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Moduli Spaces of Reflexive Sheaves of Rank 2

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Abstract. Let $\mathcal{F}$ be a coherent rank 2 sheaf on a scheme $Y \subset \mathbb{P}^n$ of dimension at least two and let $X \subset Y$ be the zero set of a section $\sigma \in H^0(\mathcal{F})$. In this paper, we study the relationship between the functor that deforms the pair $(\mathcal{F}, \sigma)$ and the two functors that deform $\mathcal{F}$ on Y, and X in Y, respectively. By imposing some conditions on two forgetful maps between the functors, we prove that the scheme structure of *e.g.,* the moduli scheme $\mathrm{M}_Y(P)$ of stable sheaves on a threefold Y at $(\mathcal{F})$, and the scheme structure at (X) of the Hilbert scheme of curves on Y become closely related. Using this relationship, we get criteria for the dimension and smoothness of $\mathrm{M}_Y(P)$ at $(\mathcal{F})$, without assuming $\mathrm{Ext}^2(\mathcal{F}, \mathcal{F}) = 0$. For reflexive sheaves on $Y = \mathbb{P}^3$ whose deficiency module $M = H^1_*(\mathcal{F})$ satisfies ${}_0\mathrm{Ext}^2(M, M) = 0$ (*e.g.,* of diameter at most 2), we get necessary and sufficient conditions of unobstructedness that coincide in the diameter one case. The conditions are further equivalent to the vanishing of certain graded Betti numbers of the free graded minimal resolution of $H^0_*(\mathcal{F})$. Moreover, we show that every irreducible component of $\mathrm{M}_{\mathbb{P}^3}(P)$ containing a reflexive sheaf of diameter one is reduced (generically smooth) and we compute its dimension. We also determine a good lower bound for the dimension of any component of $\mathrm{M}_{\mathbb{P}^3}(P)$ that contains a reflexive stable sheaf with "small" deficiency module M.

1 Introduction and Main Results

Let $Y \subset \mathbb{P}^n$ be an equidimensional, locally Cohen-Macaulay (CM), closed subscheme of dimension at least two over a field k and let $\mathcal{F}$ be a coherent rank 2 sheaf on Y. Let $\mathrm{Hilb}_{X/Y}$ be the local Hilbert functor of flat deformations $X_S \subset Y \times S$, S a local artinian k-algebra, of a codimension 2 locally CM subscheme X of Y. An effective method of studying the Hilbert scheme, $\mathrm{Hilb}^p(Y)$, of subschemes of Y with Hilbert polynomial p with respect to smoothness, dimension, and irreducibility at (X), is to look at other local deformation functors $\mathbf{D}$ over $\mathrm{Hilb}_{X/Y}$, $\mathbf{D} \to \mathrm{Hilb}_{X/Y}$, which allow a surjective tangent map $t_{\mathbf{D}} \to t_{\mathrm{Hilb}_{X/Y}} = H^0(\mathcal{N}_{X/Y})$, $\mathcal{N}_{X/Y} = (\mathcal{I}_{X/Y}/\mathcal{I}^2_{X/Y})^*$, and a corresponding injective map of obstruction spaces. We consider such deformation functors $\mathbf{D}$ that determine $\mathrm{Hilb}^p(Y)$ locally under various assumptions. In particular, we look at the functor of deforming a pair $(\mathcal{F}, \sigma)$ as well as at the functor of deforming the pair (X, ξ), where ξ is an extension as in the Serre correspondence

$$\xi; 0 \to \mathcal{O}_Y \xrightarrow{\sigma} \mathcal{F} \to \mathcal{I}_{X/Y} \otimes \mathcal{L} \to 0, \tag{1.1}$$

see [14, 16, 44–46] for the existence of such extensions. Let $\mathrm{Def}_{\mathcal{F}}$ (resp. $\mathrm{Def}_{\mathcal{F},\sigma}$) be the local deformation functor of flat deformations $\mathcal{F}_S$ of $\mathcal{F}$ (resp. $\mathcal{O}_{Y\times S} \xrightarrow{\sigma_S} \mathcal{F}_S$

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of $\mathcal{O}_Y \xrightarrow{\sigma} \mathcal{F}$). Note that we have an obvious forgetful map $p\colon \mathrm{Def}_{\mathcal{F},\sigma} \to \mathrm{Def}_{\mathcal{F}}$. A theorem of Hartshorne on the Serre correspondence of rank 2 reflexive sheaves states that via (1.1) there is a one-to-one correspondence between pairs $(\mathcal{F}, \sigma)$, where the zero set of $\sigma \in H^0(\mathcal{F})$ has codimension 2 in $\mathbb{P}^3$, and pairs (X, ξ), where $\xi \in H^0(\omega_X(4-c_1))$ generates the twisted canonical sheaf $\omega_X(4-c_1)$ except at finitely many points ([14, Thm. 4.1]). Motivated by that result, we define a natural projection $q\colon \mathrm{Def}_{\mathcal{F},\sigma} \to \mathrm{Hilb}_{X/Y}$ given by

$$(\mathcal{O}_{Y\times S} \xrightarrow{\sigma_S} \mathcal{F}_S) \to ((\operatorname{coker}\sigma_S) \otimes_{\mathcal{O}_{Y\times S}} (\mathcal{O}_{Y\times S} \otimes_{\mathcal{O}_Y} \mathcal{L}^{-1})),$$

which one may think of as determined by a (relative) Serre correspondence and the forgetful map $(X_S, \xi_S) \to (X_S)$. We shortly write $\operatorname{coker}\sigma_S \otimes \mathcal{L}^{-1}$ for $(\operatorname{coker}\sigma_S) \otimes_{\mathcal{O}_{Y\times S}} (\mathcal{O}_{Y\times S} \otimes_{\mathcal{O}_Y} \mathcal{L}^{-1})$ and we have written $Y \times S$ for $Y \times \operatorname{Spec}(S)$.

A main result of this paper (Theorem 2.1) states that if $H^0(\mathcal{O}_Y) \simeq k$ and $H^i(\mathcal{O}_Y) = 0$ for $i = 1, 2$, then $\mathrm{Ext}^1(\mathcal{I}_{X/Y} \otimes \mathcal{L}, \mathcal{F})$ is the tangent space of $\mathrm{Def}_{\mathcal{F},\sigma}$ and $\mathrm{Ext}^2(\mathcal{I}_{X/Y} \otimes \mathcal{L}, \mathcal{F})$ contains the obstructions of deforming $(\mathcal{F}, \sigma)$. Moreover,

(i) $p\colon \mathrm{Def}_{\mathcal{F},\sigma} \to \mathrm{Def}_{\mathcal{F}}$ is smooth (*i.e.*, formally smooth) provided $H^1(\mathcal{F}) = 0$, and
(ii) $q\colon \mathrm{Def}_{\mathcal{F},\sigma} \to \mathrm{Hilb}_{X/Y}$ is smooth provided $\mathrm{Ext}^2(\mathcal{F}, \mathcal{O}_Y) = 0$.

Let $\mathrm{M_Y}(P)$ be the moduli scheme of GM-stable sheaves with Hilbert polynomial P on Y. For the existence of $\mathrm{M_Y}(P)$, we refer to [16] and to Maruyama's papers [27, 28]. Note that $\mathcal{F}$ is called GM-stable if it is torsion-free, and, for every coherent subsheaf $\mathcal{F}'$ of $\mathcal{F}$ of rank one, we have the inequality $P_{\mathcal{F}'} < P_{\mathcal{F}}/2$ of Hilbert polynomials. Then, using small letters for the dimensions, *e.g.*, $h^0(\mathcal{F}) = \dim H^0(\mathcal{F})$ and $\mathrm{ext}^i(\mathcal{F}, \mathcal{F}) = \dim \mathrm{Ext}^i(\mathcal{F}, \mathcal{F})$ and supposing $H^1(\mathcal{F}) = 0$, $\mathrm{Ext}^2(\mathcal{F}, \mathcal{O}_Y) = 0$, $H^i(\mathcal{L}^{-1}) = 0$ for $i = 0, 1, 2$ and ω_Y invertible, we prove that

$$\mathrm{ext}^1(\mathcal{F}, \mathcal{F}) - \hom(\mathcal{F}, \mathcal{F}) + h^0(\mathcal{F}) = h^0(\mathcal{N}_{X/Y}) - 1 + h^0(\omega_X \otimes \omega_Y^{-1} \otimes \mathcal{L}^{-1}) \text{ and}$$

$$\dim_{(\mathcal{F})} \mathrm{M_Y}(P) + h^0(\mathcal{F}) = \dim_{(X)} \mathrm{Hilb}^p(Y) + h^0(\omega_X \otimes \omega_Y^{-1} \otimes \mathcal{L}^{-1}),$$

supposing $\mathcal{F}$ to be GM-stable in the latter formula. It follows that $\mathrm{M_Y}(P)$ is smooth at $(\mathcal{F})$ if and only if $\mathrm{Hilb}^p(Y)$ is smooth at (X). Moreover, $\mathcal{F}$ is a generic sheaf of some component of $\mathrm{M_Y}(P)$ if and only if X is generic in $\mathrm{Hilb}^p(Y)$, see Theorem 2.1 for further details. Note that all $\mathrm{Ext}^i(\cdot\,,\,\cdot)$-groups above are global Ext-groups of $\mathcal{O}_Y$-Modules.

Let $F = H^0_*(\mathcal{F}) := \oplus H^0(\mathcal{F}(v))$, $M = H^1_*(\mathcal{F})$ and $E = H^2_*(\mathcal{F})$. If ${}_0\mathrm{Hom}(F, M) = 0$ and Y is arithmetically Cohen-Macaulay (ACM), then we show that the local graded deformation functors of F and of $(F, H^0_*(\sigma))$ are isomorphic to $\mathrm{Def}_{\mathcal{F}}$ and $\mathrm{Def}_{\mathcal{F},\sigma}$ respectively. We get the following variation of Theorem 2.1(i): p is smooth provided ${}_0\mathrm{Hom}(F, M) = 0$ and Y is ACM.

One may interpret the morphisms p and q in Theorem 2.1 as corresponding to natural projections in an incidence correspondence of schemes of corepresentable functors, connecting $\mathrm{M_Y}(P)$ closely to $\mathrm{Hilb}^p(Y)$. Under the assumptions of Theorem 2.1, the projections are smooth of known fiber dimension. Since the fiber dimensions are easy to see and the Serre correspondence is well understood ([14]), related arguments as in the theorem are used in the literature, especially to compute dimensions of or describe very specific moduli schemes (*e.g.*, [4,8,13,15,17,31,33,42] and see [45, §4] for results and a discussion). It is, however, under the mere assumptions of (i) and (ii) above that we are able to see precisely that the scheme structures of $\mathrm{M}_{\mathrm{Y}}(P)$ and $\mathrm{Hilb}^p(Y)$ are "the same". To apply Theorem 2.1, we neither need $H^1(\mathcal{N}_{X/Y}) = 0$ nor $\mathrm{Ext}^2_{\mathcal{O}_Y}(\mathcal{F},\mathcal{F}) = 0$ to prove the smoothness of the moduli schemes. This, we think, significantly distinguishes our theorem from the results and the proofs of the mentioned papers. For the complete picture, we have no better reference than a preprint of the author ([18], for the case $Y = \mathbb{P}^3$) and the paper [9] which explicitly makes use of (without proofs) and slightly extends the results of [18], and we therefore include full proofs.

As an application, we prove several results concerning smoothness and dimension of the moduli space, $\mathrm{M}_{\mathbb{P}^3}(c_1, c_2, c_3)$, of stable *reflexive* sheaves of rank 2 with Chern classes c_1, c_2, and c_3 on $\mathbb{P}^3$. In some cases, especially for $c_3 = 0$ or small c_2 or large c_3, one knows the answer, *e.g.*, see [1, 3, 4, 6, 9, 13–15, 29, 32, 33]. Much is still unknown about $\mathrm{M}_{\mathbb{P}^3}(c_1, c_2, c_3)$, see [39] for an overview of recent research. Let

$$\mathrm{ed}(\mathcal{F}) = \mathrm{ext}^1_{\mathcal{O}_Y}(\mathcal{F},\mathcal{F}) - \mathrm{ext}^2_{\mathcal{O}_Y}(\mathcal{F},\mathcal{F}).$$

If $\mathcal{F}$ is stable, then $\mathrm{ed}(\mathcal{F})$ is sometimes called the "expected dimension" of $\mathrm{M}_{\mathrm{Y}}(P)$ at $(\mathcal{F})$ and $\mathrm{ed}(\mathcal{F}) = 8c_2 - 2c_1^2 - 3$ if $Y = \mathbb{P}^3$. We prove that $\mathrm{M}_{\mathbb{P}^3}(c_1, c_2, c_3)$ is smooth at $(\mathcal{F})$, *i.e.*, that $\mathcal{F}$ is unobstructed, and we find $\dim_{(\mathcal{F})} \mathrm{M}_{\mathbb{P}^3}(c_1, c_2, c_3)$ provided we have sufficient vanishing of ${}_{\nu}\mathrm{Hom}_R(F, M)$ and ${}_{\nu}\mathrm{Hom}_R(M, E)$ for $\nu = 0$ and -4 (Theorem 3.1). This result generalizes [32], which gives the complete answer for $M = 0$. Let ${}_0\mathrm{Ext}^2_R(M, M) = 0$. Using that the composition

$$\eta\colon {}_0\mathrm{Hom}_R(F, M) \times {}_0\mathrm{Hom}_R(M, E) \longrightarrow {}_0\mathrm{Hom}_R(F, E),$$

commutes with the cup product, we show that $\mathcal{F}$ is obstructed if $\eta \neq 0$ (see [10, 22, 47]). Thanks to this result, we get that the sufficient conditions of unobstructedness of Theorem 3.1 are close (resp. equivalent) to being *necessary* conditions provided the diameter of M is small (resp. one). Since we can substitute the non-vanishing of the Hom-groups of Theorem 3.1 by the non-triviality of certain products of graded Betti numbers appearing in the minimal resolution,

$$0 \to \bigoplus_i \mathcal{O}_{\mathbb{P}}(-i)^{\beta_{3,i}} \to \bigoplus_i \mathcal{O}_{\mathbb{P}}(-i)^{\beta_{2,i}} \to \bigoplus_i \mathcal{O}_{\mathbb{P}}(-i)^{\beta_{1,i}} \to \mathcal{F} \to 0,$$

of $\mathcal{F}$, we get, as perhaps the most interesting result of the third section, that $\mathcal{F}$ is obstructed if and only if

$$\beta_{1,c} \cdot \beta_{2,c+4} \neq 0 \quad \text{or} \quad \beta_{1,c+4} \cdot \beta_{2,c+4} \neq 0 \quad \text{or} \quad \beta_{1,c} \cdot \beta_{2,c} \neq 0.$$

Here M has diameter 1 and is concentrated in degree c (*i.e.*, $M_c \neq 0$ and $M_\nu = 0$ for $\nu \neq c$. Moreover, if $\mathcal{F}$ is an unobstructed stable sheaf and $\dim_k M = r$, then the dimension of the moduli scheme $\mathrm{M}_{\mathbb{P}^3}(c_1, c_2, c_3)$ at $(\mathcal{F})$ is

$$\dim_{(\mathcal{F})} \mathrm{M}_{\mathbb{P}^3}(c_1, c_2, c_3) = 8c_2 - 2c_1^2 - 3 + {}_0\mathrm{hom}_R(F, E) + r(\beta_{1,c+4} + \beta_{2,c}),$$

see Theorem 3.6 for details. Notice that ${}_0\mathrm{hom}_R(F, E)$ is explicitly computed in Remark 4.2.

