Intuitive Approximations in Discrete Renewal Theory
Part 1: regularly varying case
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Abstract
It is usually impossible to find explicit expressions for the renewal sequence. This paper presents a simple method to approximate the renewal sequence, which covers many of the known approximations. The paper uses the ideas of Mitov and Omey (2014).

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1. Introduction
Suppose that $X, X_1, X_2, \ldots$ are i.i.d. nonnegative integer-valued random variables with p.d.f. $p_k = P(X = k), k \in \mathbb{N}_0$. The d.f. of $X$ is given by $F(x) = P(X \leq x)$ and its tail is denoted by $\overline{F}(x) = 1 - F(x)$. Throughout the paper we assume that $(p_k)$ is aperiodic: $\gcd \{k : p_k > 0\} = 1$. We also assume that $0 < \mu = E(X) < \infty$. For $n \in \mathbb{N}_0$, the partial sums $S_n$ are given by $S_0 = 0$ and $S_n = X_1 + \cdots + X_n$ for $n \geq 1$. Note that $P(S_n \leq x) = F^{*n}(x)$, which is the $n$-fold convolution of $F$, i.e. $F^{*0} = 1_{[0,\infty)}$ and $F^{*n} = F \ast F^{*n-1}$, where the convolution of two d.f. is defined by $F \ast G(x) = \int_0^x F(x - y) dG(y)$.

For briefness, we write $F^{*n}(x)$ as $F^{*n}_k$ in case $x$ is a nonnegative integer $k$. Moreover, $P(S_n = k) = p_k^{*n}$, which is the $n$-fold convolution of $(p_k)$, i.e. $p_k^{*0} = 1_{\{0\}}(k)$ and $p_k^{*n} = (p \ast p^{*n-1})_k$, where the convolution of two sequences $(a_k)$ and $(b_k)$ is defined by $(a \ast b)_k = \sum_{i=0}^k a_i b_{k-i}$. The generating function of $X$ is $\hat{P}(z) = E(z^X), |z| < 1$, and $\hat{P}(1) = 1$. The generating function of $S_n$ is given by $\hat{P}^n(z)$. Since $X$ has finite expectation, we have $\mu = \hat{P}'(1)$.

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Let $X_e$ be a random variable, independent of $X$, that has the equilibrium distribution corresponding to $X$, i.e. $p_{e,k} = P(X_e = k) = F_k/\mu$ for $k \in \mathbb{N}_0$. The generating function of $X_e$ satisfies $\hat{P}_e(z) = (1 - \hat{P}(z))/((1 - z)).$ Define $S_{e,n} = X_{e,1} + \cdots + X_{e,n}$, $\hat{P}_{e,k}$, and $F_{e,k}$ analogously as above.

The renewal sequence $(u_n)$ is defined by $u_n = \sum_{k=0}^{\infty} p_{n,k}$. The aim of the present paper is to obtain approximations for $u_n$ when $n$ is large. Therefore, all limits that appear later are taken with respect to $n \to \infty$. It is well known that $u_n \to 1/\mu$ and the main problem is to obtain precise estimates for the rate at which $u_n - 1/\mu \to 0$ or $\Delta u_n = u_{n-1} - u_n \to 0$.

In our approach, we start from the generating function of $(u_n)$, which is given by $\hat{U}(z) = \sum_{n=0}^{\infty} u_n z^n = (1 - \hat{P}(z))^{-1} = (\mu(1-z)(1-(1-\hat{P}_e(z))))^{-1}$. Using a Taylor expansion, we obtain that

$$\hat{U}(z) = \sum_{k=0}^{\infty} \hat{T}_k(z), \quad \text{with} \quad \hat{T}_k(z) = \frac{1}{\mu(1-z)}(1-\hat{P}_e(z))^k. \quad (1)$$

Formula (1) suggests the following approximations $\hat{U}_m(z)$ for $\hat{U}(z)$: $\hat{U}_m(z) = \sum_{k=0}^{m} \hat{T}_k(z)$. By inversion, this approach then leads to approximations $u_{m,n}$ for the renewal sequence $u_n$ of the form $u_{m,n} = \sum_{k=0}^{m} t_{k,n}$, where the sequence $(t_{k,n})$ has generating function $\hat{T}_k(z) = \sum_{n=0}^{\infty} t_{k,n} z^n$. In the next section we will identify $\hat{T}_k(z)$ and $(t_{k,n})$. In this paper we focus on the cases $0 \leq m \leq 3$ and show that our approximations $(u_{m,n})$ correspond to the approximations that have been published in many papers before.

