APPLICATION OF TOPOLOGY OPTIMIZATION IN A CONJUGATE HEAT TRANSFER PROBLEM

Tijs Van Oevelen¹, Martine Baelmans²

¹,² KU Leuven, Department of Mechanical Engineering
Celestijnenlaan 300A, 3001, Heverlee
¹e-mail: tijs.vanoeveren@kuleuven.be
²e-mail: martine.baelmans@kuleuven.be

Keywords: Topology optimization, Microfluidics, Thermal management, Heat sink

Abstract. In this paper topology optimization for cooling by finned heat sinks is studied. To this end, a two-layer model is elaborated based on the full 3D Navier-Stokes and heat conduction equations. The optimization is performed using the MMA method with gradients computed based on a continuous adjoint approach. Results reveal a reduction of the thermal resistance by more than a factor of 10 in comparison with a heat sink without fins.
1 Introduction

Rooted within the field of structural mechanics optimization, a vast majority of topology optimization applications emerged in this area \cite{1}. Presently, other engineering problems are being tackled as well \cite{2}. These problems obviously welcome the generic and flexible approach that topology optimization offers.

In this paper, we focus on the application of topology optimization for conjugate heat transfer problems. This application area encompasses the coupling of convection heat transfer through fluid flow with conduction heat transfer through the solid walls. Optimization problems typically arise in heat sink devices with limited wall thermal conductivity. The combined phenomena of fluid flow and heat transfer makes energy-efficient heat sink design, meeting both technical and economic design constraints, a challenging task.

The use of topology optimization in fluid flow problems started with the work of Borrvall and Petersson \cite{3}. They proposed to parametrize the topology in a Stokes flow problem by control of a penalty parameter, called ‘inverse permeability’. The method was demonstrated on a number of test problems. Several authors have since then continued the work on Stokes flow problems, focusing on the theoretical and computational aspects. A.o., Evgrafov \cite{4} addressed the problem that the lower bound of permeability should be above zero. Otherwise, the flow could be fully blocked, which makes it impossible to satisfy inflow conditions and continuity together. Wiker et al \cite{5} applied the method to an area-to-point flow problem, and introduced a filter to control the channel size. They also demonstrated that controlling viscosity in addition to permeability has very little influence. Aage et al \cite{6} demonstrated the method in a three-dimensional large-scale application. Challis and Guest \cite{7} used the level-set method to parametrize the topology, in contrast to the material distribution method that was used before. A major advantage of this method is the preservation of the discrete nature of the topology optimization problem.

Topology optimization has also been applied to Navier-Stokes flow problems. This is even more challenging due to the non-linearity of the partial-differential equations (PDE) that describe the flow. Gersborg-Hansen et al \cite{8} applied the method to design hydraulic components such as a bend, a flow-reversal and a switch. A high-level implementation for FEMLAB (now COMSOL Multiphysics) of this approach was performed by Olesen et al \cite{9}. Evgrafov \cite{10} proposed a method with slightly compressible fluid properties to deal with impenetrable inner walls. Duan et al \cite{11} used a variational level-set method to do the topology optimization of a bend and diffuser in Navier-Stokes flow. The work of Zhou and Li \cite{12} focussed on the design of microfluidic structures, such as a maximum permeability material.

Concerning heat transfer problems, we note a few examples in the area of pure conduction problems. Gersborg-Hansen et al \cite{13} demonstrated the material distribution method on a conductive area cooling problem. A similar volume-to-point objective was pursued by Mathieu-Potvin and Gosselin \cite{14} and Zhang and Liu \cite{15}. In the work by Bruns \cite{16} and Iga et al \cite{17}, topology optimization of a conduction problem with convection heat transfer boundary conditions was considered.

Convection heat transfer problems have been considered for topology optimization as well. E.g. in the work by Okkels et al \cite{18}, a cooling problem of a volumetric heat source was solved with topology optimization. Andreasen et al \cite{19} designed a microfluidic mixer for a fluid with non-uniform concentration at the inlet. More recently, several authors have continued working on convective cooling of volumetric sources, such as Matsumori et al \cite{20}, Kontoleontos et al \cite{21} and Marck et al \cite{22}. Further, Koga et al \cite{23} designed a heat sink device using topology
optimization. Their objective was maximization of ‘heat dissipation’, which yields a result similar to minimal mean temperature. Further, we have also recently applied topology optimization to the design of a heat sink with constant temperature heat source [24].

Our ambition in this work is to continue on the topological heat sink design. We wish to extend the method towards a heat source with constant heat flux distribution. This is more representative for heat sinks in electronics cooling application, as mostly the heat generation is fixed. Our goal is to design the heat sink such that it has a minimal thermal resistance.