We also show that every irreducible component V of $\mathrm{M}_{\mathbb{P}^3}(c_1, c_2, c_3)$ whose generic sheaf $\mathcal{F}$ satisfies $\operatorname{diam} M = 1$ is *reduced* (*i.e.,* generically smooth) and we determine $\dim V$ (Theorem 3.8). If $\operatorname{diam} M = m$, we give examples of moduli spaces $\mathrm{M}_{\mathbb{P}^3}(c_1, c_2, c_3)$ containing a non-reduced component for every integer $m \geq 3$. If $\operatorname{diam} M = 2$, we conjecture that the corresponding component of $\mathrm{M}_{\mathbb{P}^3}(c_1, c_2, c_3)$ is generically smooth. We also give a new formula for the dimension of *any* generically smooth irreducible component of $\mathrm{M}_{\mathbb{P}^3}(c_1, c_2, c_3)$ (Theorem 4.4). Even though some of the results of this paper may have a direct proof in which the condition "reflexive" is replaced by "torsion-free", we have chosen just to use Theorem 2.1 and the corresponding results for $\mathrm{Hilb}^p(\mathbb{P}^3)$.

1.1 Notations and Terminology

Let $R = k[X_0, X_1, \dots, X_n]$ be a graded polynomial ring over an algebraically closed field k of arbitrary characteristic with the standard grading, $\mathfrak{m} = (X_0, \dots, X_n)$ and let $Y \subset \mathbb{P}^n$ be a closed equidimensional, locally Cohen–Macaulay (CM) subscheme. We keep the other notations of the introduction. A *curve* X in $\mathbb{P}^n$ (resp. in Y) is an *equidimensional, locally CM* subscheme of $\mathbb{P} := \mathbb{P}^n$ (resp. of Y) of dimension one with sheaf ideal $\mathcal{I}_X$ (resp. $\mathcal{I}_{X/Y}$) and normal sheaf $\mathcal{N}_X = \mathcal{H}om_{\mathcal{O}_{\mathbb{P}}}(\mathcal{I}_X, \mathcal{O}_X)$ (resp. $\mathcal{N}_{X/Y} = \mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{I}_{X/Y}, \mathcal{O}_X)$ in Y). X is *unobstructed* if the Hilbert scheme is smooth at the corresponding point $(X) = (X \subset \mathbb{P}^n)$, otherwise X is *obstructed*. The Hilbert scheme of space curves of degree d and arithmetic genus g is denoted by $\mathrm{H}(d, g)$, see [12] for existence. If $\mathcal{F}$ is a coherent $\mathcal{O}_Y$-Module, we let $H^i(\mathcal{F}) = H^i(Y, \mathcal{F})$ and $h^i(\mathcal{F}) = \dim H^i(\mathcal{F})$, and we denote $\chi(\mathcal{F}) = \Sigma(-1)^i h^i(\mathcal{F})$. Then $I_X := H^0_*(\mathbb{P}, \mathcal{I}_X)$ is the saturated homogeneous ideal of X in $\mathbb{P}^n$.

Let $M = M(\mathcal{F})$ be the deficiency module $H^1_*(\mathcal{F})$. $\mathcal{F}$ is said to be unobstructed if the hull ([40]) of the local deformation functor, $\mathrm{Def}_{\mathcal{F}}$, is smooth. By *stable* we mean GM-*stable*, *i.e.,* stable in the sense of Gieseker and Maruyama in which the Hilbert polynomial (and not the first Chern class) is used to define stability (see [16, Chpt. I]). Thus a stable $\mathcal{F}$ is unobstructed if and only if $\mathrm{M}_{\mathrm{Y}}(P)$, the moduli scheme of stable sheaves with Hilbert polynomial P on Y, is smooth at $(\mathcal{F})$ ([16, Thm. 4.5.1]). The two concepts of stability are the same if $Y = \mathbb{P}^3$ and $\mathcal{F}$ is reflexive ([14, Rem. 3.1.1]). Stable sheaves are simple, *i.e.,* $\mathrm{Hom}(\mathcal{F}, \mathcal{F}) \simeq k$ ([16, Cor. 1.2.8]). Recall that a coherent sheaf $\mathcal{F}$ is reflexive if and only if $\mathcal{F} \simeq \mathcal{F}^{**}$ where $\mathcal{F}^* = \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{F}, \mathcal{O}_Y)$ (see [14]). In the case Y is a smooth threefold, we denote by $\mathrm{M}_{\mathrm{Y}}(c_1, c_2, c_3)$ the moduli scheme of stable *reflexive* sheaves of rank 2 on Y with Chern classes c_1, c_2, and c_3. Thus $\mathrm{M}_{\mathrm{Y}}(c_1, c_2, c_3)$ is open in $\mathrm{M}_{\mathrm{Y}}(P)$. For any $\mathcal{F}$ of $\mathrm{M}_{\mathrm{Y}}(c_1, c_2, c_3)$, there exists an exact sequence (1.1) after replacing $\mathcal{F}$ by some $\mathcal{F}(t)$. As mentioned, (1.1) defines a one-to-one correspondence between pairs $(\mathcal{F}, \sigma)$, where σ vanishes in codimension 2 and pairs (X, ξ), where ξ generates $\omega_X \otimes \omega_Y^{-1} \otimes \mathcal{L}^{-1}$ almost everywhere (the Hartshorne–Serre correspondence, see [14, Thm. 4.1] and [44, Thm. 1]).

A sheaf $\mathcal{F}$ of rank 2 on $\mathbb{P}^3$ is said to be Buchsbaum if $\mathfrak{m} \cdot M(\mathcal{F}) = 0$. We define the diameter of $M(\mathcal{F})$ (or of $\mathcal{F}$) by $\operatorname{diam} M(\mathcal{F}) = c - b + 1$, where $b = \min\{n | h^1(\mathcal{F}(n)) \neq 0\}$ and $c = \max\{n | h^1(\mathcal{F}(n)) \neq 0\}$ and by $\operatorname{diam} M(\mathcal{F}) = 0$

if $M(\mathcal{F}) = 0$. The diameter of a curve C, $\operatorname{diam} M(C)$, is correspondingly defined. A curve in a sufficiently small open irreducible subset of $\mathrm{H}(d,g)$ (small enough to satisfy all the openness properties that we want to pose) is called a *generic* curve of $\mathrm{H}(d,g)$, and accordingly, if we state that a generic curve has a certain property, then there is a non-empty open irreducible subset of $\mathrm{H}(d,g)$ of curves having this property. A *generization* $C' \subset \mathbb{P}^3$ of $C \subset \mathbb{P}^3$ in $\mathrm{H}(d,g)$ is a generic curve of some irreducible subset of $\mathrm{H}(d,g)$ containing (C). In the same way, we use the word generic and generization for a stable sheaf. By an irreducible component of $\mathrm{H}(d,g)$ or $\mathrm{M}_{\mathbb{P}^3}(c_1,c_2,c_3)$ we always mean a *non-embedded* irreducible component.

For any graded R-module N, we have the right derived functors $H^i_{\mathfrak{m}}(N)$ and ${}_v\mathrm{Ext}^i_{\mathfrak{m}}(N,-)$ of $\Gamma_{\mathfrak{m}}(N) = \bigoplus_v \ker(N_v \to \Gamma(\mathbb{P},\tilde{N}(v)))$ and $\Gamma_{\mathfrak{m}}(\mathrm{Hom}_R(N,-))_v$ respectively (see [11, Exp. VI]). We use small letters for the k-dimension and subscript v for the homogeneous part of degree v, *e.g.*, ${}_v\mathrm{ext}^i_{\mathfrak{m}}(N_1,N_2) = \dim {}_v\mathrm{Ext}^i_{\mathfrak{m}}(N_1,N_2)$, for graded R-modules N_i of finite type. There is a spectral sequence ([11, Exp. VI])

$$E_2^{p,q} = {}_v\mathrm{Ext}^p_R(N_1, H^q_{\mathfrak{m}}(N_2)) \Rightarrow {}_v\mathrm{Ext}^{p+q}_{\mathfrak{m}}(N_1,N_2) \tag{1.2}$$

($\Rightarrow$ means "converging to") and a duality isomorphism ([21, Thm. 1.1]);

$${}_v\mathrm{Ext}^{i+1}_{\mathfrak{m}}(N_2,N_1) \simeq {}_{-v}\mathrm{Ext}^{n-i}_R(N_1, N_2(-n-1))^{\vee}, \tag{1.3}$$

where $(-)^{\vee} = \mathrm{Hom}_k(-,k)$, generalizing the Gorenstein duality ${}_vH^{i+1}_{\mathfrak{m}}(M) \simeq {}_{-v}\mathrm{Ext}^{n-i}_R(M,R(-4))^{\vee}$. These groups fit into a long exact sequence ([11, Exp. VI])

(1.4)
$$\to {}_v\mathrm{Ext}^i_{\mathfrak{m}}(N_1,N_2) \to {}_v\mathrm{Ext}^i_R(N_1,N_2) \to \mathrm{Ext}^i_{\mathcal{O}_{\mathbb{P}}}(\tilde{N}_1,\tilde{N}_2(v)) \to {}_v\mathrm{Ext}^{i+1}_{\mathfrak{m}}(N_1,N_2) \to$$

which *e.g.*, relates the deformation theory of $X \subset \mathbb{P}^3$, described by $H^{i-1}(\mathcal{N}_X) \simeq \mathrm{Ext}^i_{\mathcal{O}_{\mathbb{P}}}(\mathcal{I}_X,\mathcal{I}_X)$ for $i = 1,2$, to the deformation theory of the homogeneous ideal $I = I_X$ (or equivalently of $A = R/I$), described by ${}_0\mathrm{Ext}^i_R(I_X,I_X)$, in the following exact sequence

$$\begin{aligned}{}_v\mathrm{Ext}^1_R(I,I) \hookrightarrow H^0(\mathcal{N}_C(v)) \to {}_v\mathrm{Ext}^2_{\mathfrak{m}}(I,I) \xrightarrow{\alpha} {}_v\mathrm{Ext}^2_R(I,I) \to H^1(\mathcal{N}_C(v))&\\ \to {}_v\mathrm{Ext}^3_{\mathfrak{m}}(I,I) \to 0&\end{aligned} \tag{1.5}$$

(see [22, §2]). Let $M(X) = H^2_{\mathfrak{m}}(I)$. Note that, in this situation, Charles Walter proved that the map $\alpha\colon {}_v\mathrm{Ext}^2_{\mathfrak{m}}(I,I) \simeq {}_v\mathrm{Hom}_R(I, H^2_{\mathfrak{m}}(I)) \to {}_v\mathrm{Ext}^2_R(I,I)$ of (1.5) factorizes via ${}_v\mathrm{Ext}^2_R(M(X),M(X))$ in a natural way. The factorization is in fact given by a certain edge homomorphism of the spectral sequence (1.2) with $N_1 = M(X)$, $N_2 = I$ and $p+q = 4$ (see [10, Thm. 2.5]). We frequently refer to [22] and all results we use from [22] (except possibly [22, Ex. 3.12]) are true without the characteristic zero assumption of the field quoted for that paper.

2 The Scheme Structure in the Serre Correspondence

In this section, we will prove the basic Theorem 2.1 and its variations. Moreover, we give some applications and examples of moduli schemes $\mathrm{M}_{\mathrm{Y}}(c_1,c_2,c_3)$ in the case

$Y = \mathbb{P}^3$. In particular, we show that some $\mathrm{M}_{\mathbb{P}^3}(c_1, c_2, c_3)$ contains a non-reduced component.

The local deformation functors $\mathrm{Def}_{\mathcal{F}}$, $\mathrm{Def}_{\mathcal{F},\sigma}$, and $\mathrm{Hilb}_{X/Y}$ of the introduction were defined on the category $\underline{l}$ whose objects are local artinian k-algebras S with residue field k and whose morphisms are homomorphisms of local rings over k. There is also another local deformation functor on $\underline{l}$ associated with (1.1):

$$\mathrm{Def}_{X/Y,\xi}(S) = \left\{ \left(X_S \subset Y_S, \xi_S\right) \mid (X_S \subset Y_S) \in \mathrm{Hilb}_{X/Y}(S) \text{ and } \xi_S \otimes_S k = \xi \right\},$$

where $Y_S := Y \times S$, $\mathcal{L}_S := \mathcal{O}_{Y_S} \otimes_{\mathcal{O}_Y} \mathcal{L}$ and $\xi_S \in \mathrm{Ext}^1(\mathcal{I}_{X_S/Y_S} \otimes \mathcal{L}_S, \mathcal{O}_{Y_S})$. We have the following main result about the relationship of dimensions and scheme structures in the Serre correspondence.