2. The sequences $(t_{k,n})$ and $(\delta_{k,n})$

2.1. Expressions for $(t_{k,n})$ and $(\delta_{k,n})$

We first identify $\hat{T}_k(z)$. If $k = 0$, then (1) gives $\hat{T}_0(z) = 1/(\mu(1-z))$, which shows that $T_0(z)$ is the generating function of $t_{0,n} = 1/\mu$. For $k \geq 1$, the binomial expansion in (1) yields

$$\hat{T}_k(z) = \frac{1}{\mu(1-z)} \sum_{i=0}^{k} \binom{k}{i} (-1)^i \hat{P}_e^i(z)$$

$$= \frac{-1}{\mu(1-z)} \sum_{i=1}^{k} \binom{k}{i} (-1)^i (1 - \hat{P}_e^i(z)).$$

Since $\hat{P}_e^i(z)$ is the generating function of $S_{e,i}$, we have $(1 - \hat{P}_e^i(z))/(1 - z) = \sum_{n=0}^{\infty} F_{e,n} z^n$. We therefore obtain the following result.
Lemma 1. For \( n \geq 1 \), let \( \delta_{k,n} = t_{k,n-1} - t_{k,n} \). Then \( t_{0,n} = 1/\mu, \delta_{0,n} = 0 \) and, if \( k \geq 1 \),

\[
t_{k,n} = -\frac{1}{\mu} \sum_{i=1}^{k} \binom{k}{i} (-1)^i \hat{F}^{s_i}_{e,n},
\]

(2)

\[
\delta_{k,n} = -\frac{1}{\mu} \sum_{i=1}^{k} \binom{k}{i} (-1)^i \hat{p}^{s_i}_{e,n}.
\]

(3)

We now consider in detail the cases \( k = 1, 2, 3 \).

Lemma 2. For \( k \geq 2 \), let \( R^{e}_{k,n} = \hat{F}^{s_k}_{e,n} - k\hat{F}^{e}_{e,n} \) and \( r^{e}_{k,n} = p^{s_k}_{e,n} - kp^{e}_{e,n} \). Then

\[
t_{1,n} = \frac{\hat{F}^{e}_{e,n}}{\mu}, \quad \delta_{1,n} = \frac{p^{e}_{e,n}}{\mu}
\]

\[
t_{2,n} = -\frac{R^{e}_{2,n}}{\mu}, \quad \delta_{2,n} = -\frac{r^{e}_{2,n}}{\mu}
\]

\[
t_{3,n} = \frac{(R^{e}_{3,n} - 3R^{e}_{2,n})}{\mu}, \quad \delta_{3,n} = \frac{(r^{e}_{3,n} - 3r^{e}_{2,n})}{\mu}
\]

Proof. The results follow directly from (2) and (3).

2.2. Asymptotic behaviour of \((t_{k,n})\) and \((\delta_{k,n})\), \(1 \leq k \leq 3\).

In order to discuss the asymptotic behaviour of \((t_{k,n})\) and \((\delta_{k,n})\), we recall some basic definitions and properties of regularly varying sequences.

