Dynamics and kneading sequences of skew tent maps

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Abstract
We study the dynamics of the two-parameter family of skew tent maps, namely how the change of the behavior of the map is depending on parameters. So we decompose the parameter space by the nature of periodic points. Especially we prove that the form of attracting orbit is of stair type and give an explicit proof of the monotonicity of kneading sequence.

1 Introduction

Dynamics of the family of skew tent maps has been studied in, for example, [ITN79] and [MV91].

According to J. Milnor[Mil85], ”attractor” is defined as the set which attracts almost all orbits in the neighborhood. To say precisely, the set $A$ is called attractor of $f$ when it satisfies the following four conditions:

$(1)$ $A$ is a closed set.

$(2)$ $f(A) = A$.

$(3)$ The set $A$ has a neighborhood $U$ such that $\lim_{n \to \infty} f^n(x) \in A$ for almost all $x \in U$.

$(4)$ There exists a point $x \in A$ whose orbit is dense in $A$.

If attractor $A$ is a set of $k$ points, then each point in $A$ is mapped to itself under $f^k$ and called attracting $k$-periodic point.
Suppose that $x$ is a $k$-periodic point of $f$ and $f$ is differentiable on the neighborhood of orbit of $x$. If $|(f^k)'(x)| < 1$, then $x$ is attractive.

According to R. L. Devaney [Dev89], the dynamics of $f$ defined on $X$ is said to be chaotic if $f$ satisfies the following:

1. $f$ has sensitive dependance on the initial condition; there exists $\delta > 0$ such that, for any $x \in X$ and any neighborhood $U_x$, there exists $y \in U_x$ and $n \geq 0$ such that $|f^n(x) - f^n(y)| > \delta$.

2. $f$ is topologically transitive; for any pair of open sets $U, V \subset X$ there exists $n \geq 0$ such that $f^n U \cap V \neq \emptyset$.

3. periodic points are dense in $X$.

For two maps $f : V \rightarrow V$ and $g : U \rightarrow U$, if there exists a homeomorphism $h$ such that $h \circ f = g \circ h$, then $f$ and $g$ are said to be topologically conjugate, denoted by $f \sim g$. The map $h$ is called topologicall conjugacy.

For the next family of skew tent maps on $\mathbb{R}$, a quite different bifurcation occurs, compared with the one which occurs for the logistic family $g_u(x) = ux(1-x)$ with $3 < u < 4$.

$$F_{\mu}(x) = \begin{cases} 
ax + \mu, & \text{for } x \leq 0 \\
-bx + \mu, & \text{for } x \geq 0 
\end{cases}$$

where $0 < a < 1$, $b > 1$.

For $\mu = 1$, $-2$, where $a = \frac{1}{2}$, $b = 6$.

First let us fix $a, b$ and think $\mu$ as the parameter. Then a bifurcation occurs at $x = 0$ and $\mu = 0$ (see [NY95]). If we take $a = \frac{1}{2}$ and $b = 3.2$ (resp. 4.14, 4.42 or 5.5), bifurcation diagrams are given in from Fig.2 to Fig. 5. In Fig.2, the family exhibits a bifurcation from an attracting fixed point to attracting 3-periodic points. In each Fig.3, 4, 5, the family exhibits a bifurcation from an attracting fixed point to attracting intervals on which $F_{\mu}$ is chaotic. From these figures, we can see that there are various types
with respect to what bifurcation occurs at $\mu = 0$. On the other hand, for any $\mu > 0$, there does not occur new bifurcation as long as the parameter pair $(a, b)$ is fixed. It is because $F_\mu \sim F_1$ for all $\mu > 0$.

Figure 2: $a = \frac{1}{2}$, $b = 3.2$. $F_\mu$ exhibits bifurcation from an attracting fixed point to attracting 3-periodic points.

Figure 3: $a = \frac{1}{2}$, $b = 4.14$. $F_\mu$ exhibits bifurcation from an attracting fixed point to six attracting intervals. $F_\mu$ is chaotic on those intervals.

In this paper, we will study the characterization of an attracting periodic orbit of skew tent map. After this, for example, we will see when the bifurcation like as Fig. 2 occurs.

We remark that putting the results in the sequel paper [ITO] in this volume together, we can completely obtain the partition of $D$ in terms of the characters of dynamics of $f_{a,b}$.

For our purpose, we have only to fix $\mu = 1$ and analyze the dynamics of $f_{a,b}$ with the following parameter plane $D$:

$$D = \{(a,b); \ a > 0, \ b > 1, \ a + b \geq ab\}.$$  

$$f_{a,b}(x) = \begin{cases}  
  ax + 1, & \text{for } x \leq 0 \\
  -bx + 1, & \text{for } x \geq 0 
\end{cases}$$, where $(a,b) \in D$.

We expand the range of the parameter $a$ from $\{0 < a < 1\}$ to $\{a > 0\}$. If $a + b < ab$, almost all orbits tend to $-\infty$; if $a + b \geq ab$, then the orbit of $x$ in $R \setminus I_{a,b}$ tends to either the interval $[f_{a,b}^2(0), f_{a,b}(0)]$ or $-\infty$, or $x$ is a fixed point itself. Therefore the restriction $\{a + b \geq ab\}$ is reasonable.
Figure 4: $a = \frac{1}{2}, b = 4.42$. $F_\mu$ exhibits bifurcation from an attracting fixed point to three attracting intervals. $F_\mu$ is chaotic on those intervals.

Figure 5: $a = \frac{1}{2}, b = 5.5$. $F_\mu$ exhibits bifurcation from an attracting fixed point to one attracting interval. $F_\mu$ is chaotic on the interval.

