| Citation          | G. Hilhorst, G. Pipeleers, J. Swevers, (2013), Reduced-Order Multi-Objective H-infinity Control of an Overhead Crane Test Setup
| Archived version | Author manuscript: the content is identical to the content of the published paper, but without the final typesetting by the publisher |
| Published version| Not yet available |
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| IR               | [https://lirias.kuleuven.be/handle/123456789/411045](https://lirias.kuleuven.be/handle/123456789/411045) |

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Reduced-Order Multi-Objective $\mathcal{H}_\infty$ Control of an Overhead Crane Test Setup

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Abstract—A novel convex approach for reduced-order multi-objective controller synthesis is presented, and experimentally validated on an overhead crane test setup. For this setup, a discrete-time linear time-invariant model is identified. Starting from a full-order controller for the identified model, reduced-order multi-objective $\mathcal{H}_\infty$ controllers are designed by successively solving convex optimization problems. Experimental validation results on the test setup sustain the practical potential of the proposed controller design approach.

I. INTRODUCTION

The general objective of a good controller design is to stabilize a given system while one or more performance specifications are satisfied. A very popular and effective approach to the aforementioned problem is, amongst others, $\mathcal{H}_\infty$ control. Methods for $\mathcal{H}_\infty$ control have emerged since the 1980s, and despite their theoretical maturity, still constitute an important research branch.

Based on Lyapunov theory and the notion of dissipativity, convex reformulations of many analysis and control problems have been derived [1], [2], [3], [4]. The corresponding feasibility or optimization problems involve linear matrix inequalities (LMIs) and are efficiently solved by interior-point methods, which guarantee convergence to a global optimal solution within polynomial time [5]. Specifically, both the single- and multi-objective $\mathcal{H}_\infty$ analysis problem for linear time-invariant (LTI) systems can directly be formulated as an LMI optimization problem. However, the corresponding synthesis problems generally invoke bilinear matrix inequalities (BMIs) which, as opposed to LMIs, are nonconvex and thus hard to solve. For the synthesis of single-objective $\mathcal{H}_\infty$ full-order dynamic output feedback controllers for LTI systems, two well-known approaches exist that transform the BMI problem into an equivalent LMI condition. These approaches are characterized by respectively a nonlinear change of variables [2] and by elimination of the controller variables [4].

As motivated in [6], [7], breaking up the classical $\mathcal{H}_\infty$ control problem (see for instance [8]) into a multi-objective $\mathcal{H}_\infty$ control problem facilitates the weighting function design. The formulation of appropriate weighting functions in classical $\mathcal{H}_\infty$ control design is often a laborious task, while considering multiple $\mathcal{H}_\infty$ performance specifications constitutes a more versatile and intuitive approach. Since the incorporation of multiple performance criteria is often desirable, several multi-objective control design methods have been developed [2], [9] and validated [6]. As the design of full-order multi-objective LTI controllers is intrinsically a nonconvex problem, conservatism is introduced in the latter approaches to derive convex sufficient conditions.

In this paper, the reduced-order multi-objective $\mathcal{H}_\infty$ control problem for discrete-time LTI dynamics is considered. Reduced-order controllers are advantageous because of their high reliability and low implementation costs. The existence of a convex reformulation related to this problem is unknown, even when only one performance specification is imposed [10]. Despite the lack of a convex reformulation, several approaches have been developed for reduced-order controller design [11]. Those include solving the BMI problem directly [12], solving a nonconvex reformulation in terms of an LMI plus a rank constraint [13], [14], [15], or setting up convex sufficient conditions [16], [17], [18].

A recently developed LMI approach for reduced-order controller design is adopted [19], and extended to the multi-objective case. Consecutively, this approach is used to design reduced-order multi-objective $\mathcal{H}_\infty$ controllers for an overhead crane test setup. The resulting controllers are successfully experimentally validated, demonstrating the practical potential of the controller design methodology.