2.2.1. Regularly varying sequences

A sequence of real numbers \((a_n)\) is regularly varying at infinity and with real index \( \alpha \) if \( a_n > 0 \) for \( n \) large and if

\[
\lim_{x \to \infty} \frac{a[n]}{a[x]} = y^\alpha, \quad \forall y > 0.
\]

Notation: \((a_n) \in RS(\alpha)\). We write \((a_n) \in RS\) if \((a_n) \in RS(\alpha)\) for some \( \alpha \in \mathbb{R} \). If \((a_n) \in RS(\alpha)\), then (4) holds locally uniformly in \( y > 0 \); see sections 1.2 and 1.9 in Bingham et al. (1987). From this it follows that \( RS \subset LS \), where a sequence of real numbers \((a_n)\) is in the class \( LS \) if \( a_n > 0 \) for \( n \) large and if \( a_{n+1}/a_n \to 1 \). If \((a_n) \in RS(\alpha)\), then for each \( \epsilon > 0 \) we can find constants \( A, B, x^\circ \) so that

\[
Ay^{\alpha-\epsilon} \leq \frac{a[n]}{a[x]} \leq By^{\alpha+\epsilon}, \quad \forall x \geq x^\circ, \forall y \geq 1,
\]

cf. section 1.5 in Bingham et al. (1987) or Proposition 1.7 in Geluk and de Haan (1987). For sequences \((a_n)\) and \((b_n)\), we use the following notations:
\( a_n = O(1)b_n \) means that \( \limsup_{n \to \infty} a_n / b_n < \infty \);

\( a_n = o(1)b_n \) means that \( \limsup_{n \to \infty} a_n / b_n = 0 \);

\( a_n \approx b_n \) means that \( a_n = O(1)b_n \) and \( b_n = O(1)a_n \);

\( a_n \sim b_n \) means that \( \lim_{n \to \infty} a_n / b_n = 1 \).

Note that for \((a_n) \in RS(\alpha)\) we have \( a_n = o(1)n^{\alpha+\epsilon} \). If \( \alpha + \beta < -1 \), then Karamata’s theorem states that \( \sum_{n=0}^\infty n^\beta a_n < \infty \) and

\[
\sum_{k=n}^\infty k^\beta a_k \sim -\frac{n^{\beta+1}a_n}{\alpha + \beta + 1},
\]

(5)
cf. section 1.6 in Bingham et al. (1987). Using local uniform convergence, it also follows that for \( B \geq A > 0 \) we have

\[
\sum_{k=[Bn]}^{[An]} k^\beta a_k \approx n^{\beta+1}a_n.
\]

(6)

The next lemma will be used in the sections below.

**Lemma 3.** Suppose that \((a_n) \in RS(-\alpha)\) and \((b_n) \in RS\).

(i) If \( \alpha > 1 \), then \( \sum_{k=0}^{[n/2]} b_{n-k}a_k / b_n \to \sum_{k=0}^\infty a_k \);

(ii) If \( \alpha > 2 \), then \( \sum_{k=0}^{[n/2]} \sum_{i=0}^k b_{n-i}a_k / b_n \to \sum_{k=0}^\infty ka_k \).

**Proof.** Note that \( \alpha > 1 \) implies that \( \sum_{k=0}^\infty a_k < \infty \) and that \( \alpha > 2 \) implies that \( \sum_{k=0}^\infty ka_k < \infty \).

(i) Because \((b_n) \in LS\), we have \( b_{n-k}/b_n \to 1 \) for all \( k \). Since \( 0 \leq k \leq [n/2] \), \([n/2] \leq n-[n/2] \leq n-k \leq n\). From \((b_n) \in RS\) now follows that

\[
\frac{b_{n-k}}{b_n} \leq \sup_{1/2 \leq x \leq 1} \frac{b_{[nx]}}{b_n} \leq C, \quad n \geq n^0.
\]

We can use Lebesgue’s theorem to conclude that

\[
\frac{1}{b_n} \sum_{k=0}^{[n/2]} b_{n-k}a_k \to \sum_{k=0}^\infty a_k.
\]
(ii) As in (i) we have $b_{n-i}/b_n \to 1$, for all $i$, and for each fixed $k$ we have 
$$\sum_{i=0}^{k} b_{n-i}/b_n \to k.$$ Now we have 
$$\frac{1}{b_n} \sum_{i=0}^{k} b_{n-i} \leq \sup_{n-k \leq j \leq n} \frac{b_j}{b_n} \leq Ck, \quad n \geq n^o.$$ 

We can again use Lebesgue’s theorem to obtain result (ii). \(\square\)

2.2.2. Asymptotic behaviour of $(t_{1,n})$ and $(\delta_{1,n})$

Recall from Lemma 2 that $t_{1,n} = F_{e,n}/\mu$ and $\delta_{1,n} = p_{e,n}/\mu = F_n/\mu^2$. We now apply Karamata’s theorem, cf. (5), to obtain the following result.

