Families of Artinian and one-dimensional algebras

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Abstract

The purpose of this paper is to study families of Artinian or one dimensional quotients of a polynomial ring R with a special look to level algebras. Let $\mathrm{GradAlg}^H(R)$ be the scheme parametrizing graded quotients of R with Hilbert function H. Let $B \to A$ be any graded surjection of quotients of R with Hilbert function $H_B = (1, h_1, ..., h_j, ...)$ and H_A respectively. If $\dim A = 0$ (resp. $\dim A = \operatorname{depth} A = 1$) and A is a "truncation" of B in the sense that $H_A = (1, h_1, ..., h_{j-1}, \alpha, 0, 0, ...)$ (resp. $H_A = (1, h_1, ..., h_{j-1}, \alpha, \alpha, \alpha, ...)$) for some $\alpha \le h_j$, then we show there is a close relationship between $\mathrm{GradAlg}^{H_A}(R)$ and $\mathrm{GradAlg}^{H_B}(R)$ concerning e.g. smoothness and dimension at the points (A) and (B) respectively, provided B is a complete intersection or provided the Castelnuovo-Mumford regularity of A is at least 3 (sometimes 2) larger than the regularity of B. In the complete intersection case we generalize this relationship to "non-truncated" Artinian algebras A which are compressed or close to being compressed. For more general Artinian algebras we describe the dual of the tangent and obstruction space of deformations in a manageable form which we make rather explicit for level algebras of Cohen-Macaulay type 2. This description and a linkage theorem for families allow us to prove a conjecture of Iarrobino on the existence of at least two irreducible components of $\mathrm{GradAlg}^H(R)$, $H = (1, 3, 6, 10, 14, 10, 6, 2)$, whose general elements are Artinian level algebras of type 2.

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1 Introduction

The main goal of this paper is to contribute to the classification of Artinian and one dimensional graded quotients of a polynomial ring R in n variables (of degree one) over an algebraically closed field k. In particular we study the scheme $\mathrm{GradAlg}^H(R) = \mathrm{GradAlg}(H)$ which parametrizes graded quotients A of R of $\operatorname{depth} A \ge \min(1, \dim A)$ and with Hilbert function H. $\mathrm{GradAlg}^H(R)$ is the representing object of a correspondingly defined functor of flat families and it may be non-reduced. Thus $\mathrm{GradAlg}^H(R)$ may be different from the parameter spaces studied by Iarrobino, Gotzmann and others who study the "same" scheme with the reduced scheme structure. In our approach we try to benefit from having a well described tangent and obstruction space of $\mathrm{GradAlg}^H(R)$ at (A) at our disposal.

An important technique in determining $\mathrm{GradAlg}(H_A)$ is to take a graded surjection $B \to A$ of quotients of R with Hilbert functions H_B and H_A respectively, and, under certain conditions, make the relationship between $\mathrm{GradAlg}(H_A)$ and $\mathrm{GradAlg}(H_B)$ as tight as possible. We review some results of this technique in Section 1. If $B = R/I_B$, let $N_B := \mathrm{Hom}_R(I_B, B)$ and let $\mathrm{reg}(B) = \mathrm{reg}(I_B) - 1$ be the Castelnuovo-Mumford regularity of B. Let

$$... \to \oplus_{i=1}^{r_2} R(-n_{2,i}) \to \oplus_{i=1}^{r_1} R(-n_{1,i}) \to R \to B \to 0$$

be the minimal resolution and let $\epsilon(A/B) = \sum_{i=1}^{r_1} [H_B(n_{1,i}) - H_A(n_{1,i})]$. Our main results in Section 2 apply to $\mathrm{GradAlg}(H_A)$ where A is one-dimensional. We prove (cf. Theorem 12)

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Theorem 1. *Let R be a polynomial k-algebra and let $B = R/I_B \to A = R/I_A$, $I_A \neq 0$, be a graded morphism such that A is Cohen-Macaulay of dimension one and* $\operatorname{depth} B \geq 1$ *and such that $X := \operatorname{Proj}(A) \hookrightarrow Y := \operatorname{Proj}(B)$ is a local complete intersection of codimension $r \geq 0$. Let $H_A(v) = s$ for $v >> 0$ and suppose either*

(a) I_B is generated by a regular sequence (allowing $R = B$), or

(b) $B_v \to A_v$ is an isomorphism for all $v \leq \max_i\{n_{2,i}\}$ and $\dim R - \dim B \geq 2$.

Moreover suppose there is an integer j such that $B_v \simeq A_v$ for all $v \leq j-1$ and such that I_A is $(j+1)$-regular (i.e. $\operatorname{reg}(A) \leq j$, *or equivalently, $H_A(v) = s$ for $v \geq j$). Then* $\dim(N_A)_0 = \dim(N_B)_0 + rs - \epsilon(A/B)$, *and*

$$\dim_{(A)} \operatorname{GradAlg}^{H_A}(R) = \dim_{(B)} \operatorname{GradAlg}^{H_B}(R) + rs - \epsilon(A/B) \ .$$

In particular A is unobstructed as a graded R-algebra (i.e. $\operatorname{GradAlg}^{H_A}(R)$ *is smooth at (A)) if and only if B is unobstructed as a graded R-algebra.*

One may look upon the conditions on j above as assuming the minimal free resolution of $I_{A/B} := I_A/I_B$ to be semi-linear (close to being linear, cf. (5)), and the conditions of (b) and (a) as requiring this j to be large enough in the case Y is not a complete intersection (CI).

Theorem 1 is what we need to treat the case where X consists of s points in generic position on Y in the sense that H_A is the truncation of H_B at the level s and the points are distinct. Indeed Geramita et al. ([13]) defines such a truncation by

$$H_A(i) = \inf\{H_B(i), s\} \quad \text{for} \;\; i \geq 0 \ ,$$

and they show that there exists a reduced scheme X on Y with truncated Hilbert function H_A provided Y is reduced and consists of more than s points. We prove (cf. Corollary 14)

Corollary 2. *Let $Y = \operatorname{Proj}(B)$, $B = R/I_B$, be a reduced scheme consisting of more than s points, and let $X = \operatorname{Proj}(A)$ be s points (avoid Sing$Y$) of codimension r in generic position on Y. Let j be the smallest number such that $H_A(j) \neq H_B(j)$. If Y is not a CI, suppose $j \geq \operatorname{reg}(I_B) + 2$. Then* $\dim(N_A)_0 = \dim(N_B)_0 + rs - \epsilon(A/B)$, *and*

$$\dim_{(A)} \operatorname{GradAlg}^{H_A}(R) = \dim_{(B)} \operatorname{GradAlg}^{H_B}(R) + rs - \epsilon(A/B) \ .$$

Hence A is unobstructed as a graded R-algebra iff B is unobstructed as a graded R-algebra.

Moreover in Corollary 2 (and Theorem 1) we may allow the codimension r to vary along the s points, say such that the i-th point has codimension c_i in Y (Y need not be equidimensional). Then Corollary 2 holds if we replace rs by $\sum_{i=1}^{s} c_i$.

The analogue of Theorem 1 for Artinian algebras is the main result of Section 3 (cf. Theorem 29).

Theorem 3. *Let R be a polynomial k-algebra and let $B = R/I_B \to A = R/I_A$ be a graded morphism such that A is Artinian and* $\operatorname{depth} B \geq \min(1, \dim B)$, *and suppose either*

(a) I_B is generated by a regular sequence (allowing $R = B$), or

(b) $B_v \to A_v$ is an isomorphism for all $v \leq \max_i\{n_{2,i}\}$ and $\dim R - \dim B \geq 2$.

Let F be a free B-module such that $F \to I_{A/B}$ is surjective and minimal, and suppose there is an integer j such that the degrees of minimal generators of the B-module $\ker(F \to I_{A/B})$ *are strictly greater than j (e.g. $B_v \simeq A_v$ for all $v \leq j-1$) and such that I_A is $(j+1)$-regular (i.e. $A_{j+1} = 0$). Then* $\dim(N_A)_0 = \dim(N_B)_0 + \dim {}_0\operatorname{Hom}_B(F, A) - \epsilon(A/B)$, *and*

$$\dim_{(A)} \operatorname{GradAlg}^{H_A}(R) = \dim_{(B)} \operatorname{GradAlg}^{H_B}(R) + \dim {}_0\operatorname{Hom}_B(F, A) - \epsilon(A/B) \ .$$

In particular A is unobstructed as a graded R-algebra if and only if B is unobstructed as a graded R-algebra.

In Proposition 42 we show an improvement of Theorem 3(a) in the case "$B_v \simeq A_v$ for all $v \le j-1$". In this case we can skip the condition $A_{j+1} = 0$, or equivalently $(K_A)_{-j-1} = 0$ (K_A the canonical module) provided the minimal resolution of K_A has no relations in degree greater or equal to j. This generalization applies to algebras which are compressed or close to being compressed. In the compressed case the dimension and the smoothness of $\mathrm{GradAlg}(H_A)$ coincide with the results of [24].

Theorem 3 applies nicely to Artinian truncations and more generally to Artinian quotients A with Hilbert function $H_A = (1, h_1, h_2, ..., h_{j-1}, \alpha, 0, 0, ..)$ where $H_B = (1, h_1, h_2, ..., h_{j-1}, h_j, h_{j+1}, ...)$ and $\alpha \le h_j$. In that case the relationship between $\mathrm{GradAlg}(H_A)$ and the open subset $\mathrm{GradAlg}(H_B)_\eta$ of $\mathrm{GradAlg}(H_B)$ consisting of points (B) where $\mathrm{reg}(I_B) \le \eta$, may be described by an incidence correspondence

$$\begin{array}{ccc} \mathrm{GradAlg}(H_B, H_A)_\eta & \xrightarrow{q} & \mathrm{GradAlg}(H_B)_\eta \subset \mathrm{GradAlg}(H_B) \\ \downarrow^{p} & & \\ \mathrm{GradAlg}(H_A) & & \end{array} \tag{1}$$

where p and q are the natural projections (cf. (8) and Proposition 33 for details).

Proposition 4. *Let $H_B = (1, h_1, h_2, ...)$ be the Hilbert function of an algebra $B \neq R$ satisfying* $\operatorname{depth} B \ge 1$*, and let j, $\eta \le j-2$ and $\alpha \le h_j$ be non-negative integers. Let $H_A = (1, h_1, ..., h_{j-1}, \alpha, 0, 0, ..)$ and look to the maps p and q in* (1)*. Then*

(i) q is smooth and surjective with connected fibers, of fiber dimension $\alpha(h_j - \alpha)$, and

(ii) p is an isomorphism onto an open subscheme of $\mathrm{GradAlg}(H_A)$.

In particular the incidence correspondence (1) *determines a well-defined injective application π from the set of irreducible components W of* $\mathrm{GradAlg}(H_B)_\eta$*, to the set of irreducible components V of* $\mathrm{GradAlg}(H_A)$ *whose general elements satify the Weak Lefschetz property. In this application the generically smooth components correspond. Indeed $V = \pi(W)$ is the closure of $p(q^{-1}(W))$, and we have*

$$\dim V = \dim W + \alpha(h_j - \alpha) \ .$$

Also Theorem 1 allows a corollary very similar to Proposition 4 in the one dimensional case (cf. Proposition 19).

In Section 4 we characterize the tangent and obstruction space of $\mathrm{GradAlg}^H(R)$ at an Artinian algebra (A). Note that if A is Gorenstein with socle degree j and $S_2 I_A$ is the second symmetric power of I_A, then the k-dual of the obstruction space is by [32], Thm. 11 isomorphic to the kernel of the natural map $(S_2 I_A)_j \to ({I_A}^2)_j$, or equivalently, to the cokernel of $(\Lambda^2 I_A)_j \to (I_A \otimes I_A)_j \simeq \mathrm{Tor}_1^R(I_A, K_A)_0$. This result generalizes to the following result (cf. Theorem 36), in which $\mathrm{H}_2(R, A, K_A)$ is the algebra homology, cf. (2).

Theorem 5. *Let $R \to A = R/I_A$ be a graded Artinian quotient with Hilbert function H. Then $\dim(I_A \otimes_R K_A)_0$ is the dimension of the tangent space of* $\mathrm{GradAlg}^H(R)$ *at (A), and the dual of ${}_0\mathrm{H}_2(R, A, K_A)$ contains the obstructions of deforming A as a graded R-algebra. In particular* $\mathrm{GradAlg}^H(R)$ *is smooth at (A) provided the natural "antisymmetrization" map*

$$I_A \otimes_R I_A \otimes_R K_A \to \mathrm{Tor}_1^R(I_A, K_A)$$

(cf. (17)*) is surjective in degree zero.*

Since we in Theorem 44 show that the parameter space $\mathrm{L}(H)$ of level algebras, introduced in [9] through the ideas of [26], is sufficiently close to being an open subscheme of $\mathrm{GradAlg}^H(R)$, we get that Theorem 5 holds if we everywhere replace $\mathrm{GradAlg}^H(R)$ by $\mathrm{L}(H)$ at a level algebra (A). Note that the tangent space of $\mathrm{L}(H)$ is already well described in [9]. Moreover if A is Artinian of codimension 3 in R, we show that the obstruction space is contained in the dual of $(N_A)_{-3}$ and we make explicit a formula which is a lower bound for the dimension of any irreducible component of $\mathrm{GradAlg}^H(R)$ (Proposition 39 and Corollary 40).

Finally we look to type 2 level algebras $A = R/ann(F_1, F_2)$ where F_1 and F_2 are forms of the same degree s in the "dual" polynomial algebra of R. Such algebras are studied in [25], and an extended draft of [25] determines the tangent space of $\mathrm{GradAlg}^H(R)$ at (A). If $H_A(i) = \min\{\dim R_i, H_{A_1}(i)+ H_{A_2}(i)\}$ for any i, then $\{F_1, F_2\}$ is said to be complementary [25]. Using Theorem 5 we describe the tangent and obstruction space of $\mathrm{GradAlg}^H(R)$ at (A) in the following way (see Proposition 45).

Proposition 6. *Let $\{F_1, F_2\}$ be complementary forms of degree s, and let $A = R/I_A$ be the Artinian level quotient with Hilbert function H given by $I_A = ann(F_1, F_2)$. Let $I_{A_i} = ann(F_i)$. Then $(I_A/I_A \cdot I_{A_1})_s \oplus (I_A/I_A \cdot I_{A_1})_s$ is the dual of the tangent space of $\mathrm{GradAlg}^H(R)$ at (A), and ${}_s\mathrm{H}_2(R, A, A_1) \oplus {}_s\mathrm{H}_2(R, A, A_2)$ is the dual of a space containing the obstructions of deforming A as a graded R-algebra. In particular if the sequences*

$$I_A \otimes_R I_A \xrightarrow{\lambda} I_A \otimes I_{A_i} \to I_A \cdot I_{A_i}$$

where $\lambda(x \otimes y) = x \otimes y - y \otimes x$, are exact for $i = 1$ and 2, then $\mathrm{GradAlg}(H_A)$ is unobstructed at (A) and we have $\dim_{(A)} \mathrm{GradAlg}^{H_A}(R) = \sum_{i=1}^{2} \dim(I_A/I_A \cdot I_{A_i})_s$.

Then we use Proposition 6 and a linkage theorem (Theorem 24) to prove a conjecture of A. Iarrobino, appearing in the draft of [25], namely that $\mathrm{L}(H)$ with $H = (1, 3, 6, 10, 14, 10, 6, 2)$ contains at least two irreducible components whose general elements are level quotients of type 2 (Example 49). Once having one example of such a phenomena, we produce infinitely many by liaison (Remark 50). Even though this conjecture was open until now, Iarrobino and Boij have in a joint work already constructed other examples of reducible $\mathrm{L}(H)$ whose general elements are type 2 level quotients, one with $H = (1, 3, 6, 10, 14, 18, 20, 20, 12, 6, 2)$, and they have got a doubly infinite series of such components [5].

In this paper we give many examples to illustrate our results, some of them with the use of Macaulay 2. Among examples of particular interest, in addition to the proven conjecture, we mention Example 21 of two irreducible components of $\mathrm{GradAlg}(H)$ (and of $\mathrm{PGor}(H)$) whose intersection contains Artinian Gorenstein algebras, and Example 41 of two components of $\mathrm{GradAlg}(H)$ with $H = (1, 3, 6, 6, 3, 1)$ whose general elements of both components are Artinian licci algebras.

Together with co-authors we have in several papers ([30], [32], [34], [35], [36] and [33] which makes a correction to [36], Ch. 10) studied the scheme $\mathrm{GradAlg}^H(R)$ and its subset $\mathrm{PGor}(H)$ which parametrizes Gorenstein quotients. The latter is essentially an open subscheme of the former (cf. [32], Thm. 11, or Theorem 44 of this paper). One will see from [34], [36] and [33] that our cohomological methods often require the quotients to have depth at least two (in which case $\mathrm{GradAlg}^H(R)$ is locally isomorphic to the usual Hilbert scheme, see Propostion 8). In [32] and partially in [34], we were, however, able to treat Gorenstein algebras $A = B/I_B$ of any dimension satisfactorily, utilizing properties of the canonical module K_B. In the present paper we take a major further step in generalizing our notable cohomological and infinitesimal approach to treat non-Gorenstein low dimensional algebras.