The paper is organized as follows. First, the general problem is formulated and some important definitions are introduced in Section II. Then a theoretical extension to existing results is presented in Section III. Section IV discusses the experimental validation on an overhead crane test setup. Finally, the conclusions are given and future work is proposed in Section V.

Notation: Since different controller orders are used throughout this paper, the dependence of a system or matrix $X$ on the controller order $q$ is indicated by $X^{(q)}$. $I_n$ denotes the identity matrix of dimension $n$ and $0_{m \times n}$ denotes a zero matrix of dimension $m \times n$. The subscripts are omitted whenever the dimensions can be inferred from the context. The transpose of a matrix $X$ (or $X^{(q)}$) is written as $X'$ (or $X'^{(q)}$). The set of $n \times n$ symmetric matrices is defined as $\mathbb{S}^n$. A star ($\ast$) denotes symmetric terms in matrix inequalities.

*This work benefits from K.U.Leuven-BOF PFV/10/002 Center-of-Excellence Optimization in Engineering (OPTEC), and from the project IWT-SBO 80032 (LeCoPro) of IWT-Flanders.
**Goele Pipeleers is a Postdoctoral Fellow of the Research Foundation - Flanders (FWO-Vlaanderen).
II. PROBLEM STATEMENT

Consider the finite-dimensional discrete-time multiple-input multiple-output LTI state-space realization

\[
G: \begin{cases}
    x(k+1) = A x(k) + B_o w(k) + B_u u(k), \\
    z(k) = C_x x(k) + D_{zw} w(k) + D_{zu} u(k), \\
    y(k) = C_y x(k) + D_{wy} w(k),
\end{cases}
\]

where \( x \in \mathbb{R}^{n_x} \) is the state, \( u \in \mathbb{R}^{n_u} \) the control input, \( w \in \mathbb{R}^{n_w} \) the exogenous input, \( z \in \mathbb{R}^{n_z} \) the regulated output, and \( y \in \mathbb{R}^{n_y} \) the measured output. The discrete time is denoted by \( k \). Under a mild well-posedness condition (see [20]), the direct feedthrough matrix \( D_{wu} \) may be set to 0.

The objective is to compute dynamic output feedback controllers of fixed order \( q \), where \( 0 \leq q \leq n_x \), that stabilize \( G \) and meet various closed-loop performance specifications. A controller of order \( q \) is denoted by

\[
C(q): \begin{cases}
    x_c(q)(k+1) = A_c(q) x_c(q)(k) + B_c(q) y(k), \\
    u(k) = C_c(q) x_c(q)(k) + D_c(q) y(k),
\end{cases}
\]

where \( x_c(q) \in \mathbb{R}^q \). Setting \( q = 0 \) results in a static output feedback.

A performance channel \( j \) can be selected by defining appropriate matrices \( L_j \) and \( R_j \), such that \( w = R_j w_j \) and \( z_j = L_j z \), and considering the dynamics for that specific channel:

\[
G_j: \begin{cases}
    x(k+1) = A x(k) + B_o R_j w_j(k) + B_u u(k), \\
    z_j(k) = L_j C_x x(k) + L_j D_{zw} R_j w_j(k) + L_j D_{zu} u(k), \\
    y(k) = C_y x(k) + D_{wy} R_j w_j(k),
\end{cases}
\]

The interconnection of the plant \( G_j \) with a controller \( C(q) \) yields the closed-loop dynamics

\[
H_j(q): \begin{cases}
    \dot{x}(q)(k+1) = A(q) \dot{x}(q)(k) + B(q) w_j(k), \\
    z_j(k) = C(q) \dot{x}(q)(k) + D(q) w_j(k),
\end{cases}
\]

for optimization channel \( j \), where

\[
\dot{x}(q) = \begin{bmatrix} x_c(q) \\ e(q) \end{bmatrix} \in \mathbb{R}^{n_x+q}
\]