Theorem 2.1 *Let Y be an equidimensional, locally CM closed, subscheme of $\mathbb{P}^n$ of dimension* $\dim Y \geq 2$ *and suppose $H^0(\mathcal{O}_Y) \simeq k$ and $H^i(\mathcal{O}_Y) = 0$ for $i = 1, 2$. Moreover, suppose there exists an exact sequence* (1.1)*, where $X \subset Y$ is an equidimensional, locally CM, closed subscheme of codimension 2 in Y and $\mathcal{L}$ is an invertible $\mathcal{O}_Y$-Module. Let $\mathcal{I}_{X/Y} = \ker(\mathcal{O}_Y \to \mathcal{O}_X)$. Then*

(a) $\mathrm{Ext}^1_{\mathcal{O}_Y}(\mathcal{I}_{X/Y} \otimes \mathcal{L}, \mathcal{F})$ *is the tangent space of* $\mathrm{Def}_{\mathcal{F},\sigma}$ *and* $\mathrm{Ext}^2_{\mathcal{O}_Y}(\mathcal{I}_{X/Y} \otimes \mathcal{L}, \mathcal{F})$ *contains the obstructions of deforming* $(\mathcal{F}, \sigma)$*. Moreover* $\mathrm{Def}_{\mathcal{F},\sigma} \simeq \mathrm{Def}_{X/Y,\xi}$ *are isomorphic on $\underline{l}$ and*

  (i) $p\colon \mathrm{Def}_{\mathcal{F},\sigma} \to \mathrm{Def}_{\mathcal{F}}$ *is smooth (*i.e., *formally smooth) provided $H^1(\mathcal{F}) = 0$, and*

  (ii) $q\colon \mathrm{Def}_{\mathcal{F},\sigma} \to \mathrm{Hilb}_{X/Y}$ *is smooth provided* $\mathrm{Ext}^2_{\mathcal{O}_Y}(\mathcal{F}, \mathcal{O}_Y) = 0$.

(b) *Suppose $H^1(\mathcal{F}) = 0$,* $\mathrm{Ext}^2_{\mathcal{O}_Y}(\mathcal{F}, \mathcal{O}_Y) = 0$*, and that ω_Y is invertible. Then*

$$\mathrm{ext}^1_{\mathcal{O}_Y}(\mathcal{F}, \mathcal{F}) - \mathrm{hom}_{\mathcal{O}_Y}(\mathcal{F}, \mathcal{F}) + h^0(\mathcal{F}) = h^0(\mathcal{N}_{X/Y}) - 1 + h^0(\omega_X \otimes \omega_Y^{-1} \otimes \mathcal{L}^{-1}) - \sum_{i=0}^{2} (-1)^i h^i(\mathcal{L}^{-1}).$$

*Suppose in addition that $\mathcal{F}$ is stable (*i.e., *GM-stable) and $H^i(\mathcal{L}^{-1}) = 0$ for $i = 0, 1, 2$. Then*

$$\dim_{(\mathcal{F})} \mathrm{M_Y}(P) + h^0(\mathcal{F}) = \dim_{(X)} \mathrm{Hilb}^p(Y) + h^0(\omega_X \otimes \omega_Y^{-1} \otimes \mathcal{L}^{-1}).$$

It follows that $\mathrm{M_Y}(P)$ is smooth at $(\mathcal{F})$ if and only if $\mathrm{Hilb}^p(Y)$ is smooth at (X). Furthermore, $\mathcal{F}$ is a generic sheaf of some irreducible component of $\mathrm{M_Y}(P)$ if and only if X is generic in some irreducible component of $\mathrm{Hilb}^p(Y)$.

Remark 2.2 Under the assumptions of Theorem 2.1(a), we get that $H^1(\mathcal{F}) \simeq H^1(\mathcal{I}_{X/Y} \otimes \mathcal{L})$ and $\mathrm{Ext}^2_{\mathcal{O}_Y}(\mathcal{F}, \mathcal{O}_Y) \simeq \mathrm{Ext}^2_{\mathcal{O}_Y}(\mathcal{I}_{X/Y} \otimes \mathcal{L}, \mathcal{O}_Y)$ by using (1.1). Moreover, if the dualizing sheaf ω_Y is invertible (*i.e., Y* locally Gorenstein), then

$$\mathrm{Ext}^2_{\mathcal{O}_Y}(\mathcal{F}, \mathcal{O}_Y)^\vee \simeq H^{\dim Y - 2}(\mathcal{F} \otimes \omega_Y) \quad \text{and}$$

$$\mathrm{Ext}^2_{\mathcal{O}_Y}(\mathcal{I}_{X/Y} \otimes \mathcal{L}, \mathcal{O}_Y)^\vee \simeq H^{\dim Y - 2}(\mathcal{I}_{X/Y} \otimes \mathcal{L} \otimes \omega_Y).$$

In the case where Y is locally Gorenstein and the closed immersion $Y \hookrightarrow \mathbb{P}^n$ induces an isomorphism $\mathrm{Pic}(Y) \simeq \mathrm{Pic}(\mathbb{P}^n) = \mathbb{Z}$, we will use this isomorphism to look at the first Chern class c_1 as an integer, *i.e.*, $\mathcal{L} \simeq \wedge^2 \mathcal{F} = \mathcal{O}_Y(c_1)$. Then (1.1) takes the form

$$\xi; 0 \to \mathcal{O}_Y \xrightarrow{\sigma} \mathcal{F} \to \mathcal{I}_{X/Y}(c_1) \to 0. \tag{2.1}$$

Moreover, put $\omega_Y = \mathcal{O}_Y(e)$. By Remark 2.2, Theorem 2.1 immediately implies the following.

Corollary 2.3 *Suppose, in addition to the general assumptions of Theorem* 2.1*, that Y is locally Gorenstein and that $Y \hookrightarrow \mathbb{P}^n$ induces an isomorphism* $\mathrm{Pic}(Y) \simeq \mathrm{Pic}(\mathbb{P}^n)$*. Then* $\mathrm{Ext}^1_{\mathcal{O}_Y}(\mathcal{I}_{X/Y}(c_1), \mathcal{F})$ *is the tangent space of* $\mathrm{Def}_{\mathcal{F},\sigma}$ *and* $\mathrm{Ext}^2_{\mathcal{O}_Y}(\mathcal{I}_{X/Y}(c_1), \mathcal{F})$ *contains the obstructions of deforming* $(\mathcal{F}, \sigma)$*. Moreover*

(i) $p\colon \mathrm{Def}_{\mathcal{F},\sigma} \to \mathrm{Def}_{\mathcal{F}}$ *is smooth provided* $H^1(\mathcal{I}_{X/Y}(c_1)) = 0$*, and*

(ii) $q\colon \mathrm{Def}_{\mathcal{F},\sigma} \to \mathrm{Hilb}_{X/Y}$ *is smooth provided* $H^{\dim Y - 2}(\mathcal{I}_{X/Y}(c_1 + e)) = 0$.

Furthermore, suppose $H^i(\mathcal{O}_Y(-c_1)) = 0$ *for* $i = 0, 1, 2$,

$$H^1(\mathcal{I}_{X/Y}(c_1)) = 0, \quad H^{\dim Y - 2}(\mathcal{I}_{X/Y}(c_1 + e)) = 0$$

and that $\mathcal{F}$ is a stable sheaf. Then

$$\mathrm{ext}^1_{\mathcal{O}_Y}(\mathcal{F}, \mathcal{F}) + h^0(\mathcal{F}) = h^0(\mathcal{N}_{X/Y}) + h^0(\omega_X(-c_1 - e)),$$

$$\dim_{(\mathcal{F})} \mathrm{M}_Y(P) + h^0(\mathcal{F}) = \dim_{(X)} \mathrm{Hilb}^p(Y) + h^0(\omega_X(-c_1 - e)),$$

and $\mathcal{F}$ is unobstructed (*resp. generic in* $\mathrm{M}_Y(P)$) *if and only if X is unobstructed* (*resp. generic in* $\mathrm{Hilb}^p(Y)$).

Somehow we may look upon the corollary as the Hartshorne–Serre correspondence for flat families. We do not, however, need $\mathcal{F}$ to be reflexive (only torsion-free as one may easily deduce from (2.1)).

We shortly return to the proof of Theorem 2.1. First, we give an example to see that conditions as in Corollary 2.3 are needed for comparing the structure of $\mathrm{H}(d, g)$ and $\mathrm{M}_{\mathbb{P}^3}(c_1, c_2, c_3)$, while the same example "twisted" leads to a non-reduced component of $\mathrm{M}_{\mathbb{P}^3}(c_1, c_2, c_3)$ once the conditions of the theorem are satisfied. Below we will use the following result ([20, Prop. 3.2]). Let C and X be two space curves that are algebraically linked by a complete intersection of two surfaces of degrees f and g (a c.i. of type (f, g)), see [30] for the theory on linkage. If we suppose

$$H^1(\mathcal{I}_C(v)) = 0 \text{ for } v = f, g, f - 4 \text{ and } g - 4, \tag{2.2}$$

then C is unobstructed (resp. generic) if and only if X is unobstructed (resp. generic), and we have

$$\dim_{(C)} \mathrm{H}(d, g) + h^0(\mathcal{I}_C(f)) + h^0(\mathcal{I}_C(g)) = \dim_{(X)} \mathrm{H}(d', g') + h^0(\mathcal{I}_X(f)) + h^0(\mathcal{I}_X(g)).$$

Example 2.4 The generic curve C of Mumford's well-known example of a non-reduced component of $\mathrm{H}(14, 24)$ satisfies $H^1(\mathcal{I}_C(v)) = 0$ for $v \neq 3, 4, 5$ ([36]). Moreover, there is a c.i. of type $(6, 6)$ containing C whose linked curve is smooth. Hence by the result mentioned in (2.2), the linked curve is the general curve X of a non-reduced component of $\mathrm{H}(22, 56)$ of dimension 88. We leave to the reader to verify that X is subcanonical ($\omega_X \simeq \mathcal{O}_X(5)$) and satisfies $H^1(\mathcal{I}_X(v)) = 0$ for $v \neq 3, 4, 5$.

(a) If we take a general element of $H^0(\mathcal{O}_X) \simeq H^0(\omega_X(-5)) \simeq \mathrm{Ext}^1(\mathcal{I}_X, \mathcal{O}_{\mathbb{P}^3}(-9))$, we get an extension

$$\xi; 0 \to \mathcal{O}_{\mathbb{P}^3} \xrightarrow{\sigma} \mathcal{E} \to \mathcal{I}_X(9) \to 0$$

in which $\mathcal{E}$ is a stable vector bundle with $c_1 = 9$ and $c_1(\mathcal{E}(-5)) = -1$, $c_2(\mathcal{E}(-5)) = 2$. It is well known that $\mathrm{M}_{\mathbb{P}^3}(-1, 2, 0)$ is smooth [15], *i.e.,* $\mathcal{E}$ is unobstructed while X is obstructed. The assumption $H^1(\mathcal{I}_X(c_1 + e)) = 0$ of Corollary 2.3 is, however, not satisfied. Indeed, $H^1(\mathcal{I}_X(c_1 + e)) = H^1(\mathcal{I}_X(5)) \neq 0$.

(b) If we take a general global section of $\mathcal{O}_X(3) \simeq \omega_X(-2)$, we get an extension

$$\xi; 0 \to \mathcal{O}_{\mathbb{P}^3} \xrightarrow{\sigma} \mathcal{F} \to \mathcal{I}_X(6) \to 0$$

in which $\mathcal{F}$ is a stable reflexive sheaf belonging to $\mathrm{M}_{\mathbb{P}^3}(6, 22, 66) \simeq \mathrm{M}_{\mathbb{P}^3}(0, 13, 66)$. Since all assumptions of Corollary 2.3 are satisfied, we conclude that $\mathcal{F}$ is the general point of a non-reduced component of $\mathrm{M}_{\mathbb{P}^3}(0, 13, 66)$ of dimension $-h^0(\mathcal{F}) + \dim_{(X)} \mathrm{H}(22, 56) + h^0(\omega_X(-2)) = -8 + 88 + 21 = 101$. Note that, in this case, we have $\mathrm{ed}(\mathcal{F}) = 8c_2 - 2c_1^2 - 3 = 101$, *i.e.,* the component is non-reduced of the least possible dimension.

Example 2.5 Here we apply Corollary 2.3 directly to Mumford's example of a generic obstructed curve C of $\mathrm{H}(14, 24)$.

(a) If we take a general element of $H^0(\omega_C(2)) \simeq \mathrm{Ext}^1(\mathcal{I}_C, \mathcal{O}_{\mathbb{P}^3}(-2))$, we get an extension

$$\xi; 0 \to \mathcal{O}_{\mathbb{P}^3} \xrightarrow{\sigma} \mathcal{F} \to \mathcal{I}_C(2) \to 0$$

in which $\mathcal{F}$ is a stable reflexive sheaf with $c_1 = 2$ and $c_1(\mathcal{F}(-1)) = 0$, $c_2(\mathcal{F}(-1)) = 13$. The assumptions $H^1(\mathcal{I}_C(c_1)) = 0$, $H^1(\mathcal{I}_C(c_1 - 4)) = 0$ of Corollary 2.3 are satisfied and we get a non-reduced component of $\mathrm{M}_{\mathbb{P}^3}(0, 13, 74)$ of dimension

$$-h^0(\mathcal{F}) + \dim_{(C)} \mathrm{H}(14, 24) + h^0(\omega_C(2)) = -1 + 56 + 51 = 106.$$

(b) If we take a general global section of $H^0(\omega_C(3)) \simeq \mathrm{Ext}^1(\mathcal{I}_C, \mathcal{O}_{\mathbb{P}^3}(-1))$, we get, by Corollary 2.3 (with $c_1 = 1$), an extension where $\mathcal{F}$ is a stable reflexive sheaf belonging to a non-reduced component of $\mathrm{M}_{\mathbb{P}^3}(1, 14, 88) \simeq \mathrm{M}_{\mathbb{P}^3}(-1, 14, 88)$ of dimension $-1 + 56 + 65 = 120$.