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1.1 Preliminaries

Let R be a polynomial k-algebra in n variables of degree 1 where k is algebraically closed. In the following we focus on the scheme parametrizing Artinian graded quotients B of R, as well as the scheme parametrizing closed schemes $Y = \mathrm{Proj}(B)$ in $\mathbb{P} = \mathbb{P}^{n-1}$, with fixed Hilbert function H. Both schemes are denoted by $\mathrm{GradAlg}^H(R)$. When we write $Y = \mathrm{Proj}(B)$, we always take B as the homogeneous coordinate ring of Y and the Hilbert function of B, or Y, as $H_B(v) = H_Y(v) := \dim B_v$. Now if $H(v) = \dim B_v$ does not vanish for $v >> 0$, we call $\mathrm{GradAlg}^H(R)$ the postulation Hilbert scheme because this name seems to be most common, at least when it is endowed with its reduced scheme structure and $\dim B = 1$ (cf. [15], [26]). Since $\mathrm{GradAlg}^H(R)$ is the representing object of a certain functor of flat deformations, it may be non-reduced. We continue denoting it by $\mathrm{GradAlg}^H(R)$, to make it clear that it may be non-reduced.

Now we recall the definition of $\mathrm{GradAlg}^H(R)$. Let $\mathrm{Hilb}^p(\mathbb{P})$ be the Hilbert scheme ([17]) parametrizing closed subschemes Y of $\mathbb{P} = \mathrm{Proj}(R)$ with Hilbert polynomial $p \in \mathbb{Q}[t]$. The k-point of $\mathrm{Hilb}^p(\mathbb{P})$ which corresponds to Y is denoted by (Y). A closed subscheme Y of $\mathbb{P}$ is called *unobstructed* if $\mathrm{Hilb}^p(\mathbb{P})$ is smooth at (Y).

Let $\mathrm{GradAlg}(H) := \mathrm{GradAlg}^H(R)$ be the stratum of $\mathrm{Hilb}^p(\mathbb{P})$ given by deforming $Y = \mathrm{Proj}(B) \subset \mathbb{P}$ with constant Hilbert function $H_B = H$ (more precisely its functor deforms the *homogeneous coordinate ring*, $B = R/I_B$, of Y flatly), cf. [30] or [32]. $\mathrm{GradAlg}^H(R)$ allows a natural scheme structure whose tangent (resp. "obstruction") space at (Y) is ${}_0\mathrm{Hom}_B(I_B/I_B^2, B) \simeq {}_0\mathrm{Hom}_R(I_B, B)$ (resp. ${}_0\mathrm{H}^2(R, B, B)$), i.e. it is given by deforming B as a graded R-algebra ([28], Thm. 1.5). In the case $H(v)$ does not vanish for large v (i.e. B is non-Artinian), we may look upon $\mathrm{GradAlg}^H(R)$ as parametrizing graded R-quotients, $R \to B$, satisfying $\mathrm{depth}_{\mathfrak{m}} B \geq 1$ and with Hilbert function $H_B = H$. If B is Artinian, $\mathrm{GradAlg}^H(R)$ still represents a functor parametrizing graded R-quotients with Hilbert function $H_B = H$ (see [32], Prop. 9 and Thm. 11). B is called *unobstructed as a graded R-algebra* if and only if (iff) $\mathrm{GradAlg}(H_B)$ is smooth at (B), i.e. at (Y).

Remark 7. *(a) It follows from a theorem of Pardue (Thm. 34 of [47], cf. [15] for the codimension 2 case) that* $\mathrm{GradAlg}(H)$ *is a connected scheme (see also [40]).*

(b) Note also that Ragusa-Zappala's result for zero-schemes ([49]), that different minima of the set of graded Betti numbers yield different components of $\mathrm{GradAlg}(H)$, *is valid for any H and R. This follows from the representability of the functor which defines* $\mathrm{GradAlg}(H)$ *([32], Prop. 9), the semicontinuity of the graded Betti numbers and Ragusa-Zappala's proof because we may avoid the "very flatness" argument in their proof by using the flatness of the representing object of the mentioned functor. Note that the set of graded Betti numbers is partially ordered because* each *Betti number (i.e. the number of minimal generators of a fixed degree of some finitely generated syzygy R-module) obey semicontinuity and may decrease under generization. Thus different minima may occur. In particular incomparable sets (i.e. sets without a common minimum) of graded Betti numbers lead to different components in general (see [43] for a discussion).*

The following comparison result is due to Ellingsrud ([12]) in the case $\mathrm{depth}_{\mathfrak{m}} B \geq 2$, see [28], Thm. 3.6 and Rem. 3.7 for the general case. Below $s(I_B)$ is the minimal degree of the minimal generators of I_B. Note that the openness statements follow easily from the first isomorphism by the semicontinuity of $\dim \mathrm{H}^1(Y, \widetilde{I_B}(v))$.

Proposition 8. *Let $B = R/I_B$ satisfy* $\mathrm{depth}_{\mathfrak{m}} B \geq 1$ *and let $Y = \mathrm{Proj}(B)$. Then*

$$\mathrm{GradAlg}^H(R) \simeq \mathrm{Hilb}^p(\mathbb{P}) \quad \text{at} \quad (Y) \ ,$$

provided ${}_0\mathrm{Hom}_R(I_B, \mathrm{H}^1_{\mathfrak{m}}(B)) = 0$ *(e.g. provided* $\mathrm{depth}_{\mathfrak{m}} B \geq 2$*). In particular the open sets*

$$U(H) := \{(B) \in \mathrm{GradAlg}^H(R) | \ {}_v\mathrm{H}^1_{\mathfrak{m}}(B) = 0 \text{ for every } v \geq s(I_B)\}$$

and $\{(B) \in \mathrm{GradAlg}^H(R) \mid \mathrm{depth}_{\mathfrak{m}} B \geq 2\}$ *of* $\mathrm{GradAlg}^H(R)$ *are also open in* $\mathrm{Hilb}^p(\mathbb{P})$.

Here $\mathrm{depth}_{\mathfrak{m}} M$ (or just $\mathrm{depth}\, M$, M finitely generated) denotes the length of a maximal M-sequence in the irrelevant maximal ideal $\mathfrak{m}$, and $\mathrm{H}^i_{\mathfrak{m}}(-)$ is the right derived functor of the functor of sections with support in $\mathrm{Spec}(B/\mathfrak{m})$. Note that $\mathrm{depth}_{\mathfrak{m}} M \geq r$ iff $\mathrm{H}^i_{\mathfrak{m}}(M) = 0$ for $i < r$, cf. [19]. A Cohen-Macaulay B-module M satisfies $\mathrm{depth}\, M = \dim M$ by definition. If B is Cohen-Macaulay of codimension c in R and $K_B = \mathrm{Ext}^c_R(B, R(-n))$ is the canonical module of B, we know by Gorenstein duality that the v-graded piece of $\mathrm{H}^i_{\mathfrak{m}}(M)$ satisfies (cf. [19])

$$_v\mathrm{H}^i_{\mathfrak{m}}(M) \simeq \ _{-v}\mathrm{Ext}^{n-c-i}_B(M, K_B)^\vee \ .$$

Two graded quotients, R/J and R/J', are said to be linked by a complete intersection if there exists a homogeneous complete intersection ideal L such that $J = L : J'$ and $J' = L : J$ (with $L \subseteq J \cap J'$). The relationship of being linked generates the equivalence relation, "linkage". $B = R/I_B$ is said to be licci (and hence Cohen-Macaulay) if it is in the linkage class of a complete intersection (cf. [44] for a survey).

The algebraic (co)homology groups $\mathrm{H}_2(R, B, M)$ and $\mathrm{H}^2(R, B, M)$ may be described as follows. The former group is given by an exact sequence

$$0 \to \mathrm{H}_2(R, B, M) \to \mathrm{H}_1 \otimes_B M \to G_1 \otimes_R M \to I_B/I_B^2 \otimes_B M \to 0. \tag{2}$$

in which G_1 is R-free, $G_1 \twoheadrightarrow I_B$ is surjective and minimal, and $\mathrm{H}_1 = \mathrm{H}_1(I_B)$ is the degree one Koszul homology of I_B [39]. For the graded group $\mathrm{H}^2(R, B, M)$ we only remark that by [1], Prop. 16.1, and [39], there are injections

$$_0\mathrm{Ext}^1_B(I_B/I_B^2, M) \hookrightarrow \ _0\mathrm{H}^2(R, B, M) \hookrightarrow \ _0\mathrm{Ext}^1_R(I_B, M) \tag{3}$$

A quotient $B = R/I_B$ of codimension $c := \dim R - \dim B$ in R has a minimal R-free resolution of the following form (cf. [11])

$$... \to G_c \to ... \to G_1 \to R \to B \to 0 \ , \qquad G_j = \oplus_{i=1}^{r_j} R(-n_{j,i}) \tag{4}$$

and B is Cohen-Macaulay (CM) iff $G_{c+1} = 0$. The function $\max_i\{n_{j,i}\} - j$ is increasing as a function in j if B is CM. If B is Artinian (i.e. $c = n$), then $\max_i\{n_{c,i}\} - c$ is the socle degree of B. More generally the Castelnuovo-Mumford regularity of I_B is given by $\mathrm{reg}(I_B) = \max_{\{j,i\}} \{n_{j,i} - j + 1\}$ and $\mathrm{reg}(B) = \mathrm{reg}(I_B) - 1$ (cf. [44], p. 8). In particular

$$\max_i\{n_{j,i}\} \leq \mathrm{reg}(I_B) + j - 1 \quad \text{for any } j.$$

If $G_{c+1} = 0$ and G_c has rank 1 (resp. $G_c = R(-s)^t$), then B is Gorenstein (resp. level of type t). In these cases B is a compressed Artinian R-algebra if H_B (i.e. $H_B(v)$ for any v) is as large as possible for a fixed socle degree and fixed type (cf. [24] for existence).

An R-module M of projective dimension $t - 1$ is said to have a **semi-linear** (resp. linear) resolution provided the minimal resolution of M has the following form

$$0 \to R(-j-t)^{\beta_t} \oplus R(-j-t+1)^{\alpha_t} \to ... \to R(-j-1)^{\beta_1} \oplus R(-j)^{\alpha_1} \to M \to 0 \tag{5}$$

(resp. with $\alpha_i = 0$ for all i). With B as in (4) and $B \to A \simeq B/I_{A/B}$ a graded surjection, we define

$$\epsilon = \epsilon(A/B) = \sum_{i=1}^{r_1} \dim(I_{A/B})_{n_{1,i}} = \sum_{i=1}^{r_1} \left[H_B(n_{1,i}) - H_A(n_{1,i})\right] \tag{6}$$

where H_B and H_A are the Hilbert functions of B and A. If B is a complete intersection (CI), allowing $R = B$, then I_B/I_B^2 and the normal module $N_B = (I_B/I_B^2)^*$ are R-free of rank $r_1 \geq 0$, and

$$\dim_{(B)} \mathrm{GradAlg}^{H_B}(R) = \dim(N_B)_0 = \sum_{i=1}^{r_1} H_B(n_{1,i})$$

In Lemma 9 and Theorem 12 of the next section we look upon the special case $B = R$ as a CI with $r_1 = 0$. Throughout we pass to small letters to denote the k-vector space dimension of the (co)homology groups involved, e.g. for any $i \geq 0$,

$$\mathrm{h}^i(\widetilde{M}) = \dim \mathrm{H}^i(\widetilde{M}),\ {}_v\mathrm{h}^i(R,B,M) = \dim\ {}_v\mathrm{H}^i(R,B,M),\ {}_v\mathrm{ext}^i_B(M,N) = \dim\ {}_v\mathrm{Ext}^i_B(M,N).$$

Lemma 9. *Let $R \to B = R/I_B \to A \simeq B/I_{A/B}$ be graded morphisms, let $c = \dim R - \dim B$ and suppose either*

(a) I_B is generated by a regular sequence (allowing $R = B$), or

(b) $c \geq 2$ and $B_v \to A_v$ is an isomorphism for all $v \leq \max_i\{n_{2,i}\}$.

Then ${}_0\mathrm{H}^2(R,B,I_{A/B}) = 0$ and ${}_0\mathrm{hom}_R(I_B,I_{A/B}) = \epsilon(A/B)$. Moreover $\epsilon(A/B) = 0$ if (b) holds.

Proof. If B is a CI, then it is well known that ${}_0\mathrm{H}^2(R,B,I_{A/B}) = 0$, and moreover that ${}_0\mathrm{H}^1(R,B,I_{A/B}) \simeq {}_0\mathrm{Hom}_R(I_B/I_B^2, I_{A/B}) \simeq (\oplus_i I_{A/B}(n_{1,i}))_0$ and we get the lemma in this case. In (b) it suffices by (3) to show ${}_0\mathrm{Ext}^i_R(I_B,I_{A/B}) = 0$ for $i \leq 1$. Applying ${}_0\mathrm{Hom}_R(-,I_{A/B})$ to the minimal resolution of I_B deduced from (4), we conclude by the assumptions of (b). $\square$

The following Proposition is a part of Prop. 4 of [32] and is used quite often in this paper. Below $\mathrm{GradAlg}(H_B,H_A)$ is the representing object of the functor deforming surjections $B \to A$ of graded quotients of R of positive depth (for non-Artinian quotients) and with Hilbert functions H_B and H_A of B and A respectively. Then there exist natural projection morphisms $p : \mathrm{GradAlg}(H_B,H_A) \to \mathrm{GradAlg}(H_A)$, induced by $p((B \to A)) = (A)$, and $q : \mathrm{GradAlg}(H_B,H_A) \to \mathrm{GradAlg}(H_B)$, induced by $q((B \to A)) = (B)$, which under the assumptions of Prop. 4 of [32] have nice properties. Recall that A is called *H_B-generic* if there is an *open* subset $U \ni (A)$ of $\mathrm{GradAlg}^{H_A}(R)$ such that every $(A') \in U$ belongs to $\operatorname{im} p$. Now since the surjectivity of the natural map ${}_0\mathrm{Hom}_B(I_B,B) \to {}_0\mathrm{Hom}_R(I_B,A)$ together with the injectivity of ${}_0\mathrm{H}^2(R,B,B) \to {}_0\mathrm{H}^2(R,B,A)$ is equivalent to

$${}_0\mathrm{H}^2(R,B,I_{A/B}) = 0$$

by the long exact sequence of algebra cohomology, we may state [32], Prop. 4 (i), resp. (ii) as (i), resp. (ii) of the Proposition below.

Proposition 10. *Let $B = R/I_B \to A \simeq B/I_{A/B}$ be a graded morphism of quotients of R.*

(i) If ${}_0\mathrm{H}^2(B,A,A) = 0$, (e.g. ${}_0\mathrm{Ext}^1_B(I_{A/B},A) = 0$), then the projection $q : \mathrm{GradAlg}(H_B,H_A) \to \mathrm{GradAlg}(H_B)$ induced by $q((B \to A)) = (B)$ is smooth with fiber dimension ${}_0\mathrm{hom}_B(I_{A/B},A)$ at $(B \to A)$.

(ii) If ${}_0\mathrm{H}^2(R,B,I_{A/B}) = 0$, then the projection $p : \mathrm{GradAlg}(H_B,H_A) \to \mathrm{GradAlg}(H_A)$ induced by $p((B \to A)) = (A)$ is smooth with fiber dimension ${}_0\mathrm{hom}_R(I_B,I_{A/B})$ at $(B \to A)$. In particular A is H_B-generic.

Corollary 11. *Let $B \to A$ be a graded surjection of quotients of R. If ${}_0\mathrm{H}^2(B,A,A) = 0$ and ${}_0\mathrm{H}^2(R,B,I_{A/B}) = 0$, then A is H_B-generic, and we have*

$$\dim(N_A)_0 + {}_0\mathrm{hom}_R(I_B,I_{A/B}) = \dim(N_B)_0 + {}_0\mathrm{hom}_B(I_{A/B},A)\ , \text{ and}$$

$$\dim_{(A)} \mathrm{GradAlg}(H_A) + {}_0\mathrm{hom}_R(I_B,I_{A/B}) = \dim_{(B)} \mathrm{GradAlg}(H_B) + {}_0\mathrm{hom}_B(I_{A/B},A)\ .$$

Hence A is unobstructed as a graded R-algebra iff B is unobstructed as a graded R-algebra.

Proof. Using Proposition 10(i) we get

$$\dim_{(B\to A)} \operatorname{GradAlg}(H_B, H_A) = \dim_{(B)} \operatorname{GradAlg}(H_B) + {}_0\hom_B(I_{A/B}, A)$$

while (ii) implies $\dim_{(B\to A)} \operatorname{GradAlg}(H_B, H_A) = \dim_{(A)} \operatorname{GradAlg}(H_A) +_0 \hom_R(I_B, I_{A/B})$ which gives one of the dimension formulas. Since smooth morphisms imply surjective tangent maps of their tangent spaces and since the Hom-groups of Proposition 10 are the tangent spaces of the fibers, we can argue as above to get the other dimension formula. □

2 Families of one dimensional quotients of R.

In this section we focus on families of zero schemes in $\mathbb{P} = \mathbb{P}^{n-1}$ with fixed Hilbert function H, i.e. we study the (possibly non-reduced) postulation Hilbert scheme $\operatorname{GradAlg}^H(R)$ where $H(v)$ is a constant for $v >> 0$.

If $Y \subset \mathbb{P} = \operatorname{Proj} R$ is a closed subscheme and $X = \operatorname{Proj}(A)$ is obtained by choosing s points in generic position on $Y = \operatorname{Proj}(B)$ (see the paragraph before Corollary 2 for a definition), the main theorem of this section implies that A and B are simultaneously unobstructed as graded algebras and $\dim_{(A)} \operatorname{GradAlg}^{H_A}(R)$ and $\dim_{(B)} \operatorname{GradAlg}^{H_B}(R)$ are closely related (Theorem 12, Corollary 14). Even though this result may seem new as stated, it is a straightforward consequence of Theorem 9.16 of [36] if Y is a curve. In this section we generalize the result to any scheme Y. In Proposition 19 we extend the result to families, and we finish by a theorem on linkage of families (Theorem 24).