Hence we find that it is sufficient to examine the dynamics of $f_{a,b}$ on $I_{a,b}$, where we denote $I_{a,b} = [f_{a,b}^2(0), f_{a,b}(0)]$.

## 2 Attracting region of $D$

In this section, we shall characterize the regions where $f_{a,b}$ has an attracting orbit.

For $k \geq 2$, Set

$$D_k = \left\{ (a, b) \in D; \frac{1}{a} + \cdots + \frac{1}{a^{k-2}} < b \leq \frac{1}{a} + \cdots + \frac{1}{a^{k-1}} \right\},$$

$$D_k^A = \left\{ (a, b) \in D_k; a^{k-1}b \leq 1 \right\}.$$

**Proposition 1** A boundary of $D_k^A$ is determined by algebraic curves. These curves have the following defining equations:

$$a^{k-1}b = 1, \quad b = 1 + \frac{1}{a} + \cdots + \frac{1}{a^{k-2}}.$$

These curves intersect at only one point $P_k$, and $\lim_{k \to \infty} P_k = \left(\frac{1}{2}, \infty\right)$. 

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Proof. The equation $a^{k-1} + \cdots + a = 1$ has a unique positive root, and so $P_k$ is unique. Obviously, it is smaller than 1. Since $1 + \frac{1}{2} + \frac{1}{2^2} + \cdots = 1$, the root converges to $a = \frac{1}{2}$ as $k \to \infty$. And the value $b$ diverges to $\infty$. □

Figure 6: The domains $D_k^4$ and boundary curves

For our purpose, we refer some results from the paper [ITN79].

First, let us prepare the following intervals and subdomains of $D$.

$I_L = [f^2(0), 0]$, $I_R = [0, f(0)]$,
$D_0 = \{(a, b) \in D; \ ab > 1, \ a + b \geq ab^2\}$,
$D_1 = \{(a, b) \in D; \ a + b < ab^2, \ b < 1 + \frac{1}{a}\}$.

For $k \geq 3$, 

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\[ D_k^B = \{(a,b) \in D_k; a^{k-1}b > 1, a+b \geq a^{k-1}b^2\}, \]
\[ D_k^* = D_k \setminus (D_k^A \cup D_k^B), \]
\[ D^* = \{(a,b) \in D; a > 1, a+b < ab^2\}. \]

In the last two equations above, we note that \( D_k^B = D_0 \) and \( D_k^* = D_1 \) for \( k = 2 \).

In the following Fact 1 and Fact 2, we suppose that \((a, b)\) belongs in the interior of \( D_k \) for \( k \geq 3 \).

**Fact 1** Set

\[
x_+ = \frac{1-a-b+a^{k-1}b}{(1-a)(1+a^{k-1}b)}, \quad x^* = \frac{-a^{k-2}b^2+b^2-b+ab+1-a}{(1-a)(1-a^{k-2}b^2)},
\]
\[
x_0 = \frac{-1+a^{k-2}}{a^{k-2}(1-a)}, \quad \text{and} \quad I_0 = [f_{a,b}^2(0), x^*].
\]

Then the point \( x_0 \) is mapped to \( x = 0 \) under \( f_{a,b}^{k-2} \), and both \( x_+ \) and \( x^* \) are \( k \)-periodic points of \( f_{a,b} \) with \( f_{a,b}^2(0) < x_+ < x_0 < x^* \). Moreover, if \((a, b)\) is in \( D \setminus D_k^* \), then almost all points in \( I_{a,b} \) tend to the interval \( I_0 \) under iteration of \( f_{a,b} \).

**Fact 2** In order to analyze the dynamics of \( f_{a,b}^k \vert_{I_0} \), a function \( g_{\alpha,\beta} \) is introduced as follows:

\[
g_{\alpha,\beta}(x) = \begin{cases} 
\alpha x, & \text{for } 0 \leq x \leq \frac{1}{\alpha} \\
-\beta x + \frac{\alpha+\beta}{\alpha}, & \text{for } \frac{1}{\alpha} \leq x \leq 1
\end{cases}
\]

where \((\alpha, \beta)\) such that \( \alpha > 1, \beta > 0 \) and \( \frac{1}{\alpha} + \frac{1}{\beta} \geq 1 \). As \( I_0 \) is invariant under \( f_{a,b}^k \), \( f_{a,b}^k \vert_{I_0} \) is topologically conjugate to \( g_{\alpha,\beta} \) with some \( \beta < 1 \). If \( \beta < 1 \), then the orbits of almost all points in \([0,1]\) tend to the fixed point \( x = \frac{\alpha+\beta}{\alpha(\beta+1)} \) under the iteration of \( g_{\alpha,\beta} \). The attracting fixed point for \( g_{\alpha,\beta} \) corresponds to \( x_+ \) for \( f_{a,b}^k \vert_{I_0} \). Therefore almost all orbits in \( I_0 \) under \( f_{a,b}^k \) tend to \( x^* \).

According to J. Milnor and W. Thurston in [MT88], we will define *turning point* as follows.

Let \( f \) be a continuous map on the interval \( I = [c_0, c_1] \). We suppose that \( I \) is decomposed into finite maximal intervals, \( J_1 = [c_0, c_1], \ldots, J_t = [c_{t-1}, c_t] \).
with \( c_0 < c_1 < \cdots < c_t \), and each restriction \( f|_{I_j} \) is monotone map. Then these endpoints \( c_0, c_1, \ldots, c_t \) will be called turning points of \( f \).