**Lemma 4.** (i) If $(F_n) \in RS(-\alpha)$ and $\alpha > 1$, then $\mu < \infty$, $\delta_{1,n} = F_n/\mu^2$, and $t_{1,n} \sim nF_n/(\mu^2(\alpha - 1))$.

(ii) If $(p_n) \in RS(-\alpha)$ and $\alpha > 2$, then $\mu < \infty$, $F_n \sim np_n/(\alpha - 1)$, $\delta_{1,n} \sim np_n/(\mu^2(\alpha - 1))$, and $t_{1,n} \sim n^2p_n/(\mu^2(\alpha - 1)(\alpha - 2))$.

**Remark 1.** From $t_{1,n} = F_{e,n}/\mu$ and $\delta_{1,n} = F_n/\mu^2$ (cf. Lemma 2), we also obtain that 
$$\sum_{n=1}^{\infty} n^r t_{1,n} = \frac{1}{\mu} \sum_{n=1}^{\infty} n^r F_{e,n} \leq CE(X_{e}^{r+1}),$$ 
$$\sum_{n=1}^{\infty} n^r \delta_{1,n} = \frac{1}{\mu^2} \sum_{n=1}^{\infty} n^r F_n \leq CE(X_{e}^{r+1}).$$

2.2.3. Asymptotic behaviour of $(t_{2,n})$ and $(\delta_{2,n})$

Recall from Lemma 2 that $t_{2,n} = -R_{2,n}/\mu$ and $\delta_{2,n} = -r_{2,n}/\mu$, where $R_{2,n} = P(X_{e,1}+X_{e,2} > n) - 2P(X_{e} > n)$ and $r_{2,k} = P(X_{e,1} + X_{e,2} = n) - 2P(X_{e} = n)$, with $X_{e,1}$ and $X_{e,2}$ i.i.d. copies of $X_{e}$.

**Two lemmas.** In the first result we study more generally the asymptotic behaviour of $R_{2,n}$ and $r_{2,n}$, where 
$$R_{2,n} = P(X+Y > n) - P(X > n) - P(Y > n),$$ 
$$r_{2,n} = P(X+Y = n) - P(X = n) - P(Y = n),$$

and where $X$ and $Y$ are independent discrete r.v. with $a_n = P(X = n)$ and $b_n = P(Y = n)$, $n \geq 0$. For further use, starting from $(a_n)$, we define
\[ \delta_n^a = a_{n-1} - a_n, \quad n \geq 1. \] Clearly, for \( j > i \), we have
\[ \sum_{n=i+1}^{j} \delta_n^a = a_i - a_j \quad \text{and} \quad \sum_{n=i+1}^{\infty} \delta_n^a = a_i. \]

We use a similar notation for \((b_n)\). Note that \( R_{2,n} = \sum_{k=n+1}^{\infty} r_{2,k} \) and that \( P(X + Y = n) = (a \ast b)_n \). First we consider \( R_{2,n} \) and \((a \ast b)_n\).

**Lemma 5.** Assume that \((a_n) \in RS(-\alpha)\) and \((b_n) \in RS(-\beta)\), with \( \alpha > 2 \) and \( \beta > 2 \).

(i) We have \( E(X) + E(Y) < \infty \) and
\[ R_{2,n} = b_n E(X) + a_n E(Y) + o(1) a_n + o(1) b_n; \]
(ii) \( (a \ast b)_n = a_n + b_n + o(1) a_n + o(1) b_n \).