A zero-dimensional closed scheme $X \hookrightarrow Y$ is said to be a local complete intersection (l.c.i) of codimension $(r_1, ..., r_t)$ with respect to $X = X_1 \cup ... \cup X_t$ if X can be written as a disjoint union $X = X_1 \cup ... \cup X_t$ where, for each i, the ideal $(\mathcal{J}_{X/Y})_x$ is generated by an $\mathcal{O}_{Y,x}$-regular sequence of length r_i for every $x \in X_i$. If r_i are equal for all i, say $r_i = r$, we simply say $X \hookrightarrow Y$ is an l.c.i of codimension r. Note that in the case $r_i = 0$, then X_i is mapped isomorphically onto an open subscheme of Y. Below $N_B := \operatorname{Hom}_B(I_B/I_B^2, B)$ is the normal module of B in R, $\epsilon(A/B)$ is defined in (6), $n_{2,j}$ in (4) and the H_B-genericity of A is defined in the text before Proposition 10.

Theorem 12. *Let R be a polynomial k-algebra and let $B = R/I_B \to A = R/I_A$, $I_A \neq 0$, be a graded morphism such that A is Cohen-Macaulay of dimension one and* $\operatorname{depth}_{\mathfrak{m}} B \geq 1$, *and such that $X := \operatorname{Proj}(A) \hookrightarrow Y := \operatorname{Proj}(B)$ is a local complete intersection of codimension $(r_1, ..., r_t)$ with respect to $X = X_1 \cup ... \cup X_t$. Let $H_A(v) = s$ and $H_{X_i}(v) = s_i$ for $v >> 0$ (so $s = \sum_i s_i$) and suppose either*

(a) I_B is generated by a regular sequence (allowing $R = B$), or

(b) $B_v \to A_v$ is an isomorphism for all $v \leq \max_i\{n_{2,i}\}$ and $\dim R - \dim B \geq 2$.

Moreover suppose there is an integer j such that $B_v \simeq A_v$ for all $v \leq j-1$ and such that I_A is $(j+1)$-regular (or equivalently, such that $H_A(v) = H_B(v)$ for $v \leq j-1$ and $H_A(v) = s$ for $v \geq j$). Then A is H_B-generic, $\dim(N_A)_0 = \dim(N_B)_0 + \sum_i r_i s_i - \epsilon(A/B)$, *and*

$$\dim_{(A)} \operatorname{GradAlg}^{H_A}(R) = \dim_{(B)} \operatorname{GradAlg}^{H_B}(R) + \sum_i^t r_i s_i - \epsilon(A/B) .$$

In particular A is unobstructed as a graded R-algebra if and only if B is unobstructed as a graded R-algebra.

Remark 13. *Theorem 12 applies to quotients $B \to A \simeq B/I_{A/B}$ where the mapping cone construction produces the minimal resolution of A from the free resolution of B and a* semi-linear *resolution of $I_{A/B}$ (modulo redundant terms). For instance if $M := I_{A/B}$ and $t = r-1$ in (5) (i.e.*

depth $I_{A/B}=2$), then $B_v \simeq A_v$ for all $v \le j-1$ and I_A is $(j+1)$-regular, and Theorem 12 applies. Also in the case depth $I_{A/B}=1$ in which the "$G_{r-1} \to \ldots \to G_1$"-part of the minimal resolution $0 \to G_r \to \ldots \to G_1 \to I_{A/B} \to 0$ is semi-linear, then Theorem 12 applies because the contribution from G_r becomes redundant in the minimal resolution of A. Thus the condition on j of Theorem 12 essentially requires $I_{A/B}$ to have a semi-linear resolution; in the non-CI case j must be large enough to have (b) fulfilled (e.g. $j \ge \operatorname{reg}(I_B)+2$).

Proof. It is enough to prove that A is H_B-generic and the two dimension formulas. Due to Corollary 11 it suffices to show ${}_0\mathrm{H}^2(B,A,A)=0$ and ${}_0\hom_B(I_{A/B},A)=\sum_i r_i s_i$, as well as ${}_0\mathrm{H}^2(R,B,I_{A/B})=0$ and ${}_0\hom_R(I_B,I_{A/B})=\epsilon(A/B)$. The latter follows from Lemma 9. Let $\mathcal{N}_{X/Y}$ the normal sheaf of $X \hookrightarrow Y$. Since $\dim X=0$ and the composition $x \hookrightarrow X \hookrightarrow Y$ is a local complete intersection for any $x \in X$, then the sequence (9.6) in the proof of Thm. 9.16 of [36] is still exact (cf. [28], Lem. 3.5 and (3.3)) and may be written as

$$0 \to {}_0\mathrm{H}^1(B,A,A) \to \mathrm{H}^0(\mathcal{N}_{X/Y}) \to {}_0\mathrm{Hom}_R(I_{A/B},\mathrm{H}^1_{\mathfrak{m}}(A)) \to {}_0\mathrm{H}^2(B,A,A) \to 0 \ .$$

Hence ${}_0\mathrm{H}^2(B,A,A)=0$ and ${}_0\mathrm{Hom}_B(I_{A/B},A) \simeq {}_0\mathrm{H}^1(B,A,A) \simeq \mathrm{H}^0(\mathcal{N}_{X/Y})$ provided

$${}_0\mathrm{Hom}_R(I_{A/B},\mathrm{H}^1_{\mathfrak{m}}(A))=0 \ .$$

Since $\mathrm{H}^1_{\mathfrak{m}}(A)_v \simeq \mathrm{H}^2_{\mathfrak{m}}(I_A)_v \simeq {}_{-v}\mathrm{Ext}^{n-2}_R(I_A,R(-n))^\vee$, we get that $\mathrm{H}^1_{\mathfrak{m}}(A)_v=0$ for $v \ge j$ by the $(j+1)$-regularity of I_A. Note that since $0 \to A_v \to \mathrm{H}^0(\mathcal{O}_X(v)) \to \mathrm{H}^1_{\mathfrak{m}}(A)_v \to 0$ is exact, it follows that the vanishing of $\mathrm{H}^1_{\mathfrak{m}}(A)_v=0$ for $v \ge j$ is equivalent to $H_A(v)=\dim A_v = s$ for $v \ge j$. Now, using $(I_{A/B})_{j-1}=0$ we conclude that ${}_0\mathrm{Hom}_R(I_{A/B},\mathrm{H}^1_{\mathfrak{m}}(A))=0$.

Hence it suffices to show $\dim \mathrm{H}^0(\mathcal{N}_{X/Y})=\sum_i r_i s_i$. Since $\mathrm{Supp}(X)$ is finite, we know that $\mathrm{h}^0(\mathcal{O}_{X_i})=\sum_{x\in \mathrm{Supp}(X_i)} \mathrm{length}(O_{X_i,x})=s_i$. Using that $\mathcal{N}_{X/Y,x}$ is a free $\mathcal{O}_{X,x}$-module of rank r_i for any $x \in \mathrm{Supp}(X_i)$, we conclude by

$$\mathrm{h}^0(\mathcal{N}_{X/Y})=\sum_i^t \sum_{x\in\mathrm{Supp}(X_i)} \mathrm{length}(\mathcal{N}_{X/Y,x})=\sum_i^t \sum_{x\in\mathrm{Supp}(X_i)} r_i\cdot \mathrm{length}(O_{X_i,x})=\sum_i^t r_i s_i \ .$$

□

Theorem 12 is precisely what we need to treat the case where X consists of s points in generic position on Y (i.e. H_A is the truncation of H_B at the level s and the points are distinct). Indeed if we define the truncation of H_B at the level s by

$$H_A(i)=\inf\{H_B(i),s\} \quad \text{for} \quad i \ge 0 \ ,$$

then a theorem of Geramita-Maroscia-Roberts ([13]) show that there exists a reduced scheme X on Y with truncated Hilbert function H_A as above provided Y is reduced and consists of more than s points. Denoting the singular locus of Y by $SingY$, we get

Corollary 14. *Let $Y=\mathrm{Proj}(B)$, $B=R/I_B$, be a reduced scheme consisting of more than s points, and let $X=\mathrm{Proj}(A)$ be s points (avoid $SingY$) in generic position on Y. Let j be the smallest number such that $H_A(j)\neq H_B(j)$. If Y is not a CI, suppose $j \ge \max_i\{n_{2,i}\}+1$ (e.g. $j \ge \operatorname{reg}(I_B)+2$). Then $X \hookrightarrow Y$ is an l.c.i of codimension $(r_1,\ldots,r_t)$ with respect to some decomposition $X=X_1\cup\ldots\cup X_t$. Moreover A is H_B-generic, $\dim(N_A)_0=\dim(N_B)_0+\sum_i r_i s_i \ -\epsilon(A/B)$, and*

$$\dim_{(A)} \mathrm{GradAlg}^{H_A}(R)=\dim_{(B)} \mathrm{GradAlg}^{H_B}(R)+\sum_i^t r_i s_i \ -\epsilon(A/B) \ .$$

Hence A is unobstructed as a graded R-algebra iff B is unobstructed as a graded R-algebra.

Proof. Since $\mathcal{O}_{Y,x}$ and $\mathcal{O}_{X,x}$ are regular local rings for any $x \in X$, it follows that $X \hookrightarrow Y$ is an l.c.i of codimension as in Corollary 14. By the definition of s generic points, $(I_{A/B})_{j-1} = 0$. Since $H_A(v) \neq H_A(j)$ for $v \geq j$, it follows that I_A is $(j+1)$-regular (and j-regular if $s = H_A(j-1)$). Then Theorem 12 applies supposing j large enough. $\square$

Remark 15. *It is well known that the Hilbert polynomial $p_B(x)$ equals $H_B(x)$ for all $x \geq \operatorname{reg}(I_B)-1$. Thus the number $j \geq \operatorname{reg}(I_B)+2$ of Corollary 14 is so large that $p_B(x) = H_B(x)$ for $x \geq j-3$. In particular using Corollary 14 with say $j = \operatorname{reg}(I_B)+2$, we get an algebra A with Hilbert function $H_A(x) = H_B(x) = p_B(x)$ for $x \in \{j-3, j-2, j-1\}$ and $H_A(x) = s$ for $x \geq j$.*

Example 16. *(an obstructed one-dimensional level algebra A with $H_A = (1,5,9,13,13,13,...)$).*

The subset of the Hilbert scheme $\mathrm{Hilb}^{dx+1-g}(\mathbb{P}^4)$ consisting of rational normal curves of degree $d = 4$ has been thoroughly studied ([42], [46]). Indeed this subset forms a smooth, irreducible open subscheme of $\mathrm{Hilb}^{4x+1}(\mathbb{P}^4)$ whose closure V is an irreducible component of dimension $5d+1 = 21$. All arithmetically CM (ACM) curves are contained in the component V by [42]. Moreover the normal sheaf of the general curve Y_g of V satisfies $\mathrm{H}^1(\mathcal{N}_{Y_g}) = 0$, while for instance $Y = \mathrm{Proj}(B)$, the union of four lines meeting at a point, belongs to the same component V and satisfies $\dim \mathrm{H}^1(\mathcal{N}_Y) = 3$ (cf. [36], Rem. 9.9), i.e. Y is an obstructed reduced ACM curve. Both curves have the same graded Betti numbers, e.g.

$$0 \to R(-4)^3 \to R(-3)^8 \to R(-2)^6 \to R \to B \to 0\ .$$

Since the locus of ACM curves in $\mathrm{GradAlg}(H)$ is open in $\mathrm{Hilb}^{4x+1}(\mathbb{P}^4)$ by Proposition 8, then V corresponds to an irreducible component of $\mathrm{GradAlg}(H)$ to which (Y_g) and (Y) belong. Let $X = \mathrm{Proj}(A)$ (resp. $X_g = \mathrm{Proj}(A_g)$) be obtained by choosing $s \geq 13$ generic points on Y (resp. Y_g). Since $\dim B_v = 4v+1$ for $v \geq 0$, we see that Corollary 14 applies for $j \geq 4$. It follows that A_g is unobstructed while A is obstructed as graded R-algebras and

$$\dim_{(A_g)} \mathrm{GradAlg}(H_{A_g}) = \dim(N_{A_g})_0 = \mathrm{h}^0(\mathcal{N}_{Y_g}) + s = 21 + s$$

(resp. $\dim(N_A)_0 = \dim(N_B)_0 + s = 24 + s$). In particular if $s = 13$, then $H_A = H_{A_g} = (1,5,9,13,13,13,...)$, and it is straightforward to see that A_g and A are level algebras with the same graded Betti numbers, e.g. the minimal resolution of I_A is

$$0 \to R(-7)^4 \to R(-6)^{12} \oplus R(-4)^3 \to R(-5)^{12} \oplus R(-3)^8 \to R(-4)^4 \oplus R(-2)^6 \to I_A \to 0.$$

Corollary 14 applies also to families of reduced schemes Y which are not necessarily equidimensional.

Example 17. *Let $H(x) = 3x+1$ for $x \geq 0$, so $H = (1,4,7,10,13,...)$. If $Y_1 = \mathrm{Proj}(B_1) \subset \mathbb{P}^3$ is a twisted cubic curve and $Y_2 = \mathrm{Proj}(B_2)$ is the union of a plane space curve C of degree 3 and a point P outside the plane containing C, then it is easy to see that both curves belong to the same stratum $\mathrm{GradAlg}(H)$ of the Hilbert scheme $\mathrm{Hilb}^{3x+1}(\mathbb{P}^3)$. We claim they belong to two different components of $\mathrm{GradAlg}(H)$. Indeed (Y_1) belongs to a 12-dimensional irreducible component of $\mathrm{GradAlg}(H)$, and using $\mathcal{N}_{Y_2} \simeq \mathcal{N}_C \oplus \mathcal{N}_P$ and that $C \hookrightarrow \mathbb{P}^3$ and $P \hookrightarrow \mathbb{P}^3$ are CI, we easily get $\mathrm{h}^0(\mathcal{N}_{Y_2}) = 15$ and $\mathrm{H}^1(\mathcal{N}_{Y_2}) = 0$. Invoking Proposition 8 we see that (Y_2) belongs to a 15-dimensional irreducible component of $\mathrm{GradAlg}(H)$, cf. [48] for a complete description of $\mathrm{Hilb}^{3x+1}(\mathbb{P}^3)$. The minimal resolution of I_{B_2} (resp. I_{B_1}) is of the form*

$$0 \to R(-4) \to R(-4) \oplus R(-3)^3 \to R(-3) \oplus R(-2)^3 \to I_{B_2} \to 0 \tag{7}$$

(resp. of the form (7) where both $R(-4)$ and two of $R(-3)$ are removed). Since Corollary 14 applies for $j \geq 5$, let $X_1 = \mathrm{Proj}(A_1)$ (resp. $X_2 = \mathrm{Proj}(A_2)$) be obtained by choosing $s \geq 13$ generic

points on Y_1 (resp. Y_2); on Y_2 we must choose P as one of the s generic points to get the right Hilbert function. It follows that A_i are unobstructed as graded R-algebras for $i = 1$ and 2 and that $\dim_{(A_1)} \operatorname{GradAlg}(H') = 12 + s$ *where* $H' = (1, 4, 7, ..., 3j-2, s, s, ...)$, $3j - 2 \le s < 3j + 1$. *Since $X_2 \hookrightarrow Y_2$ is an l.c.i of codimension $(1, 0)$ with respect to the decomposition $X_2 = C_2 \cup P$ where C_2 consists of $s - 1$ points, we get*

$$\dim_{(A_2)} \operatorname{GradAlg}(H') = 15 + s - 1 = 14 + s \ .$$

Hence we get two different components of $\operatorname{GradAlg}(H')$. *Finally if $s = 13$ it is straightforward to see that A_2 has the minimal resolution*

$$0 \to R(-7)^3 \oplus R(-4) \to R(-6)^6 \oplus R(-4) \oplus R(-3)^3 \to R(-5)^3 \oplus R(-3) \oplus R(-2)^3 \to I_{A_2} \to 0.$$

Once the connection between $\operatorname{GradAlg}(H_A)$ and $\operatorname{GradAlg}(H_B)$ for s generic points X on Y is as nice as described in Corollary 14, one may also ask if their irreducible components correspond exactly and similar questions. E.g., may we look upon A_g of Example 16 as the general element of an irreducible component of $\operatorname{GradAlg}(H_{A_g})$? To see that the answer is yes we use some ideas of [32].

Definition 18. *Inside* $\operatorname{GradAlg}^H(R)$, $H \neq H_R$, *we look to the following open subsets,* $\operatorname{Smc}(H)$ *(resp.* $\operatorname{SmCM}(H)$*), consisting of points (R/I) such that* $\operatorname{Proj}(R/I)$ *is a smooth geometrically connected scheme (resp. smooth and arithmetically CM). Here "points" should be considered as "Ω-points" where Ω is an overfield of k. Moreover let* $\operatorname{Smc}(H)_\eta$ *be the open subset of* $\operatorname{Smc}(H)$ *consisting of points (R/I) where the Castelnuovo-Mumford regularity satisfies* $\operatorname{reg}(I) \le \eta$. *Similarly we let* $\operatorname{CI}(H)$ *(resp.* $\operatorname{CM}(H)$*) consist of points (R/I) where I is generated by a regular sequence (resp. R/I is CM).*

Now let

$$\operatorname{SmCM}(H_B, H_A)_\eta := p^{-1}(\operatorname{SmCM}(H_A)) \cap q^{-1}(\operatorname{Smc}(H_B)_\eta)$$

where $q : \operatorname{GradAlg}(H_B, H_A) \to \operatorname{GradAlg}(H_B)$ and $p : \operatorname{GradAlg}(H_B, H_A) \to \operatorname{GradAlg}(H_A)$ are the two natural projection morphisms. (e.g. $q((B \to A)) = (B)$). Denoting the following restrictions of p and q by the same letters, we get a diagram (incidence correspondence)

$$\begin{array}{ccc} \operatorname{SmCM}(H_B, H_A)_\eta & \xrightarrow{q} & \operatorname{Smc}(H_B)_\eta \subset \operatorname{GradAlg}(H_B) \\ \downarrow^{p} & & \\ \operatorname{GradAlg}(H_A) & & \end{array} \tag{8}$$

Proposition 19. *Let H_B be the Hilbert function of some smooth connected curve* $\operatorname{Proj}(B)$, $B \neq R$, *and let H_A be its truncated Hilbert function at the level s, i.e.* $H_A(i) = \inf\{H_B(i), s\}$ *for* $i \ge 0$. *Let* $j = \min\{i | H_A(i) \neq H_B(i)\}$, *let* $\eta \le j - 2$ *and look to the maps p and q in* (8). *Then*

(i) q is smooth and surjective and its fibers are geometrically connected, of fiber dimension s, and

(ii) p is an isomorphism onto an open subscheme of $\operatorname{GradAlg}(H_A)$.