(c) If we take a general global section of $H^0(\omega_C(-2)) \simeq \mathrm{Ext}^1(\mathcal{I}_C, \mathcal{O}_{\mathbb{P}^3}(-6))$, we get, by Corollary 2.3 (with $c_1 = 6$), an extension where $\mathcal{F}$ is a semistable obstructed reflexive sheaf belonging to the moduli space of semistable sheaves $\overline{\mathrm{M}}_{\mathbb{P}^3}(6, 14, 18) \simeq$

$\overline{\mathrm{M}}_{\mathbb{P}^3}(0,5,18)$. Even though $\mathcal{F}$ is obstructed, *i.e.,* the hull of the local deformation functor is singular, we do not yet know the hull's precise relationship to the local ring $O_{\overline{M},(\mathcal{F})}$ of $\overline{\mathrm{M}}_{\mathbb{P}^3}(0,5,18)$ at $(\mathcal{F})$, and we are not able to state whether $O_{\overline{M},(\mathcal{F})}$ is singular or not.

Proof of Theorem 2.1 (a) Using Laudal's results ([23]) for the local deformation functor of deforming a category, we *claim* that $\mathrm{Ext}^1(\mathcal{I}_{X/Y}\otimes\mathcal{L},\mathcal{F})$ is the tangent space of $\mathrm{Def}_{\mathcal{F},\sigma}$ and that $\mathrm{Ext}^2(\mathcal{I}_{X/Y}\otimes\mathcal{L},\mathcal{F})$ contains the obstructions of deforming $(\mathcal{F},\sigma)$. Indeed, letting $\underline{e}$ be the category consisting of two objects, $\mathcal{O}_Y$ and $\mathcal{F}$, and one nontrivial morphism σ, it follows from [23, Thm. 4.1.14] that there are cohomology groups $A^{(\cdot)}(\underline{e})$ such that $A^1(\underline{e})$ is the tangent space of $\mathrm{Def}_{\mathcal{F},\sigma}$ and $A^2(\underline{e})$ contains the obstructions of deforming $(\mathcal{F},\sigma)$. Moreover, thanks to [23, Lem. 3.1.7] (see [23, p. 155] to see how Lemma 3.1.7 applies to a category similar to $\underline{e}$), there is a spectral sequence

$$E_2^{p,q}=\varprojlim{}^{(p)}\left\{\begin{array}{ccc}\mathrm{Ext}^q(\mathcal{F},\mathcal{F}) & & \mathrm{Ext}^q(\mathcal{O}_Y,\mathcal{O}_Y)\\ & \searrow^{\alpha^q}\quad\swarrow & \\ & \mathrm{Ext}^q(\mathcal{O}_Y,\mathcal{F}) & \end{array}\right\}$$

converging to $A^{(\cdot)}:=A^{(\cdot)}(\underline{e})$. Here both arrows correspond to natural maps induced by the section $\mathcal{O}_Y\xrightarrow{\sigma}\mathcal{F}$ and $\varprojlim^{(p)}$ is the right derived functor of $\varprojlim$ over the category $\underline{e}$, see [24, § 2] for another example. Since $E_2^{p,q}=0$ for $p\geq 2$, we get the exact sequence

$$0\to E_2^{1,q-1}\to A^q\to E_2^{0,q}\to 0.$$

Moreover, $\mathrm{Ext}^q(\mathcal{O}_Y,\mathcal{O}_Y)=0$ for $0<q<3$ by the assumption $H^i(\mathcal{O}_Y)=0$ for $i=1,2$, and we get $E_2^{0,q}=\ker\alpha^q$ and $E_2^{1,q}=\operatorname{coker}\alpha^q$ for $q>0$. Observe also that $E_2^{1,0}=\operatorname{coker}\alpha^0$ because $k\simeq H^0(\mathcal{O}_Y)\simeq\mathrm{Hom}(\mathcal{O}_Y,\mathcal{O}_Y)\subset\mathrm{Hom}(\mathcal{F},\mathcal{F})$. We therefore have an exact sequence

$$0\to\operatorname{coker}\alpha^{q-1}\to A^q\to\ker\alpha^q\to 0$$

for any $q>0$. Combining with the long exact sequence

$$(2.3)\qquad \to\mathrm{Hom}(\mathcal{F},\mathcal{F})\xrightarrow{\alpha^0}H^0(\mathcal{F})\to\mathrm{Ext}^1(\mathcal{I}_{X/Y}\otimes\mathcal{L},\mathcal{F})\xrightarrow{p^1}\mathrm{Ext}^1(\mathcal{F},\mathcal{F})\xrightarrow{\alpha^1}$$
$$H^1(\mathcal{F})\to\mathrm{Ext}^2(\mathcal{I}_{X/Y}\otimes\mathcal{L},\mathcal{F})\xrightarrow{p^2}\mathrm{Ext}^2(\mathcal{F},\mathcal{F})\xrightarrow{\alpha^2}H^2(\mathcal{F})$$

deduced from $0\to\mathcal{O}_Y\to\mathcal{F}\to\mathcal{I}_{X/Y}\otimes\mathcal{L}\to 0$, we get the claim.

(i) From (2.3) and the proven claim that leads to the fact that p^1 (resp. p^2) is the tangent map (resp. a map of obstruction spaces, mapping obstructions to obstructions) of p, we get the smoothness of p, since p^1 is surjective and p^2 is injective. We will, however, give an independent proof that one may use (slightly changed) to prove the remark below.

Let $(T,\mathfrak{m}_T)\to(S,\mathfrak{m}_S)$ be a small artinian surjection (*i.e.,* of local artinian k-algebras with residue fields k whose kernel $\mathfrak{a}$ satisfies $\mathfrak{a}\cdot\mathfrak{m}_T=0$). To prove the (formal) smoothness of p, we must, by definition, show that the map

$$\mathrm{Def}_{\mathcal{F},\sigma}(T)\to\mathrm{Def}_{\mathcal{F},\sigma}(S)\times_{\mathrm{Def}_{\mathcal{F}}(S)}\mathrm{Def}_{\mathcal{F}}(T)$$

is surjective. Let $\sigma_S \colon \mathcal{O}_{Y\times S} \to \mathcal{F}_S$ be a deformation of σ to S and let $\mathcal{F}_T$ be a deformation of $\mathcal{F}_S$ to T. It suffices to find a map $\sigma_T \colon \mathcal{O}_{Y\times T} \to \mathcal{F}_T$ such that $\sigma_T \otimes_T id_S = \sigma_S$, *i.e.*, we must prove that $H^0(\mathcal{F}_T) \to H^0(\mathcal{F}_S)$ is surjective. Taking global sections of the short exact sequence

$$0 \to \mathcal{F} \otimes_k \mathfrak{a} \simeq \mathcal{F}_T \otimes_T \mathfrak{a} \to \mathcal{F}_T \to \mathcal{F}_S \to 0\,,$$

we get the surjectivity because $H^1(\mathcal{F}) \otimes_k \mathfrak{a} = 0$.

(ii) Again we have a long exact sequence

(2.4)

$$\to \operatorname{Ext}^1(\mathcal{I}_{X/Y} \otimes \mathcal{L}, \mathcal{O}_Y) \to \operatorname{Ext}^1(\mathcal{I}_{X/Y} \otimes \mathcal{L}, \mathcal{F}) \xrightarrow{q^1} \operatorname{Ext}^1(\mathcal{I}_{X/Y} \otimes \mathcal{L}, \mathcal{I}_{X/Y} \otimes \mathcal{L}) \to$$

$$\operatorname{Ext}^2(\mathcal{I}_{X/Y} \otimes \mathcal{L}, \mathcal{O}_Y) \to \operatorname{Ext}^2(\mathcal{I}_{X/Y} \otimes \mathcal{L}, \mathcal{F}) \xrightarrow{q^2} \operatorname{Ext}^2(\mathcal{I}_{X/Y} \otimes \mathcal{L}, \mathcal{I}_{X/Y} \otimes \mathcal{L}) \to$$

containing maps q^1 (resp. q^2) that we may interpret as the tangent map (resp. a map of obstruction spaces, which maps obstructions to obstructions) of q. Indeed, since $\mathcal{E}xt^1(\mathcal{I}_{X/Y}, \mathcal{I}_{X/Y}) \simeq \mathcal{N}_{X/Y}$ and $\mathcal{H}om(\mathcal{I}_{X/Y}, \mathcal{I}_{X/Y}) \simeq \mathcal{O}_Y$, the assumption $H^i(\mathcal{O}_Y) = 0$ for $i = 1, 2$ and the spectral sequence relating global and local Ext-groups show $\operatorname{Ext}^1(\mathcal{I}_{X/Y}, \mathcal{I}_{X/Y}) \simeq H^0(\mathcal{N}_{X/Y})$ and the injectivity of $\operatorname{Ext}^2(\mathcal{I}_{X/Y}, \mathcal{I}_{X/Y}) \hookrightarrow H^1(\mathcal{N}_{X/Y})$ (see [47] or [43]; the case $Y = \mathbb{P}^3$ was in fact proved in [18]), as well as

$$\operatorname{Ext}^i(\mathcal{I}_{X/Y} \otimes \mathcal{L}, \mathcal{I}_{X/Y} \otimes \mathcal{L}) \simeq \operatorname{Ext}^i(\mathcal{I}_{X/Y}, \mathcal{I}_{X/Y}) \tag{2.5}$$

for $i = 1, 2$. Hence we get the smoothness of q because q^1 is surjective and q^2 is injective by the assumption $\operatorname{Ext}^2(\mathcal{I}_{X/Y} \otimes \mathcal{L}, \mathcal{O}_Y) \simeq \operatorname{Ext}^2(\mathcal{F}, \mathcal{O}_Y) = 0$. We will, however, again give an independent proof using the definition of smoothness.

Let $T \to S$, $\mathfrak{a}$ and $\sigma_S \colon \mathcal{O}_{Y\times S} \to \mathcal{F}_S$ be as in the proof of (i) above. Let $\mathcal{G}_S = \operatorname{coker} \sigma_S$ and let $\mathcal{G}_T$ be a deformation of $\mathcal{G}_S$ to T. By the theory of extensions, it suffices to show that the natural map

$$\operatorname{Ext}^1(\mathcal{G}_T, \mathcal{O}_{Y\times T}) \to \operatorname{Ext}^1(\mathcal{G}_S, \mathcal{O}_{Y\times S})$$

is surjective. Modulo isomorphisms we refind this map in the middle of the long exact sequence

$$\to \operatorname{Ext}^1(\mathcal{G}_T, \mathcal{O}_{Y\times T} \otimes_T \mathfrak{a}) \to \operatorname{Ext}^1(\mathcal{G}_T, \mathcal{O}_{Y\times T}) \to \operatorname{Ext}^1(\mathcal{G}_T, \mathcal{O}_{Y\times S})$$
$$\to \operatorname{Ext}^2(\mathcal{G}_T, \mathcal{O}_{Y\times T} \otimes_T \mathfrak{a})$$

Since $\operatorname{Ext}^2(\mathcal{G}_T, \mathcal{O}_{Y\times T} \otimes_T \mathfrak{a}) \simeq \operatorname{Ext}^2(\mathcal{I}_{X/Y} \otimes \mathcal{L}, \mathcal{O}_Y) \otimes_k \mathfrak{a} = 0$, by assumption we get the smoothness.

To see that $\operatorname{Def}_{\mathcal{F},\sigma}(S) \simeq \operatorname{Def}_{X/Y,\xi}(S)$ are isomorphic, take a deformation $\mathcal{O}_{Y\times S} \xrightarrow{\sigma_S} \mathcal{F}_S$ of $\mathcal{O}_Y \xrightarrow{\sigma} \mathcal{F}$. Since $\mathcal{F}_S$ is flat, so are $\operatorname{coker} \sigma_S$ and $(\operatorname{coker} \sigma_S) \otimes_{\mathcal{O}_{Y_S}} (\mathcal{O}_{Y_S} \otimes_{\mathcal{O}_Y} \mathcal{L}^{-1})$. The former, $\operatorname{coker} \sigma_S$, fits into a short exact sequence starting with $0 \to \mathcal{O}_{Y\times S} \xrightarrow{\sigma_S} \mathcal{F}_S$, *i.e.*, we get an extension ξ_S satisfying $\xi_S \otimes_S k = \xi$. The latter is a flat deformation of $\mathcal{I}_{X/Y}$. Thanks to the isomorphism $\mathrm{Ext}^1(\mathcal{I}_{X/Y}, \mathcal{I}_{X/Y}) \simeq H^0(\mathcal{N}_{X/Y})$ and the injectivity $\mathrm{Ext}^2(\mathcal{I}_{X/Y}, \mathcal{I}_{X/Y}) \hookrightarrow H^1(\mathcal{N}_{X/Y})$ above, one knows that a deformation of $\mathcal{I}_{X/Y}$ defines a deformation of X in Y, *i.e.*, we get an element $(X_S \subset Y_S) \in \mathrm{Hilb}_{X/Y}$ and hence we get $(X_S \subset Y_S, \xi_S) \in \mathrm{Def}_{X/Y,\xi}(S)$. This defines a map $\mathrm{Def}_{\mathcal{F},\sigma}(S) \to \mathrm{Def}_{X/Y,\xi}(S)$. Since the morphism the other way is just an obvious forgetful map, we get a functorial isomorphism $\mathrm{Def}_{\mathcal{F},\sigma}(S) \simeq \mathrm{Def}_{X/Y,\xi}(S)$, as claimed in the theorem.