We obtain the following result for the turning point \( x = 0 \) of the skew tent maps. This result is quite different from the case of differentiable map.

**Proposition 2** The periodic turning point of \( f_{a,b} \) is not attractive.

*Proof.* Put \( f_{a,b} = f \). If the turning point of \( f^j \) for some \( j \geq 1 \) is a fixed point, then the orbit of \( x = 0 \) and \( f^2(0) \) also have the period \( j \). For each region in \( D \), we will prove the claim.

- \((a, b) \in D_0\)

There exist subintervals of \( I_{a,b} \)

\[ L_0 = [f^2(0), f^4(0)], \quad L_1 = [f^3(0), f(0)] \]

such that \( L_0 \cap L_1 = \emptyset, fL_0 = L_1 \) and \( fL_1 = L_0 \). There is no attracting periodic point in the interval \( (f^4(0), f^3(0)) \) since each orbit in \( I_{a,b} \setminus \{1 - \frac{b}{a}\} \) tends to \( L_0 \) or \( L_1 \). If an attracting periodic point exists in \( L_0 \cup L_1 \), it must have even period as \( L_0 \) and \( L_1 \) are mapped to each other under \( f \). Let us suppose the orbit of \( x = 0 \) has \( 2j \)-period. Then on any differentiable point \( x \) in \( L_0 \cup L_1 \), \( |(f^{2j})'(x)| \geq (ab)^j > 1 \) since \((a, b) \in D_0 \) and \( fL_0 \subset I_R \). Hence even if \( x = 0 \) is a periodic point, it is not attractive. The graph of \( f^{2j} \) on the neighborhood of \( x = 0 \) is like as Fig. 7 or Fig. 8.

- \((a, b) \in D_1\)

If \( f \) has \( j \)-periodic orbits, then we have \( j \geq 2 \) and \( |(f^j)'(x)| > 1 \) for the point \( x \) on which \( f \) is differentiable, since \( fI_L \subset I_R \) and \( b > 1 \). Hence even if \( x = 0 \) is a periodic point, it is not attractive. The graph of \( f^j \) on the neighborhood of \( x = 0 \) is like as Fig. 7 or Fig. 8.
• \((a, b) \in D_k^B\) for \(k \geq 3\)

Recall that there exists \(k\)-periodic point of \(f\) \(x^*\) and the interval \(C_0 = [f^2(0), x^*]\) is \(f^k\)-invariant. Therefore the orbit of \(f^2(0)\) has a period with the form \(j \times k\) for some \(j \geq 0\), so does the turning point \(0\). Because, for any orbit, at most \(k - 1\) successive images can stay on the interval \(I_L\), \(|(f^{kj})'(x)| \geq (a^{k-1}b)^j > 1\) for the point \(x\) on which \(f\) is differentiable. Hence even if \(x = 0\) is a periodic point, it is not attractive. The graph of \(f^{jk}\) is like as Fig. 7 or Fig. 8.

• \((a, b) \in D_k^A\) for \(k \geq 3\)

It is easy to see that \([f^2(0), x^*]\) is not invariant. Hence \(f^2(0)\) is \(j\)-periodic point with \(j \geq k\). Because, for any orbit, at most \(k - 1\) successive images can stay on the interval \(I_L\), \(|(f^{kj})'(x)| \geq (a^{k-1}b)^j > 1\) for the point \(x\) on which \(f\) is differentiable. Hence even if \(x = 0\) is a periodic point, it is not attractive. The graph of \(f^{jk}\) is like as Fig. 7 or Fig. 8.

• \((a, b) \in D_k^A\) and \(a^{k-1}b \neq 1\)

For \(k \geq 3\), if \(a^{k-1}b \neq 1\), the orbit of \(f^2(0)\) under \(f^k\) is attracted to \(x^*\) by Fact 1. This contradicts that turning point is periodic.

For \(k = 2\), it is easy to check by setting \(x^* = \frac{1 - b}{1 + ab}\) that \(x^*\) is 2-periodic point and that \(f^2(0)\) tend to \(x^*\) under \(f^2_{a,b}\). Therefore turning point cannot be attractive and periodic.

• \((a, b) \in D_k^A\) and \(a^{k-1}b = 1\)

For both case of \(k = 2\) and \(k \geq 3\), the orbit of \(x = 0\) has period 2\(k\). But the graph of \(f^{2k}\) is like as Fig. 9 or Fig. 10, it is not attractive in the sense of [Mil85].

\(\square\)
Definition The periodic orbit will be called of stair type when it satisfies the following conditions:

\[ O(x) = \{x, f(x), \ldots, f^{k-1}(x)\}, \]
\[ x (= f^k(x)) < f(x) < \ldots < f^{k-2}(x) \leq 0 < f^{k-1}(x). \]

We remark that the following result is implicitly stated in [ITN79].

Proposition 3 Skew tent map \( f_{a,b} \) has attractive periodic orbit if and only if \( (a,b) \) is in the interior of \( D_A \).

Proof. If \( (a, b) \) is in the interior of \( D_A \) for some \( k \geq 3 \), then \( f_{a,b} \) has an attractive \( k \)-periodic point \( x^* \) by Fact 2. If \( (a, b) \) is in \( D_A \), by setting \( x^* = \frac{1 - b}{1 + ab} \), it is proved that \( x^* \) is 2-periodic point and that almost all points in \( I_{a,b} \) tend to \( x^* \) under \( f^2_{a,b} \). And the orbit is of stair type.