In particular the correspondence (8) *determines a well-defined injective application π from the set of irreducible components W of* $\operatorname{Smc}(H_B)_\eta$, *to the set of irreducible components V of* $\operatorname{GradAlg}(H_A)$, *in which generically smooth components correspond. Indeed $V = \pi(W)$ is the closure of $p(q^{-1}(W))$, and we have*

$$\dim V = \dim W + s \ .$$

Proof. (i) By Geramita et al. [13] we get the surjectivity of q. Since we showed ${}_0\mathrm{H}^2(B, A, A) = 0$ in Theorem 12, the smoothness of q follows immediately from Proposition 10(i). To show that the fibers of q are (geometrically) connected, one may simply look at the fiber as the variation of s generic points on a fixed Y, i.e. as a non-empty dense set of Y^s. This set is irreducible since Y is irreducible, and we conclude as claimed.

(ii) In Lemma 9 we showed ${}_0\mathrm{Ext}^i_R(I_B, I_{A/B}) = 0$ for $i \leq 1$ assuming $j \geq \mathrm{reg}(I_B) + 2$. By Proposition 10(ii) this implies that p is smooth and unramified. It is easy to see that $j \geq \mathrm{reg}(I_B)+1$ implies that p is injective (in fact, universally injective or "radiciel"), cf. Lemma 7(a) of [32]. Hence we get (ii) by [18], Thm. 17.9.1. Now combining (i) and [20], Prop. 1.8, we get that $q^{-1}(W)$ is an irreducible component of $\mathrm{SmCM}(H_B, H_A)_\eta$. The application π is therefore well defined, and it is injective by (ii). Finally since q is smooth and p is an open immersion, we easily get the dimension formulas. □

Remark 20. *If we in Proposition 19 drop the assumption* $\dim \mathrm{Proj}(B) = 1$ *and maintain the other assumptions, we still get that q is smooth and that p is an isomorphism onto an open subscheme (but the irreducibility of* $q^{-1}(W)$ *may fail).*

Now we consider an example of several components of $\mathrm{GradAlg}(H_A)$, which one may, as in [43], distinguish by the incomparability of the set of graded Betti numbers (Remark 7). Applying, however, Proposition 19 to our example we can also describe as well the graded Betti numbers of some algebras in the *intersection of the two components*. Below ΔH is the first difference of H, i.e. $\Delta H(v) = H(v) - H(v-1)$ for any integer v, in which case ΔH_A is of the form $(1, c_1, c_2, ..., c_t, 0, 0, ...)$ (with $c_t \neq 0$) if A is one-dimensional. In this case we often write ΔH_A as

$$\Delta H_A = (1, c_1, c_2, ..., c_t) \ ,$$

i.e. as the so-called *h-vector* of A. In the Artinian case, the h-vector of A coincides with the Hilbert function H_A provided we write H_A in the form; $H_A = (1, h_1, h_2, ..., h_t)$, with $h_t \neq 0$ and $h_j = 0$ for $j > t$.

Example 21. *In [50] C. Walter gives examples of infinitely many Hilbert schemes of space curves containing* obstructed *smooth curves of maximal rank. Indeed his example of a smooth space curve* Y *of the lowest degree (i.e. the curve with Hilbert polynomial* $p(x) = 33x - 116$ *which we consider below) was independently discovered by Bolondi et al [6] and it was the first example of an obstructed curve of maximal rank which was detected. In [6] we showed that* $\mathrm{Hilb}^{33x-116}(\mathbb{P}^3)$ *contains at least two irreducible components whose intersection contains* (Y)*. Since the curve* $Y = \mathrm{Proj}(B)$ *is of maximal rank, we have* ${}_0\mathrm{Hom}_R(I_B, \mathrm{H}^1_{\mathfrak{m}}(B)) = 0$*, and Proposition 8 applies. It follows that the corresponding algebra B is* obstructed *as a graded algebra since* $\mathrm{Hilb}^{33x-116}(\mathbb{P}^3)$ *is not smooth at* (Y)*. Indeed* (B) *sits in the intersection of two irreducible components* W_1 *and* W_2*, both of dimension* $4d = 132$*, of the postulation Hilbert scheme of space curves* $\mathrm{GradAlg}(H_B)$*, cf. [35], ex. 35.*

In [35] we also considered the minimal resolution of B as well as the minimal resolution of the general elements B_1 *and* B_2 *of* W_1 *and* W_2 *respectively. Indeed*

$$0 \to G_3 = R(-9) \to R(-10)^2 \oplus R(-9) \oplus R(-8)^4 \to R(-9) \oplus R(-8) \oplus R(-7)^5 \to I_B \to 0$$

is exact and we get the minimal resolution of B_1 *(resp.* B_2*) by making the factor* $R(-9)$ *redundant in two different ways, i.e. by removing this factor from the leftmost (*G_3*) and the middle term (*G_2*), (resp. from* G_2 *and rightmost term* G_1*). The Castelnuovo-Mumford regularity for all three curves satisfies* $\mathrm{reg}(I) = 9$*, and the Hilbert function of all algebras is*

$$(1, 4, 10, 20, 35, 56, 84, 115, 148, 181, 214, ...) \ .$$

Thus taking $s \geq 214$ *points* $X = \mathrm{Proj}(A)$ *on* Y *in general position and correspondingly for the others, then Proposition 19 applies with* $j \geq 11$*. Or more precisely, both* W_1 *and* W_2 *and its* *intersection essentially belong to* $\mathrm{Smc}(H_B)_9 \subset \mathrm{GradAlg}(H_B)$*, and Proposition 19 applies to (every element* $\mathrm{Proj}(B')$ *of)* $\mathrm{Smc}(H_B)_9$ *and an* s*-dimensional linear space of choices of* s *generic points on* $\mathrm{Proj}(B')$*. Hence for each* $s \geq 214$ *it follows that* X *is in the intersection of two irreducible components* V_1 *and* V_2 *of the postulation Hilbert scheme* $\mathrm{GradAlg}(H_A)$ *of dimension* $\dim V_i = 132+s$ *for* $i = 1,2$*. In the special case* $s = 214$ *we have* $\Delta H_A = (1,3,6,10,15,21,28,31,33,33,33,0,...)$*, and it is not difficult to see that the minimal resolution of* I_A *is*

$$0 \to R(-13)^{33} \oplus G_3 \to R(-12)^{66} \oplus G_2 \to R(-11)^{33} \oplus G_1 \to I_A \to 0.$$

and that the minimal resolutions of the corresponding general elements $\mathrm{Proj}(A_i)$ *of* V_i *are obtained by removing the free factor* $R(-9)$ *from* G_3 *and* G_2 *(resp. from* G_2 *and* G_1*). Looking to the corresponding sets of graded Betti numbers of* A_1 *and* A_2 *we see that they are incomparable.*

We finish this section by recalling some known results about the postulation Hilbert scheme $\mathrm{GradAlg}^H(R)$, consisting of zero-dimensional schemes $\mathrm{Proj}(A)$ of degree s. Since we have observed that ${}_0\mathrm{H}^2(R,A,A) = 0$ implies the smoothness of $\mathrm{GradAlg}^H(R)$ at (A), we remark that the smoothness results of Remark 22(i) (when A is generically a CI) and of Remark 22(ii) also follow from works of Herzog, Buchweitz-Ulrich and Huneke ([21], [8] and [22]). Moreover for Remark 22(iii) we remark that Buchweitz's thesis [7], or [8], show that a generically CI licci quotient is unobstructed. Now in addition to Theorem 12 and Proposition 19, we have

Remark 22. *(i) If* $\mathrm{Proj}(R) = \mathbb{P}^2$*, then Gotzmann ([15]) shows that* $\mathrm{GradAlg}^H(R)$ *is irreducible and he finds its dimension ([26], Thm. 5.21 and Thm. 5.51). It is smooth by licciness and say (iii) below (or by [15] provided* $\mathrm{GradAlg}^H(R)$ *is reduced). As indicated by Iarrobino-Kanev ([26], Remark to Thm. 5.51), the dimension formula given in [33], Rem. 4.4, holds in this case ([33], Rem. 4.6).*

(ii) If $\mathrm{Proj}(R) = \mathbb{P}^3$*, then the the open part of* $\mathrm{GradAlg}^H(R)$ *consisting of Gorenstein quotients is irreducible (cf. [10]), of known dimension by ([31], Remark to Thm. 2.6) and smooth by say (iii) below. This dimension formula is included in [30], Thm. 2.3 with a proof (which also takes care of the Artinian case). [30], Prop. 3.1 contains a second "dual" dimension formula for the same parameter space.*

(iii) Let $\mathrm{Proj}(R) = \mathbb{P}^n$ *and let* A *and* A' *be two graded CM quotients algebraically linked by a CI* B *of type* $(a_1,...,a_m)$ *with resolution* (4)*. By [30], Prop. 1.7, then* A *and* A' *are simultaneously unobstructed as graded algebras, and we have*

$$\dim_{(A)} \mathrm{GradAlg}^{H_A}(R) - \sum_{i=1}^{m} H_A(a_i) = \dim_{(A')} \mathrm{GradAlg}^{H_{A'}}(R) - \sum_{i=1}^{m} H_{A'}(a_i)\ .$$

(iv) Let $B = R/I_B$ *be a graded, generically Gorenstein CM quotient with canonical module* K_B *and let* A *be the Gorenstein algebra given by a regular section of* $\sigma \in (K_B^*)_t$ *for some integer* t*, i.e. given by a graded exact sequence* $0 \to K_B(-t) \xrightarrow{\sigma} B \to A \to 0$*.*

a) If B *is licci, then* A *is unobstructed as a graded* R*-algebra (indeed* $\mathrm{H}^2(R,A,A) = 0$*), and,*

$$\dim_{(A)} \mathrm{GradAlg}^{H_A}(R) = \dim_{(B)} \mathrm{GradAlg}^{H_B}(R) + \dim(K_B^*)_t - 1 - \delta(B)_{-t}$$

where $\delta(B)_v = {}_v\hom_B(I_B/I_B^2, K_B) - {}_v\mathrm{ext}^1_B(I_B/I_B^2, K_B)$*.*

b) If $\mathrm{Proj}(B)$ *is locally Gorenstein and* $t >> 0$*, then* A *and* B *are simultaneously unobstructed as graded algebras, and the dimension formula of a) holds (with* $\delta(B)_{-t} = 0$*).*

This theorem is true in arbitrary dimension of B*. It is proved in [32], Thm. 16 and is a substantial generalization of some of the statements of (ii) above because, when we apply it to a CM* B *of codimension two (necessarily licci), we get the dimension formula of (ii) by [32], Ex. 26. The preprint [34] contains further generalizations of this theorem.*

By (iii) we see that CI-linkage preserves the smoothness of the parameter spaces. Due to [29], Prop. 3.4 it also preserves the irreducibility of the linked family. To define the linked family, let $\mathrm{CICM}(H_B, H_A)$ consist of points $(B \to A)$ such that B is CI and A is CM, i.e.

$$\mathrm{CICM}(H_B, H_A) := p^{-1}(\mathrm{CM}(H_A)) \cap q^{-1}(\mathrm{CI}(H_B))$$

where $p : \mathrm{GradAlg}(H_B, H_A) \to \mathrm{GradAlg}(H_A)$ (resp. q) is the second (resp. first) projection morphism (e.g. $q((B \to A)) = (B)$). In the case $\dim A = \dim B$ (not necessarily equal to one) and $(B \to A) \in \mathrm{CICM}(H_B, H_A)$, the linked algebra is defined by $A' := B/\operatorname{Hom}_B(A, B)$. This definition extends to families and preserves flatness [29]. Indeed by [29], Thm. 2.6 there is an isomorphism τ of schemes and obvious second projection morphisms p and p' fitting into

$$\begin{array}{ccc} \tau : \mathrm{CICM}(H_B, H_A) & \xrightarrow{\simeq} & \mathrm{CICM}(H_B, H_{A'}) \\ \downarrow^{p} & & \downarrow^{p'} \\ \mathrm{GradAlg}(H_A) & & \mathrm{GradAlg}(H_{A'}) \end{array} \tag{9}$$

where τ is given by sending $(B_1 \to A_1)$ to $(B_1 \to A_1' := B_1/\operatorname{Hom}_{B_1}(A_1, B_1))$.

Definition 23. *Let the Hilbert polynomials p_B and p_A (corresponding to H_B and H_A respectively) have the same degree (≥ -1) and let U be a locally closed subset of* $\operatorname{im} p$ *in* (9). *Then the H_B-linked family of U is*

$$U' := p'(\tau(p^{-1}(U))$$

Theorem 24. *In* (9) *the morphisms p and p' are smooth and their fibers are geometrically connected, of fiber dimension $\epsilon(A/B)$ at $(B \to A)$ and $\epsilon(A'/B)$ at $(B \to A')$ respectively. In particular the H_B-linked family U' is irreducible (resp. open in* $\mathrm{GradAlg}(H_{A'})$*) if and only if U is irreducible (resp. open in* $\mathrm{GradAlg}(H_A)$*).*

Proof. The proof of [30], Prop. 1.7 takes care of the smoothness of p and p' and their fiber dimension. It remains to prove the connectedness of the fibers since the other conclusions then follow easily. The connectedness is, however, a straightforward consequence of the proof of Theorem 1.16 of [29] (that part of the proof doesn't require $\deg p > 0$ and it is easy to reformulate it for the Artinian case as well), cf. [41], Ch. VII for similar results. □

Of course Theorem 24 implies Remark 22(iii) above. We may use Theorem 24 to see that many other properties are by preserved by linkage. Indeed subsets of $\mathrm{GradAlg}(H)$ for which the members allow the same sequence of CI-linkages which ends in a CI, is irreducible. It does not mean that the subset of $\mathrm{GradAlg}(H)$ of licci quotients is irreducible, as the following example shows.

Example 25. *We claim* $\mathrm{GradAlg}^H(k[x,y,z,w])$ *with* $\Delta H = (1,3,6,6,3,1)$ *contains (at least) two irreducible components whose general elements are licci. Of course, the general element of one of the components is an arithmetically Gorenstein scheme consisting of* 20 *points, with minimal resolution*

$$0 \to R(-8) \to R(-5)^4 \oplus R(-4) \to R(-4) \oplus R(-3)^4 \to R \to A_1 \to 0.$$

We get A_1 by starting with a CI of type $(1,1,2)$ *and then perform general CI-linkages of type* $(2,3,3)$ *and* $(4,3,3)$. *It follows that the component is generically smooth of dimension* 44 *by using Remark 22(iii), or Theorem 24, twice.*

To get the other component, we start with a point $\mathrm{Proj}(A)$, *i.e. a CI of type* $(1,1,1)$, *and we proceed by performing six general CI linkages of type* $(1,2,3)$, $(2,2,4)$, $(2,3,4)$, $(3,4,4)$, $(3,4,5)$, $(3,3,5)$, *in this order. (The first five linkages are the same as for the level algebra 64] in the appendix*

C of [14]; hence GradAlg(H) *with* $\Delta H = (1,3,6,7,6,2)$ *also contains two "licci" components. By experimenting with Macaulay 2 we have learned that the three linkages;* $(2,3,4)$, $(3,4,4)$, $(3,4,5)$ *above may be reduced to one single linkage;* $(3,3,4)$*). We get in this way an open subset* U *of* GradAlg(H) *of algebras* A_2 *with minimal resolution*

$$0 \to R(-8) \oplus R(-7) \oplus R(-6) \to R(-7) \oplus R(-6) \oplus R(-5)^5 \to R(-5) \oplus R(-3)^4 \to R \to A_2 \to 0.$$

Since the Betti numbers do not coincide with the general element A_1 *of the other component, the closure of* U *must be a generically smooth component of dimension* 44 *by Theorem 24.*

(This example holds correspondingly for codimension 3 quotients with h-vector $(1,3,6,6,3,1)$ *in a polynomial ring of any dimension. For the Artinian case, see Example 41).*

3 Families of Artinian R-quotients (possibly Gorenstein).

In this section we look to families of Artinian algebras A of Hilbert function $H = H_A$, i.e. we study the scheme GradAlg(H) in the Artinian case with a special look to level and Gorenstein Artinian quotients. In particular we give examples of codimension 4 (resp. 3) quotients where GradAlg(H) has at least two components with a Gorenstein (resp. level) algebra belonging to the intersection of the two components. Moreover we notice that almost all the results of the preceding section (cf. Remark 22) are known in the Artinian case, except Theorem 12 and Proposition 19, whose corresponding Artinian counterparts are the main new results of this section (Theorem 29 and Proposition 33). Of course there are a few changes to Remark 22, mostly concerned with references, and we include some further results. To summarize,

Remark 26. *(i) Iarrobino shows that* $\mathrm{GradAlg}^H(k[x,y])$ *is irreducible ([23], Thm. 2.9) and he finds the dimension ([23], Thm. 2.12 and Thm. 3.13). It is smooth by licciness (or by [23], Thm. 2.9 provided* $\mathrm{GradAlg}^H(k[x,y])$ *is reduced). Also in this case, the dimension formula given in [33], Rem. 4.4, holds (by the indicated argument of [33], Rem. 4.6).*

(ii) If $R = k[x,y,z]$*, then the open part of* $\mathrm{GradAlg}^H(R)$ *consisting of Gorenstein quotients is irreducible ([10]) and smooth of known dimension ([30], Thm. 2.3). See also [22], Cor 4.9 for the smoothness.*

(iii) of Remark 22 holds as stated in Remark 22.