is a closed-loop state vector. By defining the matrices

\[
\begin{bmatrix}
    \tilde{A}_j(q) & \tilde{B}_j(q) \\
    \tilde{C}_j(q) & \tilde{D}_j(q)
\end{bmatrix}
\begin{bmatrix}
    \tilde{D}_{ju}(q) & \tilde{D}_{yj}(q)
\end{bmatrix}
\begin{bmatrix}
    A_c(q) & B_c(q) \\
    C_c(q) & D_c(q)
\end{bmatrix}
\]

and the controller parameter

\[
\Theta(q) := \begin{bmatrix}
    A_c(q) & B_c(q) \\
    C_c(q) & D_c(q)
\end{bmatrix},
\]

the affine dependence of the closed-loop matrices on \( \Theta(q) \) is expressed as

\[
\begin{bmatrix}
    A(q) & B(q) \\
    C(q) & D(q)
\end{bmatrix}
\begin{bmatrix}
    \tilde{A}_j(q) & \tilde{B}_j(q) \\
    \tilde{C}_j(q) & \tilde{D}_j(q)
\end{bmatrix}
\begin{bmatrix}
    \tilde{D}_{ju}(q) & \tilde{D}_{yj}(q)
\end{bmatrix}
\begin{bmatrix}
    A_c(q) & B_c(q) \\
    C_c(q) & D_c(q)
\end{bmatrix} = \Theta(q) \begin{bmatrix}
    A(q) & B(q) \\
    C(q) & D(q)
\end{bmatrix}.
\]
$X_{12j}, X_{22j}$ and $X_{32j}$ are independent results in maximum freedom while retaining convexity of the optimization problem (7).

**LMI Procedure**

Whenever the difference between $p$ and $q$ is large, the optimization problem (7) can be solved successively by breaking down $p - q$ in steps of $\Delta r$, instead of going from $p$ to $q$ in one step. This leads to the following procedure:

1) Suppose that a multi-objective $H_\infty$ controller of order $p$ is given. If not, compute a full-order controller with the method from [2] and set $p := n_c$. Choose a step size $\Delta r$, and set $r := p - \Delta r$.
2) Determine a multi-objective $H_\infty$ controller of order $r$, using the controller of order $r + \Delta r$. This is achieved by setting $p := r + \Delta r$ and $q := r$ in the LMIs (5), and solving optimization problem (7).
3) Update $r := \max\{r - \Delta r, q\}$.
4) Repeat steps 2 and 3 until $r = q$.

**IV. APPLICATION**

This section is devoted to the experimental validation of the LMI procedure proposed in Section III, on an overhead crane test setup. After a brief model description in Section IV-A, the controller design objectives and simulation results are discussed in Section IV-B. Finally, Section IV-D elaborates on the experimental results, validating the practical potential of the controller design approach.

**A. Model Description**

A schematic representation of the overhead crane is depicted in Figure 1. The crane setup consists a cart that is driven by a motor, and a load that is connected to the cart with a cable. Actuation of the motor causes a change in the horizontal cart position, resulting in a swinging load. The fixed cable length $L$ equals 0.45m, which yields a resonance frequency of 0.74Hz. The actuator is connected to an amplifier with a built-in velocity feedback controller, such that a reference velocity $u$ [m/s] is the input. The horizontal cart position $x_c$ [m] and the angle $\phi$ [rad] are each measured by an encoder. The horizontal load position $y$ [m] is taken as the system output, and is derived from the sensor measurements (assuming that the cable is stiff) as

$$y = x_c + L \sin(\phi).$$

Whenever the angle remains small, $\sin(\phi) \approx \phi$, such that the load position can accurately be approximated by

$$y \approx x_c + L\phi.$$

Using this approximation, assuming perfect velocity tracking and neglecting friction forces, the transfer function from reference velocity to load position is

$$\frac{Y(s)}{U(s)} = \frac{g}{s(Ls^2 + g)}.$$

where $g$ denotes the gravitational constant.