Now we prove that only if \( (a, b) \) is in the interior of \( D_A \), \( f_{a,b} \) has an attracting \( k \)-periodic point. By Proposition 2, the orbit of turning point is not attractive. Therefore in neighborhood of the periodic orbit, there exists an open interval where the itinerary is same. This cannot happen if \( (a, b) \notin \bigcup_{k=2}^{\infty} D_k \) by virtue of Lemma 2.1 in [ITN79]. Hence if \( f_{a,b} \) has an attracting periodic orbit, there exists unique \( k \) such that \( (a,b) \in D_k \). On the other hand, for \( (a,b) \) on the boundary of \( D_A \), there is no attractor in the sense of [Mil85] (see Fig. 9 and Fig. 10).

For the case of differentiable map with one critical point, the following result is well known by using the method of Schwarzian derivative (see for example, [Dev89], p.74). But despite of the indefferentiablity, it is worthy of notice that we have the same result.
Lemma 1 \( f_{a,b} \) has at most one attracting periodic orbit.

Proof. Suppose that \( f_{a,b} \) has attracting \( k \)-periodic orbit. By Proposition 3, it is the orbit of \( x_* \). By considering this and Fact 2, \( f_{a,b} \) has at most one attractive periodic orbit.

By the argument above, the attracting periodic orbit of skew tent map is characterized as follows.

Theorem The attracting periodic orbit of \( f_{a,b} \) is of stair type.

Proof. The proof will immediately follow by Lemma 3 and Lemma 1. \( \square \)

3 Monotonicity of kneading sequences

We first introduce the concept of symbolic dynamics.

For \( x \in I_{a,b} \) and \( j \geq 0 \), we associate \( f_{a,b}^j(x) \) with a symbol \( s^j(x) \) as follows:

\[
s^j(x) = \begin{cases} 
L, & \text{if } f_{a,b}^j(x) < 0 \\
C, & \text{if } f_{a,b}^j(x) = 0 \\
R, & \text{if } f_{a,b}^j(x) > 0
\end{cases}
\]

For \( x \in I_{a,b} \), the sequence \( \{s^j(x)\}_{i \geq 0} \) is called itinerary and denoted as \( S_{a,b}(x) \). Kneading sequence of \( f_{a,b} \) is defined as \( S_{a,b}(1) \) and will be denoted by \( K(a,b) \). If there exists \( i \geq 0 \) such that \( f_{a,b}^i(1) = 0 \), then we regard that \( K(a,b) \) is a finite sequence, which is ended with \( C \).

In this section we will mention the monotonicity property of kneading sequence in the domain

\[
\tilde{D} = \{(a,b) \in D; a \geq 1\}.
\]

Let us define the order for parameter pairs as follows, according to M. Misiurewicz and E. Visinescu [MV91]:

\((a,b) \succ (a',b') \iff a' \geq a, b' \geq b\), and at least one of these inequalities is strict.

Kneading sequences are monotone increasing with respect to this order.

Monotonicity Theorem (Theorem A in [MV91]) For \( (a',b'), (a,b) \) in \( \tilde{D} \) with \( (a',b') \succ (a,b) \), it holds that \( K(a',b') \succ K(a,b) \).
This theorem is already proved in [MV91]. M. Misiurewicz and E. Visinescu showed the claim by using the estimation of topological entropy. But we shall reprove it by using only their results for $D^*$ in [MV91], and renormalization method, not via the topological entropy. For that purpose, we show Proposition 5, Proposition 6 and Proposition 7.

3.1 Preliminaries

In order to prove our Monotonicity Theorem, we summarize some definitions and facts for symbolic dynamical systems.

- Finite symbolic sequence $M$ is said to be even (resp. odd) if $M$ has an even (resp. odd) number of symbol $R$.

- Order between two itineraries is defined as follows:

  Set $I = I_0I_1 \cdots I_nI_{n+1} \cdots$, and $J = J_0J_1 \cdots J_nJ_{n+1} \cdots$.

  Assume that $I_k = J_k$ for $0 \leq k \leq n$, $I_{n+1} \neq J_{n+1}$.

  $I \prec J \iff \begin{cases} I_{n+1} < J_{n+1}, & \text{if } I_0I_1 \cdots I_n \text{ is even} \\ I_{n+1} > J_{n+1}, & \text{if } I_0I_1 \cdots I_n \text{ is odd} \end{cases}$

  where $L < C < R$.

- Symbolic sequence $M$ is said to be admissible if $M$ is an infinite sequence of $L$ and $R$, or if it is a finite sequence of $L$ and $R$, ended with $C$. 

Figure 11: The domain $\tilde{D}$
• Symbolic sequence $M$ is said to be maximal if it is admissible and $\sigma^k(M) \preceq M$ for all $k \geq 0$, where $\sigma$ is shift map i.e.

$$\sigma(M) = (M_1M_2\ldots) \text{ for } M = (M_0M_1\ldots).$$

3.2  $*$-product

According to P. Collet and J. P. Eckmann [CE85], we define $*$-product.

