot}} \) and \( N \) behave approximately in parallel, even well before the asymptotic region. An explanation for this might be sought along the following lines. For the relation \( R = \sqrt{N} \) to be meaningful, it is necessary to have a reasonably well-defined quantity \( R \) for the radius of the proton. While this exists in the black disc limit, at present (LHC) energies the proton has a considerable “grey” or semi-transparent area (viewed in impact parameter \( b \)).

We would like to argue, however, that there is likely a well-defined radius for elastic scattering before there is one for the total cross section. This is because the elastic and total scattering at a given \( b \) are given by the same quantity, \( (1 - \eta(b)) \), squared for elastic scattering and linear for total cross section. (In these arguments we take the scattering amplitude to be purely imaginary, and \( \eta(b) = e^{-\chi(b)} \), with \( \chi \) the imaginary eikonal.) The quantity \( (1 - \eta(b)) \) is the ‘opacity’ at \( b \) and varies between an ‘inner region’ where it is close to 1 and an ‘edge’ where it falls to 0. Since the square of a number less than one is smaller than the number itself, the transition from 1 to 0 will tend to be more abrupt in the squared, elastic case, and a radius \( R \) will be more well defined. This argument is also supported by the presence of the diffraction minimum [9], which would be washed out if there were not a relatively well-defined ‘edge’. Hence it does not seem implausible that \( \sigma_{\text{tot}} \sim N \) works for \( \sigma_{\text{tot}} \) before it does for \( \sigma_{\text{tot}} \).

Assuming that Eq. (1) is true, as our results appear to indicate, it is naturally possible that it can arise from models other than that of Ref. [1]. But in that picture for the expanding proton radius, we can now evaluate the step length \( R_0 \) in the random walk from our numerical results. This would be \( R_0 = \sqrt{\frac{\sigma_{\text{tot}}}{\pi}} \), using \( \sigma = \pi R^2 = \pi R_0^2 N \). From Eq. (5), using either the inelastic or elastic ratio.

\[ R_0 = \sqrt{\frac{0.16 \text{ mb}}{\pi}} = 0.071 \text{ f.} \]  
(7)

In interpreting this result one should bear in mind that it is based on using the cross section per pion. If, as is likely, the pions are not the primary particles emitted, but rather come from some fewer numbers of parent particles, then \( R_0 \) for the primary emission will be correspondingly larger. In QCD, where the primary emission would be of gluons, one would insert a factor for the ‘number of pions per gluon’.

The uncertainties due to our estimate of the multiplicity via Eq. (6) would be reduced if direct measurements of the total multiplicity are obtained from the LHC. We hope these will become available shortly. Similarly, data from the full LHC energy of 14 TeV will help to test and fix our parameterizations.

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References

[6] See Fig. 20 (top) of S. Chatrchyan, et al., CMS Collaboration, Study of the inclusive production of charged pions, kaons, and protons in pp collisions at \( \sqrt{s} = 0.9 \times 2.76 \), and 7 TeV, Eur. Phys. J. C 72 (2012) 2164, arXiv:1207.4724 [hep-ex]. For the numerical values for \( dN/dy \), see hepdata.cedar.ac.uk/view/ins1123177 [hepdata.cedar.ac.uk], last table.