We will describe the simulation model in section 2. The optimization methodology is explained in section 3. Subsequently, we will present our results in section 4. Conclusions are summarized in section 5.

2 Model

We focus on the typical thermal management problem depicted in Figure 1(a). It comprises of a flat heat source of size $L \times W$ with a uniformly distributed heat production $\dot{Q}''$. It is desired to keep the temperature of this heat source everywhere as low as possible. This is accomplished by direct contact with a heat sink. The heat sink absorbs heat from the heat source and transfers it to a coolant that flows through the heat sink cooling channels.

The heat sink consists of two layers. The bottom layer with height $H_b$ is a solid material base plate in contact with the heat source and has no holes. This layer provides mechanical strength and good thermal contact between heat source and heat sink. The top layer with height $H_t$ is penetrated with holes or channels for the coolant to flow through. The micro- and macro-scale geometric properties of the interface between the heat fins and coolant determine to a large extent the transfer rates of energy and momentum. Therefore, the top layer is crucial for achieving improved heat sink performance. Topology optimization of this top layer is our goal.

The heat sink domain consists of two sub-domains $\Omega_t$ and $\Omega_b$, which represent the top and bottom layer respectively. They are defined by:

$$\Omega_t = \Omega \times (H_b, H_b + H_t) \subset \mathbb{R}^3,$$

$$\Omega_b = \Omega \times (0, H_b) \subset \mathbb{R}^3,$$

where $\Omega = (0, L) \times (0, W) \subset \mathbb{R}^2$ is the surface of the heat source, as shown schematically in Figure 1(b).
2.1 Flow and heat transfer model equations

The heat sink performance evaluation is based on a numerical model. This model describes the stationary flow and heat transfer through the heat sink, keeping our optimization challenge in mind. The bottom layer consists of solid material and is left unchanged by the optimization, a pure heat diffusion equation suffices here. In contrast, the top layer consists of both a solid phase (heat fins) and a liquid phase (coolant), without a priori knowledge of their distribution. We will use a mixed solid-fluid model in view of the material distribution method for the topology optimization of the top layer. This means that the same set of partial-differential equations is used in both phases, with spatially varying material properties, see e.g. [25].

In particular, the following set of equations is applicable for the top layer:

\[
\nabla \cdot (\mathbf{v}) = 0, \tag{3}
\]

\[
\nabla \cdot (\rho \mathbf{v} \mathbf{v}) - \nabla \cdot (\mu \nabla \mathbf{v}) + \nabla p + \frac{\mu}{\kappa} \mathbf{v} = 0, \tag{4}
\]

\[
\nabla \cdot (\rho c T_t \mathbf{v}) - \nabla \cdot (k_t \nabla T_t) = 0. \tag{5}
\]

These three equations govern the conservation of mass, momentum and energy. The state variables are the velocity \( \mathbf{v} \), pressure \( p \) and temperature \( T_t \). \( \rho \), \( \mu \) and \( c \) are respectively the mass density, the dynamic viscosity and the specific heat capacity of the coolant. These material parameters are constant throughout the domain. \( \kappa \) is the permeability of the flow and depends locally on the design, i.e. \( \kappa = \infty \) where the domain contains fluid and 0 in the solid material. As such, the model collapses to the steady, incompressible Navier-Stokes equations in the fluid and zero velocity in the solid. \( k_t \) is the heat conductivity in the top layer, which equals \( k_t \) in the fluid phase and \( k_s \) in the solid phase respectively.

In addition, the energy equation holds in the bottom layer:

\[
-\nabla \cdot (k_b \nabla T_b) = 0, \tag{6}
\]

where \( T_b \) represents the temperature state and \( k_b \) the heat conductivity in the bottom layer. \( k_b \) is constant and equals \( k_s \) everywhere.

Energy conservation holds at the interface between the two layers, hence the following conditions apply at \( z = H_b \):

\[
T_t = T_b = T_i, \tag{7}
\]

\[
-k_t \frac{\partial T_t}{\partial z} = -k_b \frac{\partial T_b}{\partial z} = \dot{Q}_i''. \tag{8}
\]

2.2 Two-dimensional reduced model

The aforementioned model offers a reasonably accurate physical representation of the heat sink. In order to reduce the computational effort in a first approach, the 3D model equations are averaged over the height of the respective layer. Contributions from the eliminated vertical coordinate direction are then modelled based on assumed approximate profiles. These profiles correspond to fully developed flow and heat transfer between parallel plates with spacing \( H_t \) for the top layer, and a linear temperature profile for the bottom layer. The resulting two-
The dimensional model is, with the $\nabla$-operators understood in a two-dimensional way:

$$\nabla \cdot (\langle \mathbf{v} \rangle_t) = 0, \quad (9)$$

$$\nabla \cdot (K_{cm} \rho \langle \mathbf{v} \rangle_t \langle \mathbf{v} \rangle_t) - \nabla \cdot (\mu \nabla \langle \mathbf{v} \rangle_t) + \nabla \langle p \rangle_t + \alpha \langle \mathbf{v} \rangle_t = 0, \quad (10)$$

$$\nabla \cdot (K_{ce} \rho c \langle \mathbf{v} \rangle_t \langle \mathbf{v} \rangle_t) - \nabla \cdot (k_t \nabla \langle \mathbf{v} \rangle_t) + \nabla \langle p \rangle_t + \alpha \langle \mathbf{v} \rangle_t = 0, \quad (11)$$

$$-\nabla \cdot (k_b \nabla \langle T \rangle_b) + h_H \frac{\partial}{\partial x} (\langle T \rangle_b - \langle T \rangle_t) = 0, \quad (12)$$

where $\langle \cdot \rangle_t, \langle \cdot \rangle_b$ denote the averaging operations over the height of the top layer and bottom layer respectively. The parameters that are introduced while averaging are:

$$\alpha = \mu \left( \frac{1}{\kappa} + \frac{K_{cm}}{H_t^2} \right), \quad (13)$$

$$h = \frac{h_t h_b}{h_t + h_b}, \quad (14)$$

with $h_j = k_j \frac{K_{ej}}{H_j}$, $j \in \{ t, b \}$. Note that both parameters are spatially variable and depend on the heat sink topology. The factor $\alpha$ was termed ‘inverse permeability’ by Borrvall and Petersson [3]. $h$ is the total heat transfer coefficient between top and bottom layers. The interface heat flux then satisfies:

$$\dot{Q}''_i = h (T_b - T_t). \quad (15)$$

The K-coefficients are defined as follows:

$$K_{cm} = \langle \zeta (\tilde{z})^2 \rangle_t, \quad K_{cm} = -\left\langle \frac{\partial^2 \zeta (\tilde{z})}{\partial \tilde{z}^2} \right\rangle_t, \quad (16)$$

$$K_{ce,t} = \langle \zeta (\tilde{z}) \xi (\tilde{z}) \rangle_t, \quad K_{ce,t} = -\left\langle \frac{\partial^2 \xi (\tilde{z})}{\partial \tilde{z}^2} \right\rangle_t, \quad (17)$$

where $\tilde{z} = \frac{z}{H_t}$ is the dimensionless $z$-coordinate. $\zeta (\tilde{z})$ and $\xi (\tilde{z})$ are respectively the velocity and top layer temperature profiles, defined by:

$$\zeta (\tilde{z}) = \frac{v_x (\tilde{z})}{\langle v_x \rangle_t} = \frac{v_y (\tilde{z})}{\langle v_y \rangle_t}, \quad \xi (\tilde{z}) = \frac{T_t (\tilde{z}) - T_i}{\langle T \rangle_t - \langle T \rangle_t}, \quad (18)$$

where $T_i$ is the corresponding interface temperature. Corresponding to the fully developed flow and constant flux heat transfer profiles, the following values for the coefficients emerge: $K_{cm} = 1.2, K_{cm} = 12, K_{ce,t} = 52, K_{ce,t} = 70$. These values are in agreement with data from Shah and London [26]. The dimensionless heat transfer coefficient in the bottom layer $K_{ab}^n$ is equal to 2, which follows from a linear temperature profile.

In the remainder of the text, the averaging notations $\langle \cdot \rangle$ are omitted. The following boundary conditions are applied for the two-dimensional model. See also Figure [1(b)].

**Inlet:**

$$p = \Delta p, \quad v_y = 0, \quad T_t = T_{t_{inlet}}^\text{inlet}, \quad k_b \frac{\partial T_b}{\partial x} = 0, \quad (19)$$
Outlet:

\[ p = 0, \quad v_y = 0, \quad k_t \frac{\partial T_t}{\partial x} = 0, \quad k_b \frac{\partial T_b}{\partial x} = 0, \quad (20) \]

Wall:

\[ v_x = 0, \quad v_y = 0, \quad k_t \frac{\partial T_t}{\partial y} = 0, \quad k_b \frac{\partial T_b}{\partial y} = 0. \quad (21) \]

2.3 Implementation

The equations are solved using the finite volume method on a structured Cartesian grid. The momentum equations are discretized using a standard staggered grid approach with hybrid scheme discretization [27] and implicit Euler false time step integration.

Because the Reynolds number in microfluidics is typically low, no turbulence model is needed and solving the continuity and momentum equations in fully coupled way leads to a fast and robust simulation method. After the flow field has been computed, the energy equations are solved in one linear system solve at once.

The numerical method to solve the partial-differential equations is implemented in Matlab.

3 Optimization

In this section, we discuss the formulation of the optimization problem and the optimization method.