**Proof.** Let \( m = \lfloor n/2 \rfloor \).

(i) We have \( R_{2,n} = I + II + III - IV \), where
\[
I = P(X + Y > n, m < Y \leq n, X \leq m) \\
II = P(X + Y > n, m < X \leq n, Y \leq m) \\
III = P(m < X \leq n, m < Y \leq n) \\
IV = P(X > n, Y > n).
\]

First we consider \( I \). We clearly have
\[
I = \sum_{k=0}^{m} P(n - k < Y \leq n) P(X = k) = \sum_{k=0}^{m} \sum_{i=0}^{k-1} b_{n-i} a_k.
\]

Using Lemma 3 (ii) we obtain that \( I/b_n \to \sum_{k=0}^{\infty} k a_k = E(X) \). In a similar way, we obtain that \( II/a_n \to E(Y) \). Now consider \( III \). Since \( m = \lfloor n/2 \rfloor \) and \((a_n), (b_n) \in RS\), we have, cf. (6),
\[
III = \sum_{i=m+1}^{n} a_i \sum_{j=m+1}^{n} b_j \approx n^2 a_n b_n.
\]

Since \( \alpha > 2 \), it follows that \( n^2 a_n \to 0 \) and \( III = o(1) b_n \). Finally, for \( IV \) we have \( P(X > n) \approx n a_n \) and \( P(Y > n) \approx n b_n \), cf. (5), and it therefore follows that \( IV \approx n^2 a_n b_n = o(1) a_n \). Combining the estimates for \( I, II, III, \) and \( IV \), result (i) follows.
(ii) We have

\[(a * b)_n = \sum_{k=0}^{m} a_k b_{n-k} + \sum_{k=0}^{n-m-1} a_{n-k} b_k = I + II.\]

Using Lemma 3 (i) we have \(I/b_n \to \sum_{k=0}^{\infty} a_k = 1\) and \(II/a_n \to 1.\) \(\square\)

In our next result we discuss \(r_{2,n}\).

**Lemma 6.** Suppose that \((\delta_n^a) \in RS(-\alpha)\) and \((\delta_n^b) \in RS(-\beta)\), with \(\alpha > 3\) and \(\beta > 3\). Then \(r_{2,n} = \delta_n^b E(X) + \delta_n^a E(Y) + o(\delta_n^a) + o(\delta_n^b)\).

**Proof.** Using \(m = [n/2]\), we have \(r_{2,n} = I + II\), where

\[I = \sum_{k=0}^{m} a_k b_{n-k} - b_n \quad \text{and} \quad II = \sum_{k=0}^{n-m-1} a_{n-k} b_k - a_n.\]

First consider \(I\). We have

\[I = \sum_{k=1}^{m} a_k (b_{n-k} - b_n) - b_n \sum_{k=m+1}^{\infty} a_k = I_A - I_B.\]

Using \(b_{n-k} - b_n = \sum_{i=0}^{k-1} \delta_{n-i}^b\), we have \(I_A = \sum_{k=1}^{m} a_k \sum_{i=0}^{k-1} \delta_{n-i}^b\). Lemma 3 (ii) shows that \(I_A/\delta_n^b \to E(X)\). Now consider \(I_B\). Using \(\sum_{k=n+1}^{\infty} \delta_k^b = b_n\), \((\delta_n^b) \in RS(-\beta)\) with \(\beta > 3\), we have \(b_n \approx n \delta_n^b\) and \((b_n) \in RS(1 - \alpha)\). Similarly, \(a_n \approx n \delta_n^a\) and \((a_n) \in RS(1 - \alpha)\). Since \(\alpha - 1 > 2\), it follows that \(\sum_{k=n+1}^{\infty} a_k \approx n \delta_n^a \approx n^2 \delta_n^a\). Combining these estimates, it follows that

\[I_B \approx n \delta_n^b \sum_{k=m+1}^{\infty} a_k \approx n \delta_n^b m^2 \delta_n^a.\]

Since \(m = [n/2]\), \((\delta_n^a) \in RS\), and \(\beta > 3\), we conclude that \(I_B \approx n^3 \delta_n^b \delta_n^a = o(1) \delta_n^a\). We similarly find that \(II/\delta_n^a \to E(Y)\). \(\square\)

**Asymptotic behaviour of \((t_{2,n})\) and \((\delta_{2,n})\).** We apply Lemma 5 and Lemma 6, with \(a_n = b_n = p_{e,n}\) and \(\delta_n^a = p_{e,n-1} - p_{e,n} = (\mathcal{F}_{n-1} - \mathcal{F}_n)/\mu = p_n/\mu\).