(b) To prove the first dimension formula, we continue (2.3) to the left. Using $H^1(\mathcal{F}) = 0$, we get

$$\sum_{i=0}^{1}(-1)^{i+1}\,\mathrm{ext}^i(\mathcal{I}_{X/Y}\otimes\mathcal{L},\mathcal{F}) = \mathrm{ext}^1(\mathcal{F},\mathcal{F}) - \hom(\mathcal{F},\mathcal{F}) + h^0(\mathcal{F}),$$

while (2.4) (continued), $\mathrm{Ext}^2(\mathcal{F}, \mathcal{O}_Y) = 0$, $\mathcal{H}om(\mathcal{I}_{X/Y}, \mathcal{I}_{X/Y}) \simeq \mathcal{O}_Y$ and (2.5) show

$$\sum_{i=0}^{1}(-1)^{i+1}\,\mathrm{ext}^i(\mathcal{I}_{X/Y}\otimes\mathcal{L},\mathcal{F}) = \mathrm{ext}^1(\mathcal{I}_{X/Y},\mathcal{I}_{X/Y}) - 1 + \sum_{i=0}^{1}(-1)^{i+1}\,\mathrm{ext}^i(\mathcal{I}_{X/Y}\otimes\mathcal{L},\mathcal{O}_Y).$$

Since $\mathcal{E}xt^1(\mathcal{I}_{X/Y}, \mathcal{I}_{X/Y}) \simeq \mathcal{N}_{X/Y}$, it remains to show

$$\sum_{i=0}^{1}(-1)^{i+1}\,\mathrm{ext}^i(\mathcal{I}_{X/Y}\otimes\mathcal{L},\mathcal{O}_Y) = h^0(\omega_X\otimes\omega_Y^{-1}\otimes\mathcal{L}^{-1}) - \sum_{i=0}^{2}(-1)^i h^i(\mathcal{L}^{-1}).$$

Since $\mathcal{H}om(\mathcal{I}_{X/Y}\otimes\mathcal{L}, \mathcal{O}_Y) \simeq \mathcal{L}^{-1}$, $\mathcal{E}xt^1(\mathcal{I}_{X/Y}, \mathcal{O}_Y) \simeq \omega_X \otimes \omega_Y^{-1}$ and $\mathrm{Ext}^2(\mathcal{I}_{X/Y} \otimes \mathcal{L}, \mathcal{O}_Y) = 0$ (Remark 2.2), we get $\hom(\mathcal{I}_{X/Y}\otimes\mathcal{L}, \mathcal{O}_Y) = h^0(\mathcal{L}^{-1})$ and

$$\mathrm{ext}^1(\mathcal{I}_{X/Y}\otimes\mathcal{L},\mathcal{O}_Y) = h^0(\omega_X\otimes\omega_Y^{-1}\otimes\mathcal{L}^{-1}) + h^1(\mathcal{L}^{-1}) - h^2(\mathcal{L}^{-1})$$

by the spectral sequence relating global and local Ext-groups, and we get the first dimension formula.

Finally, to see the last dimension formula (resp. the genericness property), let $U \subset \mathrm{Hilb}^p(Y)$ be a small enough open (resp. small enough open irreducible) subscheme containing (X) and let $\mathcal{I}_{X_U/Y_U}$ be the sheaf ideal of $X_U \subset Y_U := Y \times U$, the universal object of $\mathrm{Hilb}^p(Y)$ restricted to U. Let $\mathcal{L}_U := \mathcal{L} \otimes_{\mathcal{O}_Y} \mathcal{O}_{Y\times U}$. Using (2.4), which takes the form

$$0 \to H^0(\mathcal{O}_Y) \to \mathrm{Ext}^1(\mathcal{I}_{X/Y}\otimes\mathcal{L},\mathcal{O}_Y) \to \mathrm{Ext}^1(\mathcal{I}_{X/Y}\otimes\mathcal{L},\mathcal{F}) \xrightarrow{q^1} \mathrm{Ext}^1(\mathcal{I}_{X/Y},\mathcal{I}_{X/Y}) \to 0,$$

and recalling that q^1 is the tangent map of $q\colon \mathrm{Def}_{\mathcal{F},\sigma} \to \mathrm{Hilb}_{X/Y}$ and that q is smooth, we can look upon the fiber of q as $\mathrm{Ext}^1(\mathcal{I}_{X/Y} \otimes \mathcal{L}, \mathcal{O}_Y)/k$. In the same way, since $\mathcal{F}$ is stable and hence simple, we can use the exact sequence (2.3) to see that the fiber of p is isomorphic to $H^0(\mathcal{F})/k$. Hence we get the second dimension formula since the functor $\mathrm{Def}_{\mathcal{F}}$ is pro-represented by the completion of the local ring of $\mathrm{M_Y}(P)$ at $(\mathcal{F})$ ([16, Thm. 4.5.1]). More precisely, the family $D := \mathbb{P}(\mathrm{Ext}^1(\mathcal{I}_{X_U/Y_U} \otimes \mathcal{L}_U, \mathcal{O}_{Y_U})^\vee) \to U$ parametrizes exactly extensions as in (1.1) over U, and the definition of a moduli space implies the existence of a morphism $\overline{p}\colon D \to \mathrm{M}_Y(P)$ whose corresponding local homomorphism at $(\mathcal{F}, \sigma)$ and $(\mathcal{F})$ induces p. Note that $\overline{p}$ is smooth at $(X \subset Y, \xi)$ and hence maps the generic points of D onto generic points of $\mathrm{M}_Y(P)$. This also proves that $\mathcal{F}$ is a generic sheaf of some component of $\mathrm{M}_Y(P)$ if and only if X is generic in some component of $\mathrm{Hilb}^p(Y)$. Note that, just by comparing the two dimension formulas, we get the statement on the smoothness of the theorem, and we are done. ∎

Remark 2.6 (a) Suppose Y is ACM and let $B := H^0_*(\mathcal{O}_Y)$. Applying $H^0_*(-)$ onto (1.1), we get an exact sequence

$$0 \to B \xrightarrow{H^0_*(\sigma)} F \to \operatorname{coker}(H^0_*(\sigma)) \to 0$$

inducing a long exact sequence (*) as in (2.3) in which we have replaced the global Ext-groups of sheaves with the corresponding graded ${}_0\mathrm{Ext}$-groups. Similar to $\mathrm{Def}_{\mathcal{F}}$ (resp. $\mathrm{Def}_{\mathcal{F},\sigma}$), we may define local deformation functors Def_F (resp. $\mathrm{Def}_{F,H^0_*(\sigma)}$) on $\underline{l}$ of flat graded deformations F_S of F (resp. $B \otimes_k S \xrightarrow{H^0_*(\sigma_S)} F_S$ of $B \xrightarrow{H^0_*(\sigma)} F$). There is a natural forgetful map $p_0\colon \mathrm{Def}_{F,H^0_*(\sigma)} \to \mathrm{Def}_F$ whose tangent map fits into (*) and corresponds to p^1 in (2.3). Since ${}_0\mathrm{Ext}^1_B(B, F) = 0$ in (*), it follows that

$$p_0\colon \mathrm{Def}_{F,H^0_*(\sigma)} \to \mathrm{Def}_F$$

is smooth by the first (*i.e.,* the cohomological) proof of Theorem 2.1(i), above.

(b) Suppose Y is ACM and ${}_0\mathrm{Hom}_B(F, M) = 0$. Then we claim that $\mathrm{Def}_{\mathcal{F}} \simeq \mathrm{Def}_F$. Indeed, by (1.4),

$$0 \to {}_0\mathrm{Ext}^1_B(F, F) \to \mathrm{Ext}^1_{\mathcal{O}_Y}(\mathcal{F}, \mathcal{F}) \to {}_0\mathrm{Ext}^2_{\mathfrak{m}}(F, F) \to {}_0\mathrm{Ext}^2_B(F, F) \to \mathrm{Ext}^2_{\mathcal{O}_Y}(\mathcal{F}, \mathcal{F})$$

is exact and ${}_0\mathrm{Ext}^2_{\mathfrak{m}}(F, F) = {}_0\mathrm{Hom}_B(F, M)$ by (1.2). Hence we get the claim by the cohomological argument used in Theorem 2.1(i). In the same way (or directly), we can prove that $\mathrm{Def}_{\mathcal{F},\sigma} \simeq \mathrm{Def}_{F,H^0_*(\sigma)}$. It follows that the morphism $p\colon \mathrm{Def}_{\mathcal{F},\sigma} \to \mathrm{Def}_{\mathcal{F}}$ of Theorem 2.1 is smooth.

3 Reflexive Sheaves on $\mathbb{P}^3$ of Small Diameter

As an application we concentrate on $\mathrm{M}_Y(c_1, c_2, c_3)$ with $Y = \mathbb{P}^3$. A main result of this section states that in the diameter one case the obstructedness of $\mathcal{F}$ is equivalent to the non-vanishing of certain products of graded Betti numbers of the free graded minimal resolution of $H^0_*(\mathcal{F})$ (Theorem 3.6). We also show that generic diameter one sheaves are unobstructed and we determine the dimension of the corresponding component (Theorem 3.8). We end this section with a conjecture for generic sheaves of diameter 2.

Recalling the notions $F = H^0_*(\mathcal{F})$, $M = H^1_*(\mathcal{F})$, $E = H^2_*(\mathcal{F})$ and $\mathrm{ed}(\mathcal{F}) := \mathrm{ext}^1(\mathcal{F}, \mathcal{F}) - \mathrm{ext}^2(\mathcal{F}, \mathcal{F}) = 8c_2 - 2c_1^2 - 3$, we first consider sufficient conditions of unobstructedness.

Theorem 3.1 *Let $\mathcal{F}$ be a reflexive sheaf of rank 2 on $\mathbb{P}^3$, and suppose that one of the following conditions holds:*

(i) ${}_{\nu}\mathrm{Hom}_R(F, M) = 0$ *for* $\nu = 0$ *and* $\nu = -4$*;*
(ii) ${}_{\nu}\mathrm{Hom}_R(M, E) = 0$ *for* $\nu = 0$ *and* $\nu = -4$*;*
(iii) ${}_0\mathrm{Hom}_R(F, M) = 0$, ${}_0\mathrm{Hom}_R(M, E) = 0$ *and M is unobstructed as a graded module* (e.g., ${}_0\mathrm{Ext}^2_R(M, M) = 0$).

Then $\mathcal{F}$ is unobstructed. Moreover, if ${}_0\mathrm{Ext}^i_R(M, M) = 0$ *for* $i \geq 2$ *and $\mathcal{F}$ is stable, then* $\mathrm{M}_{\mathbb{P}^3}(c_1, c_2, c_3)$ *is smooth at* $(\mathcal{F})$ *and its dimension at* $(\mathcal{F})$ *is*

$$\dim_{(\mathcal{F})} \mathrm{M}_{\mathbb{P}^3}(c_1, c_2, c_3) = \mathrm{ed}(\mathcal{F}) + {}_{-4}\mathrm{hom}_R(F, M) + {}_{-4}\mathrm{hom}_R(M, E) + {}_0\mathrm{hom}_R(F, E)\,.$$

Furthermore, ${}_{-4}\mathrm{hom}_R(M, E) = {}_0\mathrm{ext}^1_R(F, M)$ *and* ${}_{-4}\mathrm{hom}_R(M, E) + {}_0\mathrm{hom}_R(F, E) = {}_{-4}\mathrm{ext}^1_R(F, F)$.

Note that Theorem 3.1 applies to prove unobstructedness if $M = 0$ (this case is known by [32]). The natural application of Theorem 3.1 is to sheaves whose graded modules M are concentrated in a few degrees, *e.g.,* $\mathrm{diam}\, M \leq 2$. For such modules we can prove more, namely that the sufficient conditions of unobstructedness of Theorem 3.1 are quite close to being necessary conditions. Indeed, if the diameter of M is one, they are necessary! Moreover, in such cases a minimal resolution of $\mathcal{F}$ is often sufficient for computing the Hom-groups in the theorem (see also Lemma 3.4).

To find necessary conditions, we consider the *cup product* or, more precisely, its "images" in ${}_0\mathrm{Hom}_R(F, E)$, ${}_{-4}\mathrm{Hom}_R(F, M)^\vee$ and ${}_{-4}\mathrm{Hom}_R(M, E)^\vee$ via some natural maps, see [10, 22, 47] and [25, § 2]. Here we only include the cup product factorization given by (a) and hence (b)(i) below, for which there is a proof in [22, Prop. 3.6] of the corresponding result for curves using Walter's factorization of α in (1.5). We remark that this result for curves, to our knowledge now, was first proved by Fløystad (an easy consequence of [10, Prop. 2.13]). For similarly generalizing the cases (ii) and (iii) of (b), we refer to [22, Prop. 3.8]. Note that the necessary conditions in (b) apply to many other sheaves than to those of diameter one (*i.e.,* those with $M' = 0$), *e.g.,* they apply to Buchsbaum sheaves and to sheaves obtained by liaison addition ([30]) of curves.

Proposition 3.2 *Let $\mathcal{F}$ be a reflexive sheaf of rank 2 on $\mathbb{P}^3$ and suppose that* ${}_0\mathrm{Ext}^2_R(M, M) = 0$.

(a) *If the natural morphism*

$${}_0\mathrm{Hom}_R(F, M) \times {}_0\mathrm{Hom}_R(M, E) \longrightarrow {}_0\mathrm{Hom}_R(F, E)$$

(given by the composition) is non-zero, then $\mathcal{F}$ is obstructed.