For the purpose of control, a more accurate 3rd order discrete-time model is identified using multisine excitation.

![Fig. 1. Schematic representation of the overhead crane.](image1)

![Fig. 2. Closed-loop system for $H_\infty$ control of the overhead crane.](image2)

To this end, five frequency response functions (FRFs) are measured and averaged. Subsequently, the following LTI model with sampling period 0.02s is fitted:

$$x(k+1) = \begin{bmatrix} 2.9910 & -2.9905 & 0.9995 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u(k),$$

$$y(k) \cdot 10^3 = \begin{bmatrix} 0.6571 & -1.272 & 0.7173 \end{bmatrix} x(k) + 3.408u(k).$$

**B. Controller Design**

The first step in the controller design procedure is to properly define the control objectives. In words, the imposed objectives are to move the load to a desired reference position $r$ as fast as possible, while avoiding excitation of the resonance frequency. Uncertainty in both the model and the measurements should be taken into account.

The closed-loop setup is displayed in Figure 2. $P$ denotes the identified LTI model, and $C(\phi)$ is a controller of order $q$ to be designed, such that the closed-loop dynamics are stable and some performance specifications from $r$ to $z_1$ and $z_2$ are satisfied. The transfer functions $W_S$ and $W_T$ are weights on the sensitivity function $S$, respectively the complementary sensitivity function $T$.

Using the relative uncertainty of the five measured FRFs with respect to the fitted FRF, a weighting function $W_T$ is derived that characterizes the required level of robustness for the crane model. Assuming that the model $P$ is subject to multiplicative uncertainty, and following the lines on pages 275-276 of [21], the closed-loop uncertain system is robustly stable if

$$\|W_T\|_{\infty} < 1,$$  \hspace{1cm} (8)

where $|W_T|$ is an upper bound for the maximum relative model uncertainty. Figure 3 shows both the maximum relative uncertainty and the selected upper bound $|W_T|$ as a
function of frequency. The resulting weighting function $W_T$ corresponds to a third order LTI model.

In addition, a first order sensitivity weight

$$W_S = \frac{s/U + 1}{s + L}$$  \hspace{1cm} (9)$$

is selected to optimize performance, with $L = -40\text{dB}$ and $U = 20\text{dB}$. Note that $|1/W_S| \approx L$ at low frequencies, while $|1/W_S| \approx U$ at high frequencies.

Accordingly, the generalized plant is of order $3 + 3 + 1 = 7$. For perfect reference tracking, $S(0) = 0$ is required. However, a low frequency dummy pole is included in $W_S$ to avoid numerical issues. In the synthesis step, this dummy pole appears in the controller, together with a zero at $s = 0$ due to integrating action in the plant. To clean up the controller, a pole-zero cancellation corresponding to the dummy pole and the zero at $s = 0$ is applied, resulting in an open-loop transfer function with integrating action. The controller resulting from full-order synthesis will thus be of order 6.

The control objective is to maximize closed-loop bandwidth, while satisfying (8). Let $\gamma_1$ and $\gamma_2$ denote upper bounds on $|W_S|_\infty$ respectively $|W_T|_\infty$. A full-order controller is computed by minimizing $\gamma_1$ subject to a set of LMIs (see [2]) and imposing the robustness bound $\gamma_2 < 1$. For reduced-order controller design, the problem boils down to the convex optimization problem

\begin{align}
\text{minimize} \quad & \gamma_1 \\
\text{subject to:} \quad & \text{LMI (5),} \quad j = 1, 2, \\
& \gamma_1 < b_1, \quad \gamma_2 < 1. \hspace{1cm} (10)
\end{align}

An optional bound $b_1$ is incorporated to bound the feasible set associated with the optimization problem. It is worth to remark that the multi-objective formulation with constraints $|W_S|_\infty < \gamma_1$ and $|W_T|_\infty < \gamma_2$ is always less conservative than the mixed-sensitivity formulation

$$\begin{bmatrix} W_S/\gamma_1 \\ W_T/\gamma_2 \end{bmatrix} < 1.$$

### C. Numerical Evaluation

Table I shows the results for full- and reduced-order multi-objective controller designs. For each controller order $q$ in the first column, the resulting closed-loop $\mathcal{H}_\infty$ norms and upper-bounds are given. The tuning parameter $b_1$ is also shown. In addition, the Bode magnitude plots of the complementary sensitivities and the controllers are shown in Figure 4 respectively Figure 5.