3.1 Optimization problem formulation

The numerical representation of the heat sink topology is parametrized with the material distribution method [25]. This means that the strict separation between the two phases is relaxed, such that the material fractions can vary in a continuous way. The distribution of material is represented by the porosity \( \varepsilon(x) \), which is defined as the local volume fraction of liquid:

\[ \varepsilon(x) = \frac{\text{liquid volume}}{\text{total volume}} \quad \text{in a small neighbourhood around } x. \quad (22) \]

A mapping from the porosity towards the variable parameters \( \alpha(\varepsilon) \) and \( k_t(\varepsilon) \) in the model (9)–(12) is provided. These mappings define the effect of the design variable \( \varepsilon \) on the model. They satisfy the condition that for strict 0/1-solutions, i.e. \( \varepsilon \in \{0, 1\} \), the model collapses to the equations for strictly separated phases. The specific profile between the extremal points, i.e. for \( \varepsilon \in (0, 1) \), has a controlling effect on the properties of the optimal design.

We have adopted here the \( q \)-interpolation method proposed by Borrvall & Petersson [3] to map porosity \( \varepsilon \) onto the inverse permeability \( \alpha \). The penalization parameter \( q \) in this method affects the amount of grey material in the end result. Larger values of \( q \) yield less grey material. It is proven in [3] that a linear interpolation for \( \alpha \) ensures a strict black/white result, for the problem of minimizing the total potential power of the flow with limited fluid volume. However, they advise the value \( q = 0.1 \), as it yields satisfactory results and avoids the problem of local minima.

\[ \alpha(\varepsilon) = \alpha_{\text{max}} + (\alpha_{\text{min}} - \alpha_{\text{max}}) \frac{1 + q}{\varepsilon + q}. \quad (23) \]
The mapping from porosity to the thermal conductivity is taken as a linear interpolation:

\[ k_t(\varepsilon) = \varepsilon k_f + (1 - \varepsilon) k_a. \]  
(24)

Note that we use a constant value for \( \mu \). This could in principle also be taken as a function of \( \varepsilon \). However, Wiker et al [5] concluded in their study that this has no additional benefits.

We use following objective functional for the topology optimization of the heat sink:

\[
\begin{align*}
\min_{\phi, \varepsilon} \ J(\phi, \varepsilon) &= \int_{\Omega} (T_b(x) - T_{\text{target}}^b)^2 \, dx, \\
\text{subject to:}
\end{align*}
\]
(25)

where \( \phi = (v, p, T_t, T_b)^T \) is the state vector.

This objective functional is chosen as an approximation to thermal resistance minimization. The thermal resistance is defined as the ratio of maximal temperature difference to total heat transfer rate:

\[ R_{th} = \frac{T_{\text{max}}^b - T_{\text{inlet}}^t}{Q' LW}. \]  
(26)

### 3.2 Optimization method

The gradient of the reduced objective functional is needed as input for the optimization algorithm. The reduced objective functional is:

\[ \hat{J}(\varepsilon) = J(\phi(\varepsilon), \varepsilon), \]  
(27)

where \( \phi(\varepsilon) \) is the solution of the model equations for a given \( \varepsilon \), in short notation:

\[
\begin{pmatrix}
H_t C(\phi) \\
H_t M(\phi, \varepsilon) \\
H_t E_t(\phi, \varepsilon) \\
H_b E_b(\phi, \varepsilon)
\end{pmatrix}
= A(\phi, \varepsilon) = b =
\begin{pmatrix}
0 \\
0 \\
0 \\
0
\end{pmatrix},
\]
(28)

where \( C(\phi), M(\phi, \varepsilon), E_t(\phi, \varepsilon) \) and \( E_b(\phi, \varepsilon) \) are respectively the continuity, momentum and both energy equations as given in (9)–(12). The multiplication factors \( H_t \) and \( H_b \), representing the height of the related layer, are deliberately used in order to preserve consistency with the three-dimensional origin of the heat sink model. This way, adjoint equations are obtained that resemble most the original model equations.

The derivative of the reduced objective functional is calculated with the adjoint method, see e.g. [28]. We consider the derivation of the adjoint equations for a general objective functional:

\[ J(\phi, \varepsilon) = \int_{\Omega} [H_t f_t(x) + H_b f_b(x)] \, dx + \int_{\partial \Omega} \left[ H_t g_t(s) + H_b g_b(s) \right] \, ds. \]  
(29)