(b) *Suppose M admits a decomposition* $M = M' \oplus M_{[t]}$ *as R-* modules*, where the diameter of* $M_{[t]}$ *is one and supported in degree t. Then $\mathcal{F}$ is obstructed provided*

(i) ${}_0\mathrm{Hom}_R(F, M_{[t]}) \neq 0$ *and* ${}_0\mathrm{Hom}_R(M_{[t]}, E) \neq 0$*, or*
(ii) ${}_{-4}\mathrm{Hom}_R(F, M_{[t]}) \neq 0$ *and* ${}_0\mathrm{Hom}_R(M_{[t]}, E) \neq 0$*, or*
(iii) ${}_0\mathrm{Hom}_R(F, M_{[t]}) \neq 0$ *and* ${}_{-4}\mathrm{Hom}_R(M_{[t]}, E) \neq 0$.

Proof of Theorem 3.1 and Proposition 3.2 To prove the results, we may replace $\mathcal{F}$ by $\mathcal{F}(j)$ for $j \gg 0$ because, for both results, the assumptions as well as the conclusions hold for $\mathcal{F}$ if and only if they hold for $\mathcal{F}(j)$. In particular, we may assume $H^1(\mathcal{F}(v)) = 0$ for $v \leq 0$ and hence $\mathrm{Ext}^2(\mathcal{F}, \mathcal{O}_{\mathbb{P}^3})^\vee \simeq H^1(\mathcal{F}(-4)) = 0$. It follows that the maps p and q of Theorem 2.1 are smooth. From the Hartshorne–Serre correspondence we get an exact sequence

$$0 \to R \to F \to I_X(c_1) \to 0,$$

which implies $M = H^1_*(\mathcal{F}) \simeq H^1_*(\mathcal{I}_X(c_1))$. We also get the exact sequence

$$0 \to E = H^2_*(\mathcal{F}) \to H^2_*(\mathcal{I}_X(c_1)) \to H^3_*(\mathcal{O}_{\mathbb{P}^3}).$$

Using these sequences and $H^1(\mathcal{I}_X(c_1+v)) = 0$ for $v \leq 0$, we get

$$\begin{aligned} &{}_v\mathrm{Hom}_R(F, M) \simeq {}_v\mathrm{Hom}_R(I_X, H^1_*(\mathcal{I}_X)) \text{ and} \\ &{}_v\mathrm{Hom}_R(M, E) \simeq {}_v\mathrm{Hom}_R(H^1_*(\mathcal{I}_X), H^2_*(\mathcal{I}_X)) \end{aligned} \tag{3.1}$$

for $-4 \leq v \leq 0$ because ${}_v\mathrm{Hom}_R(R, M) = 0$ and

$${}_v\mathrm{Hom}_R(M, H^3_*(\mathcal{O}_{\mathbb{P}^3})) \simeq {}_v\mathrm{Ext}^4_{\mathfrak{m}}(M, R) \simeq M^\vee_{-v-4} = 0$$

by (1.2) and (1.3). Now recall that in [22] we proved results similar to Theorem 3.1 and Proposition 3.2 for the unobstructedness (resp. obstructedness) of X with the difference that the Hom-groups, ${}_v\mathrm{Hom}_R(H^i_*(\mathcal{F}), H^{i+1}_*(\mathcal{F}))$ for $\mathcal{F}$ were exchanged by the corresponding groups, ${}_v\mathrm{Hom}_R(H^i_*(\mathcal{I}_X), H^{i+1}_*(\mathcal{I}_X))$ for $\mathcal{I}_X$. Therefore, (3.1) and Theorem 2.1 show that $\mathcal{F}$ is unobstructed in Theorem 3.1 (resp. obstructed in Proposition 3.2) because X is unobstructed (resp. obstructed) by [22, Thm. 2.6] (resp. [22, Prop. 3.6 and Thm. 3.2] and Remark 3.3.

To prove the dimension formula, we suppose ${}_0\mathrm{Ext}^i_R(M, M) = 0$ for $2 \leq i \leq 4$. With this assumption the map α in (1.5) is zero for $v = 0$ by Walter's observation. Note that there is a corresponding connecting map $\alpha(N_1, N_2) : {}_0\mathrm{Ext}^2_{\mathfrak{m}}(N_1, N_2) \to {}_0\mathrm{Ext}^2_R(N_1, N_2)$ appearing in (1.4). Indeed, $\alpha = \alpha(I_X, I_X)$.

We *claim* that $\alpha(F, F) = 0$. To prove it we use the functoriality of the sequence (1.4) and $\alpha = 0$. Since the natural map ${}_0\mathrm{Ext}^2_R(I_X(c_1), F) \to {}_0\mathrm{Ext}^2_R(I_X(c_1), I_X(c_1))$ is an isomorphism by (3.1),

$$\begin{aligned} {}_0\mathrm{Ext}^2_R(I_X(c_1), F)^\vee &\simeq {}_{-4}\mathrm{Ext}^2_{\mathfrak{m}}(F, I_X(c_1)) \simeq {}_{-4}\mathrm{Hom}_R(F, M), \text{ and} \\ {}_0\mathrm{Ext}^2_R(I_X(c_1), I_X(c_1))^\vee &\simeq {}_{-4}\mathrm{Ext}^2_{\mathfrak{m}}(I_X, I_X) \simeq {}_{-4}\mathrm{Hom}_R(I_X, H^1_*(\mathcal{I}_X)), \end{aligned}$$

see (1.2) and (1.3), we get $\alpha(I_X(c_1), F) = 0$. In a similar way, the natural map ${}_0\mathrm{Ext}^2_{\mathfrak{m}}(I_X(c_1), F) \to {}_0\mathrm{Ext}^2_{\mathfrak{m}}(F, F)$ is an isomorphism (*i.e.,* both groups are naturally isomorphic to ${}_0\mathrm{Hom}_R(F, M)$ by (1.2)), and we get the claim from $\alpha(I_X(c_1), F) = 0$.

Now, using the fact that the projective dimension of F is 2, the proven claim, and (1.4), we get an exact sequence

$$0 \to {}_0\mathrm{Ext}^2_R(F, F) \to \mathrm{Ext}^2_{\mathcal{O}_{\mathbb{P}}}(\mathcal{F}, \mathcal{F}) \to {}_0\mathrm{Ext}^3_{\mathfrak{m}}(F, F) \to 0.$$

As above, we have ${}_0\mathrm{Ext}^2_R(F,F)^\vee \simeq {}_{-4}\mathrm{Hom}_R(F,M)$ and similarly ${}_0\mathrm{Ext}^3_{\mathfrak{m}}(F,F)^\vee \simeq {}_{-4}\mathrm{Ext}^1_R(F,F)$ by (1.3). We get $\mathrm{ext}^2_{\mathcal{O}_{\mathbb{P}}}(\mathcal{F},\mathcal{F}) = {}_{-4}\mathrm{hom}_R(F,M) + {}_{-4}\mathrm{ext}^1_R(F,F)$. Using (1.2), we get an exact sequence

$$0 \to {}_0\mathrm{Ext}^1_R(F,M) \to {}_0\mathrm{Ext}^3_{\mathfrak{m}}(F,F) \to {}_0\mathrm{Hom}_R(F,E) \to {}_0\mathrm{Ext}^2_R(F,M) \to \tag{3.2}$$

and hence ${}_{-4}\mathrm{ext}^1_R(F,F) = {}_0\,\mathrm{ext}^1_R(F,M) + {}_0\mathrm{hom}_R(F,E)$, because

$$\begin{aligned} {}_0\,\mathrm{Ext}^2_R(F,M) \simeq {}_{-4}\mathrm{Ext}^2_{\mathfrak{m}}(M,F)^\vee \simeq {}_{-4}\mathrm{Hom}_R(M,M)^\vee \simeq {}_0\mathrm{Ext}^4_{\mathfrak{m}}(M,M) \\ \simeq {}_0\mathrm{Ext}^4_R(M,M) = 0 \end{aligned} \tag{3.3}$$

by (1.2) and (1.3). By the arguments of (3.3), we also get

$${}_{-4}\mathrm{Ext}^1_R(M,M)^\vee \simeq {}_0\mathrm{Ext}^3_R(M,M) = 0 \quad \text{and}$$
$${}_0\mathrm{ext}^1_R(F,M) = {}_{-4}\mathrm{ext}^3_{\mathfrak{m}}(M,F) = {}_{-4}\mathrm{hom}_R(M,E),$$

and putting things together, we are done. ■

Remark 3.3 Theorem 2.6(iii) of [22] actually proves a slightly weaker statement than needed to prove Theorem 3.1(iii). However, putting different results of *e.g.,* [22] together, we get what we want. Indeed, we claim that a curve $X \subset \mathbb{P}^3$ is unobstructed provided

$${}_0\mathrm{Hom}_R(I_X, H^1_*(\mathcal{I}_X)) = 0, \quad {}_0\mathrm{Hom}_R(H^1_*(\mathcal{I}_X), H^2_*(\mathcal{I}_X)) = 0, \tag{3.4}$$

and $H^1_*(\mathcal{I}_X)$ is unobstructed as a graded module (*e.g.,* ${}_0\,\mathrm{Ext}^2_R(H^1_*(\mathcal{I}_X), H^1_*(\mathcal{I}_X)) = 0$). This is mainly a consequence of results proven in [26] by Martin-Deschamps and Perrin. Indeed, their smoothness theorem for the morphism from the Hilbert scheme of constant cohomology, $H(d,g)_{cc}$, onto the scheme of "Rao modules" ([26, Thm. 1.5, p. 135]) combined with their tangent space descriptions (pp. 155–156), or more precisely combined with [22, Prop. 2.10], which states that the vanishing of the two Hom-groups in (3.4) leads to an isomorphism $H(d,g)_{cc} \simeq H(d,g)$ at (X), we conclude easily.

We can compute the number ${}_0\mathrm{hom}_R(F,E)$ in terms of the graded Betti numbers $\beta_{j,i}$ of F;

$$0 \to \bigoplus_i R(-i)^{\beta_{3,i}} \to \bigoplus_i R(-i)^{\beta_{2,i}} \to \bigoplus_i R(-i)^{\beta_{1,i}} \to F \to 0 \tag{3.5}$$

(sheafifying, we get the "resolution" of $\mathcal{F}$ in the introduction), by using the following result.

Lemma 3.4 *Let $\mathcal{F}$ be a reflexive sheaf of rank 2 on $\mathbb{P}^3$, and suppose ${}_{-4}\mathrm{Hom}_R(F,F) = 0$. Then*

$${}_0\mathrm{hom}_R(F,E) = \sum_i (\beta_{1,i} - \beta_{2,i} + \beta_{3,i}) \cdot (h^2(\mathcal{F}(i)) - h^3(\mathcal{F}(i))).$$

Proof Recall $E := H^2_*(\mathcal{F}) \simeq H^3_{\mathfrak{m}}(F)$. If we apply ${}_{\nu}\mathrm{Hom}_R(-, E)$ to the minimal resolution (3.5), we get a complex

$$(3.6)\quad 0 \to {}_0\mathrm{Hom}_R(F, H^2_*(\mathcal{F})) \to \bigoplus_i H^2(\mathcal{F}(i))^{\beta_{1,i}} \to \bigoplus_i H^2(\mathcal{F}(i))^{\beta_{2,i}} \to \bigoplus_i H^2(\mathcal{F}(i))^{\beta_{3,i}} \to 0.$$

Since the alternating sum of the dimension of the terms in a complex equals the alternating sum of the dimension of its homology groups, it suffices to show that ${}_0\mathrm{Ext}^1_R(F, E) = 0$, ${}_0\mathrm{Ext}^2_R(F, E) \simeq {}_0\mathrm{Hom}_R(F, H^3_*(\mathcal{F}))$, and

$$(3.7)\qquad {}_0\mathrm{hom}_R(F, H^3_*(\mathcal{F})) = \sum_i (\beta_{1,i} - \beta_{2,i} + \beta_{3,i}) \cdot h^3(\mathcal{F}(i)).$$

Using (1.2) and that ${}_0\mathrm{Ext}^4_{\mathfrak{m}}(F, F) \simeq {}_{-4}\mathrm{Hom}_R(F, F)^\vee = 0$ by assumption, we get ${}_0\mathrm{Ext}^1_R(F, H^3_{\mathfrak{m}}(F)) = 0$ and an exact sequence

$$0 \to {}_0\,\mathrm{Hom}_R(F, H^4_{\mathfrak{m}}(F)) \to {}_0\mathrm{Ext}^2_R(F, H^3_{\mathfrak{m}}(F)) \to {}_0\mathrm{Ext}^5_{\mathfrak{m}}(F, F) \to {}_0\,\mathrm{Ext}^1_R(F, H^4_{\mathfrak{m}}(F)) \to 0.$$

Since we have ${}_0\mathrm{Ext}^5_{\mathfrak{m}}(F, F) = 0$, the proof is complete provided we can prove (3.7). To this end, it is sufficient to see that (3.6), with H^2 replaced by H^3, is exact. Since we have ${}_0\mathrm{Ext}^i_{\mathfrak{m}}(F, F) = 0$ for $i = 5, 6$, by duality, we get ${}_0\,\mathrm{Ext}^i_R(F, H^4_{\mathfrak{m}}(F)) = 0$ for $i = 1, 2$ by (1.2), and we are done. ■

Remark 3.5 For later use we remark that if we apply ${}_0\mathrm{hom}_R(-, M)$, $M = H^1_*(\mathcal{F})$ to (3.5), we get

$$\sum_{i=0}^{2} (-1)^i\, {}_0\mathrm{ext}^i(F, M) = \sum_i (\beta_{1,i} - \beta_{2,i} + \beta_{3,i}) \cdot h^1(\mathcal{F}(i)),$$

see (3.6). Suppose $\mathcal{F}$ is reflexive and ${}_{-4}\mathrm{Hom}_R(F, F) = 0$. Using (1.2) as in (3.2) and the proof above, we get

$$\sum_{i=2}^{3} (-1)^i\, {}_0\mathrm{ext}^i_{\mathfrak{m}}(F, F) = \sum_{i=0}^{2} (-1)^i\, {}_0\mathrm{ext}^i(F, M) - {}_0\mathrm{hom}_R(F, E).$$

Hence we have

$$\sum_{i=2}^{3} (-1)^i\, {}_0\mathrm{ext}^i_{\mathfrak{m}}(F, F) = \sum_i (\beta_{1,i} - \beta_{2,i} + \beta_{3,i}) \cdot (h^1(\mathcal{F}(i)) - h^2(\mathcal{F}(i)) + h^3(\mathcal{F}(i))).$$

It is easy to substitute the non-vanishing of the Hom-groups of Theorem 3.1 by the non-triviality of certain graded Betti numbers in the minimal resolution of F. Indeed, we have the following.