The results in Table I confirm that the sufficient LMI conditions yield stabilizing controllers that satisfy the robustness bound for all orders $0 \leq q \leq 6$. $\Delta r = 1$ is selected in the LMI procedure, hence the controller order is decreased successively. Although more conservative $\mathcal{H}_\infty$ upper bounds result for lower controller orders, the same closed-loop performance is achieved for $4 \leq q \leq 6$, confirming that a controller of order 4 is sufficient for this application. Note that the high order controllers invert the resonance frequency of the plant (see [22] and references therein), which is perfectly legitimate in this case since this frequency has been reliably identified. We observed that decreasing $b_1$ leads to lower closed-loop $\mathcal{H}_\infty$ norms. For controllers of order 3 and lower, the bandwidth decreases drastically. The intuition behind this is that the resonance frequency can no longer be sufficiently suppressed by the controller, see Figure 5. As depicted in Figure 4, this makes it hard to avoid violation of the robustness bound, since a peak will appear in the complementary sensitivity function. Roughly speaking, a decrease of the controller gain is necessary to satisfy the robustness bound, and this comes at the cost of a decrease in bandwidth. To obtain feasible solutions for $0 \leq q \leq 3$, it was necessary to manually decrease the gain of the controller that was used as a starting point in the optimization problem. Although not explicitly enforced, all controllers achieve a modulus margin larger than 0.60.

### D. Experimental Validation

This section discusses the experimental validation of the designed reduced-order controllers (see Section IV-B) on the overhead crane test setup. For all the controller orders $0 \leq q \leq 6$, the corresponding closed-loop experimental responses to a smoothed step are determined. This allows an intuitive comparison between the performance of different controllers, as well as a comparison between simulations and experimental results.
The closed-loop simulated and experimental responses to a smoothed step of 0.5m are depicted in Figure 6 for the controllers of order $4 \leq q \leq 6$, and show that the experimental responses are subject to more overshoot compared to the simulations. On the other hand, the experimental reference tracking is better than expected. It is worth remarking that the performance of the three controllers is similar, which confirms that controllers of order higher than four are unnecessarily complex for this application. This observation strongly sustains the value of the proposed controller design approach.

Considering the responses for the controller orders $0 \leq q \leq 3$ as shown in Figure 7, it is seen that the performance compared to higher order controllers is severely deteriorated. Despite this poor performance, which was to be expected, experiments reveal that the low order controllers are still stabilizing.

V. CONCLUSIONS

A novel reduced-order multi-objective $H_{\infty}$ controller design approach is presented, and experimentally validated on an overhead crane test setup. The experimental results confirm that, starting from a full-order controller, successive reduction of the controller order leads to promising results. Namely, despite the conservatism inherent in the LMI approach, reduced-order controllers with comparable performance to the full-order controller are obtained, and successfully validated experimentally.

The design of reduced-order linear parameter-varying (LPV) and robust controllers for LPV systems is future work. In addition, the extension of the reduced-order synthesis conditions for continuous-time LTI systems is planned.