**Theorem 7.** (i) If \((p_{e,n}) \in RS(-\alpha)\), with \(\alpha > 2\), then \(t_{2,n} \sim -2 \mu_e p_{e,n}/\mu\).

(ii) If \((p_n) \in RS(-\alpha)\), with \(\alpha > 3\), then \(p_{e,n} \sim np_n/((\mu(\alpha - 1))\), \(\delta_{2,n} \sim -2 \mu_e p_n/\mu^2\), and \(t_{2,n} \sim -2 np_n \mu_e/((\mu^2(\alpha - 1))\).
Proof. Recall from Lemma 2 that \( t_{2,n} = -R_{2,n}^e/\mu \) and \( \delta_{2,n} = -r_{2,n}^e/\mu \).

(i) Lemma 5 (i) shows that \( R_{2,n}^e \sim 2\mu e p_{e,n} \).

(ii) \((p_n) \in RS(-\alpha), \alpha > 3, \) and (5) yield \( p_{e,n} = F_n/\mu \sim np_n/(\mu(\alpha - 1)) \).

This also shows that \((p_{e,n}) \in RS(-\alpha - 1)\). The result for \( t_{2,n} \) thus
follows from (i). Since \((\delta_{a,n}) = (p_n/\mu) \in RS(-\alpha) \) and \( \alpha > 3 \), Lemma 6
shows that \( \delta_{2,n} = -r_{2,n}^e/\mu \sim -2\delta_{a,n}^e/\mu = -2p_n\mu e/\mu^2 \).

\[ \sum_{n=1}^{\infty} n^{r-1} |t_{2,n}| \leq C \sum_{n=1}^{\infty} n^{r} |R_{2,n}^e| \leq C \left( E(X_e + Y_e)^{r+1} + EX_e^{r+1} \right). \]

2.2.4. Asymptotic behaviour of \( (t_{3,n}) \)

Recall from Lemma 2 that \( t_{3,n} = (R_{3,n}^e - 3R_{2,n}^e)/\mu \).

One lemma. We first discuss, more generally, the asymptotic behaviour of
\( A_n = R_{3,n} - 3R_{2,n} \), where \( R_{2,n} = P(X + Y > n) - 2P(X > n), R_{3,n} = P(X + Y + Z > n) - 3P(X > n) \), and \( X, Y \) and \( Z \) are i.i.d. with \( a_n = P(X = n) \).

We now use \( \delta_n = a_{n-1} - a_n \).

Lemma 8. If \((\delta_n) \in RS(-\alpha) \) with \( \alpha > 3 \), then \( A_n/\delta_n \to 3 \left( E(X) \right)^2 \).

Proof. Direct calculations show that
\[ A_n = \sum_{k=0}^{n} R_{2,n-k}a_k - R_{2,n} = I + II, \]

where (using \( m = [n/2] \))
\[ I = \sum_{k=0}^{m} R_{2,n-k}a_k - R_{2,n} \quad \text{and} \quad II = \sum_{k=m+1}^{n} R_{2,n-k}a_k. \]

Clearly,
\[ I = \sum_{k=0}^{m} (R_{2,n-k} - R_{2,n})a_k - R_{2,n}\sum_{k=m+1}^{\infty} a_k = I_A - I_B \]

and
\[ II = \sum_{k=0}^{n-m-1} R_{2,k}(a_{n-k} - a_n) + a_n\sum_{k=0}^{n-m-1} R_{2,k} = II_A + II_B. \]
First consider $I_A$. We have $R_{2,n-k} - R_{2,n} = \sum_{j=0}^{k-1} r_{2,n-j}$. Lemma 6 yields $r_{2,n} \sim 2\delta_n E(X) \in RS(-\alpha)$. Lemma 3 (ii) shows that $I_A/\delta_n \rightarrow 2(E(X))^2$. Next consider $I_B$. Lemma 5 (i) shows that $R_{2,n} \approx a_n$. Since $(a_n) \in RV(1-\alpha)$, it follows that $I_B \approx a_n \sum_{k=0}^{\infty} a_k \approx n\delta_n^2$. Since $a_n \approx n\delta_n$ and $\alpha > 3$, we obtain that $I_B = o(\delta_n)$.