(iv) of Remark 22 holds as well. One may make a little progress to (iv,b) by stating it as:

b) If Proj(B) *is a locally Gorenstein zero-scheme of degree* s *and if* $t \geq 2\,\mathrm{reg}(I_B)$*, then* A *and* B *are simultaneously unobstructed as graded algebras, and the dimension formula of (iv, a) holds (with* $\delta(B)_{-t} = 0$ *and* $\dim(K_B^*)_t = s$*, cf. [32], Rem. 22). We formulate this using the ideas of Proposition 19 as Theorem 27 below ([32], Prop. 23, cf. [34] for further generalizations).*

(v) One may, via the Macaulay correspondence, consider the set $PS(s,j,n)$ *of Gorenstein quotients obtained from the set of homogeneous polynomials of degree* j *in the "dual" polynomial ring, of the form*

$$f = L_1^j + ... + L_s^j \ ,$$

where L_i *are general enough linear forms and* s *is fixed. If* H_A *(which we denote by* $H(s,j,n)$*) contains a subsequence of the form* (s,s,s)*, then the closure of* $PS(s,j,n)$ *in* $\mathrm{GradAlg}^{H_A}(R)$ *determines by Macaulay duality a generically smooth irreducible component of* $\mathrm{GradAlg}^{H_A}(R)$ *of known dimension ([26], Thm. 4.10A and Thm. 1.61, see Thm. 4.13 for similar results when* H_A *does not contain such a subsequence).*

(vi) In the interesting Gorenstein Artinian codimension 4 case, there is a structure theorem when $H_A = (1,4,7,h,i,...)$ *with* $3h - i - 17 \geq 0$*, allowing us to describe well the corresponding (generically*

smooth) irreducible component of $\mathrm{GradAlg}^{H_A}(R)$ *([26]). In [37] Johannes Kleppe comes up with classes of generically smooth components of known dimension of a similar nature.*

(vii) Compressed Artinian algebras of fixed socle degrees belong to an irreducible generically smooth component of known dimension by [24], Thm. IIB.

To accomplish Remark 26(iv,b), let $U_t \subset \mathrm{GradAlg}(H')$ be an open subscheme consisting of points (B) such that B is CM and such that $\mathrm{Proj}(B)$ is a locally Gorenstein zero-scheme of degree s satisfying $H_B = H'$ and $\mathrm{reg}(I_B) \le t/2$. Recall that a regular section of $\sigma \in (K_B^*)_t$ defines a graded Gorenstein algebra A given by the exact sequence $0 \to K_B(-t) \xrightarrow{\sigma} B \to A \to 0$. Let $q : \mathrm{GradAlg}(H_B, H_A) \to \mathrm{GradAlg}(H_B)$ be the first projection and let $q^{-1}(U_t)_{reg}$ be the intersection of $q^{-1}(U_t)$ by the space of those quotients $(B \to A)$ which correspond to regular sections of $(K_B^*)_t$. Then we have a diagram where we have restricted the two natural projection morphisms q and $p : \mathrm{GradAlg}(H_B, H_A) \to \mathrm{GradAlg}(H_A)$ to $q^{-1}(U_t)_{reg}$:

$$\begin{array}{ccc} q^{-1}(U_t)_{reg} & \xrightarrow{q_{res}} & U_t \subset \mathrm{GradAlg}(H_B) \\ \downarrow^{p_{res}} & & \\ \mathrm{GradAlg}(H_A) & & \end{array} \tag{10}$$

Theorem 27. *With notations as above, then*

(i) $q_{res} : q^{-1}(U_t)_{reg} \to U_t$ *is smooth and surjective, and its fibers are geometrically connected of fiber dimension* $s-1$, *and*

(ii) p_{res} *is an isomorphism onto an open subscheme of* $\mathrm{GradAlg}(H_A)$.

In particular the correspondence (10) *determines a well-defined injective application* π *from the set of irreducible components* W *of* U_t*, to the set of irreducible components* V *of* $\mathrm{GradAlg}(H_A)$*, in which generically smooth components correspond. Indeed* $V = \pi(W)$ *is the closure of* $p_{res}(q_{res}^{-1}(W))$*, and we have*

$$\dim V = \dim W + s - 1 \ .$$

Proof (also of Remark 26(iv,b)). These results are almost exactly the second part of Thm. 16 (cf. Rem. 22) and Thm. 24 (cf. Prop. 23) of [32] with a slight improvement for the case $\dim B = 1$. Indeed in replacing "$t >> 0$" by "$t \ge 2\,\mathrm{reg}(I_B)$" we assumed in [32], Rem. 22 and Prop. 23(iii) that B was generically syzygetic (e.g. $\mathrm{Proj}(B)$ locally licci). This was needed in Prop. 23(iii) to show that p_{res} was smooth. Since, however, $K_B(-t) \simeq I_{A/B}$ and the R-dual of (4) is a free resolution of $K_B(1)$, we have $(I_{A/B})_v = 0$ for $v \ge t + 1 - \mathrm{reg}(I_B)$. Hence we may use Lemma 9 and Proposition 10(ii) of this paper to see that p_{res} is smooth under the assumption $t \ge 2\,\mathrm{reg}(I_B)$ without requiring B to be generically syzygetic. Note also that q_{res} is surjective by [4], Thm. 3.2, (cf. [32], Rem. 22). Since the remaining part of Theorem 27 was proved in [32] (Prop. 23, Thm. 24, cf. Rem. 25(a)), we get the theorem. Moreover note that Remark 26(iv,b) follows from Theorem 27 since we may get $\delta(B)_{-t} = 0$ and $\dim(K_B^*)_t = s$ from Proposition 10 (or by directly using [32], Rem. 14(a) and the first part of Rem. 22). Hence also in Remark 26(iv,b) it suffices to suppose $t \ge 2\,\mathrm{reg}(I_B)$ without requiring B to be generically syzygetic. □

Now we illustrate Theorem 27. The benefit of using Theorem 27 instead of Remark 26(iv,b) is clear because it is a statement about the whole subscheme $U_t \subset \mathrm{GradAlg}(H')$ and not only about a point in U_t. E.g. note that if we apply (iv,b) to the two components of $\mathrm{GradAlg}(H')$ of Example 17, say with $s = 13$ and $t \ge 10$ to simplify, we get two components of $\mathrm{PGor}(H)$, or of $\mathrm{GradAlg}(H)$, with

$$H = (1, 4, 7, 10, 13, 13, ..., 13, 10, 7, 4, 1)$$

of dimension 37 and 39 where the number 13 occurs $t - 7$ times. The existence of such components for $\mathrm{PGor}(H)$ is now well known ([3], see also [27]). Since, however, $\mathrm{GradAlg}(H)$ is connected there are graded Artinian quotients belonging to the intersection of the components of $\mathrm{GradAlg}(H)$. But are there Gorenstein quotients in this intersection? The answer would be yes if the intersection of the two components of $\mathrm{GradAlg}(H')$ of Example 17 contains points (B) such that $\mathrm{Proj}(B)$ is a locally Gorenstein zero-scheme because we then could apply Theorem 27! We doubt that there exists such a quotient B, i.e. we expect that the intersection of the two mentioned components of $\mathrm{PGor}(H)$ is empty (cf. Piene-Schlessinger's characterization of the intersection of the two components described in Example 34). Here is an example where we somehow control the intersection.

Example 28. *(Two components of* $\mathrm{PGor}(H)$ *with non-empty intersection)*

In Example 21 we showed the existence of an algebra, which we now call B whose corresponding point (B) of the postulation Hilbert scheme, $\mathrm{GradAlg}(H_B)$*, sat in the intersection of two irreducible components V_1 and V_2 of* $\mathrm{GradAlg}(H_B)$ *of dimension* $\dim V_i = 132 + s = 346$ *for* $i = 1, 2$. *The element (B) as well as the two general elements (B_i) of V_i were obtained by taking $s = 214$ generic points on certain curves of* $\mathrm{Hilb}^{33x-116}(\mathbb{P}^3)$. *One of the curves had minimal resolution*

$$0 \to G_3 = R(-9) \to G_2 = R(-10)^2 \oplus R(-9) \oplus R(-8)^4 \to G_1 = R(-9) \oplus R(-8) \oplus R(-7)^5 \to I$$

Moreover $H_B = H_{B_i} = (1, 4, 10, 20, 35, 56, 84, 115, 148, 181, 214, 214, 214, ...)$ *and the minimal resolution of I_B (resp. of I_{B_1}, or I_{B_2}) was*

$$0 \to R(-13)^{33} \oplus G_3 \to R(-12)^{66} \oplus G_2 \to R(-11)^{33} \oplus G_1 \to I_B \to 0 \tag{11}$$

(resp. (11) *in which the factor $R(-9)$ from G_3 and G_2, or from G_2 and G_1, were removed).*

Since we have $\mathrm{reg}(I_B) = \mathrm{reg}(I_{B_i}) = 11$, *we may use Theorem 27 to get, for every $t \geq 22$, two generically smooth irreducible components of* $\mathrm{PGor}(H_A)$ *of dimension* $132 + s + s - 1 = 559$ *whose intersection is non-empty, i.e. the intersection contains an* obstructed *Gorenstein Artinian algebra whose h-vector is the $(t + 1)$-tuple*

$$H_A = (1, 4, 10, 20, 35, 56, 84, 115, 148, 181, 214, 214, ..., 214, 181, 148, ..., 4, 1)$$

where the number 214 *occurs $t - 19$ times. The corresponding sets of graded Betti numbers of the general elements, A_1 and A_2, of the two components turn out to be incomparable because the factors $R(-9)$ (and $R(-t + 5)$) appearing in the resolution of I_A become redundant in different ways in the resolutions I_{A_1} and I_{A_2}. Of course, for every $s \geq 214$ we can construct similar examples.*

Now we prove the analogue of Theorem 12, which is the main result of this section.

Theorem 29. *Let R be a polynomial k-algebra and let $B = R/I_B \to A = R/I_A$ be a graded morphism such that A is Artinian and* $\mathrm{depth}_{\mathfrak{m}} B \geq \min(1, \dim B)$*, and suppose either*

(a) I_B is generated by a regular sequence (allowing $R = B$), or

(b) $B_v \to A_v$ is an isomorphism for all $v \leq \max_i\{n_{2,i}\}$ and $\dim R - \dim B \geq 2$.

Let F be a free B-module such that $F \to I_{A/B}$ is surjective and minimal, and suppose there is an integer j such that the degrees of minimal generators of the B-module $\ker(F \to I_{A/B})$ *are strictly greater than j (e.g. $B_v \simeq A_v$ for all $v \leq j - 1$) and such that I_A is $(j+1)$-regular (i.e. $A_{j+1} = 0$). Then A is H_B-generic,* $\dim(N_A)_0 = \dim(N_B)_0 + {}_0\hom_B(F, A) - \epsilon(A/B)$ *, and*

$$\dim_{(A)} \mathrm{GradAlg}^{H_A}(R) = \dim_{(B)} \mathrm{GradAlg}^{H_B}(R) + {}_0\mathrm{hom}_B(F, A) - \epsilon(A/B) \ .$$

In particular A is unobstructed as a graded R-algebra if and only if B is unobstructed as a graded R-algebra.

At least in the case $\operatorname{depth}_{\mathfrak{m}} B \geq 1$, the natural application of Theorem 29 is the same as for Theorem 12; the minimal resolution of A should be the one of B in addition to a semi-linear contribution coming from $I_{A/B}$ via the mapping cone construction, cf. Remark 32.

Proof. All we need to prove is the two dimension formulas. Due to Corollary 11 it suffices to show ${}_0\mathrm{H}^2(R,B,I_{A/B}) = 0$ and ${}_0\hom_R(I_B, I_{A/B}) = \epsilon(A/B)$ together with

$$\dim\ {}_0\mathrm{Hom}_B(I_{A/B}, A) = {}_0\hom_B(F, A) \quad \text{and} \quad {}_0\mathrm{Ext}^1_B(I_{A/B}, A) = 0\ , \tag{12}$$

because the latter of (12) implies ${}_0\mathrm{H}^2(B,A,A) = 0$. By Lemma 9 it suffices to prove (12). Let

$$F' \to F \to I_{A/B} \to 0 \tag{13}$$

be the first terms of a B-free minimal resolution of $I_{A/B}$. Applying ${}_0\mathrm{Hom}_B(-,A) = 0$ onto (13) and using $A_{j+1} = 0$, we get ${}_0\mathrm{Hom}_B(I_{A/B}, A) \simeq {}_0\mathrm{Hom}_B(F, A)$ and ${}_0\mathrm{Ext}^1_B(I_{A/B}, A) = 0$ by the assumption, and we are done. □

Remark 30. *By the long exact sequence of algebra cohomology, we have the exact sequence*

$$\to {}_0\mathrm{H}^2(B,A,A) \to {}_0\mathrm{H}^2(R,A,A) \to {}_0\mathrm{H}^2(R,B,A) \to\ .$$

Since it is well known that $\mathrm{H}^2(R,B,-) = 0$ *if Theorem 29(a) holds and since we have* ${}_0\mathrm{H}^2(B,A,A) = 0$ *by the proof above, it follows that we in Theorem 29(a) have*

$${}_0\mathrm{H}^2(R,A,A) = 0\ .$$

Remark 31. *A natural choice of j in Theorem 29 such that $(I_{A/B})_{j-1} = 0$ and such that (b) holds, is $j \geq \operatorname{reg}(I_B)+2$, in which case we have that I_A is $(j+1)$-regular iff $I_{A/B}$ is $(j+1)$-regular, and that $H_A(x) = H_B(x) = p_B(x)$ for $x \geq j-3$, cf. Remark 15. Since it then follows that $B \simeq A$ provided B is Artinian, the (only) real application of Theorem 29(b) seems to be in the case* $\operatorname{depth}_{\mathfrak{m}} B \geq 1$. *It is, however, natural to use Theorem 29(a) also when* $\operatorname{depth}_{\mathfrak{m}} B = 0$.

Remark 32. *Suppose* $\operatorname{depth}_{\mathfrak{m}} B \geq 1$ *and that $I_{A/B}$ is $(j+1)$-regular, and look to*

$$0 \to I_{A/B} \to B \to A \to 0\ . \tag{14}$$

Since $\mathrm{H}^0_{\mathfrak{m}}(I_{A/B}) = 0$, *we have* $\operatorname{depth}_{\mathfrak{m}} I_{A/B} \geq 1$, *i.e. $pd_R(I_{A/B}) \leq n-1$ and in fact $pd_R(I_{A/B}) = n-1$ since $pd(A) = n$. A mapping cone construction applied to* (14) *in which we use the minimal resolutions of $I_{A/B}$ and B, leads easily to an R-free resolution of A. Moreover if $I_{A/B}$ admits a semi-linear resolution, then $(I_{A/B})_{j-1} = 0$, and conversely provided* $\operatorname{reg}(I_{A/B}) = j+1$. *Note that A becomes a level algebra if $I_{A/B}$ admits a linear resolution. In particular, the natural application of Theorem 29(b) is the same as for Theorem 12, cf. Remark 13, i.e. the minimal resolution of A should be the one of B in addition to a semi-linear contribution coming from $I_{A/B}$ via the mapping cone construction.*

Theorem 29 applies nicely to Artinian truncations and more generally to Artinian quotients A with Hilbert function $H_A = (1, h_1, h_2, ..., h_{j-1}, \alpha, 0, 0, ..)$ where $H_B = (1, h_1, h_2, ..., h_{j-1}, h_j, h_{j+1}, ...)$ is the Hilbert function of B and $\alpha \leq h_j$. To see it let, in a very similar way to what we did in Definition 18 and Proposition 19, $\mathrm{GradAlg}(H)_\eta$ (resp. $\mathrm{GradAlg}(H_B, H_A)_\eta$) be the open subset of $\mathrm{GradAlg}(H)$ consisting of (R/I) (resp. $(B = R/I \to A)$) where the Castelnuovo-Mumford regularity satisfies $\mathrm{reg}(I) \leq \eta$. Then we have a diagram as in (8) where we now restrict the natural projection morphism $q : \mathrm{GradAlg}(H_B, H_A) \to \mathrm{GradAlg}(H_B)$ and p to $\mathrm{GradAlg}(H_B, H_A)_\eta$;

$$\begin{array}{ccc} \mathrm{GradAlg}(H_B, H_A)_\eta & \xrightarrow{q} & \mathrm{GradAlg}(H_B)_\eta \subset \mathrm{GradAlg}(H_B) \\ \downarrow^{p} & & \\ \mathrm{GradAlg}(H_A) & & \end{array} \tag{15}$$

Below we restrict to the case $B \neq R$, even though the proposition holds for $B = R$ with the following small changes; skip the lower indices η in (15) and drop the assumption $\eta \leq j-2$ below, and note that $\mathrm{GradAlg}(H_R)$ is a smooth scheme consisting of one point. Recalling that an Artinian quotient A has the Weak Lefschetz property if the multiplication map $A_v \xrightarrow{\cdot\, l} A_{v+1}$ by a general linear form l is either injective or surjective for every v, we have

Proposition 33. *Let $H_B = (1, h_1, h_2, ...)$ be the Hilbert function of an algebra $B \neq R$ satisfying $\mathrm{depth}_{\mathfrak{m}} B \geq 1$ and let j, $\eta \leq j-2$ and $\alpha \leq h_j$ be non-negative integers. Let $H_A = (1, h_1, h_2, ..., h_{j-1}, \alpha, 0, 0, ..)$ and look to the maps p and q in (15). Then*

(i) q is smooth and surjective with geometrically connected fibers, of fiber dimension $\alpha(h_j - \alpha)$, and

(ii) p is an isomorphism onto an open subscheme of $\mathrm{GradAlg}(H_A)$.