For a finite sequence $A$ of $R$, $L$ and an admissible sequence $B$, $A * B$ is defined as follows:

1. If $A$ is even and $B$ is infinite $B = B_0B_1\ldots$,
   $$A * B = AB_0AB_1AB_2\ldots.$$
2. If $A$ is odd and $B$ is infinite $B = B_0B_1\ldots$,
   $$A * B = A\bar{B}_0A\bar{B}_1A\bar{B}_2\ldots.$$
3. If $A$ is even and $B$ is finite $B = B_0B_1\ldots B_{n-1}C$,
   $$A * B = AB_0AB_1\ldots AB_{n-1}AC.$$
4. If $A$ is odd and is $B$ is finite $B = B_0B_1\ldots B_{n-1}C$,
   $$A * B = A\bar{B}_0A\bar{B}_1\ldots AB_{n-1}AC.$$

Here we define $\bar{B}_i$ as follows

$$\bar{B}_i = \begin{cases} L_i & \text{if } B_i = R \\ R_i & \text{if } B_i = L \end{cases}$$

We remark that $*$-product holds associative law:

If $A_1C$, $A_2C$ and $B$ is admissible and $AC = A_1 * (A_2C)$, then it is proved that $A_1 * (A_2 * B) = A * B$. Hence the sequence $R^n$ is well defined such that

$$R^n * B = R * (R * (\ldots R * (R * B)) \ldots).$$

The sequence $R^\infty$ is defined to be $\lim_{n \to \infty} R^n$.

Example.
\[ R^{*1} = R, \]
\[ R^{*2} = RLR, \]
\[ R^{*3} = RLR RLR, \]
\[ R^{*4} = RLR RLR RLR RLR. \]

**Remark.** Related to \( R^{*n} \), similar sequences are introduced in [Dev89].

\[ \tau_0 = R, \]
\[ \tau_1 = RL, \]
\[ \tau_2 = RLR, \]
\[ \vdots \]
\[ \tau_{n+1} = \tau_n \hat{\tau}_n, \text{ where } \hat{\tau}_n = t_0 \cdots t_k \text{ when } \tau_n = t_0 \cdots t_k. \]

Note that \( R^{*n} \) is equal to the sequence whose last entry is removed from \( \tau_{n+1} \).

**Proposition 4** The length of \( R^{*n} \) is odd for all \( n \). Moreover \( R^{*n} \) is an odd (resp. even) sequence if \( n \) is odd (resp. even).

**Proof.** For the first part of the proposition, we can show that the length of \( R^{*n} \) is odd for \( n \) by induction of \( n \). For the second part, we shall show the claim by induction of \( n \).

(i) When \( n = 1 \), \( R^{*n} = R \) and it is odd.

(ii) When \( n = 2 \), \( R^{*2} = RLR \) and it is even.

(iii) For \( n \leq 2k \), we assume the claim holds.

Recall that \( R^{*2k+1}C = R \ast (R^{*2k}C) \).

Let us denote \( R^{*2k}C = S_0 \cdots S_{2j}C \) for some \( j \geq 0 \) since the length of \( R^{*2k} \) is odd. By assumption, \( R^{*2k} \) is even and has an odd number of \( L \). It follows that \( S_0 \cdots S_{2j} \) is odd. Recall that

\[ R \ast (R^{*2k}C) = RS_0 \cdots RS_{2j}RC. \]

And this sequence is odd. Hence \( R^{*2k+1} \) is odd.

(iv) Let us check that \( R^{*2k+2} \) is even. Recall that \( R^{*2k+2}C = R \ast (R^{*2k+1}C) \).

We can denote \( R^{*2k+1}C = T_0 \cdots T_{2i}C \) for some \( i \geq 0 \) since the length of \( R^{*2k+1} \) is odd. By (iii), \( R^{*2k+1} \) is odd and has an even number of \( L \). It follows that \( T_0 \cdots T_{2i} \) is odd. Recall that

\[ R \ast (R^{*2k+1}C) = R\hat{T}_0 \cdots R\hat{T}_{2i}RC. \]

And this sequence is even. Hence \( R^{*2k+2} \) is even.
By (i)∼(iv), the proof is done.

**Proposition 5** Let $A$ and $B$ be symbolic sequences with $A \triangleright B$. Then for all $n \geq 1$, it holds that $R^n \star A \triangleright R^n \star B$.

**Proof.** Set $A = A_0A_1 \cdots A_{k-1}A_k \cdots$, $B = B_0B_1 \cdots B_{k-1}B_k \cdots$.

Let $A_j = B_j$, for $0 \leq j \leq k - 1$ and $A_k \neq B_k$.

(I) If $n$ is an even number, then $R^n$ is an even sequence.

If $A_0 \cdots A_{k-1} = B_0 \cdots B_{k-1}$ is even, then $A_k > B_k$ since $A \triangleright B$.

Recall that

$$R^n \star A = R^nA_0 \cdots R^nA_{k-1}R^nA_k \cdots.$$  

Then it is easy to see that $R^nA_0 \cdots R^nA_{k-1}R^nA_k$ is even. Hence we have $R^n \star A \triangleright R^n \star B$.

If $A_0 \cdots A_{k-1} = B_0 \cdots B_{k-1}$ is odd, then $A_k < B_k$ since $A \triangleright B$.

Recall that

$$R^n \star A = R^nA_0 \cdots R^nA_{k-1}R^nA_k \cdots.$$  

Therefore $R^nA_0 \cdots R^nA_{k-1}R^nA_k$ is odd. And we have that

$R^n \star A \triangleright R^n \star B$.

(II) If $n$ is an odd number, then $R^n$ is an odd sequence.

(a) Assume that $k$ is an even number.

If $A_0 \cdots A_{k-1} = B_0 \cdots B_{k-1}$ is even, then $A_k > B_k$ since $A \triangleright B$.

Moreover we have that $\bar{A}_0 \cdots \bar{A}_{k-1} = \bar{B}_0 \cdots \bar{B}_{k-1}$ is even and that $\bar{A}_k < \bar{B}_k$. By recalling that

$$R^n \star A = R^n\bar{A}_0 \cdots R^n\bar{A}_{k-1}R^n\bar{A}_k \cdots,$$

it is easy to see that $R^n\bar{A}_0 \cdots R^n\bar{A}_{k-1}R^n\bar{A}_k$ is odd. Hence we have that $R^n \star A \triangleright R^n \star B$.