The subscripts (t and b) denote the domain (top and bottom layer respectively) on which the functions \( f_j \) and \( g_j \) \( (j \in \{t, b\}) \) are defined.
The adjoint equations are obtained from the Lagrangian functional $\mathcal{L}(\phi, \varepsilon, \phi^*)$, defined below:

$$
\mathcal{L}(\phi, \varepsilon, \phi^*) = \mathcal{J}(\phi, \varepsilon) - \int_{\Omega} p^* C(\phi) \, dx - \int_{\Omega} \mathbf{v}^* \cdot \mathbf{M}(\phi, \varepsilon) \, dx
$$

$$
- \int_{\Omega} T_t^* E_t(\phi, \varepsilon) \, dx - \int_{\Omega_b} T_b^* \left( E_b(\phi, \varepsilon) - \frac{\dot{\mathcal{Q}}}{H_b} \right) \, dx
$$

$$
= \mathcal{J}(\phi, \varepsilon) - \int_{\Omega} \phi^* \cdot (\mathbf{A}(\phi, \varepsilon) - \mathbf{b}) \, dx,
$$

where $\phi^* = (\mathbf{v}^*, p^*, T_t^*, T_b^*)^T$ are the so-called adjoint variables, acting as Lagrange multipliers for the model equations.

The adjoint equations are retrieved from setting the derivative of $\mathcal{L}$ with respect to the state variables $\phi$ to zero for all possible variations $\phi'$. Subscripted notation is used to denote the derivatives:

$$
\mathcal{L}_\phi(\phi, \varepsilon, \phi^*)\phi' = \mathcal{J}_\phi(\phi, \varepsilon)\phi' - \int_{\Omega} \phi^* \cdot A_\phi(\phi, \varepsilon)\phi' \, dx = 0, \quad \forall \phi'
$$

(32)

By partial integration of the last term, spatial derivatives are shifted from $\phi'$ to $\phi^*$ to get the adjoint operator $A_\phi^*$:

$$
\int_{\Omega} \phi^* \cdot A_\phi(\varepsilon, \phi)\phi' \, dx \equiv \int_{\Omega} \phi' \cdot A_\phi^*(\varepsilon, \phi)\phi^* \, dx + BT,
$$

(33)

where $BT$ are boundary integrals that appear as a result of the partial integration. Since $\mathcal{L}_\phi\phi'$ has to vanish for all $\phi'$, we can let the domain integrals vanish to get the adjoint equations:

$$
-\nabla \cdot (\mathbf{v}^*) = \frac{\partial f_t}{\partial p},
$$

(34)

$$
-K_v^m \rho \mathbf{v} \cdot (\nabla \mathbf{v}^* + (\nabla \mathbf{v}^*)^T) - \nabla \cdot (\mu \nabla \mathbf{v}^*) - \nabla p^* + \alpha(\varepsilon) \mathbf{v}^* + K_v^c \rho c T_t^* \nabla T_t = \frac{\partial f_t}{\partial \mathbf{v}},
$$

(35)

$$
-K_{v, t}^c \rho c \mathbf{v} \cdot \nabla T_t^* - \nabla \cdot (k_t(\varepsilon) \nabla T_t^*) + \frac{h(\varepsilon)}{H_t} (T_t^* - T_t^b) = \frac{\partial f_t}{\partial T_t},
$$

(36)

$$
-\nabla \cdot (k_b \nabla T_b^*) + \frac{h(\varepsilon)}{H_b} (T_b^* - T_t^b) = \frac{\partial f_b}{\partial T_b},
$$

(37)

These equations constitute the adjoint PDE system $A_\phi^*(\phi, \varepsilon)\phi^* = f_\phi$. The same can be done to make the boundary integrals vanish, which yields the boundary conditions for this adjoint PDE:

Inlet:

$$
p^* + K_v^m \rho \mathbf{v} \cdot \mathbf{v}^* + K_v^m \rho v_n v_n^* + \mu \frac{\partial v_n^*}{\partial n} = \frac{\partial q_t}{\partial n}, \quad v_t^* = 0,
$$

(38)

$$
T_t^* = 0,
$$

(39)
Outlet:
\[ p^* + K^m_c \rho v \cdot v^* + K^m_c \rho v_n v^*_n + \mu \frac{\partial v^*_n}{\partial v_n} = \frac{\partial q_h}{\partial v_n}, \]
\[ \partial g_t \partial v^*_n, v^*_t = 0, \quad (40) \]
\[ K^e_c \rho c v_n T^*_t + k^t \frac{\partial T^*_t}{\partial n} = \frac{\partial q_k}{\partial T_t}, \quad k^b \frac{\partial T^*_b}{\partial n} = \frac{\partial q_b}{\partial T_b}, \quad (41) \]

Wall:
\[ v^*_n = 0, \quad v^*_t = 0, \quad (42) \]
\[ k^t \frac{\partial T^*_t}{\partial n} = \frac{\partial q_k}{\partial T_t}, \quad k^b \frac{\partial T^*_b}{\partial n} = \frac{\partial q_b}{\partial T_b}. \quad (43) \]

The subscripts \( n \) and \( t \) are used to denote the normal and tangential velocity components. \( n \) is the outward normal coordinate.