In particular the incidence correspondence (15) determines a well-defined injective application π from the set of irreducible components W of $\mathrm{GradAlg}(H_B)_\eta$, to the set of irreducible components V of $\mathrm{GradAlg}(H_A)$ whose general elements satisfy the Weak Lefschetz property. In this application the generically smooth components correspond. Indeed $V = \pi(W)$ is the closure of $p(q^{-1}(W))$, and we have

$$\dim V = \dim W + \alpha(h_j - \alpha) \ .$$

Proof. (i) To any point (B') of $\mathrm{GradAlg}(H_B)_\eta$, let $A' := \oplus_{i=0}^{j-1} B'_i \oplus V_j$ where V_j is an α-dimensional quotient of B'_j. This shows that q is surjective. Moreover we get the smoothness of q from Proposition 9(i) since ${}_0\mathrm{H}^2(B, A, A) = 0$ by the proof of Theorem 29. To show that the fibers of q are (geometrically) connected, one may look upon the fiber as the Grassmannian of α-dimensional quotients of B'_j. Since the Grassmannian is irreducible, we conclude easily.

(ii) Since the proof of the Weak Lefschetz property is standard ($\mathrm{depth}_{\mathfrak{m}} B \geq 1$), the proof is the same as for (ii) of Proposition 19. □

We will call an Artinian algebra A with Hilbert function H_A as in Proposition 33 with $\alpha = 0$ an *Artinian truncation in degree j*. Moreover, by Remark 31, we normally need $j \geq \mathrm{reg}(I_B) + 2$ for some B to use Proposition 33 with $\mathrm{GradAlg}(H_B)_{j-2}$ non-empty. Having several irreducible components with a non-empty intersection in $\mathrm{GradAlg}(H_B)_\eta$, we get exactly the same type of irreducible components with a non-empty intersection for their Artinian truncations in a fixed degree j (for every $j \geq \eta + 2$) in $\mathrm{GradAlg}(H_A)$ (for instance, the B and the components given by the B_i of Example 21, we leave the details to the reader). We finish this section by another example.

Example 34. *(obstructed Artinian level algebra with Hilbert function $(1, 4, 7, 10, 13, 0, 0, ...)$).*

We have seen that $Y_1 = \mathrm{Proj}(B_1) \subset \mathbb{P}^3$, a twisted cubic curve and $Y_2 = \mathrm{Proj}(B_2)$, a union of a plane space curve C of degree 3 and a point P outside the plane containing C, correspond to two different irreducible components of the stratum $\mathrm{GradAlg}(H)$ of the Hilbert scheme $\mathrm{Hilb}^{3x+1}(\mathbb{P}^3)$ where $H = (1, 4, 7, 10, 13, ...)$. Indeed (Y_1) belongs to a 12-dimensional irreducible component of $\mathrm{GradAlg}(H)$ while (Y_2) belongs to a 15-dimensional irreducible component of $\mathrm{GradAlg}(H)$. Using Piene-Schlessinger's Theorem from [48] to see a complete description of $\mathrm{Hilb}^{3x+1}(\mathbb{P}^3)$, we also get *that the general element of the intersection of the two components (which is 11-dimensional) is a curve* $Y = \mathrm{Proj}(B)$ *with an embedded point. The minimal resolution of* $I = I_B$ *or* I_{B_2} *(resp.* I_{B_1}*) are of the form*

$$0 \to R(-4) \to R(-4) \oplus R(-3)^3 \to R(-3) \oplus R(-2)^3 \to I \to 0 \tag{16}$$

(resp. of the form (16) *where both* $R(-4)$ *and two of* $R(-3)$ *are removed). Hence the regularity of* I_B *and* I_{B_i} *for* $i = 1$ *and* 2 *is at most* 3*, i.e. the two components and its intersection essentially belong to* $\mathrm{GradAlg}(H)_3$*. Applying Proposition 33 for* $j \geq 5$ *and* $\eta = 3$ *and to any* $\alpha \leq 3j+1$*, we get two irreducible components* V_i *of* $\mathrm{GradAlg}(H_A)$ *with a well described non-empty intersection. Indeed let* $X_1 = \mathrm{Proj}(A_1)$ *and* $X_2 = \mathrm{Proj}(A_2)$ *(resp.* $X = \mathrm{Proj}(A)$*) be obtained by modding out by* $h_j - \alpha$ *linearly independent forms of* $(B_1)_j$ *and* $(B_2)_j$ *(resp.* B_j*) and all forms of degree* $j+1$*. It follows that* A_i *are unobstructed as graded* R*-algebras for* $i = 1$ *and* 2 *and that* $\dim_{(A_1)} \mathrm{GradAlg}(H') = 12 + \alpha(h_j - \alpha)$ *and* $\dim_{(A_2)} \mathrm{GradAlg}(H') = 15 + \alpha(h_j - \alpha)$ *where* $H' = (1, 4, 7, ... 3j-5, 3j-2, \alpha, 0, 0, ...)$*. Moreover* (A) *is a singular point of* $\mathrm{GradAlg}(H')$*and belongs to the* $11 + \alpha(h_j - \alpha)$*-dimensional intersection of the components. Finally if* $\alpha = 0$ *and* $j = 5$ *it is straightforward to see that the free terms of a minimal resolution of* A_2 *(and* A*) are*

$$0 \to R(-8)^{13} \to R(-7)^{42} \oplus R(-4) \to R(-6)^{45} \oplus R(-4) \oplus R(-3)^3 \to R(-5)^{16} \oplus R(-3) \oplus R(-2)^3$$

4 Tangent and obstruction spaces of Artinian families.

In this section we consider graded Artinian algebras with a special look to level quotients of $k[x, y, z]$. Note that, in most cases, results such as Theorem 29, Proposition 33 and Remark 26 do not apply because their assumptions limit their applications considerably. We can, however, still analyze $\mathrm{GradAlg}^H(R)$ at a point (A) infinitesimally by means of its tangent and obstruction spaces and a certain obstruction morphism, cf. [38]. In the following we make these spaces more explicit by duality (Theorem 36), and since we show that the parameter space of level schemes, $\mathrm{L}(H)$, of [9] is sufficiently close to being an open subscheme of $\mathrm{GradAlg}^H(R)$ (cf. Theorem 44), we can use our results to study $\mathrm{L}(H)$. In particular we study in detail the level type 2 algebras which correspond to a pencil of forms by apolarity ([25]), and we prove in Example 49 a conjecture of Iarrobino on the existence of several irreducible components of $\mathrm{L}(H)$ when $H = (1, 3, 6, 10, 14, 10, 6, 2)$.

Indeed inside $\mathrm{GradAlg}^H(R)$ there is an open set, possibly empty, consisting of graded Artinian Gorenstein quotients $R \to A$ with Hilbert function H (which essentially is the scheme $\mathrm{PGor}(H)$, see [32]). An elementary way of finding the obstruction space of $\mathrm{PGor}(H)$ is to compute the kernel of the natural surjection

$$\eta_j : (S_2 I_A)_j \to ({I_A}^2)_j$$

from the second symmetric power to the second power of I_A in the socle degree j of A. Indeed, up to duality, this kernel is isomorphic to ${}_0\mathrm{H}^2(R, A, A)$, the obstruction space of $\mathrm{PGor}(H)$. To generalize this result to any Artinian A, we remark that $\ker \eta_j$ is isomorphic to the cokernel of the natural morphism $(\Lambda_2 I_A)_j \to \mathrm{Tor}_2^R(A, A)_j$ (at least if $char(k) \neq 2$). This formulation allows a generalization to any Artinian A. Indeed there is a special product, given as an antisymmetrization map ([1], Prop. 24.1),

$$\mathrm{Tor}_1^R(A, A) \otimes_A \mathrm{Tor}_1^R(A, K_A) \to \mathrm{Tor}_2^R(A, K_A) \tag{17}$$

with cokernel $\mathrm{H}_2(R, A, K_A)$. Up to duality we will show that the zero degree piece of this cokernel is the obstruction space of $\mathrm{GradAlg}^H(R)$ at (A). To prove it we need a variation of the following spectral sequence

$$\mathrm{Ext}_A^p(\mathrm{H}_q(R, A, A), M) \implies \mathrm{H}^{p+q}(R, A, M)$$

(cf. [1], Prop. 16.1). Keeping also the spectral sequence ([21], Satz 1.2)

$$\mathrm{Ext}^p_C(\mathrm{Tor}^A_q(M,K_C),K_C) \implies \mathrm{Ext}^{p+q}_A(M,C) \tag{18}$$

in mind (C a CM quotient of A with canonical module K_C), the following result is not surprising

Proposition 35. *If $B \to A \to C$ are quotients of R of arbitrary dimension and if C is CM with canonical module K_C, then there is a spectral sequence converging to $\mathrm{H}^*(B,A,C)$;*

$$'E_2^{p,q} := \mathrm{Ext}^q_C(\mathrm{H}_p(B,A,K_C),K_C) \implies \mathrm{H}^{p+q}(B,A,C)\ . \tag{19}$$

In particular if C is a graded Artinian algebra, then there is a degree-preserving isomorphism

$$\mathrm{Hom}_C(\mathrm{H}_q(B,A,K_C),K_C) \simeq \mathrm{H}^q(B,A,C)\ .$$

Proof. One knows that $\mathrm{Hom}_A(M,C) \simeq \mathrm{Hom}_C(M \otimes_A K_C, K_C)$, M an A-module. Using this we can prove our proposition in the usual way, i.e. by considering the double complex

$$K_{*,*} = \mathrm{Hom}_C(\mathrm{Diff}(B,A_*,A)\otimes_A K_C, I_*)$$

where $0 \to K_C \to I_*$ is an injective resolution of the C-module K_C and $\mathrm{Diff}(B,A_*,A) := \Omega_{A_*/B}\otimes_{A_*} A$ is the complex of Kähler differentials based on a simplicial resolution, A_*, of the B-algebra A (as in [1], Prop. 17.1, so each A_i is a polynomial ring over B). If we in $''E_2^{p,q}$ first take homology of $K_{*,*}$ with respect to the second variable (i.e. I_*), we get $''E_2^{p,0} = \mathrm{H}^p(B,A,C)$ and $''E_2^{p,q} = 0$ for $q > 0$ because $\mathrm{Ext}^q_C(K_C,K_C) = 0$ for $q > 0$ by Cohen-Macaulayness and the fact that each $\mathrm{Diff}(B,A_i,A)\otimes_A C$ is C-free. Calculating $'E_2^{p,q}$ by reversing the order, i.e. by first taking homology with respect to the first variable, we get (19). Finally since K_C is an injective C-module in the Artinian case, we are done. $\square$

Theorem 36. *Let $R \to A = R/I_A$ be a graded Artinian quotient with Hilbert function H. Then the dual of $(I_A \otimes_R K_A)_0$ is the tangent space of $\mathrm{GradAlg}^H(R)$ at (A), and the dual of ${}_0\mathrm{H}_2(R,A,K_A)$ contains the obstructions to deforming A as a graded R-algebra. Moreover*

$$\dim\,(I_A \otimes_R K_A)_0 - {}_0\mathrm{h}_2(R,A,K_A) \le \dim_{(A)}\mathrm{GradAlg}(H) \le \dim\,(I_A \otimes_R K_A)_0.$$

In particular $\mathrm{GradAlg}^H(R)$ is smooth at (A) provided the natural map

$$I_A \otimes_R I_A \otimes_R K_A \to \mathrm{Tor}^R_1(I_A,K_A)$$

of (17), i.e. the map ζ concretely described in (20) below, is surjective in degree zero.

Proof. Since it is known that the tangent (resp. "obstruction") space of $\mathrm{GradAlg}(H_A)$ at (A) is ${}_0\mathrm{H}^1(R,A,A) = {}_0\mathrm{Hom}_A(I_A/I_A^2,A) \simeq {}_0\mathrm{Hom}_R(I_A,A)$ (resp. ${}_0\mathrm{H}^2(R,A,A)$) by [28], Thm. 1.5, we get the description in Theorem 36 of these spaces by Proposition 35. Then the left inequality of the dimension formula follows rather easily from [38], Thm. 4.2.4 while the right inequality is trivial. Hence we get all conclusions of the theorem once we have shown that the surjectivity in (17) and the surjectivity of ζ in (20) are equivalent. Indeed $\mathrm{Tor}^R_2(A,K_A) \simeq \mathrm{Tor}^R_1(I_A,K_A)$ and $\mathrm{Tor}^R_1(A,K_A) \simeq I_A \otimes_R K_A$ and the map of (17) is just the natural map $\zeta : I_A \otimes_R I_A \otimes_R K_A \to \mathrm{Tor}^R_1(I_A,K_A)$ uniquely described in the following way. Let $0 \to N \to F \to K_A \to 0$ be a short exact sequence where F is A-free. Applying $I_A \otimes_R (-)$ onto this sequence we get an injection $\mathrm{Tor}^R_1(I_A,K_A) \hookrightarrow I_A \otimes N$ which together with the surjection $F \twoheadrightarrow K_A$ lead to the composition

$$I_A \otimes_R I_A \otimes_R F \twoheadrightarrow I_A \otimes_R I_A \otimes_R K_A \xrightarrow{\zeta} \mathrm{Tor}^R_1(I_A,K_A) \hookrightarrow I_A \otimes_R N \tag{20}$$

given by $x \otimes y \otimes \omega \to x \otimes (y\omega) - y \otimes (x\omega) \in I_A \otimes_R N$ (cf. [2], Prop. 9, p. 204 for details). $\square$

Remark 37. *Let M be a graded A-module and let $0 \to N \to F \to M \to 0$ be a graded exact sequence where F is A-free. Arguing as in the proof above, we see that ${}_v\mathrm{H}_2(R,A,M)$ is the homology in degree v of* (21) *below. Hence it vanishes if and only if the sequence*

$$I_A \otimes_R I_A \otimes_R F \xrightarrow{\lambda} I_A \otimes_R N \to I_A \otimes_R F \ , \tag{21}$$

where $\lambda(x \otimes y \otimes \omega) = x \otimes (y\omega) - y \otimes (x\omega)$, is exact in degree v ([2], Prop. 9, p. 204).

Remark 38. *Let $A = R/I_A$ be a graded Artinian algebra and let M be a finitely generated R-module. Using* (18) *we get*

$$\mathrm{Hom}_A(\mathrm{Tor}_q^R(M,K_A),K_A) \simeq \ \mathrm{Ext}_R^q(M,A) \ .$$

Thus $(I_A \otimes_R K_A)_v$ (resp. ${}_v\mathrm{Tor}_1^R(I_A,K_A)$) is dual to ${}_{-v}\mathrm{Hom}_R(I_A,A)$ (resp. ${}_{-v}\mathrm{Ext}_R^1(I_A,A)$) and the dual of the degree v part of (17) *augmented by ${}_v\mathrm{H}_2(R,A,K_A)$ yields an exact sequence*

$${}_{-v}\mathrm{H}^2(R,A,A) \hookrightarrow \ {}_{-v}\mathrm{Ext}_R^1(I_A,A) \to \ {}_{-v}\mathrm{Hom}_R(I_A \otimes_R I_A, A)$$

where the left injective map must be the right inclusion of (3) *in degree $-v$.*

In the codimension 3 case it turns out that ${}_{-v}\mathrm{Ext}_R^1(I_A,A)$ is also dual to ${}_{v-3}\mathrm{Hom}_R(I_A,A)$:

Proposition 39. *Let $R \to A = R/I_A$, $R = k[x,y,z]$ be a graded Artinian quotient with Hilbert function H and minimal resolution*

$$0 \to \oplus_{i=1}^{r_3} R(-n_{3,i}) \to \oplus_{i=1}^{r_2} R(-n_{2,i}) \to \oplus_{i=1}^{r_1} R(-n_{1,i}) \to R \to A \to 0 \ . \tag{22}$$

Then ${}_v\mathrm{Ext}_R^i(I_A,A) \simeq \ {}_v\mathrm{Tor}_{1-i}^R(I_A,K_A(3))$ for $0 \le i \le 1$ and $N_A := \mathrm{Hom}_R(I_A,A)$ satisfies

$$\dim(N_A)_v - \ {}_v\mathrm{ext}_R^1(I_A,A) = \sum_{j=1}^{3}\sum_{i=1}^{r_j}(-1)^{j-1}H(n_{j,i}+v) - H(-v-3) \ .$$

Hence ${}_v\mathrm{Ext}_R^1(I_A,A)$ is dual to $(N_A)_{-v-3}$ for every v. Moreover if $(N_A)_{-3} = 0$, then ${}_0\mathrm{H}^2(R,A,A) = 0$ and $\mathrm{GradAlg}(H)$ is smooth at (A) of dimension $\sum_{j=1}^{3}\sum_{i=1}^{r_j}(-1)^{j-1}H(n_{j,i})$.