Now consider $II_A$. Using $a_{n-k} - a_n = \sum_{i=0}^{k-1} \delta_{n-i}$ and $(\delta_i) \in RS(-\alpha) \subset LS$, Lemma 3 (ii) gives $II_A/\delta_n \rightarrow \sum_{k=0}^{\infty} k\delta_{2,k}$. Remark 3 below shows that $\sum_{k=0}^{\infty} k\delta_{2,k} = (E(X))^2$. Finally consider $II_B$. Since $\sum_{n=0}^{\infty} R_{2,n} = 0$, we have $II_B = -a_n \sum_{k=0}^{\infty} R_{2,k}$. Using $R_{2,n} \approx a_n$, cf. Lemma 5, and $(a_n) \in RV(1-\alpha)$ with $\alpha > 3$, we have $II_B/\delta_n \approx n\delta_n^2/\delta_n \approx n^3\delta_n \rightarrow 0$.

**Remark 3.** For the sequence $(R_{2,n})$ we have

$$\hat{R}_2(z) = \sum_{n=0}^{\infty} z^n R_{2,k} = -\frac{(1 - \hat{P}(z))^2}{1 - z}. $$

If $\mu < \infty$, we have

$$\lim_{z \rightarrow 1} \hat{R}_2(z) = -\lim_{z \rightarrow 1} \frac{1 - \hat{P}(z)}{1 - z} = -\mu \times 0 = 0,$$

or $\hat{R}_2(1) = 0$. Using this result, we get

$$\hat{R}'_2(1) = \lim_{z \rightarrow 1} \frac{\hat{R}_2(1) - \hat{R}_2(z)}{1 - z} = \lim_{z \rightarrow 1} \frac{(1 - \hat{P}(z))^2}{1 - z} = \mu^2.$$

**Asymptotic behaviour of $(t_{3,n})$.** Applying Lemma 8 gives the following result.

**Theorem 9.** If $(p_n) \in RS(-\alpha)$ with $\alpha > 3$, then $t_{3,n}/p_n \rightarrow 3(\mu_e/\mu)^2$.

### 3. Consecutive approximations for $(u_n)$

Since $u_{m,n} = \sum_{k=0}^{m} t_{k,n}$ and $\Delta u_{m,n} = u_{m,n-1} - u_{m,n} = \sum_{k=1}^{m} \delta_{k,n}$, Lemma 2 yields the following approximations for $u_n$ and $\Delta u_n = u_{n-1} - u_n$:

\begin{align*}
&u_{0,n} = \frac{1}{\mu}, \\
&u_{1,n} = u_{0,n} + \frac{1}{\mu} F_e,n, \\
&u_{2,n} = u_{1,n} - \frac{1}{\mu} R_{e,n}^e, \\
&u_{3,n} = u_{2,n} + \frac{1}{\mu} (R_{e,n}^e - 3R_{e,n}^e).
\end{align*}

(7) \quad (8) \quad (9) \quad (10)
3.1. The approximation \((u_{0,n})\)
Blackwell's theorem states that the renewal sequence satisfies \(u_n \to 1/\mu\) and \(\Delta u_n \to 0\). In our case, (7) shows that \(u_{0,n} = 1/\mu\) and \(\Delta u_{0,n} = 0\).