Theorem 3.6 *Let $\mathcal{F}$ be a reflexive sheaf of rank 2 on $\mathbb{P}^3$, let $M = H^1_*(\mathcal{F})$, and suppose $M \neq 0$ is of diameter 1 and concentrated in degree c. Then $\mathcal{F}$ is obstructed if and only if*

$$\beta_{1,c} \cdot \beta_{2,c+4} \neq 0 \quad \textit{or} \quad \beta_{1,c+4} \cdot \beta_{2,c+4} \neq 0 \quad \textit{or} \quad \beta_{1,c} \cdot \beta_{2,c} \neq 0 \,.$$

Moreover, if $\mathcal{F}$ is an unobstructed stable sheaf and $\dim_k M = r$, then the dimension of the moduli scheme $\mathrm{M}_{\mathbb{P}^3}(c_1, c_2, c_3)$ at $(\mathcal{F})$ is

$$\dim_{(\mathcal{F})} \mathrm{M}_{\mathbb{P}^3}(c_1, c_2, c_3) = 8c_2 - 2c_1^2 - 3 + {}_0\mathrm{hom}_R(F, E) + r(\beta_{1,c+4} + \beta_{2,c}).$$

Before proving Theorem 3.6, we remark that we have the following result.

Proposition 3.7 *Let $\mathcal{F}$ be a reflexive sheaf of rank 2 on $\mathbb{P}^3$ and suppose $M \neq 0$ is of diameter 1. Then $\mathcal{F}$ is obstructed if and only if at least one of the following conditions holds:*

(i) ${}_0\mathrm{Hom}_R(F, M) \neq 0$ *and* ${}_0\mathrm{Hom}_R(M, E) \neq 0$,
(ii) ${}_{-4}\mathrm{Hom}_R(F, M) \neq 0$ *and* ${}_0\mathrm{Hom}_R(M, E) \neq 0$,
(iii) ${}_0\mathrm{Hom}_R(F, M) \neq 0$ *and* ${}_{-4}\mathrm{Hom}_R(M, E) \neq 0$.

Proof Indeed, if $\mathcal{F}$ is obstructed, then it is a simple reformulation of Theorem 3.1 to see that we have either (i) or (ii) or (iii). The converse follows immediately from Proposition 3.2 by letting $M' = 0$. ∎

Proof of Theorem 3.6 By applying ${}_v\mathrm{Hom}_R(-, M)$ to the minimal resolution (3.5), we get

$$(3.8) \qquad {}_0\mathrm{hom}_R(F, M) = r\beta_{1,c} \quad \text{and} \quad {}_{-4}\mathrm{hom}_R(F, M) = r\beta_{1,c+4}$$

because $\mathfrak{m} \cdot M = 0$. Moreover, we have ${}_{-v-4}\,\mathrm{Ext}^1_R(F, M)^\vee \simeq {}_v\mathrm{Hom}_R(M, E)$ by (1.3) and (1.2). Computing ${}_{-v-4}\,\mathrm{Ext}^1_R(F, M)$ via the minimal resolution (3.5) of F as in (3.8), we get

$${}_0\mathrm{hom}_R(M, E) = r\beta_{2,c+4} \quad \text{and} \quad {}_{-4}\mathrm{hom}_R(M, E) = r\beta_{2,c} \,.$$

Since $r \neq 0$, we get the unobstructedness criterion and the dimension formula of Theorem 3.6 from Proposition 3.7 and Theorem 3.1. ∎

Theorem 3.8 *Every irreducible component V of $\mathrm{M}_{\mathbb{P}^3}(c_1, c_2, c_3)$ whose generic sheaf $\mathcal{F}$ satisfies* $\operatorname{diam} M \leq 1$ *is* reduced *(*i.e., *generically smooth). Moreover, if* $\dim_k M = r$, *then*

$$\dim V = 8c_2 - 2c_1^2 - 3 + {}_0\mathrm{hom}_R(F, E) + r(\beta_{1,c+4} + \beta_{2,c}).$$

Proof By replacing $\mathcal{F}$ by $\mathcal{F}(j)$ for $j \gg 0$ (see the proof of Theorem 3.1(i)), we can use the Hartshorne–Serre correspondence to get a corresponding curve X such that all assumptions of Corollary 2.3 are satisfied. Hence X is generic and $\operatorname{diam} H^1_*(\mathcal{I}_X) \leq 1$. Since it is proved in [22, Cor. 4.3] that a generic curve X of diameter at most one is unobstructed, it follows by Corollary 2.3 that $\mathcal{F}$ is unobstructed, *i.e.,* that the corresponding component of $\mathrm{M}_{\mathbb{P}^3}(c_1, c_2, c_3)$ is generically smooth. Since the dimension formula follows from Theorem 3.6 (and from Theorem 3.1 if $M = 0$), we are done. ∎

Example 3.9 (*char* $k = 0$) Using some results of Chang on Ω-resolutions of Buchsbaum curves ([2] or [47, Thm. 4.1]), one shows that there exists a smooth connected curve X of diameter 1 satisfying $h^0(\mathcal{I}_X(e)) = 1$, $h^1(\mathcal{I}_X(e)) = r$, $h^1(\mathcal{O}_X(e)) = b$, $h^1(\mathcal{O}_X(v)) = 0$ for $v > e$ and with $e = 1 + b + 2r$ and Ω-resolution

$$0 \to \mathcal{O}_{\mathbb{P}}(-2)^{3r-1} \oplus \mathcal{O}_{\mathbb{P}}(-4)^b \to \mathcal{O}_{\mathbb{P}} \oplus \Omega^r \oplus \mathcal{O}_{\mathbb{P}}(-3)^{b-1} \to \mathcal{I}_X(e) \to 0$$

for every pair (r, b) of positive integers (cf. [22, Ex. 3.12]). Moreover, the degree and genus of X are $d = \binom{e+4}{2} - 3r - 7$ and $g = (e+1)d - \binom{e+4}{3} + 5$. Recalling that Ω corresponds to the first syzygy in the Koszul resolution of the regular sequence $\{X_0, X_1, X_2, X_3\}$, we get an exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}}(-4) \to \mathcal{O}_{\mathbb{P}}(-3)^4 \to \mathcal{O}_{\mathbb{P}}(-2)^6 \to \Omega \to 0\,.$$

Hence we can use the mapping cone construction to show that there is a resolution

$$\begin{aligned}(3.9)\quad 0 \to \mathcal{O}_{\mathbb{P}}(-4)^r \to \mathcal{O}_{\mathbb{P}}(-4)^b \oplus \mathcal{O}_{\mathbb{P}}(-3)^{4r} \oplus \mathcal{O}_{\mathbb{P}}(-2)^{3r-1} \\ \to \mathcal{O}_{\mathbb{P}}(-2)^{6r} \oplus \mathcal{O}_{\mathbb{P}} \oplus \mathcal{O}_{\mathbb{P}}(-3)^{b-1} \to \mathcal{I}_X(e) \to 0,\end{aligned}$$

where we may possibly skip the factor $\mathcal{O}_{\mathbb{P}}(-2)^{3r-1}$ (thus reducing $\mathcal{O}_{\mathbb{P}}(-2)^{6r}$ to $\mathcal{O}_{\mathbb{P}}(-2)^{3r+1}$) to get a minimal resolution. Instead of looking into this problem, we will illustrate [22, Thm. 4.1], which makes a deformation theoretic improvement to a theorem of Rao ([38, Thm. 2.5]). Indeed, since the composition of the leftmost non-trivial map in (3.9) with the projection onto $\mathcal{O}_{\mathbb{P}}(-2)^{3r-1}$ is zero by Rao's theorem, there is, by [22, Thm. 4.1], a deformation with constant cohomology and Rao module to a curve that makes $\mathcal{O}_{\mathbb{P}}(-2)^{3r-1}$ redundant (no matter whether the original factor was redundant or not)! So we certainly may skip the factor $\mathcal{O}_{\mathbb{P}}(-2)^{3r-1}$ and reduce $\mathcal{O}_{\mathbb{P}}(-2)^{6r}$ to $\mathcal{O}_{\mathbb{P}}(-2)^{3r+1}$, at least after a deformation (to a curve that we still denote by X).

Now, by the Hartshorne–Serre correspondence, there is a reflexive sheaf $\mathcal{F}$ given by

$$0 \to \mathcal{O}_{\mathbb{P}} \xrightarrow{\sigma} \mathcal{F} \to \mathcal{I}_X(e+4) \to 0,$$

which, combined with the Horseshoe lemma [48], leads to the following minimal resolution of $\mathcal{F}$,

$$(3.10)\quad 0 \to \mathcal{O}_{\mathbb{P}}^r \to \mathcal{O}_{\mathbb{P}}^b \oplus \mathcal{O}_{\mathbb{P}}(1)^{4r} \to \mathcal{O}_{\mathbb{P}}(2)^{3r+1} \oplus \mathcal{O}_{\mathbb{P}}(4) \oplus \mathcal{O}_{\mathbb{P}}(1)^{b-1} \oplus \mathcal{O}_{\mathbb{P}} \to \mathcal{F} \to 0\,.$$

Note that $h^1(\mathcal{F}(-4)) = h^1(\mathcal{I}_X(e)) = r$, *i.e.*, the number c of Theorem 3.6 is $c = -4$. From (3.10), we see that $\beta_{2,0} = b \neq 0$ and $\beta_{1,-4} = 1$. By Theorem 3.6, $\mathcal{F}$ is obstructed.

Computing Chern classes c_i of $\mathcal{F}$, we get $c_1 = e + 4$, $c_2 = d = \binom{c_1}{2} - 3r - 7$ and $c_3 = \binom{c_1}{3} - \binom{c_1}{2}(3r+7) + 6r + 22$. The simplest case is $(r, b) = (1, 1)$, which yields a reflexive sheaf $\mathcal{F}$ whose normalized sheaf $\mathcal{F}(-4)$ is semistable and with Chern classes

$(c_1', c_2', c_3') = (0, 2, 4)$ (the corresponding curve X has $d = 18$, $g = 39$ and is Sernesi's example of an obstructed curve, see [41] or [5], see also [35], which thoroughly studies $\overline{M}_{\mathbb{P}^3}(0, 2, 4)$ and [34], which uses Sernesi's example to show the existence of a stable rank 3 obstructed vector bundle). For $(r, b) \neq (1, 1)$, then $e > 4$ and we see easily that the obstructed sheaves constructed above are stable. If $(r, b) = (2, 1)$, then the normalized sheaf has Chern classes $(c_1', c_2', c_3') = (0, 7, 24)$, while $(r, b) = (1, 2)$ yields stable sheaves with $(c_1', c_2', c_3') = (-1, 6, 22)$. One may show that all curves corresponding to the sheaves of the case $(r, 1)$ satisfy $h^1(\mathcal{N}_X) = 1$. The ideal of the local ring of $\mathrm{H}(d, g)$ at (X) is generated by a single element, which is irreducible for $r > 1$, see [22, Ex. 3.12, (3.16)]. As in the curve case, we expect that the corresponding point $(\mathcal{F})$, in every case with $r > b = 1$, belongs to a unique irreducible component of $\mathrm{M}_{\mathbb{P}^3}(c_1, c_2, c_3)$, while $(\mathcal{F})$, for sheaves with $r \leq b$ and $b \geq 2$, sits in the intersection of exactly two irreducible components of $\mathrm{M}_{\mathbb{P}^3}(c_1, c_2, c_3)$, see [22, Prop. 4.6], which applies to all curves appearing in this example.

In Examples 2.4 and 2.5, the diameter of M of the obstructed generic sheaves is 3. Combining the results of this paper with the large number of non-reduced components one may find in [19], we can easily produce similar examples for every $\operatorname{diam} M \geq 3$. Indeed, as is well known, a smooth cubic surface $X \subset \mathbb{P}^3$ satisfies $Pic(X) \simeq \mathbb{Z}^{\oplus 7}$. It follows from the main theorem of [19] (or of [37]) that the general curve that corresponds to $(3\alpha, \alpha^5, 2) \in \mathbb{Z}^{\oplus 7}$ is the generic curve of a non-reduced component of $\mathrm{H}(d, g)$ for every $\alpha \geq 4$. (Mumford's example in [36] corresponds to $\alpha = 4$.) The diameter is $2\alpha - 5$. In the same way, the general curve that corresponds to $(3\alpha + 1, \alpha^5, 2) \in \mathbb{Z}^{\oplus 7}$ is the generic curve of a non-reduced component of $\mathrm{H}(d, g)$ with $\operatorname{diam} M = 2\alpha - 4$ for every $\alpha \geq 4$. Using Corollary 2.3 for $c_1 = 2$, we get non-reduced components of $\mathrm{M}_{\mathbb{P}^3}(c_1, c_2, c_3)$ for every $\operatorname{diam} M(\mathcal{F}) \geq 3$, $\mathcal{F}$ the generic sheaf. Thanks to Theorem 3.8, there is only one value of $\operatorname{diam} M(\mathcal{F})$ left, and we expect the following to be true.

Conjecture 3.10 *Every irreducible component of* $\mathrm{M}_{\mathbb{P}^3}(c_1, c_2, c_3)$ *whose generic sheaf* $\mathcal{F}$ *satisfies* $\operatorname{diam} M = 2$ *is* reduced *(*i.e., *generically smooth).*

There is some evidence for this conjecture, namely that *every* Buchsbaum curve of diameter at most 2 admits a generization in $\mathrm{H}(d, g)$ that is unobstructed ([22, Cor. 4.4]), *i.e.,* belongs to a generically smooth irreducible component. By the arguments in the proof of Theorem 3.8, *every* Buchsbaum sheaf of diameter at most 2 must belong to a generically smooth irreducible component of some $\mathrm{M}_{\mathbb{P}^3}(c_1, c_2, c_3)$.