The adjoint partial-differential system is solved numerically in a similar fashion as the state problem. In principle, all adjoint equations can be solved at once due to the linearity of the adjoint problem. However, we adopt the same solution strategy as for the state problem, but in reversed order. First, the adjoint energy equations are solved for \( T^*_t \) and \( T^*_b \). Then, the adjoint flow equations are solved for \( v^* \) and \( p^* \), where the last term on the left-hand side of Eq. (35) acts as a constant source term. The presence of \( T^*_t \) in this term imposes the reversed solution order. This reflects the nature of the adjoint method, which passes information in reversed order through the model, i.e. from output to input.

The gradient of the reduced cost functional is now calculated as the derivative of the Lagrangian with respect to \( \varepsilon \), using the solutions \( \phi \) and \( \phi^* \) of the state and adjoint equations:
\[ \hat{J}_\varepsilon(\varepsilon') = L_\varepsilon(\phi, \varepsilon, \phi^*)\varepsilon' \]
\[ = \int_\Omega \left( H_t \frac{\partial^2 f_t}{\partial \varepsilon^2} - H_t \frac{\partial \alpha}{\partial \varepsilon} v \cdot v^* - H_t \frac{\partial k^t}{\partial \varepsilon} \nabla T_t \cdot \nabla T^*_t + \frac{\partial h}{\partial \varepsilon}(T_t - T_b)(T^*_t - T^*_b) \right) \varepsilon' dx. \]
\[ = \nabla_x \hat{\mathcal{J}} \quad (45) \]

The adjoint method can be used for the optimization problem (25) by identification of the general objective functional (29) with the objective functional of (25).

The solution of the optimization problem (25) is found iteratively by the method of moving asymptotes (MMA). The MMA has been introduced by Svanberg [29] and is widely adopted for topology optimization problems. In each iteration, a convex, separable approximating sub-problem is generated and solved. The solution of the sub-problem is used for a new approximation. This procedure generates a sequence of approximate solutions that converges towards the solution of the original problem (25).

To avoid getting stuck in suboptimal local minima, we adopt a continuation approach for the penalization parameter \( q \) in Eq. (23). A similar continuation approach is advised in [3]. However, we increase \( q \) gradually, instead of in a few larger steps. The following procedure is used:
\[ \beta = \left( \frac{q_{\text{final}}}{q_{\text{init}}} \right)^{\frac{1}{Nq-1}}, \quad (46) \]
\[ q_{k+1} = \min( \beta q_k, q_{\text{final}} ). \quad (47) \]
Table 1: Performance comparison of initial and optimized designs.

<table>
<thead>
<tr>
<th></th>
<th>$\mathcal{J}$ (K²m³)</th>
<th>$T_{b}^{\text{max}}$ (K)</th>
<th>$R_{\text{th}}$ (K/W)</th>
<th>$\dot{m}$ (g/s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Initial ($\varepsilon = 1$)</td>
<td>$1.72 \cdot 10^{-3}$</td>
<td>293</td>
<td>2.93</td>
<td>101</td>
</tr>
<tr>
<td>Optimized</td>
<td>$3.34 \cdot 10^{-6}$</td>
<td>19.2</td>
<td>0.192</td>
<td>3.49</td>
</tr>
</tbody>
</table>

$q_{\text{init}}$ is the initial value of $q$, i.e. at iteration $k = 1$. $q_{\text{final}}$ is the desired final value of $q$. $N_q$ is the amount of iterations after which the algorithm should reach the value $q_{\text{final}}$. This allows to continue the algorithm for $k > N_q$, without further increasing $q$.

4 Results

The method described in the previous sections has been applied to a number of test cases. First, we’ll establish a reference case. Next, the influence of two important parameters is investigated and discussed, namely the temperature target $T_{b}^{\text{target}}$ in the objective functional, see (25), and the thickness of the bottom layer $H_b$.

4.1 Heat sink design: reference case

The reference case is defined by the following parameters. The size of the heat sink is $L \times W = 1\text{cm} \times 1\text{cm}$. The thickness of the top and bottom layers is respectively $H_t = 500\mu\text{m}$ and $H_b = 200\mu\text{m}$. The heat sink is made of silicon ($k_s = 149\text{W/m} \cdot \text{K}$). The coolant is pure water ($k_f = 0.598\text{W/m} \cdot \text{K}$, $\mu = 1.004 \cdot 10^{-3}\text{Pa} \cdot \text{s}$, $\rho = 998\text{kg/m}^3$, $c = 4180\text{J/kg} \cdot \text{K}$). The static pressure drop over the heat sink is $\Delta p = 10\text{kPa}$. The heat flux $\dot{Q}''$ of the heat source is $100\text{W/m}^2$. The inlet temperature $T_{\text{inlet}}$ is set to $0\text{K}$, so all temperatures should be interpreted as differences with respect to the inlet. The value of the target temperature is set to $0\text{K}$.