If $A_0 \cdots A_{k-1} = B_0 \cdots B_{k-1}$ is odd, then $A_k < B_k$ since $A \triangleright B$.

Moreover we have that $\bar{A}_0 \cdots \bar{A}_{k-1} = \bar{B}_0 \cdots \bar{B}_{k-1}$ is odd and that $\bar{A}_k > \bar{B}_k$. By recalling that

$$R^n \star A = R^n\bar{A}_0 \cdots R^n\bar{A}_{k-1}R^n\bar{A}_k \cdots,$$
it is easy to see that
\[ R^n \bar{A}_0 \cdots R^n \bar{A}_{k-1} R^n \] is even. Hence we have that
\[ R^n \star A \succ R^n \star B. \]

(b) Assume that \( k \) is an odd number.
If \( A_0 \cdots A_{k-1} = B_0 \cdots B_{k-1} \) is even, then \( A_k > B_k \) since \( A \succ B \).
Moreover we have \( \bar{A}_0 \cdots \bar{A}_{k-1} = \bar{B}_0 \cdots \bar{B}_{k-1} \) is odd and that
\( \bar{A}_k < \bar{B}_k \). By recalling that
\[ R^n \star A = R^n \bar{A}_0 \cdots R^n \bar{A}_{k-1} R^n \bar{A}_k \cdots, \]
it is easy to see that \( R^n \bar{A}_0 \cdots R^n \bar{A}_{k-1} R^n \bar{A}_k \) is odd. Hence we have that \( R^n \star A \succ R^n \star B \).
If \( A_0 \cdots A_{k-1} = B_0 \cdots B_{k-1} \) is odd, then \( A_k < B_k \) since \( A \succ B \).
Moreover we have that \( \bar{A}_0 \cdots \bar{A}_{k-1} = \bar{B}_0 \cdots \bar{B}_{k-1} \) is even and that
\( \bar{A}_k > \bar{B}_k \). By recalling that
\[ R^n \star A = R^n \bar{A}_0 \cdots R^n \bar{A}_{k-1} R^n \bar{A}_k \cdots, \]
it follows that
\[ R^n \bar{A}_0 \cdots R^n \bar{A}_{k-1} R^n \] is odd. Hence we conclude that
\[ R^n \star A \succ R^n \star B. \]

\[ \square \]

3.3 Monotonicity in \( D^* \)

M. Misiurewicz and E. Visinescu proved in [MV91] that \( K(a', b') > K(a, b) \) for
\( (a', b'), (a, b) \in D^* \) such that \( (a', b') \succ (a, b) \), where \( D^* \) is defined by
\[ D^* = \{(a, b) \in D ; a + b < ab^2, a > 1\}. \]

This domain \( D^* \) is characterized by the following lemma.

**Fact 3** (Lemma 2.1 in [MV91])
\( (a, b) \in D^* \Leftrightarrow K(a, b) \succ RLR^\infty. \)

![Figure 12: The shadowed part is domain of \( D^* \).](image)
Using a parameter $t \in [0, 1]$, set
\[ f_t = f_{a't+a(1-t)}, b't+b(1-t). \]

First, the monotonicity of kneading sequence is shown as to parameter $t$.

**Fact 4** (Lemma 4.1 in [MV91]) If $0 \leq v < \omega \leq 1$ then $K(f_v) < K(f_\omega)$, where $K(f_t)$ denotes kneading sequence of $f_t$.

**Fact 5** (Lemma 4.2 in [MV91]) $K(f_t)$ is not constant on $[0, 1]$.

From the two lemmas above, monotone increasing property of kneading sequence is proved in $D^*$.

**Fact 6** (Proposition 4.3 in [MV91]) If $(a, b)$ and $(a', b')$ are in $D^*$ with $(a, b) \prec (a', b')$, then it holds that $K(a, b) \prec K(a', b')$.

### 3.4 Renormalization and $\star$-product

The aim of this section is to prove Monotonicity Theorem by using only renormalization method. For the details of renormalization and prime sequence, see the sequel paper [ITO] in this volume.

**Proposition 6** Let $(a, b)$ be in $D$. The following three conditions are equivalent mutually.

(i) $(a, b) \in D_0$.

(ii) There exists a unique number $m \geq 1$ and a prime sequence $B$ whose length is longer than 2 such that $K(a, b) = R^m \star B$.

(iii) There exists some number $m \geq 1$ such that $\varphi^m(a, b) \in D^*$, where $\varphi(a, b) = (b^2, ab)$.

Furthermore, there exist closed subintervals of $I_{a, b} \{I_i\}_{i=0, \ldots, 2^m-1}$ such that their interiors are disjoint mutually, $f_{a, b}I_i = I_{i+1}$ for $0 \leq i \leq 2^m - 2$ and $f_{a, b}I_{2^m-1} = I_0$, $I_{2^m-1} \ni 0$, and $f_{a, b}^{2^m} | I_i \sim f_{\varphi^m(a, b)}$.  

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Proof. Recall that $D_0 = \{(a, b) \in D; ab > 1, a + b \geq ab^2\}$.

First, we will show that (i) $\Rightarrow$ (iii).