Proof. Applying ${}_0\mathrm{Hom}_R(-,R)$ to (22) we get an R-free resolution of $K_A(3)$. Then we get the complex

$$0 \to A \to \oplus_{i=1}^{r_1} A(n_{1,i}) \to \oplus_{i=1}^{r_2} A(n_{2,i}) \to \oplus_{i=1}^{r_1} A(n_{3,i}) \to K_A(3) \to 0$$

by tensoring the resolution of $K_A(3)$ by A. Note that the map $A \to \oplus_{i=1}^{r_1} A(n_{1,i})$ is zero since its matrix is given by the generators of I_A. It follows that the homology groups of the complex

$$0 \to \oplus_{i=1}^{r_1} A(n_{1,i}) \to \oplus_{i=1}^{r_2} A(n_{2,i}) \to \oplus_{i=1}^{r_1} A(n_{3,i}) \to 0$$

are precisely $\mathrm{Tor}_{2-j}^R(A,K_A(3))$ for $0 \le j \le 2$ and that $\mathrm{Tor}_0^R(A,K_A(3)) \simeq K_A(3)$. Moreover by applying ${}_0\mathrm{Hom}_R(-,A)$ onto the minimal resolution of I_A deduced from (22), we get exactly the latter complex. Hence the homology groups of the complex are also $\mathrm{Ext}_R^j(I_A,A)$ for $0 \le j \le 2$ by the definition of Ext. In particular ${}_v\mathrm{Ext}_R^2(I_A,A) \simeq \ K_A(3)_v$ and ${}_v\mathrm{Ext}_R^j(I_A,A) \simeq \ {}_v\mathrm{Tor}_{2-j}^R(A,K_A(3)) \simeq {}_v\mathrm{Tor}_{1-j}^R(I_A,K_A(3))$ for $0 \le j \le 1$ where the last isomorphism is easily proved by tensoring the exact sequence $0 \to I_A \to R \to A \to 0$ by $K_A(3)$. Since the alternating sum of the dimension of the terms in a complex equals the alternating sum of the dimension of its homology groups, we also get the double summation formula by combining with $\dim K_A(3)_v = \dim A_{-v-3}$.

Finally we know that $(I_A \otimes K_A(3))_v$ is dual to ${}_{-v-3}\mathrm{Hom}_R(I_A,A) = (N_A)_{-v-3}$ by Remark 38 and that ${}_v\mathrm{Ext}_R^1(I_A,A) \simeq \ {}_v\mathrm{Tor}_0^R(I_A,K_A(3))$ by the first part of the proof. Hence ${}_v\mathrm{Ext}_R^1(I_A,A)$ is dual to $(N_A)_{-v-3}$ and since ${}_0\mathrm{Ext}_R^1(I_A,A)$ contains ${}_0\mathrm{H}^2(R,A,A)$ which is dual to ${}_0\mathrm{H}_2(R,A,K_A)$ by Remark 38 we conclude by Theorem 36. $\square$

Corollary 40. *With A as in Proposition 39 we have*

$$\sum_{j=1}^{3}\sum_{i=1}^{r_j}(-1)^{j-1}H(n_{j,i}) = 1-\sum_{j=1}^{3}\sum_{i=1}^{r_j}(-1)^{j-1}H(n_{j,i}-3)\ .$$

Moreover the sum on the left above, which we call $\rho(H)$, depends only upon the Hilbert function H and not upon the graded Betti numbers. We have

$$\dim_{(A)} \mathrm{GradAlg}^H(k[x,y,z]) \geq \rho(H)\ .$$

Indeed $\rho(H)$ is a lower bound for the dimension of any *irreducible (non-embedded) component of* $\mathrm{GradAlg}^H(k[x,y,z])$.

Proof. The duality of the proposition shows that

$$\dim(N_A)_v - {}_v\mathrm{ext}^1_R(I_A,A) = {}_{-v-3}\mathrm{ext}^1_R(I_A,A) - \dim(N_A)_{-v-3}\ .$$

Putting $v=0$ we get the equality of the two expressions of $\rho(H)$ of the corollary by using the corresponding formula of Proposition 39 for $v=0$ and $v=-3$. Moreover, by Theorem 36 and Remark 38, the number $\dim(N_A)_0 - \dim\ {}_0\mathrm{H}^2(R,A,A)$ is a lower bound of $\dim_{(A)} \mathrm{GradAlg}^H(k[x,y,z])$. Using Remark 38 we get $\rho(H)$ to be a possibly smaller lower bound. Finally let V be a non-embedded irreducible component of $\mathrm{GradAlg}^H(k[x,y,z])$ and let $(A') \in V$ such that $\dim_{(A')} \mathrm{GradAlg}^H(k[x,y,z]) = \dim V$. Note that the sum which defines $\rho(H)$ depends only upon the Hilbert function because the contribution from all ghost terms (i.e. common direct free factors in consecutive terms in the minimal resolution of A) or from incomparable sets of graded Betti numbers (Remark 7) sums to zero! Hence $\dim V \geq \rho(H)$ by the first part of the proof, we are done. $\square$

Example 41. *To illustrate Proposition 39, we consider $H=(1,3,6,6,3,1)$ and the two different irreducible components (now of* $\mathrm{GradAlg}^H(k[x,y,z])$*) of Example 25 whose general elements are licci. Looking to the minimal resolutions of A_i of Example 25, we get*

$$\dim(N_{A_i})_0 - {}_0\mathrm{ext}^1_R(I_{A_i},A_i) = 4H(3)-4H(5) = 20\ .$$

By Remark 38, ${}_0\mathrm{Ext}^1_R(I_{A_1},A_1)=0$ since we get ${}_0\mathrm{Tor}_1^R(I_{A_1},K_{A_1}) \hookrightarrow (I_{A_1}\otimes I_{A_1})_5 = 0$ by using $K_{A_1}\simeq A_1(5)$. (Indeed this ${}_0\mathrm{Ext}^1$ -group always vanishes in the compressed Gorenstein case.) Thus the "Gorenstein" component has dimension 20 *(also well known by [24]), while Remark 26(iii) or Theorem 24 applied to the successive linkages* $(1,2,3)$, $(2,2,4)$, $(2,3,4)$, $(3,4,4)$, $(3,4,5)$, $(3,3,5)$ *obtained from a CI of type* $(1,1,1)$*, shows that the other component is generically smooth of dimension* 21. *Thus ${}_0\mathrm{ext}^1_R(I_{A_2},A_2)=1$.*

An alternative way of describing A_2 is to specify the three generators, F_1, F_2 and F_3, of degree $3,4$ *and* 5 *respectively, in the* dual polynomial algebra *of R which we will consider more closely later in this section. Indeed if we take F_1 to be a general polynomial of degree* 3 *(i.e. an element of some open set of the irreducible parameter space of all forms of degree* 3*), F_2 to be a sum of length* 4 *of general linear forms to the* 4*-th power and F_3 to be a sum of length* 2 *of general linear forms to the* 5*-th power, we get precisely A_2 as $A_2 = R/ann(F_1,F_2,F_3)$ (verifyed by using Macaulay 2).*

Recall that if A itself admits a semi-linear R-free resolution (except possibly at the minimal generators of I_A), then

$$ {}_0\mathrm{H}^2(R,A,A) = 0 \tag{23}$$

by Remark 30 and Remark 32. This vanishing also follows from Theorem 36. Moreover using Theorem 36, we can prove a "dual" result. Indeed suppose I_A admits a semi-linear resolution except possibly at the *left* end of the resolution, i.e. suppose I_A has minimal generators only in degree j and $j+1$ and that the resolution continues by

$$0 \to G \oplus R(-j-n+1)^{\alpha_n} \to R(-j-n+1)^{\beta_{n-1}} \oplus R(-j-n+2)^{\alpha_{n-1}} \to \dots \to F_1 \to I_A \qquad (24)$$

where G is any R-free module. Here $F_1 = R(-j-1)^{\beta_1} \oplus R(-j)^{\alpha_1}$ and R is the polynomial ring $k[x_1, \dots, x_n]$ with $n \geq 2$. Then (23) holds. Using (18) we can even replace F_1 by

$$F_1 = R(-j-1)^{\beta_1} \oplus R(-j)^{\alpha_1} \oplus (\oplus_{i=1}^m R(-a_i)) \ \ , \ \ a_i < j \ \text{ for all } \ i \qquad (25)$$

where the set of generators $\{f_1, \dots, f_m\}$ which correspond to $\{a_1, \dots, a_m\}$ form a regular sequence, and still get (23), i.e.;

Proposition 42. *Let $A = R/I_A$ be a graded Artinian quotient with Hilbert function H, whose minimal resolution is given by* (24) *and* (25) *where the generators $\{f_1, \dots, f_m\}$ of I_A which correspond to $\{a_1, \dots, a_m\}$ form a regular sequence. Let $B = R/(f_1, \dots, f_m)$ (and $B = R$ if $m = 0$). Then ${}_0\mathrm{H}^2(R, A, A) = 0$ and $\mathrm{GradAlg}(H)$ is smooth at (A). Moreover*

$$\dim_{(A)} \mathrm{GradAlg}(H) = \ {}_{-n}\hom_R(G, B) - \ {}_{-n}\hom_R(G, A) + \sum_{i=1}^{m} H(a_i).$$

Proof. By the long exact sequence of algebra cohomology (Remark 30) and (3) we get ${}_0\mathrm{H}^2(R, A, A) = 0$ provided we can show $\mathrm{Ext}^1_B(I_{A/B}, A) = 0$. Continuing the long exact sequence of Remark 30 to the left we see that $\mathrm{Ext}^1_B(I_{A/B}, A) = 0$ also leads to

$$\dim(N_A)_0 = \hom_B(I_{A/B}, A) + \hom_B(I_B/I_B^2, A) \ . \qquad (26)$$

To show $\mathrm{Ext}^1_B(I_{A/B}, A) = 0$, we improve a little bit upon Theorem 36 by using (18). Indeed we have $\mathrm{Hom}_A(\mathrm{Tor}^B_q(I_{A/B}, K_A), K_A) \simeq \ \mathrm{Ext}^q_B(I_{A/B}, A)$. Hence it suffices to show ${}_0\mathrm{Tor}^R_1(I_{A/B}, K_A) = 0$. Now look to the exact sequence

$$\to R(j+n-1)^{\beta_{n-1}} \oplus R(j+n-2)^{\alpha_{n-1}} \to G^* \oplus R(j+n-1)^{\alpha_n} \to K_A(n) \to 0$$

which we tensor with $I_{A/B}(-n)$. By the definition of $\mathrm{Tor}^R_1(I_{A/B}, K_A)$, it suffices to show $(I_{A/B}(-n)(j+n-1))_0 = 0$ and $(I_{A/B}(-n)(j+n-2))_0 = 0$. This is true since $(I_{A/B})_{j-1} = 0$ by assumption. Moreover the argument also shows $(I_{A/B} \otimes K_A)_0 \simeq \dim(G^*(-n) \otimes I_{A/B})_0$. Hence we get

$${}_0\hom_B(I_{A/B}, A) = \dim \mathrm{Tor}^B_0(I_{A/B}, K_A) = \ {}_{-n}\hom_R(G, B) - \ {}_{-n}\hom_R(G, A) \ ,$$

and we conclude by (26) and the fact that $I_B/(I_B)^2 \simeq \oplus_{i=1}^m B(-a_i)$. ☐

Proposition 42 with $B = R$ applies nicely to compressed Artinian algebras. Indeed the number ${}_{-n}\hom_R(G, B) - {}_{-n}\hom_R(G, A)$ coincides with the dimension of the corresponding component given in Thm. IIB of [24]. If $B \neq R$, Proposition 42 also applies to non-compressed algebras:

Example 43. *As a special case of Proposition 42 we look to Artinian level quotients with "almost semi-linear" resolution. All level algebras below may be constructed as $A = R/ann(F_1, F_2)$ where F_1 and F_2 are forms of degree 7 in the dual polynomial algebra of R (cf. later discussion). Indeed we easily construct in this way algebras A_i with Hilbert functions $H_{A_1} = (1, 3, 6, 10, 15, 12, 6, 2)$, $H_{A_2} = (1, 3, 6, 10, 14, 12, 6, 2)$, $H_{A_3} = (1, 3, 6, 10, 13, 12, 6, 2)$ and corresponding minimal resolutions*

$$0 \to R(-10)^2 \to R(-7)^5 \oplus R(-6)^5 \to R(-5)^9 \to I_{A_1} \to 0$$

$$0 \to R(-10)^2 \to R(-7)^6 \oplus R(-6)^4 \to R(-6)^2 \oplus R(-5)^6 \oplus R(-4) \to I_{A_2} \to 0$$
$$0 \to R(-10)^2 \to R(-7)^7 \oplus R(-6)^3 \to R(-6)^4 \oplus R(-5)^3 \oplus R(-4)^2 \to I_{A_3} \to 0 \ .$$

Only A_1 is compressed, but since one may show that the minimal generators of I_{A_i} of degree 4 (which we use to define B_i) form a regular sequence, Proposition 42 applies (we have used Macaulay 2 to check it and to find the minimal resolutions). Hence the algebras A_i are unobstructed and since ${}_{-n}\hom_R(G,M) = 2 \cdot \dim M_7$ and $\dim(B_i)_7 = (i-1)\dim R_3$ for $i = 2$ and 3 (and $B_1 = R$), we get the number

$$\dim_{(A_i)} \mathrm{GradAlg}(H_{A_i}) = 2 \cdot \dim(B_i)_7 - 2 \cdot \dim H_{A_i}(7) + (i-1)H_{A_i}(4)$$

to be $68, 62, 54$ for $i = 1, 2, 3$ respectively.

To this end we consider level algebras of CM-type t. Let $\mathrm{LevAlg}(H)$ be the open set of $\mathrm{GradAlg}(H)$ (and hence open as a subscheme with its induced scheme structure) consisting of graded level quotients with Hilbert function H. Since we work with Artinian algebras there is another known scheme, $\mathrm{L}(H)$, parametrizing graded level quotients with suitable Hilbert function H, namely the determinantal loci in the Grassmannian $G(t,j)$ of t-dimensional vector spaces of forms of degree j, cut out by requiring their "catalecticant matrices" to have ranks given by the Hilbert function (see [9], and [26], Sect. 1.1 for the Gorenstein case). Then the underlying sets of closed points of $\mathrm{L}(H)$ and $\mathrm{LevAlg}(H)$ are the same by apolarity (the Macaulay correspondence), and their tangent spaces are isomorphic ([9], Thm. 2.1 for $\mathrm{L}(H)$, and [28], Thm. 1.5 for $\mathrm{GradAlg}(H)$). Since one may by the proof below see that $\mathrm{LevAlg}(H)$ and $\mathrm{L}(H)$ are in fact isomorphic as topological spaces (expected since they have the Zariski topologies and the bijection between them is natural), we have

Theorem 44. *Let $R \to A$ be a graded Artinian level quotient with Hilbert function H. Then $\dim_{(A)} \mathrm{GradAlg}^H(R) = \dim_{(A)} \mathrm{L}(H)$. Hence $\mathrm{L}(H)$ is smooth at (A) if and only if $\mathrm{GradAlg}^H(R)$ is smooth at (A). In particular $\mathrm{L}(H)$ is smooth at (A) provided ${}_0\mathrm{H}^2(R,A,A) = 0$, i.e. provided the map of (17) is surjective in degree zero, or equivalently, the displayed sequence of Remark 37 with $M = K_A$ is exact in degree zero.*

Proof. Let $V \subset \mathrm{L}(H)$ be a closed irreducible subset, and let V have the reduced scheme structure. By the definition of $\mathrm{L}(H)$, the restriction of the "universal" bundle of the Grassmannian $G(t,j)$ to V defines via apolarity a family of graded Artinian level quotients, A_V, over V with constant Hilbert function H. Since V is integral, it follows that the family (i.e. the morphism $\mathrm{Spec}(A_V) \to V$) is flat ([45], Lect. 6). Hence we have a morphism $\pi : V \to \mathrm{LevAlg}(H)$ by the universal property of $\mathrm{GradAlg}(H)$. $\pi(V)$ is irreducible and closed in $\mathrm{LevAlg}(H)$ (it is closed because an "inverse" $(\mathrm{LevAlg}(H))_{red} \to \mathrm{L}(H)$ on closed points exists, by [26], p. 249). So chains of closed irreducible subsets in $\mathrm{L}(H)$ and $\mathrm{LevAlg}(H)$ correspond, and the spaces have the same dimension. Since their tangent spaces are isomorphic, it follows that $\mathrm{GradAlg}^H(R)$ is smooth at (A) iff $\mathrm{L}(H)$ is smooth at (A). Then we conclude by Theorem 36 since the surjectivity of (17) in degree zero is equivalent to the exactness of the corresponding sequence in Remark 37. $\square$

As an application we consider certain type 2 level algebras studied by Iarrobino in [25], i.e. level algebras given by $A = R/ann(F_1, F_2)$ where F_1 and F_2 are forms of the same degree j in the dual polynomial algebra of R, upon which R acts by differentiation ("without coefficients"). Let $A_i := R/ann(F_i)$. Since we have $I_A = I_{A_1} \cap I_{A_2}$, we get an exact sequence

$$0 \to A \to A_1 \oplus A_2 \to R/(ann(F_1) + ann(F_2)) \to 0.$$

In an extended draft of [25] the author determines the tangent space of $\mathrm{LevAlg}(H)$ at such an (A) and he gives it a manageable form in the case $\{F_1,F_2\}$ is *complementary*, i.e. provided

$$H_A(i)=\min\{\dim R_i, H_{A_1}(i)+H_{A_2}(i)\} \quad \text{for any } i\ . \tag{27}$$

where $H_A=H$. Our Theorem 36 gives us, not only a tangent space which coincides with his, but it provides us also with the following manageable form of the obstruction space.