The computational grid consists of 200x200 cells. The value of the loss coefficient in the solid is taken at $\alpha_{\text{max}} = 10^{4} \frac{\text{K}}{\text{m}^2}$, which is 4 orders of magnitude larger than the value in the fluid $\alpha_{\text{min}} = \mu \frac{K^m}{H_t^2}$. The value of the interpolation parameter $q$ is initially set at 0.01, and increased every optimization iteration until it reaches $q_{\text{final}} = 10$ after 50 iterations. The algorithm is allowed to settle in 50 additional iterations. In all simulations, we observe no more significant changes to the design nor the value of the objective functional.

In the present study, we limit the test problem to the Stokes flow equations in order to avoid non-linearities in this first exploration of heat sink topology optimization. This means that the convective term in the momentum equations is discarded, as well as its consequences on the adjoint model and boundary conditions.

The design is initialized uniformly at $\varepsilon = 1$, which corresponds to a heat sink without fins, containing only fluid in the top layer. The initial performance is found in Table 1 in terms of the objective functional value, the maximal temperature, the thermal resistance and the mass flow rate $\dot{m}$.

During optimization, the porosity in the first column of cells, i.e. the inlet at the west boundary, is held fixed to $\varepsilon = 1$. This ensures a more physically meaningful result. Without this restriction, the optimizer takes advantage of the constant temperature inlet condition by placing solid material in contact with the west boundary. This enhances the heat transfer, and thus reduces thermal resistance, due to conduction through the west boundary. However in reality, the cold inlet flow is distributed with a collector, which does not actually fix the temperature at the inlet boundary. Only those parts of the inlet boundary that are in the fluid phase, can be
Figure 2: Evolution of the heat sink design through the optimization process. The material penalization parameter $q$ in (23) is gradually increased from 0.01 → 10, during the first 50 iterations.

assumed to have a constant temperature. Therefore, we ensure that the whole inlet is in the fluid phase, which resembles most the presence of a collector.

A view of the heat sink design evolution is provided in Figure 2. A greyscale representation of the porosity distribution $\varepsilon(x)$ is shown at several iteration steps (black: $\varepsilon = 0$, white: $\varepsilon = 1$). The design evolves nicely from a smooth, rather homogeneous material distribution towards a sharp image with clearly distinct phases. One can clearly see the effect of our continuation approach, in which the penalization parameter $q$ as defined in (23) is continuously increased from 0.01 towards 10 in the first 50 iterations. This increases the penalization of grey material gradually. Whereas the domain is grey almost everywhere in the first iterations, this is almost completely vanished by iteration 50.

It is also observed that after 50 iterations, the final design is already almost discovered. By iteration 100, not much has changed with respect to iteration 50.

The performance of the optimized design is given in Table I. The values of the objective functional, maximal temperature and thermal resistance have dramatically decreased with respect to the initial design. Contour plots of the state variables $v_x$, $v_y$, $p$, $T_i$ and $T_b$ are drawn in Figure 3 as well as the interface heat flux between the two layers $Q''_i$.

The whole surface of the heat sink is covered with channels of different sizes. The main flow occurs through a few larger channels, which have the least flow resistance (see Figs. 3(a), 3(b)). These larger channels are extensively interconnected by smaller channels. None of the larger channels stretches from the inlet to the outlet, which forces the flow through one of the smaller channels at some point through the heat sink. This is expected because the convection heat transfer from the fins to the fluid is stronger in smaller channels. On the other hand, smaller channels will increase the pressure drop significantly. Forcing the flow through these channels thus makes sure that all fluid passes these strong convection regions over a rather short distance.

Figure 3(d) shows the temperature in the top layer of the heat sink. Large temperature differ-
ences are clearly present between the fins and the coolant flow. Furthermore, the temperature rise from inlet to outlet can be observed. Figure 3(e) shows the temperature in the bottom layer. The temperature distribution here is more smooth, which is a result of the stronger heat conduction that takes place in this layer. Figure 3(f) displays the heat transfer between both layers, which occurs mainly at positions where the fins are placed.

### 4.2 Influence of target temperature

The objective functional measures the difference between the bottom layer temperature and a target temperature $T_{b}^{\text{target}}$ with an $L^2$-norm. This is used as an approximation to the $R_{th}$ objective. The target temperature in the reference case is 0K. Now, it is investigated whether the choice of the target temperature can be used to further lower the thermal resistance. To this end, several values of the target temperature are evaluated. The results are shown in Table 2 and Figure 4.