Let $(a, b) \in D_0$. Set $\varphi^n(a, b) = (\tilde{a}_n, \tilde{b}_n)$ and assume that $\tilde{a}_n \leq \frac{\tilde{b}_n}{\tilde{b}_{n-1}}$ for all $n \geq 1$. Fix any $k \geq 1$. By the definition of $\varphi$, we have that $\tilde{b}_{k+p} > (\tilde{a}_k \tilde{b}_k)(\tilde{b}_k)^{p-1}$ for all $p \geq 1$. If we take $p \to \infty$, then we have that $\lim_{p \to \infty} \tilde{b}_{k+p} = \infty$ and $\lim_{n \to \infty} \tilde{b}_n = \infty$. If we take $n \to \infty$ on the assumption $\tilde{a}_n \tilde{b}_n \geq \tilde{b}_1$, then we have that

$$\tilde{a}_n \leq \frac{\tilde{b}_n}{\tilde{b}_{n-1}},$$

then we have that

$$\frac{\tilde{b}_n}{\tilde{b}_{n-1}} - 1 = \frac{1}{\frac{\tilde{b}_n}{\tilde{b}_{n-1}}} \to 0.$$

This contradicts that $\tilde{a}_n > 1$ for $n \geq 2$. Hence, for each $(a, b) \in D_0$, there exists $m \geq 1$ such that $\varphi^m(a, b) \in D^*$. We remark that once we have $\varphi^m(a, b) \in D^*$, $(\varphi^{m+1}(a, b)$ does not belong to our domain $D$ any longer. For the second part, it is clear by [ITN79] and [IN97b].

Second, we will show that (iii) $\Rightarrow$ (i).

If $(a, b) \in D^A \cup D^*_k$, then it is proved that there is no subinterval where $f_{a,b}^{2m}$ is surjective, except for $I_{a,b}$. If $(a, b) \in D^B_k$ for some $k \geq 3$, then we have that $a + b < ab^2$ and that $\varphi(a, b)$ is in $\{(a, b); a + b < ab\}$. Because the map $\varphi$ is proved to be surjective on this domain, it follows that $\varphi^m(a, b) \notin D^*$ for any $m \geq 1$. Hence $(a, b)$ is in $D^B_k = D_0$.

To see that (i) $\Rightarrow$ (ii), it is clear by virtue of Lemma 5.1, Lemma 5.2 in [MV91]. Moreover, $B$ is equal to $K(\varphi^m(a, b))$ with $\varphi^m(a, b) \in D^*$. If we suppose that $B$ is not prime, then we can show a contradiction because of Theorem 3 in [ITO] and of that $\varphi^m(a, b) \in D^*$. Last, we will show that (ii) $\Rightarrow$ (i).

If $(a, b) \in D^A_2$, then we have either $K(a, b) = (RL)^\infty$ or $K(a, b) = R^\ast(RC)$. If $(a, b) \in D_1$, then we have $K(a, b) = RLRLRL\ldots$, or $K(a, b) = RLRLRLRC$.

If $(a, b) \in D_k$ for some $k \geq 3$, then we have $K(a, b) = RLL\ldots$.

Hence the kneading sequence cannot be written in the form $R^m \ast B$ for any $m \geq 1$. Therefore we have $(a, b) \in D_0$.

We remark that the function $\varphi$ is defined only on $D_0$. □
**Proposition 7** Let \((a, b), (a', b') \in \tilde{D} \setminus D^*\) such that \((a, b) \prec (a', b')\). If \(\varphi^m(a, b) \in D^*\) and \(\varphi^n(a', b') \in D^*\), then \(m \geq n\).

**Proof.** Assume that \(n > m\). Then \(\varphi^m(a, b)\) is in \(D\). But it cannot be in \(D^*\). Because of that \((a, b) \prec (a', b')\) and of the definition of \(\varphi\), we have that \(\varphi^m(a', b') \prec \varphi^m(a, b)\). Setting

\[
\varphi^m(a', b') = (\tilde{a}', \tilde{b}') \quad \text{and} \quad \varphi^m(a, b) = (\tilde{a}, \tilde{b}),
\]

we have the inequalities \(\tilde{a}' \geq \tilde{a}, \tilde{b}' \geq \tilde{b}\) and that \(\tilde{a} > \frac{\tilde{b}}{\tilde{b}^2 - 1}\) since \(\varphi^m(a, b)\) is in \(D^*\).

On the other hand, since \(s \mapsto \frac{s^2}{s^2 - 1}\) is strictly decreasing, we have

\[
\frac{\tilde{b}}{\tilde{b}^2 - 1} > \frac{\tilde{b}'}{\tilde{b}'^2 - 1} > \tilde{a}' \geq \tilde{a}.
\]

Hence this is a contradiction. Therefore we conclude that \(n \leq m\). \(\square\)

### 3.5 Proof of Monotonicity Theorem

Assume that \((a, b) \prec (a', b')\).

(i) If both \((a, b)\) and \((a', b')\) belong to \(D^*\), then the proof is already given by Fact 6.

(ii) Assume that either \((a, b)\) or \((a', b')\) belongs to \(D^*\). Then \((a', b')\) is in \(D^*\) because \((a, b) \prec (a', b')\). By virtue of Fact 3, it follows that

\[
K(a, b) \leq RLR^\infty \prec K(a', b').
\]

We have that \(K(a, b) \prec K(a', b')\) since an order relation "\(\prec\)" is total.