Proposition 45. *Let $\{F_1,F_2\}$ be complementary forms of degree j, and let $A=R/I_A$ be the Artinian level quotient with Hilbert function H given by $I_A=ann(F_1,F_2)$. Let $I_{A_i}=ann(F_i)$. Then $(I_A/I_A\cdot I_{A_1})_j\ \oplus(I_A/I_A\cdot I_{A_2})_j$ is the dual of the tangent space of $\mathrm{GradAlg}^H(R)$ at (A), and ${}_j\mathrm{H}_2(R,A,A_1)\oplus_j\mathrm{H}_2(R,A,A_2)$ is the dual of a space containing all obstructions of deforming A as a graded R-algebra. In particular if the sequences*

$$I_A\otimes_R I_A \xrightarrow{\lambda} I_A\otimes_R I_{A_i} \twoheadrightarrow I_A\cdot I_{A_i}$$

where $\lambda(x\otimes y)=x\otimes y-y\otimes x$, are exact in degree j for $i=1,2$, then $\mathrm{GradAlg}(H)$ (and $\mathrm{L}(H)$) is smooth at (A) and we have

$$\dim_{(A)}\mathrm{GradAlg}^H(R)=\sum_{i=1}^{2}\dim(I_A/I_A\cdot I_{A_i})_j$$

Remark 46. *The map $I_A\otimes_R I_A \xrightarrow{\lambda'} I_A\otimes_R I_A$, defined by $\lambda'(x\otimes y)=x\otimes y-y\otimes x$, obviously commutes with λ above. Since λ' factors via the natural surjection $I_A\otimes_R I_A\twoheadrightarrow\Lambda^2 I_A$ (in $char(k)\neq 2$), then λ also does. In $char(k)\neq 2$ the exactness of the two sequences of Proposition 45 is therefore equivalent to the exactness of*

$$\Lambda^2 I_A\to I_A\otimes_R I_{A_i}\twoheadrightarrow I_A\cdot I_{A_i}\ , \tag{28}$$

$i=1,2$, in degree j. Indeed, by Remark 37, ${}_j\mathrm{H}_2(R,A,A_i)$ is the homology of (28) in degree j. In particular if $(I_A\otimes_R I_A)_j\simeq(S_2I_A)_j$ (e.g. $(I_A\otimes_R I_A)_j\simeq({I_A}^2)_j$), then the exactness of the sequences of Proposition 45 is equivalent to $(I_A\otimes_R I_{A_i})_j\simeq(I_A\cdot I_{A_i})_j$.

Proof. Let $s(I_A)$ be the minimal degree of a minimal generator of I_A and let $\overline{A}=R/(ann(F_1)+ann(F_2))$. Since $\{F_1,F_2\}$ is complementary, we get $(\overline{A})_v=0$, i.e. $A_v\simeq(A_1)_v\oplus(A_2)_v$ for $v\geq s(I_A)$. It follows that

$$(K_{A_1})_v\oplus(K_{A_2})_v\simeq(K_A)_v$$

for $v\leq -s(I_A)$. Defining $\overline{K}$ by the long exact sequence

$$0\to\overline{K}\to K_{A_1}\oplus K_{A_2}\to K_A\to 0\ , \tag{29}$$

we get $(\overline{K})_v=0$ for $v\leq -s(I_A)$. By considering a minimal R-free resolution of I_A, it follows that

$${}_0\mathrm{Tor}_i^R(I_A,\overline{K})=0 \ \text{ for } \ i\leq 0\ . \tag{30}$$

Now applying $I_A\otimes(-)$ onto (29), or more precisely using the corresponding long exact sequence of algebra homology, we get

$${}_0\mathrm{H}_i(R,A,K_{A_1}\oplus K_{A_2})\simeq {}_0\mathrm{H}_i(R,A,K_A)$$

for $i=1$ and 2 because ${}_0\mathrm{Tor}_1^R(I_A,\overline{K})\simeq {}_0\mathrm{Tor}_2^R(A,\overline{K})\twoheadrightarrow {}_0\mathrm{H}_2(R,A,\overline{K})$ is surjective (cf. (17)) and $I_A\otimes_R\overline{K}\simeq\mathrm{H}_1(R,A,\overline{K})$, i.e. ${}_0\mathrm{H}_i(R,A,\overline{K})$ vanishes for $i=1$ and 2 by (30). Then we conclude easily by $A_i\simeq K_{A_i}(-j)$, Theorem 36 and Remark 37. Indeed we have

$$(I_A\otimes_R K_{A_i})_0\simeq (I_A\otimes_R A_i(j))_0\simeq(I_A\otimes_R R/I_{A_i})_j\simeq(I_A/I_A\cdot I_{A_i})_j$$

and we get $(I_A \otimes_R K_A)_0 \simeq \ (I_A \otimes_R K_{A_1})_0 \oplus (I_A \otimes_R K_{A_2})_0$ as well as

$$ {}_0\mathrm{H}_2(R,A,K_A) \simeq \ {}_j\mathrm{H}_2(R,A,A_1) \oplus_j \mathrm{H}_2(R,A,A_2) \ . $$

By the assumption of the exactness of the sequences and by Remark 37 (letting $F = R$ and $N = I_{A_i}$), we get the vanishing of ${}_0\mathrm{H}_2(R,A,K_A)$ and we are done. $\square$

Remark 47. *As Iarrobino points out in the draft of [25], Theorem* 4.8*A of [24] implies that if* F_1 *is any form of degree* j *and* F_2 *is a sum of length* s *of linear forms to the* j*-th power (i.e. the Hilbert function of* $A_2 = R/ann(F_2)$ *equals* $H(s,j,n)$ *of Remark 26(v)), then* $\{F_1, F_2\}$ *is complementary provided we choose the linear forms of* F_2 *general enough. It follows that* H_A *is given by* (27).

First we give an easy example which may also be treated by Proposition 42.

Example 48. *(a) Let* $H = (1,3,6,10,6,2)$*. By Remark 47 there are forms* F_1 *and* F_2 *where each* F_i *is a sum of length* 5 *of linear forms to the* 5*-th power (i.e.* $H_{A_i} = (1,3,5,5,3,1)$*) and such that the Hilbert function of* $A = R/ann(F_1,F_2)$ *is* H*. Then* A *is compressed. From the Hilbert functions we see that* $s(I_A) = 4$ *while* $s(I_{A_i}) = 2$*. Moreover, the socle degree of* A *and* A_i *are* 5*, and we get* $(I_A \otimes I_{A_i})_5 = 0$ *for* $i = 1$ *and* 2*. By Proposition 45 it follows that* $\mathrm{GradAlg}(H)$ *is unobstructed at* (A) *and we have*

$$ \dim_{(A)} \mathrm{GradAlg}(H) = \ 2 \cdot \dim(I_A)_5 = 38 \ . $$

(b) Let $H = (1,3,6,9,6,2)$*, let* F_1 *be as in (a), while we now let* F_2 *be a sum of length* 4 *of general linear forms to the* 5*-th power. Hence* $H_{A_2} = (1,3,4,4,3,1)$ *and* $H = H_A$ *where* $A = R/ann(F_1,F_2)$ *by Remark 47. From the Hilbert functions we see that* $s(I_A) = 3$ *and* $s(I_{A_i}) = 2$*. Since we easily see that* $I_A \otimes I_{A_i} \simeq I_A \cdot I_{A_i}$ *is an isomorphism in degree* 5 *for* $i = 1$ *and* 2*, we get by Proposition 45 that* $\mathrm{GradAlg}(H)$ *is unobstructed at* (A) *and that*

$$ \dim_{(A)} \mathrm{GradAlg}(H) = \ 2 \cdot \dim(I_A)_5 - dim(I_A \cdot I_{A_1})_5 - \dim(I_A \cdot I_{A_2})_5 = 35 \ . $$

Loosely speaking it is, for $i = 1,2$, the relations of $I_A \cdot I_{A_i}$ in degree j, modulo those coming from the relations of $I_A \otimes I_{A_i}$ and the generators of $\wedge^2 I_A$, which contribute to ${}_0\mathrm{H}^2(R,A,A)$. Of course the vanishing of ${}_0\mathrm{H}^2(R,A,A)$ as well as the dimension of $\mathrm{GradAlg}^H(R)$ is usually straightforward to get from Proposition 45 provided $s(I_A) + s(I_{A_i}) \geq j$ for $i = 1,2$, as in Example 48.

We finish this paper by proving a conjecture of Iarrobino, appearing in the draft of [25], namely that $\mathrm{L}(H)$ with $H = (1,3,6,10,14,10,6,2)$ contains at least two irreducible components, where one of the components contains Artinian level type 2 algebras given by 2 forms of Hilbert function $H_1 = (1,3,6,9,9,6,3,1)$ and $H_2 = (1,3,4,5,5,4,3,1)$, as in Remark 47, and the other contains level type 2 algebras constructed via 2 forms with Hilbert function $H_3 = (1,3,6,10,10,6,3,1)$ and $H_4 = (1,3,4,4,4,4,3,1)$. As pointed out in the Introduction, even though this conjecture was open until now, Iarrobino and Boij have in a joint work already constructed other examples of reducible $\mathrm{L}(H)$ whose general elements are level quotients of type 2, one with $H = (1,3,6,10,14,18,20,20,12,6,2)$, and moreover got a doubly infinite series of such components [5].

Example 49. *Let* $H = (1,3,6,10,14,10,6,2)$*. We claim that there are at least two components* V_1 *and* V_2 *of* $\mathrm{L}(H)$ *whose general elements are Artinian level type 2 algebras, that* $\dim V_1 = 46$ *and* $\dim V_2 = 47$ *and that both components are generically smooth.*

To get the component V_1 *of dimension* 46*, take* F_1 *to be a sum of length* 4 *of general linear forms to the* 7*-th power and take* F_2 *to be a general polynomial of degree 7. If* $A_i = R/ann(F_i)$ *and* $A = R/ann(F_1,F_2)$*, then* $H_{A_2} = (1,3,6,10,10,6,3,1)$*,* $H_{A_1} = (1,3,4,4,4,4,3,1)$ *and* $H_A = H$*. It suffices to show that* A *is unobstructed and that* $\dim_{(A)} \mathrm{GradAlg}(H) = 46$*. To do so we use*

Proposition 45. Indeed from the Hilbert functions we see that $s(I_A) = s(I_{A_2}) = 4$ *and* $s(I_{A_1}) = 2$. *Hence* $(I_A \otimes I_{A_2})_7 = 0$. *Moreover since* I_A *has one generator of degree* 4 *and* 8 *generators of degree* 5 *and* A_1 *is a complete intersection of type* $(2, 2, 6)$, *it follows that all relations of* $I_A \cdot I_{A_1}$ *must be of degree greater or equal to* 8. *We get that* $(I_A \otimes I_{A_1})_7 \simeq (I_A \cdot I_{A_1})_7$ *is an isomorphism of vector spaces of dimension* $2 \cdot (3 + 8) = 22$. *Hence Proposition 45 applies and we get the unobstructedness of* A *and*

$$\dim_{(A)} \mathrm{GradAlg}(H) = \ 2 \cdot \dim(I_A)_7 - dim(I_A \cdot I_{A_1})_7 = 46 \ .$$

To get the other component, let now F_1 *be a sum of length* 9 *of general linear forms to the* 7*-th power (i.e.* $H_{A_1} = (1, 3, 6, 9, 9, 6, 3, 1)$*), let* F_2 *be, say* $F_2 = x^6y+xy^6+z^7$ *and let* $A = R/ann(F_1, F_2)$. *Then the the Hilbert function of* A *is* H *by Remark 47. We claim that* A *is licci! Indeed it is easily checked by Macaulay 2 that* A *above admits the following CI-linkages to a CI of type* $(1, 1, 3)$. *We start with a general CI of type* $(4, 5, 7)$ *whose generators are contained in* I_A *and follow up by general CI-linkages of type* $(4, 5, 6)$, $(4, 4, 6)$, $(4, 4, 5)$, $(3, 3, 5)$, $(3, 3, 4)$, $(2, 2, 4)$ *and* $(2, 2, 3)$, *in this order. Then* A *is unobstructed and* $\dim_{(A)} \mathrm{GradAlg}(H) = 47$ *by Remark 26(iii) or Theorem 24 and we are done (of course, once using Macaulay 2 it is easier to see that the tangent space is* 47*-dimensional by computing* ${}_0\mathrm{ext}^1(I_A, I_A)$. *The unobstructedness of* A *is, however, not at all easy to see by Macaulay 2 computations because* ${}_0\mathrm{Ext}^1_R(I_A, A) \simeq \ {}_0\mathrm{Ext}^2(I_A, I_A)$ *is 1-dimensional and so is* ${}_0\mathrm{H}^2(R, A, A)$ *by Proposition 45 and Remark 46. Hence we really need to use that the unobstructedness property is preserved under CI-linkages, which is true by Theorem 24).*

Macaulay 2 also provides us with the following minimal resolution

$$0 \to R(-10)^2 \to R(-8)^2 \oplus R(-7) \oplus R(-6)^8 \to R(-7) \oplus R(-5)^8 \oplus R(-4) \to R \to A \to 0 \ .$$

Thanks to Theorem 24 we claim this is the minimal resolution of the general element of V_2*! Indeed by Theorem 24 we know that general CI-linkages take open sets onto open sets. Hence if we start with a general CI of type* $(1, 1, 3)$ *and reverse all the general CI-linkages above, we get a general element of* V_2 *which one may check (Macaulay 2) has the minimal resolution described above. Note the ghost-term* $R(-7)$. *This term is not present in the minimal resolution of the algebra we described in* V_1 *which, by Macaulay 2, has another ghost-term, namely* $R(-6)$. *Still we claim that the general elements of* V_1 *and* V_2 *have different* but comparable *sets of graded Betti numbers. Indeed in Iarrobino's draft [25] he also mention that there is another algebra* $A' = R/ann(F_1, F_2)$ *with* $H_{A'} = (1, 3, 6, 10, 14, 10, 6, 2)$ *obtained by taking two general enough forms* $\{F_1, F_2\}$ *with* $A_i = R/ann(F_i)$ *and* $H_{A_i} = (1, 3, 5, 7, 7, 5, 3, 1)$ *for* $i = 1, 2$. *We have used Macaulay 2 to see that if each* F_i *for* $i = 1, 2$ *is of the form* $F_i = (l1)^5 * (l2) * (l3) + (l1) * (l2)^5 * (l3)$ *where* lj *are general linear forms, then* A' *has a minimal resolution as above without ghost-terms. Hence the set of graded Betti number of the general elements of* V_1 *and* V_2 *have a common minimum and the claim is proved.*

Remark 50. *(a) We have tried to look for other examples of several "level type 2 components" of* smaller *socle degree, but have not yet fully succeeded. A promising candidate is* $H = (1, 3, 6, 9, 9, 6, 2)$ *where we get a level type 2 algebra* A *by starting with a CI of type* $(2, 2, 3)$ *and linking in one step via a CI of type* $(4, 4, 3)$. *By Remark 26(iii) we have* ${}_0\mathrm{hom}_R(I_A, A) = 33$. *Moreover we have an* $A' = R/ann(G_1, G_2)$ *with* ${}_0\mathrm{hom}_R(I_{A'}, A') = 35$ *(checked by Macaulay 2) by taking* G_1 *(resp.* G_2*) to be a sum of length* 3 *(resp.* 6*) of general linear forms to the* 6*-th power. It follows that* A *belongs to a* 33*-dimensional generically smooth component while, due to the size of the tangent spaces, there are only two possibilities for* A'. *It is either obstructed, or it is unobstructed in which case it belongs to an irreducible component different from the "licci" component. We have not yet been able to decide which of the possibilities that occur.*

(b) One may construct other examples of several "level type 2 components" of larger *socle type by taking the two components of Example 49 and performing a biliaison, starting with general CI's of*

type $(5,5,b)$ *containing the general elements of the components and follow up by general CI-linkages of type* (b,b,b), $b \geq 7$. *Using Theorem 24, we get two irreducible components of* $\mathrm{GradAlg}(H')$ *whose general elements are level type 2 quotients of socle degree* $3b-8$ *(*H' *may be computed from* $H = (1,3,6,10,14,10,6,2)$*).*

Remark 51. *If we want to compare the parameter space of type 2 codimension 3 level algebras to the corresponding space of Gorenstein algebras, we see many differences. In the level type 2 case,*

(i) the parameter space may be reducible (Example 49 and Remark 50(b)),

(ii) ${}_0\mathrm{H}^2(R,A,A)$ *may be non-vanishing (e.g. Example 49, there are many more).*

In the Gorenstein case (i) and (ii) are false. We have, however, not yet been able to find two irreducible "level type 2 codimension 3 components" with a type 2 level algebra in the intersection, nor have we been able to find an obstructed type 2 codimension 3 level algebra.

Since the general elements of the components of Example 49 have different sets of graded Betti numbers, one may look for multiple components in $\mathrm{LevAlg}(H)$ *(e.g. of "type 2") or in* $\mathrm{GradAlg}(H)$ *whose general elements have the same sets of graded Betti numbers. We have in [32], Ex. 26 and Rem. 27 described several such examples, the simplest one consists of "level type 3 codimension 3 components" (resp. "Gorenstein codimension 4 components") whose general algebras are level (resp. Gorenstein) of dimension 2 (resp. dimension one). We may truncate the algebras (Proposition 33), or better, divide by some twist of the canonical module (Theorem 27) to get many examples of e.g. multiple "Artinian Gorenstein codimension 5 components" whose general elements have the same sets of graded Betti numbers. We are, however, not aware of examples of multiple components in the same Betti stratum in the level (resp. Gorenstein) Artinian codimension* 3 *(resp.* 4*) case.*

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