Table 2 shows that the value of the objective functional $J$ reduces with increasing $T_{b}^{\text{target}}$. 

![Figure 3: Contour plots of state variables and interface heat flux after 100th iteration.](image)
This is logical because $T_b^{\text{target}}$ is brought closer to the actual temperature level in the bottom layer. There is an exception however. $\mathcal{J}$ is larger again for $T_b^{\text{target}} = 20$K.

Comparing the thermal resistance of the several cases, we observe that this is lowest for a target temperature of 10K. $T_b^{\text{max}}$ is 18.3K in that case, which is almost 1K below the reference case.

Figure 4 shows the porosity distribution for the different cases. The designs are all qualitatively similar. However, especially for $T_b^{\text{target}} = 20$K, but also for $T_b^{\text{target}} = 15$K, the designs have not fully crystallized to a sharp black/white distribution yet. This reveals that the effectiveness of the material distribution method is case-dependent.

### 4.3 Influence of bottom layer thickness

In this section, the influence of the bottom layer thickness is assessed as it is one of the crucial parameters for the heat sink design. Indeed, it determines the strength of the heat conduction within the bottom layer. This heat conduction is responsible for the redistribution of the source heat flux. Figure 3(f) shows this effect. The figure displays the heat flux distribution at the interface between the two layers. This distribution is clearly non-uniform, despite the fact the heat source is uniformly distributed.

A sensitivity study for several values of the bottom layer thickness is performed. The effect on the performance of the heat sink is non-monotonous, as shown in Table 3. This may indicate that the optimization gets stuck in local minima.

A more pronounced effect is observed on the layout of the optimized heat sink. This is shown in Figure 5. For increasing $H_b$, the overall number of channels decreases. Due to the improved heat conduction in the bottom layer, the local transport through the heat sink fins becomes less important. This allows for more global, coarser solutions, as the figure shows.

<table>
<thead>
<tr>
<th>$H_b$ (µm)</th>
<th>$\mathcal{J}$ (K²m³)</th>
<th>$T_b^{\text{max}}$ (K)</th>
<th>$R_{th}$ (K/W)</th>
<th>$\dot{m}$ (g/s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>50</td>
<td>6.67·10⁻⁷</td>
<td>19.8</td>
<td>0.198</td>
<td>3.50</td>
</tr>
<tr>
<td>100</td>
<td>1.36·10⁻⁶</td>
<td>17.4</td>
<td>0.174</td>
<td>3.74</td>
</tr>
<tr>
<td><strong>200</strong></td>
<td><strong>3.34·10⁻⁶</strong></td>
<td><strong>19.2</strong></td>
<td><strong>0.192</strong></td>
<td><strong>3.49</strong></td>
</tr>
<tr>
<td>400</td>
<td>6.18·10⁻⁶</td>
<td>16.7</td>
<td>0.167</td>
<td>4.41</td>
</tr>
<tr>
<td>800</td>
<td>1.61·10⁻⁵</td>
<td>17.2</td>
<td>0.172</td>
<td>4.12</td>
</tr>
</tbody>
</table>

Table 3: Heat sink performance for several values of $H_b$. 
5 Conclusion

In this paper, we have applied topology optimization for design of a finned heat sink, which is a typical conjugate heat transfer problem. A two-layer heat sink model is developed for the simulation of coolant flow and heat transfer, approximating the full 3D Navier-Stokes and energy equations. The heat sink design consists of solid material in the bottom layer, in contact with the heat source, and a distribution of fins in the top layer. The optimal distribution of these fins is posed as a topology optimization problem, parametrized with the material distribution method. The porosity, or volume fraction, $\varepsilon(x)$ is mapped onto physical parameters $\alpha$ and $k_t$ in the heat sink model.

The method of moving asymptotes is used to minimize a least-squares tracking objective. The continuous adjoint approach is used to efficiently compute the objective gradient. A continuation approach for the penalization parameter $q$ is proposed.

The method is tested on a reference case with Stokes flow. Based on our results, we conclude that the method works well. A reduction of the heat sink’s thermal resistance with more than a factor 10 compared to the initial design has been achieved. The optimized heat sink exhibits the expected branched flow network structure. It leads to a configuration with small but short interconnecting channels between wider channel branches thus providing optimal cooling while maintaining the desired pressure drop. Moderate effects of the target temperature $T_{\text{target}}$ and base plate thickness $H_b$ have been observed. Increasing the target temperature slightly decreases the objective functional value and the thermal resistance up to a certain level. However, too high target temperatures deteriorate the results again. Changes in base plate thickness lead to small changes in thermal resistance without a clear trend. This indicates that the solutions may be distorted by local minima. Further work will focus on addressing these issues.

REFERENCES