(iii) Assume that \((a, b)\) and \((a', b')\) both belong to \(\tilde{D} \setminus D^*\). Then by Proposition 7, their kneading sequences are written as, for some \(n \leq m\),

\[
K(a, b) = R^m \star K(\varphi^m(a, b)) \quad \text{and} \quad K(a', b') = R^n \star K(\varphi^n(a, b)).
\]

If \(m = n\), then we have that \(\varphi^n(a, b) \prec \varphi^n(a', b')\) since \(\varphi\) is an increasing function. Because \(K(\varphi^n(a, b)) \prec K(\varphi^n(a', b'))\) and from Proposition 5, we have that \(K(a, b) \prec K(a', b')\).
If \( n < m \), then we have that \( \varphi^n(a, b) \neq D^* \) and \( \varphi^n(a', b') \in D^* \). By virtue of Fact 3, it follows that
\[
K(\varphi^n(a, b)) \leq RLR^\infty < K(\varphi^n(a', b')).
\]

By Proposition 5, we have that \( K(a, b) < K(a', b') \).

\[\Box\]

4  Topological Entropy of \( f_{a,b} \)

J. Milnor and W. Thurston studied in the paper [MT88] that for piecewise monotone mapping, the kneading sequence determines its topological entropy. In this section, we will summarize some, well-known but important results and give some correct for a statement in [MV92].

4.1  Topological entropy of \( f_{a,b} \) for \((a, b) \in D\)

In the paper [MV91], the following theorem is also proved as Theorem B, and as a result, strict monotonicity property for the topological entropy is obtained as well.

**Fact 7** (Theorem B in [MV91]) If \((a, b) \in \tilde{D}\), then \( K(a, b) \in \mathcal{M} \), where \( \mathcal{M} \) denotes the set of kneading sequence of tent maps, that is,
\[
f_{a,a} = \begin{cases} 
a x, & \text{for } x < 0 \\
 a(1-x), & \text{for } x > 0
\end{cases}
\] with \( 1 < a \leq 2 \).

**Fact 8** (see for example [MW80] and [M89]) The entropy of \( f_{a,a} \) is proved to equal to \( \log a \), namely it is monotone increasing for \( a \).

By taking the above and our Monotonicity Theorem into consideration, it is found that in \( \tilde{D} \), the topological entropy is also strictly monotone increasing with respect to the order "\( \succ \)" (See Corollary in [MV91]).

We remark that the topological entropy of \( f_{a,b} \) is equal to \( \log \beta_{K(a,b)}(\mu) \) with \( \mu \) such that \( \beta_{K(a,b)}(\mu) = \mu \). This fact is implicitly shown by Fact 7 and Theorem C in [MV91]. Theorem C is as follows.

**Fact 9** (Theorem C in [MV91]) For each \( M \in \mathcal{M} \), there exists a number \( \gamma(M) \) and a continuous decreasing function \( \beta_M : (1, \gamma(M)) \rightarrow [1, \infty) \) (with one exception \( M = RL^\infty \) when \( \gamma(M) = \infty \)) such that for \((a, b) \in \tilde{D}, K(a, b) = M \) if and only if \( a = \beta_M(b) \). The function \( \gamma \) is increasing. The
graphs of the functions $\beta_M$ fill up the whole set $\tilde{D}$. Moreover the following are shown:

$$\lim_{M \to R^\infty} \gamma(M) = 1,$$

$$\lim_{M \to RL^\infty} \gamma(M) = \infty,$$

$$\lim_{b \to 1} \beta_M(b) = \infty \quad \text{if } M \supset RLR^\infty,$$

$$\lim_{b \to 1} \beta_M(b) = \gamma(\mathcal{J}) \quad \text{if } M \prec RLR^\infty,$$

$\mathcal{J}$ is given by

$$M = R \star \mathcal{J},$$

$$\beta_M(\gamma(M)) = 1 \quad \text{if } M \neq RL^\infty,$$

$$\lim_{b \to 1} \beta_M(b) = +\infty \quad \text{if } M = RL^\infty.$$

### 4.2 Topological Entropy of $f_{a,b}$ for $(a, b) \in D \setminus \tilde{D}$

Let us denote the topological entropy of $f_{a,b}$ by $h(a, b)$.

According to J. C. Marcuard and E. Visinescu, topological entropy of $f_{a,b}$ for $(a, b) \in D \setminus \tilde{D}$ is studied in [MV92]. But we claim that the following result, Corollary in [MV92] is not true:

*The function $h(a, b)$ is constant in $(D_k^B \cup D_k^A) \cap (D \setminus \tilde{D})$, for $k \geq 2$.\

**Proposition 8** The function $h(a, b)$ is not constant in $(D_k^B \cup D_k^A) \cap (D \setminus \tilde{D})$, for $k \geq 2$.

**Proof.** We can give the following counter example.

Consider the case of $k = 2$. Recall that $D_2^B \cap (D \setminus \tilde{D}) = \{(a, b) \in D_0 \text{ and } a < 1\}$ and that, for $(a, b) \in D_0$, $f_{a,b}$ is renormalizable of level, at least, two. By renormalization, namely by map $\varphi$, the parameter domain $\{(a, b) \in D_0 \text{ and } a < 1\}$ is mapped into a subdomain of $\tilde{D}$,

$$\{(a, b) \in \tilde{D}; \ a = 1, \ a = b^2, \ a + b = ab\}.$$ 

See also Fig. 13 and Fig. 14.

If $(a, b)$ is in $D_0$, then it is easily proved that $2h(a, b) = h(\varphi(a, b))$.  

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Figure 13: The shadowed part is the domain $D_2^B \cap (D \setminus \overline{D}) (0.4 \leq a \leq 1)$.

Figure 14: In this figure, the shadowed part corresponds to the domain where $(D_2^B \cap D \setminus \overline{D})$ is mapped by map $\varphi$ is depicted.

Consequently, the entropy is not constant in the domain

$$\{(a, b) \in D_0; \ a < 1\}.$$  

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