3.2. The approximation \((u_{1,n})\)
If \((F_n) \in RS(-\alpha)\) and \(\alpha > 1\), then (8) and Lemma 4 show that
\[
 u_{1,n} - \frac{1}{\mu} = \frac{F_{e,n}}{\mu} \sim \frac{nF_n}{\mu^2(\alpha - 1)}
\]
and \(\Delta u_{1,n} = \frac{F_n}{\mu^2}\). In renewal theory, the following results are known, cf. Theorems 3.1.6 and 3.1.7 in Frenk (1983):

- If \((F_n) \in RS(-\alpha)\) with \(\alpha > 1\), then
\[
 u_n - \frac{1}{\mu} \sim \frac{F_{e,n}}{\mu} \sim \frac{nF_n}{\mu^2(\alpha - 1)}.
\]
- As in Remark 1, several authors have discussed moment conditions on \(X\) or \(X_e\) that imply \(\sum_{n=1}^{\infty} n^r |u_n - 1/\mu| < \infty\) or \(\sum_{n=1}^{\infty} n^r |\Delta u_n| < \infty\). We refer to Feller (1949), Karlin (1955), Stone (1965), Stone and Wainger (1967), and Grübel (1979) for such type of results.

3.3. The approximation \((u_{2,n})\)
If \((p_n) \in RS(-\alpha)\) with \(\alpha > 3\), then (9) and Theorem 7 show that
\[
 u_{2,n} - \frac{1}{\mu} = -\frac{R_{e,n}}{\mu} \sim -\frac{2\mu_e}{\mu^2(\alpha - 1)}np_n,
\]
and
\[
 \Delta u_{2,n} - \frac{p_{e,n}}{\mu} = -\frac{r_{e,n}}{\mu} \sim -\frac{2\mu_e}{\mu^2}p_n.
\]
From Remark 2, it follows that \(E(X_e^{r+1}) < \infty\) implies that
\[
 \sum_{n=1}^{\infty} n^r \left| u_{2,n} - \frac{1}{\mu} - \frac{1}{\mu}F_{e,n} \right| < \infty.
\]
The following results are known in renewal theory:
• Theorem 3.1.12 of Frenk (1983) and Theorem 2.1.25 of Frenk (1987) provide conditions under which

\[ u_n - \frac{1}{\mu} - \frac{1}{\mu} F_{e,n} \sim -\frac{2\mu e}{\mu^2} F_n. \]

See also Rogozin (1973).

• Theorem 3.1.14 of Frenk (1983) and Theorem 2.1.16 of Frenk (1987) also consider the case where \((p_{e,n}) \in RS(-\alpha)\) with \(1 < \alpha < 2\).

• Theorem 2.1.20 in Frenk (1987) provides a sufficient condition of the type \(E(X^{r+1}) < \infty\) for

\[ \sum_{n=1}^{\infty} n^r \left| u_n - \frac{1}{\mu} - \frac{1}{\mu} F_{e,n} \right| < \infty. \]

3.4. The approximation \((u_{3,n})\)

For \((p_n) \in RS(-\alpha)\) with \(\alpha > 3\), (10) and Theorem 9 lead to the following approximation:

\[ u_{3,n} = \frac{1}{\mu} + \frac{1}{\mu} F_{e,n} - \frac{1}{\mu} R_{2,n} + \frac{3\mu^2}{\mu^2} p_n + o(1)p_n. \]

We have found no similar result for the renewal sequence in the renewal literature. The approximation \(u_{3,n}\) implicitly appears in Section 4 of Gr"ubel (1983). See also Ess"en (1973) and Ney (1981).

4. Concluding remarks

1. In this paper we studied the renewal sequence when \(p_n\) is regularly varying and thus bounded by a power of \(n\). This approach does not cover exponential cases such as \(p_n = Ce^{-n}\) or \(p_n \sim n^c e^n\), \(0 < c < 1\). In a forthcoming paper we will also study this case.

2. In many cases the condition \((a_n) \in RS(-\alpha)\) can be replaced by the condition \((a_n) \in ORS \cap LS\), where \(ORS\) is the class of sequences \((a_n)\) for which (4) is replaced by \(\forall y > 0 : a_{[x+y]} = O(a_{[x]})\) as \(x \to \infty\).

3. To study the asymptotic behaviour of \(\delta_{3,n}\) we should impose conditions on \(p_n - p_{n-1}\). The details are somewhat complicated and further research is needed here.
References