VI. APPENDIX

Theorem 1 is proved now. First, notice that

$$ U' \hat{\Theta}^{(q)} = X_j U' \Theta^{(q)} , $$

since

$$ X_j U' = U' \begin{bmatrix} X_{11} & X_{13} \\ X_{31} & X_{33} \end{bmatrix} . $$

Therefore, using the abbreviations

$$ S_1 := \left( U' \Theta^{(q)} V - \Theta^{(p)} \right) C_y^{(p)} , \quad S_2 := \left( U' \Theta^{(q)} V - \Theta^{(p)} \right) D_{yj}^{(p)} \]
and substituting
\[ \tilde{\Theta}(q) := \begin{bmatrix} X_{11} & X_{13} \\ X_{31} & X_{33} \end{bmatrix} \Theta(q) \]
in the LMI (5) yields the equivalent condition
\[ \begin{bmatrix} P_j & P_j \tilde{\Theta}(p) & P_j \tilde{B}_j(p) \\ * & P_j & 0 \\ * & * & \gamma_j I \\ * & * & * & \gamma_j I \end{bmatrix}^{T} \begin{bmatrix} S_j X_j' \\ S_j X_j' \\ \tilde{D}_{ju}(p) \\ -X_j - X_j' \end{bmatrix} \succ 0. \tag{11} \]

Defining the augmented state-space realization
\[ \tilde{\Theta}(q)_{\text{aug}} = \begin{bmatrix} \theta(p) \\ 0 \end{bmatrix}, \quad \Theta(q)_{\text{aug}, j} = \begin{bmatrix} \Theta(q) \\ 0 \end{bmatrix}, \]
\[ \Theta(q)_{\text{aug}, j} = \begin{bmatrix} \Theta_j(q) \end{bmatrix}, \quad \Theta(q)_{\text{aug}, j} = \begin{bmatrix} \Theta_j(q) \end{bmatrix}, \]
and substituting \( q = p \) in (4), the following relation is obtained
\[ \begin{bmatrix} \Theta(q)_{\text{aug}, j} \\ \Theta(q)_{\text{aug}, j} \end{bmatrix} = \begin{bmatrix} \Theta(p) \\ \Theta_j(p) \end{bmatrix} + \begin{bmatrix} \tilde{B}_j(p) \\ \tilde{D}_{ju}(p) \end{bmatrix} \left( U \Theta(q) - \Theta(p) \right) \begin{bmatrix} C_j(p) \\ \tilde{D}_{sj}(p) \end{bmatrix}. \tag{12} \]

Now define the transformation matrix
\[ T := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & S_1 & S_2 & 0 \end{bmatrix}. \]

Using (12), multiplication of (11) on the left by \( T' \) and on the right by \( T \) gives
\[ \begin{bmatrix} P & \Theta(q)_{\text{aug}} & P \Theta(q)_{\text{aug}, j} & 0 \\ * & P & 0 & \Theta(q)_{\text{aug}, j} \\ * & * & \gamma_j I & \Theta(q)_{\text{aug}, j} \\ * & * & * & \gamma_j I \end{bmatrix} \succ 0. \tag{13} \]

Since \( \Theta(q)_{\text{aug}} \) is stable if and only if \( \Theta(q) \) is stable, and the input-output behaviour of the realization \( (\Theta(q)_{\text{aug}, j}, \Theta(q)_{\text{aug}, j}, \Theta(q)_{\text{aug}, j}) \) and the system (2) is identical, matrix inequality (13) holds if and only if \( \gamma_j \) is an upper bound for \( \|H_j(q)\|_\infty \). Feasibility of LMI (5) implies that
\[ \begin{bmatrix} X_{11} & X_{13} \\ X_{31} & X_{33} \end{bmatrix} + \begin{bmatrix} X_{11} & X_{13} \\ X_{31} & X_{33} \end{bmatrix}^{T} \prec 0, \]
hence the controller transformation (6) is well-defined. Consequently, the closed-loop system (2) is Schur stable and \( \gamma_j \) is a guaranteed upper bound on its \( \mathcal{H}_\infty \) norm whenever LMI (5) is feasible.

VII. ACKNOWLEDGMENT

The authors wish to thank Pieter Janssens from the Department of Mechanical Engineering, KU Leuven, for the helpful discussions and his assistance regarding the experimental validations.

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