4 A Lower Bound of $\dim \mathrm{M}_{\mathbb{P}^3}(c_1, c_2, c_3)$.

In this section, we want to give a lower bound of the dimension of any irreducible component of $\mathrm{M}_{\mathbb{P}^3}(c_1, c_2, c_3)$ in terms of the graded Betti numbers of a minimal resolution of the graded R-module $F = H^0_*(\mathcal{F})$. The lower bound is straightforward to compute provided we know the dimension of the cohomology groups $H^i(\mathcal{F}(v))$ for any i and v. Recall that it is well known that $\mathrm{ed}(\mathcal{F}) = 8c_2 - 2c_1^2 - 3$ is a lower bound ([14, Prop. 3.4]), but there are many examples of so-called oversized irreducible components whose dimension is strictly greater that $\mathrm{ed}(\mathcal{F})$. Our lower bound is usually much closer to the actual dimension of the oversized components provided $H^1_*(\mathcal{F})$ is "small". If a component of $\mathrm{M}_{\mathbb{P}^3}(c_1, c_2, c_3)$ is generically smooth, we also include a formula for the dimension of the component, which is a sum of the lower bound and a correction number that we make explicit.

Definition 4.1 If $\mathcal{F}$ is a reflexive sheaf of rank 2 on $\mathbb{P}^3$, we let (see (3.5))

$$\delta^j = \sum_i (\beta_{1,i} - \beta_{2,i} + \beta_{3,i}) \cdot h^j(\mathcal{F}(i)).$$

Remark 4.2 If $\mathcal{F}$ is reflexive on $\mathbb{P}^3$ and ${}_{-4}\mathrm{Hom}_R(F, F) = 0$, then

$${}_0\mathrm{hom}_R(F, E) = \delta^2 - \delta^3$$

by Lemma 3.4. This makes the dimension formulas of Theorems 3.1, 3.6, and 3.8 more explicit.

Proposition 4.3 *Let $\mathcal{F}$ be a reflexive sheaf of rank* 2 *on $\mathbb{P}^3$ satisfying* ${}_{-4}\mathrm{Hom}_R(F, F) =$ 0. *Then*

$${}_0\mathrm{ext}^1_R(F, F) - {}_0\mathrm{ext}^2_R(F, F) = {}_0\mathrm{hom}_R(F, F) - \delta^0 = \mathrm{ed}(\mathcal{F}) + \delta^2 - \delta^1 - \delta^3.$$

Proof To see the equality to the left, we apply ${}_0\mathrm{Hom}_R(-, F)$ to the resolution (3.5). We get

$${}_0\mathrm{hom}_R(F, F) - {}_0\mathrm{ext}^1_R(F, F) + {}_0\mathrm{ext}^2_R(F, F) = \delta^0.$$

Moreover the right hand equality follows from (1.3) and (1.4). Indeed, we have already looked at some consequences of (1.3) in Lemma 3.4 and Remark 3.5. We have

$${}_0\mathrm{ext}^2_m(F, F) - {}_0\mathrm{ext}^3_m(F, F) = \delta^1 - \delta^2 + \delta^3$$

by Remark 3.5. Combining with the exact sequence (1.4), which implies

$$\mathrm{ed}(\mathcal{F}) = {}_0\mathrm{ext}^1_R(F, F) - {}_0\mathrm{ext}^2_R(F, F) + {}_0\mathrm{ext}^2_m(F, F) - {}_0\mathrm{ext}^3_m(F, F),$$

we get the last equality. ■

Theorem 4.4 *Let $\mathcal{F}$ be a stable reflexive sheaf of rank* 2 *on $\mathbb{P}^3$. Then the dimension of* $\mathrm{M}_{\mathbb{P}^3}(c_1, c_2, c_3)$ *at* $(\mathcal{F})$ *satisfies*

$$\dim_{(\mathcal{F})} \mathrm{M}_{\mathbb{P}^3}(c_1, c_2, c_3) \geq 1 - \delta^0 = 8c_2 - 2c_1^2 - 3 + \delta^2 - \delta^1 - \delta^3.$$

Moreover, if $\mathcal{F}$ is a generic sheaf of a generically smooth component V of $\mathrm{M}_{\mathbb{P}^3}(c_1, c_2, c_3)$ *and $M = H^1_*(\mathcal{F})$, then*

$$\dim V = 8c_2 - 2c_1^2 - 3 + \delta^2 - \delta^1 - \delta^3 + {}_{-4}\mathrm{hom}_R(F, M)$$

where ${}_{-4}\mathrm{Hom}_R(F, M)$ *is the kernel of the map*

$$\bigoplus_i H^1(\mathcal{F}(i-4))^{\beta_{1,i}} \longrightarrow \bigoplus_i H^1(\mathcal{F}(i-4))^{\beta_{2,i}}$$

induced by the corresponding map in (3.5).

Remark 4.5 Let $\mathcal{F}$ be a stable reflexive sheaf of rank 2 on $\mathbb{P}^3$ and let $M = H^1_*(\mathcal{F})$.

(i) If $M = 0$, then $\delta^1 = 0$ and we can use Theorem 3.8 and Remark 4.2 to see that the lower bound of Theorem 4.4 is equal to $\dim_{(\mathcal{F})} \mathrm{M}_{\mathbb{P}^3}(c_1, c_2, c_3)$. This coincides with [32].

(ii) If diam $M = 1$ and $\mathcal{F}$ is a generic sheaf, then the lower bound is equal to $8c_2 - 2c_1^2 - 3 + {}_0\hom_R(F, E) - \delta^1$ by Remark 4.2. We claim that $-\delta^1 = r\beta_{2,c}$. Indeed, using *e.g.,* Ω-resolutions as in Example 3.9, we easily see that $\beta_{3,i} = 0$ for $i \neq c + 4$. Since we have $r\beta_{1,c} = 0$ for a generic sheaf by [22, Cor. 4.4] and the proof of Theorem 3.8, we get the claim by the definition of δ^1. Moreover, in the diameter one case, the correction number ${}_{-4}\hom_R(F, M)$ is equal to $r\beta_{1,c+4}$. Hence we get the dimension formula of Theorem 3.8 from Theorem 4.4 in this case.

(iii) The lower bound of Theorem 4.4 is clearly better that the bound ed($\mathcal{F}$) provided $\delta^2 > \delta^1 + \delta^3$.

Proof By a general theorem of Laudal ([23, Thm. 4.2.4]), which describes the hull of a local deformation functor, we get that ${}_0\mathrm{ext}^1_R(F, F) - {}_0\mathrm{ext}^2_R(F, F) \leq \dim O_F$, where O_F is the hull of the deformation functor of the graded module F (see Remark 2.6). To get the inequality of the theorem, it suffices, by Proposition 4.3, to prove $\dim O_F \leq \dim_{(\mathcal{F})} \mathrm{M}_{\mathbb{P}^3}(c_1, c_2, c_3)$. Since we will use Corollary 2.3, we replace F by $F(v)$ for $v \gg 0$ to have the assumptions of Corollary 2.3 satisfied. It is known that the Hilbert scheme $\mathrm{H}(d, g)$ contains a subscheme $H := \mathrm{H}(d, g)_\gamma$ that is the representing object of the subfunctor of flat families of curves with fixed postulation γ. For the local deformation functors of $\mathrm{H}(d, g)$ and $\mathrm{H}(d, g)_\gamma$ at a curve (X) the latter corresponds precisely to the graded deformations of the homogeneous coordinate ring of X ([26] and recall $\gamma(v) = h^0(\mathcal{I}_X(v))$, $v \in \mathbb{Z}$, see also [22]). Hence we get

$$\dim O_{H,(X)} = \dim_{(X)} \mathrm{H}(d, g)_\gamma \leq \dim_{(X)} \mathrm{H}(d, g). \tag{4.1}$$

By Corollary 2.3, $\dim_{(\mathcal{F})} \mathrm{M}_{\mathbb{P}^3}(c_1, c_2, c_3) + h^0(\mathcal{F}) = \dim_{(X)} \mathrm{H}(d, g) + h^0(\omega_X(-c_1 + 4))$. We *claim* that

$$\dim O_F + h^0(\mathcal{F}) = \dim O_{H,(X)} + h^0(\omega_X(-c_1 + 4)). \tag{4.2}$$

This is mostly explained in Remark 2.6. Indeed, the natural forgetful map $p_0\colon \mathrm{Def}_{F,H^0_*(\sigma)} \to \mathrm{Def}_F$ is smooth and has the same fiber as the forgetful map $p\colon \mathrm{Def}_{\mathcal{F},\sigma} \to \mathrm{Def}_{\mathcal{F}}$ in Corollary 2.3 by Remark 2.6. In the same way, the corresponding graded variation of $q\colon \mathrm{Def}_{\mathcal{F},\sigma} \to \mathrm{Hilb}_{X/\mathbb{P}^3}$ is smooth by ${}_0\mathrm{Ext}^2(I_X(c_1), R) \simeq \mathrm{Ext}^2(\mathcal{I}_X(c_1), \mathcal{O}_{\mathbb{P}^3}) = 0$ and its fiber coincides with that of q, due to the isomorphism ${}_0\mathrm{Ext}^1(I_X(c_1), R) \simeq \mathrm{Ext}^1(\mathcal{I}_X(c_1), \mathcal{O}_{\mathbb{P}^3})$ (see (1.2) and (1.4)) and the arguments of Remark 2.6. This proves the claim and hence we get the inequality of the theorem.

It remains to prove that ${}_{-4}\hom_R(F, M)$ is the correction number, since the reformulation as a kernel is trivial. Let X be the generic curve of a component of $\mathrm{H}(d, g)$ which corresponds to V. Let γ be the postulation of X. Since there is a smooth open subscheme $U \ni (X)$ of $\mathrm{H}(d, g)$ of curves with postulation γ, we get $\mathrm{H}(d, g)_\gamma \cap U = \mathrm{H}(d, g) \cap U$. Hence $\mathrm{H}(d, g)_\gamma$ is smooth at (X) and we have equality in (4.1). By Corollary 2.3 and (4.2), $\dim O_F = \dim_{(\mathcal{F})} \mathrm{M}_{\mathbb{P}^3}(c_1, c_2, c_3)$ and O_F is smooth. Hence $\dim O_F = {}_0\mathrm{ext}^1_R(F,F)$ and ${}_0\mathrm{ext}^2_R(F,F)$ is the correction number by Proposition 4.3. Since we have ${}_0\mathrm{Ext}^2_R(F,F)^\vee \simeq {}_{-4}\mathrm{Hom}_R(F,M)$ by (1.3) and (1.2), the proof is complete. ■

In [22, Lem. 2.2] we proved a result similar to Proposition 4.3 for any curve X with minimal resolution

$$0 \to \bigoplus_i R(-i)^{\beta'_{3,i}} \to \bigoplus_i R(-i)^{\beta'_{2,i}} \to \bigoplus_i R(-i)^{\beta'_{1,i}} \to I_X \to 0, \tag{4.3}$$

implying that

$${}_0\mathrm{ext}^1_R(I_X,I_X) - {}_0\mathrm{ext}^2_R(I_X,I_X) = 1 - \delta^0_I = 4d + \delta^2_I - \delta^1_I, \tag{4.4}$$

where $\delta^j_I = \sum_i(\beta'_{1,i} - \beta'_{2,i} + \beta'_{3,i}) \cdot h^j(\mathcal{I}_X(i))$ and $d = \deg(X)$. Note that the difference of the ext-numbers in (4.4) is a lower bound for $\dim O_{\mathrm{H}(d,g)_\gamma,(X)}$ ([22, proof of Thm. 2.6(i)]). As a by-product of (4.1) and the proof above, we get the following.

Theorem 4.6 *Let X be a curve in $\mathbb{P}^3$. Then the dimension of $\mathrm{H}(d,g)$ at (X) satisfies*

$$\dim_{(X)} \mathrm{H}(d,g) \geq 1 - \delta^0_I = 4d + \delta^2_I - \delta^1_I.$$

Moreover, if X is a generic curve of a generically smooth component V of $\mathrm{H}(d,g)$ and $M := H^1_*(\mathcal{I}_X)$, then

$$\dim V = 4d + \delta^2_I - \delta^1_I + {}_{-4}\mathrm{hom}_R(I_X,M),$$

where ${}_{-4}\mathrm{Hom}_R(I_X,M)$ is the kernel of the map

$$\bigoplus_i H^1(\mathcal{I}_X(i-4))^{\beta'_{1,i}} \to \bigoplus_i H^1(\mathcal{I}_X(i-4))^{\beta'_{2,i}}$$

induced by (4.3).

Remark 4.7 Let X be any curve in $\mathbb{P}^3$ and let $M = H^1_*(\mathcal{I}_X)$.

(i) If $M = 0$, then $\delta^1_I = 0$ and we can use [22, Thm. 2.6] to see that the lower bound of Theorem 4.6 is equal to $\dim_{(X)} \mathrm{H}(d,g)$. This coincides with [7].

(ii) If $\operatorname{diam} M = 1$, $\dim_k M = r$ and X is a generic curve, then the lower bound is equal to $4d + \delta^2_I + r\beta'_{2,c}$ because $r\beta'_{1,c} = 0$ for a generic curve by [22, Cor. 4.4]. Moreover, in this case the "correction" number ${}_{-4}\mathrm{hom}_R(I_X,M)$ is equal to $r\beta'_{1,c+4}$. Hence we get

$$\dim V = 4d + \delta^2_I + r(\beta'_{2,c} + \beta'_{1,c+4}).$$

This coincides with the dimension formula of [22, Thm. 3.4].

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